Feedback Oscillatory Control of Roll Instability During Stall Using the LIBRA Mechanism

Mahmoud Abdelgalil* and Haithem Taha †
University of California Irvine , Irvine, CA, 92617

The goal of this paper is to design a stabilizing feedback controller of roll instability near stall. This problem becomes immensely challenging since the aileron sensitivity is vanishes and even reversed sign at stall. This challenge is overcome by employing the recently developed Lie Bracket Roll Augmentation (LIBRA) mechanism. In this mechanism,the nonlinear dynamics of the airplane near stall is exploited to achieve a rolling motion that is independent of the aileron sensitivity. Rather, it depends on the variation of the aileron sensitivity with the angle of attack which is non-zero at stall. The open loop characteristics of the LIBRA mechanism have been studied previously. The contribution of the current manuscript lies in using the LIBRA mechanism in a feedback fashion to stabilize the roll unstable dynamics near stall using a stabilization scheme based on motion planning techniques for highly oscillatory inputs.

I. Nomenclature

$C^{\omega}(\mathbb{R}^n)$	=	The set of analytic functions on $\mathbb{R}^n$
$\Gamma(\mathbb{R}^n)$	=	The set of analytic vector fields on $\mathbb{R}^n$
$T_{\boldsymbol{x}}\mathbb{R}^n$	=	The tangent space of $\mathbb{R}^n$ at the point $\boldsymbol{x}$
$\mathbb{R}^n$	=	The Euclidean space of dimension n
$\mathbb{N}$	=	The set of natural numbers
$\lvert\mathrm{I}\rvert$	=	The number of elements in the set I
$\lvert x\rvert$	=	The absolute value of the real number x

II. Introduction

Geometric control theory flourished during the 70s, 80s and 90s. The potential of the theory to generalize the results from Linear Control theory to systems in the Control-Affine form generated a lot of interest [1]. A variety of fundamental questions about the structure of nonlinear control systems, Controllability and Observability, Feedback Linearization, and other questions were framed, and satisfactorily answered in many cases [2][3][4][5]. Moreover, constructive techniques for Controller Design, Motion Planning and Stabilization were developed. The setting of Geometric Control theory is natural for nonlinear systems, especially for the control of mechanical systems [6]. Despite the fact that many fundamental issues remain [7], interest in geometric control theory has been declining since the beginning of the new century. This unfortunate decline can be attributed to the difficulty of the subject and the tendency to dismiss the theory as inapplicable for real-world problems. However, the second author demonstrated in a series of papers that geometric control tools can be used to tackle practical problems [8–12]. It was shown that tools from geometric control offer a potential roll stability recovery mechanism from the Loss of Controllability during stall and high angle of attack maneuvers [13–16]. The discovery of this Lie Bracket Roll Augmentation mechanism, dubbed **LIBRA**, sparked a strong interest in understanding the physics behind it [17], and in experimentally quantifying its effectiveness [18]. Moreover, it raised the important question of how to design controllers that utilize this mechanism. It is the purpose of this work to answer this question. In this manuscript, we introduce a control scheme based on geometric control techniques that exploits nonlinearities in a control-affine system to provide feedback authority in unactuated directions. We demonstrate the effectiveness of the proposed scheme on two classic examples in the geometric control literature in addition to a minimal model representing the LIBRA mechanism for an airplane model inside a windtunnel that has been introduced recently in [16].

*PhD Candidate, Mechanical and Aerospace Engineering, AIAA Member
†Associate Professor, Mechanical and Aerospace Engineering, AIAA Member

The remainder of this manuscript is organized as follows. The preliminary definitions and a quick review of the problem setting are provided in section III. We refer the reader to [19, 20] for a more complete treatment. The main scheme is outlined in section IV. The application of the proposed controller scheme to examples is provided in section V. Finally, the conclusion and future work are discussed in section VI.

III. Preliminaries

We begin the discussion by identifying the class of control systems of primary interest. This class is the class of control-affine systems. Let $\boldsymbol{x} \in \mathbb{R}^n$ be the state of the system. The evolution equation of a control-affine system is then a differential equation of the form:

$$\dot{\boldsymbol{x}} = \boldsymbol{f}_0 + \sum_{i=1}^{m} \boldsymbol{f}_i \, u_i, \tag{1}$$

where $\boldsymbol{f}_0 \in \Gamma(\mathbb{R}^n)$ is the drift vector field, $\boldsymbol{f}_i \in \Gamma(\mathbb{R}^n),\ i \in \{1, \ldots, m\}$ are the control vector fields, and $u_i \in \mathbb{R},\ \ \in \{1, \ldots, m\}$ are the corresponding control inputs. If $\boldsymbol{f}_0 = 0$, the system is called *driftless*. Denote the family of control vector fields by $\mathcal{F} = \{\boldsymbol{f}_i\}$. Associated to such a family is the Lie Algebra generated by its elements, which is denoted by $Lie^{\infty}(\mathcal{F})$, and obtained through repeated application of the *Lie Bracket* operation between different vector field in the family. The Lie bracket of two vector fields $\boldsymbol{f}_i, \boldsymbol{f}_j$ is denoted by $[\boldsymbol{f}_i, \boldsymbol{f}_j]$, and can be computed in coordinates as follows:

$$[\boldsymbol{f}_i, \boldsymbol{f}_j] = \frac{\partial \boldsymbol{f}_j}{\partial \boldsymbol{x}} \, \boldsymbol{f}_i - \frac{\partial \boldsymbol{f}_i}{\partial \boldsymbol{x}} \, \boldsymbol{f}_j$$

where $\frac{\partial \boldsymbol{f}_j}{\partial \boldsymbol{x}}$ represents the jacobian matrix of the map that defines the vector field $\boldsymbol{f}_j$ in the $\boldsymbol{x}$-coordinates. Another common notation for the Lie bracket is

$$\mathrm{ad}^1_{\boldsymbol{f}_i}(\boldsymbol{f}_j) = [\boldsymbol{f}_i, \boldsymbol{f}_j]$$

This notation is more compact when iterated Lie brackets are involved. Consider for instance applying the Lie bracket of $\boldsymbol{f}_i$ with $\boldsymbol{f}_j$ twice. This can be expressed as:

$$\mathrm{ad}^2_{\boldsymbol{f}_i}(\boldsymbol{f}_j) = \left[\boldsymbol{f}_i, \mathrm{ad}^1_{\boldsymbol{f}_i}(\boldsymbol{f}_j)\right] = \left[\boldsymbol{f}_i, [\boldsymbol{f}_i, \boldsymbol{f}_j]\right]$$

This also generalizes in a straightforward way to iterated Lie brackets of higher orders. In this notation, the Lie algebra $Lie^{\infty}(\mathcal{F})$ can be represented as:

$$Lie^{\infty}(\mathcal{F}) = \mathrm{span}_{C^{\omega}(\mathbb{R}^n)}\{\boldsymbol{f}_i, \mathrm{ad}^1_{\boldsymbol{f}_j}(\boldsymbol{f}_i), \ldots, \mathrm{ad}^k_{\boldsymbol{f}_j}(\boldsymbol{f}_i), \ldots\}$$

which is essentially all linear combinations, with analytic coefficients, of the control vector fields and their Lie brackets of arbitrary order. We can also define a sequence of *submodules* of $Lie^{\infty}(\mathcal{F})$:

$$Lie^{k}(\mathcal{F}) = \mathrm{span}_{C^{\omega}(\mathbb{R}^n)}\{\boldsymbol{f}_i, \mathrm{ad}^1_{\boldsymbol{f}_j}(\boldsymbol{f}_i), \ldots, \mathrm{ad}^k_{\boldsymbol{f}_j}(\boldsymbol{f}_i)\}$$

Note that $Lie^{k}(\mathcal{F})$ consists of linear combinations, with analytic coefficients, of the control vector fields and their Lie brackets with order **up to** k. Associated with each of these submodules is a *distribution*, which is essentially a smooth assignment of subspaces of the tangent space $T_{\boldsymbol{x}}\mathbb{R}^n$ at each point $\boldsymbol{x}$. The distribution can be represented as:

$$Lie^{k}(\mathcal{F})(\boldsymbol{x}) = \mathrm{span}_{\mathbb{R}}\{\boldsymbol{f}_i(\boldsymbol{x}), \mathrm{ad}^1_{\boldsymbol{f}_j}(\boldsymbol{f}_i)(\boldsymbol{x}), \ldots, \mathrm{ad}^k_{\boldsymbol{f}_j}(\boldsymbol{f}_i)(\boldsymbol{x})\} \subset T_{\boldsymbol{x}}\mathbb{R}^n$$

An important property of such a family of vector fields is encountered when the distribution $Lie^{k}(\mathcal{F})(\boldsymbol{x})$ is equal to the tangent space $T_{\boldsymbol{x}}\mathbb{R}^n$ at every point $\boldsymbol{x} \in \mathbb{R}^n$, for some finite order $k \in \mathbb{N}$. When the family of control vector fields of a control affine system satisfies this condition, the system is said to satisfy the ***Strong Lie Algebraic Rank Condition***, or **Strong LARC** for short. If a control system on $\mathbb{R}^n$ satisfies the **Strong LARC** condition, it is well-known that the system is *completely controllable*, i.e. the system can be steered from $\boldsymbol{x}_1$ to $\boldsymbol{x}_2$ in a finite time, for any two points $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathbb{R}^n$. Moreover, it is also well-known that if a system satisfies the **Strong LARC**, then any smooth curve in $\mathbb{R}^n$ can be uniformly approximated by controlled trajectories of the system. In particular, this result can be used to solve the Motion Planning Problem for a control affine system, as outlined in [19]. Suppose that a control affine system given by a drift vector field $\boldsymbol{f}_0$, and a family of control vector fields $\mathcal{F}$ such that it satisfies the **Strong LARC** with order k, i.e. the distribution $Lie^k(\mathcal{F})(\boldsymbol{x}) = T_{\boldsymbol{x}}\mathbb{R}^n,\ \forall \boldsymbol{x} \in \mathbb{R}^n$. Let $\tilde{\boldsymbol{f}}_i,\ i \in \{m+1, \ldots, r\}$ be a collection of iterated Lie brackets of the family $\mathcal{F}$ with an order less than or equal to k. Consider the *extended system* defined by:

$$\dot{\boldsymbol{x}} = \boldsymbol{f}_0 + \sum_{i=1}^{m} \boldsymbol{f}_i v^i + \sum_{i=m+1}^{r} \tilde{\boldsymbol{f}}_i v^i$$

If $r \geq n$, and the set of vector fields $\{\boldsymbol{f}_1, \ldots, \boldsymbol{f}_m, \tilde{\boldsymbol{f}}_{m+1}, \ldots, \tilde{\boldsymbol{f}}_r\}$ spans the whole tangent space at every point in $\mathbb{R}^n$, then any smooth curve in $\mathbb{R}^n$ can be made into a trajectory of the extended system. Specifically, let $r = n$ and let $\gamma : [a, b] \ni t \mapsto \gamma(t) \in \mathbb{R}^n$ be a smooth curve. Consider the extended inputs defined by

$$\boldsymbol{v}(t) = (v^1(t), \ldots, v^n(t)) = \boldsymbol{F}(\gamma(t))^{-1}\big(-\boldsymbol{f}_0(\gamma(t)) + \dot{\gamma}(t)\big)$$

where the matrix-valued map $\boldsymbol{F} : \mathbb{R}^n \ni \boldsymbol{x} \mapsto \boldsymbol{F}(x) \in \mathbb{R}^{n\times n}$ is the map that assigns to each point $\boldsymbol{x} \in \mathbb{R}^n$ the matrix whose columns are given by the values of the control vector fields and the chosen Lie brackets at that point:

$$\boldsymbol{F}(\boldsymbol{x}) = \Big[\boldsymbol{f}_1(\boldsymbol{x}) \;\Big|\; \cdots \;\Big|\; \boldsymbol{f}_m(\boldsymbol{x}) \;\Big|\; \tilde{\boldsymbol{f}}_{m+1}(\boldsymbol{x}) \;\Big|\; \cdots \;\Big|\; \tilde{\boldsymbol{f}}_n(\boldsymbol{x})\Big] \tag{2}$$

Since the vector fields are linearly independent at every point, we see that the control inputs $\boldsymbol{v}$ are well defined. Moreover, we can see that by direct substitution, the curve γ automatically satisfies the differential equation governing the control system when the inputs are given by Eq.(2). In [19], the author outlines an algorithm that produces a sequence of time varying high frequency high amplitude oscillatory *ordinary* inputs, i.e. inputs to the original system (not the extended system), provided that the extended inputs have been determined, such that the sequence of trajectories of the original system produced by the sequence of inputs uniformly converges to the trajectory of the extended system under the given extended input on compact time intervals. A detailed explanation of the algorithm is outlined in [19], and a summary of the algorithm along with examples and a framework to solve the motion planning problem was outlined in our earlier paper [20], to which we refer the reader. To avoid replicating the material in [19, 20], we will refer to the algorithm from here on as *Liu*, and define it as a mapping that converts extended inputs to ordinary inputs:

$$Liu : \mathbb{R}^n \times \mathrm{I} \times \mathbb{N} \ni (\boldsymbol{v}, t, j) \mapsto \boldsymbol{u} \in \mathbb{R}^m$$

where I is the interval on which the extended input is defined, and j is the frequency of oscillation (i.e., a parameter needed by the algorithm).

An important problem in control theory is the design of feedback control laws that stabilize an equilibrium point of the system. More concretely, let $\boldsymbol{x}_0 \in \mathbb{R}^n$ be such that the drift vector field vanishes at the point $\boldsymbol{f}_0(\boldsymbol{x}_0) = 0$. A *static* asymptotically stabilizing feedback law is a map

$$\boldsymbol{u} : \mathbb{R}^n \ni \boldsymbol{x} = (x^1, \ldots, x^n) \mapsto (u^1, \ldots, u^m) = \boldsymbol{u} \in \mathbb{R}^m$$

such that the closed loop system given by

$$\dot{x} = \boldsymbol{f}_0(\boldsymbol{x}) + \sum_{i=1}^{m} \boldsymbol{f}_i(\boldsymbol{x}) u^i(\boldsymbol{x})$$

has the point $\boldsymbol{x}_0$ as an *asymptotically stable* equilibrium point. If the system is driftless, then any point $\boldsymbol{x}_0 \in \mathbb{R}^n$ is an equilibrium point of the control affine system. In such a case, it is not always possible to design even a merely continuous static feedback control law such that the closed loop system is asymptotically stable at a given point unless the impractical, demanding condition $m = n$ is met. This well-known fact is due to Brockett [21] who formulated a necessary condition for the existence of a static continuous feedback law that asymptotically stabilizes a control system around an equilibrium point. However, it may still be possible to design time-varying control laws that asymptotically stabilize the system. In fact, many efforts have been made by several authors to design time-varying stabilizing feedback control laws [22][23][24][25][26]. Most of the results focused on driftless systems, and the proposed laws vary in the complexity of the produced trajectories and the difficulty of the design process. In this work, we outline a design technique based on the motion planning algorithm in [19] to stabilize a control-affine system with drift.

IV. Control Scheme

We assume that a control affine system in the canonical form:

$$\dot{\boldsymbol{x}} = \boldsymbol{f}_0 + \sum_{i=1}^{m} \boldsymbol{f}_i \; u^i \tag{3}$$

Let the control system satisfy the **Strong LARC** condition with iterated Lie brackets of order no more than k. Let $\{\tilde{\boldsymbol{f}}_{m+1}, \ldots, \tilde{\boldsymbol{f}}_n\}$ be the additional Lie brackets that are needed in order to satisfy the **Strong LARC**. Then, associated to the ordinary system (3) is the extended system given by:

$$\dot{\boldsymbol{x}} = \boldsymbol{f}_0 + \sum_{i=1}^{m} \boldsymbol{f}_i \; v^i + \sum_{i=m+1}^{n} \tilde{\boldsymbol{f}}_i v^i \tag{4}$$

Suppose that $\boldsymbol{f}_0(\boldsymbol{0}) = \boldsymbol{0}$, and that we would like to design a control law that asymptotically stabilizes the origin $\boldsymbol{0} \in \mathbb{R}^n$.

A. Step I:

The first step in the framework is to design a control law that asymptotically stabilizes the origin for the extended system. One such control law is given by the feedback law:

$$\boldsymbol{v}(\boldsymbol{x}) = -\boldsymbol{F}(\boldsymbol{x})^{-1}(\boldsymbol{f}_0(\boldsymbol{x}) + \lambda \boldsymbol{x}) \tag{5}$$

where $\lambda > 0$. By direct substitution, we see that this control law produces the closed loop extended system:

$$\dot{\boldsymbol{x}} = \boldsymbol{f}_0(\boldsymbol{x}) + \boldsymbol{F}(\boldsymbol{x})\boldsymbol{v}(\boldsymbol{x}) = \boldsymbol{f}_0(\boldsymbol{x}) - \boldsymbol{F}(\boldsymbol{x})\boldsymbol{F}(\boldsymbol{x})^{-1}(\boldsymbol{f}_0(\boldsymbol{x}) + \lambda \boldsymbol{x}) = -\lambda \boldsymbol{x}$$

for which the origin is clearly exponentially stable.

B. Step II:

The second step in the framework is to produce ordinary input feedback laws that "track" the closed loop trajectory of the extended system. The infinite time horizon $[0, \infty)$ is divided into intervals of equal length $T > 0$, and the problem of stabilization reduces to a sequence of independent trajectory tracking problems on the intervals. Feedback is made at the beginning of each interval where the current state of the system is used as the initial condition for the closed loop extended system over the interval. The trajectory of the closed loop extended system on each interval starting from the initial condition supplied by feedback is then used to generate the open loop extended inputs that give rise to this trajectory. Finally, these extended inputs are passed on to the map *Liu* to produce open loop ordinary inputs to be applied to the ordinary system over the duration of the interval. This scheme can be summarized in the following algorithm:

Algorithm 1: Stabilization Scheme

```
k = 0;
while true do
    Set I_k = [k T, (k + 1) T];
    Set x_k = x(kT);
    while t ∈ I_k do
        Set γ_k(t) = φ^t(x_k);
        Set v_k(t) = α(γ_k(t), t);
        Set u_k(t) = Liu(v_k(t), t, j);
        Apply the open loop control u_k(t) to the system
    end
    k = k+1;
end
```

In the above algorithm, the map $\phi^t : \mathbb{R}^n \to \mathbb{R}^n$ represents the flow along the vector field of the closed loop extended system under the feedback law $\alpha : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$. If we choose α to be given by the feedback law (5), then the flow ϕ^t is simply given by:

$$\phi^t(\boldsymbol{x}) = e^{-\lambda \mathbf{I} \; t} \cdot \boldsymbol{x}$$

where $\mathbf{I}$ is the $n \times n$ identity matrix. It can be shown that under this sampled feedback law, the origin can be made into a *practically* asymptotically stable point for the ordinary system. We will provide a proof of this result in the appendix.

V. Examples

In this section, we apply the proposed stabilization scheme on two examples. We begin by a well-known example from the geometric control literature:

A. Example 1:

The first example is the well-known Brockett integrator, for which we have the two control vector fields and their Lie bracket given by:

$$\begin{aligned} \boldsymbol{f}_1(\boldsymbol{x}) &= (1, 0, -x^2) \\ \boldsymbol{f}_2(\boldsymbol{x}) &= (0, 1, x^1) \\ [\boldsymbol{f}_1, \boldsymbol{f}_2](\boldsymbol{x}) &= (0, 0, 2) \end{aligned}$$

where $\boldsymbol{x} = (x^1, x^2, x^3) \in \mathbb{R}^3$. We also assume a drift vector field $\boldsymbol{f}_0(\boldsymbol{x}) = \boldsymbol{x}$, and the system is given in control affine form by:

$$\dot{x} = \boldsymbol{f} = \boldsymbol{f}_0 + \sum_{i=1}^{2} \boldsymbol{f}_i u^i$$

Note that the origin is an equilibrium point for the system, and that the drift vector field is destabilizing. The linearization of the system around the origin is given by the matrices:

$$\mathrm{A} = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\Big|_{\boldsymbol{x}=0, \boldsymbol{u}=0} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathrm{B} = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}}\Big|_{\boldsymbol{x}=0, \boldsymbol{u}=0} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

To investigate controllability of the linearization, we form the controllability matrix:

$$C = \left[\mathrm{B} \,\middle|\, \mathrm{AB} \,\middle|\, \mathrm{A}^2\mathrm{B}\right] = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Clearly, the linearization is not controllable since $\mathrm{rank}(C) = 2$. Moreover, the linearization is not even *stabilizable* since the third mode of the linearization is unstable and independent of the other states and the control inputs. Hence, linear control tools completely fail to tackle this problem. However, the Brockett integrator satisfies the **Strong LARC** condition since:

$$\mathrm{span}_{\mathbb{R}}\{\boldsymbol{f}_1(\boldsymbol{x}), \boldsymbol{f}_2(\boldsymbol{x}), [\boldsymbol{f}_1, \boldsymbol{f}_2](\boldsymbol{x})\} = T_{\boldsymbol{x}}\mathbb{R}^3, \ \forall \boldsymbol{x} \in \mathbb{R}^3$$

which means that the system is, in fact, completely controllable. Suppose that we would like to stabilize the origin for the Brockett integrator. We see that the matrix $\boldsymbol{F}(\boldsymbol{x})$ and its inverse will be given by:

$$\boldsymbol{F}(\boldsymbol{x}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -x^2 & x^1 & 2 \end{pmatrix}, \quad \boldsymbol{F}(\boldsymbol{x})^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{2}x^2 & -\frac{1}{2}x^1 & \frac{1}{2} \end{pmatrix}$$

Let $\lambda = 1$. The feedback law that stabilizes the origin for the extended system can now be readily computed as

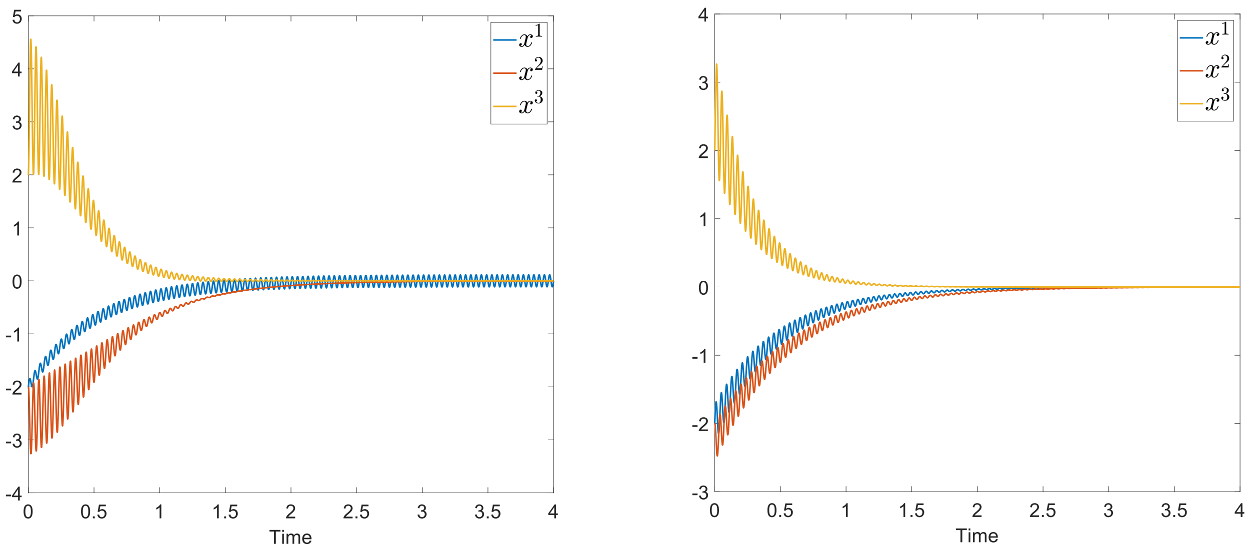
$$\alpha(\boldsymbol{x}) = -\boldsymbol{F}(\boldsymbol{x})^{-1}(\boldsymbol{f}_0(\boldsymbol{x}) + \boldsymbol{x}) = (-2x^1, -2x^2, -x^3) \tag{6}$$

For the Brockett integrator, the function *Liu* defined by the algorithm in [19] can be shown to be:

$$\boldsymbol{u} = Liu(\boldsymbol{v}, t, j) = \left(v^1 + \eta^1(v^3)\sqrt{2j}\cos(jt), v^2 + \eta^2(v^3)\sqrt{2j}\sin(jt)\right) \tag{7}$$

where $\eta^1, \eta^2 : \mathbb{R} \to \mathbb{R}$ can be any absolutely continuous functions such that

$$\eta^1(x)\eta^2(x) = x$$



(a) $\eta^1(x) = 1,\ \eta^2(x) = x$

(b) $\eta^1(x) = \sqrt{|x|},\ \eta^2(x) = \sqrt{|x|}\ \mathbf{sgn}(x)$

Fig. 1 Simulation Results for Example 1

One possible choice for these functions is $\eta^1(x) = 1,\ \eta^2(x) = x$. Another possible choice is $\eta^1(x) = \sqrt{|x|},\ \eta^2(x) = \sqrt{|x|}\ \mathrm{sgn}(x)$, where $\mathrm{sgn}(x)$ is the sign function. Both of these choices are absolutely continuous functions, and so convergence of trajectories is guaranteed by the results in [19]. However, the trajectories produced by the two are qualitatively different, especially near the origin. Specifically, the latter choice produces trajectories that seem to converge to the origin asymptotically, whereas the trajectories of the former converge to a region around the origin but keep oscillating indefinitely. Simulation results for this example are shown in Fig.(1). We used a initial condition $\boldsymbol{x}_0 = (-2, -2, 2)$, interval length $T = 0.04$ and frequency $j = 50\pi$.

B. Example 2:

The second example has a state $\boldsymbol{x} \in \mathbb{R}^5$, and two control vector fields with their Lie brackets given by:

$$\begin{aligned}
\boldsymbol{f}_1(\boldsymbol{x}) &= (1, 0, -x^2, 0, -(x^2)^2) \\
\boldsymbol{f}_2(\boldsymbol{x}) &= (1, 0, -x^1, 0, -(x^1)^2) \\
[\boldsymbol{f}_1, \boldsymbol{f}_2](\boldsymbol{x}) &= (0, 0, 2, 0, 2(x^1 + x^2)) \\
[\boldsymbol{f}_1, [\boldsymbol{f}_1, \boldsymbol{f}_2]](\boldsymbol{x}) &= (0, 0, 0, 2, 0) \\
[\boldsymbol{f}_2, [\boldsymbol{f}_1, \boldsymbol{f}_2]](\boldsymbol{x}) &= (0, 0, 0, 0, 2)
\end{aligned}$$

Let the drift vector field be $\boldsymbol{f}_0(\boldsymbol{x}) = \boldsymbol{x}$. The linearization of the system is also not controllable nor stabilizable, similar to Example 1, due to the presence of three uncontrollable and unstable modes in the linearization. However, the system satisfies the **Strong LARC** condition:

$$\mathrm{span}_{\mathbb{R}}\{\boldsymbol{f}_1(\boldsymbol{x}), \boldsymbol{f}_2(\boldsymbol{x}), [\boldsymbol{f}_1, \boldsymbol{f}_2](\boldsymbol{x}), [\boldsymbol{f}_1, [\boldsymbol{f}_1, \boldsymbol{f}_2]](\boldsymbol{x}), [\boldsymbol{f}_2, [\boldsymbol{f}_1, \boldsymbol{f}_2]](\boldsymbol{x})\} = T_{\boldsymbol{x}}\mathbb{R}^5,\ \forall \boldsymbol{x} \in \mathbb{R}^5$$

Suppose that we would like to stabilize the origin. The matrix $\boldsymbol{F}(\boldsymbol{x})$ for this system and its inverse take the form:

$$\boldsymbol{F}(\boldsymbol{x}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -x^2 & x^1 & 2 & 0 & 0 \\ 0 & (x^1)^2 & 2x^1 & 2 & 0 \\ -(x^2)^2 & 0 & 2x^2 & 0 & 2 \end{pmatrix}, \quad \boldsymbol{F}(\boldsymbol{x})^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \frac{1}{2}x^2 & \frac{1}{2}x^1 & \frac{1}{2} & 0 & 0 \\ -\frac{1}{2}x^1x^2 & 0 & -\frac{1}{2}x^1 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2}x^1x^2 & -\frac{1}{2}x^2 & 0 & \frac{1}{2} \end{pmatrix}$$

Let $\lambda = 5$. The feedback law for the extended system can be taken as in Eq.(5). The algorithm in [19] can be shown to produce the map:

$$\begin{aligned}
\boldsymbol{u} &= (u^1, u^2) = Liu(\boldsymbol{v}, t, j) \\
u^1 &= v^1 + \sqrt{2\omega_1 j}\cos(jt)\eta_1(v^3) - \sqrt[3]{8\omega_5\omega_6 j^2}\cos(\omega_8 jt)\eta_8(v^5) \\
&\quad + \sqrt[3]{8\omega_3\omega_4 j^2}\big(\cos(\omega_3 jt)\eta_3(v^4) + \cos(\omega_4 jt)\eta_4(v^4)\big) \\
u^2 &= v^2 + \sqrt{2\omega_1 j}\sin(jt)\eta_2(v^3) - \sqrt[3]{8\omega_3\omega_4 j^2}\cos(\omega_5 jt)\eta_5(v^4) \\
&\quad + \sqrt[3]{8\omega_6\omega_7 j^2}\big(\cos(\omega_6 jt)\eta_6(v^5) + \cos(\omega_7 jt)\eta_7(v^5)\big)
\end{aligned}$$

where the functions η_i are absolutely continuous functions that satisfy:

$$\eta_1(x)\eta_2(x) = x, \quad \eta_3(x)\eta_4(x)\eta_5(x) = x, \quad \eta_6(x)\eta_7(x)\eta_8(x) = x$$

and the constants ω_i are given by:

$$\omega_1 = 2,\ \omega_3 = 1,\ \omega_4 = 3,\ \omega_5 = 4,\ \omega_6 = 5,\ \omega_7 = 7,\ \omega_8 = 12$$

Once again, the choice for the functions η_i determines the qualitative behavior of the trajectories as they approach the origin. The first choice, referred to as choice A, is:

$$\begin{array}{llll}
\eta_1(x) = 1 & \eta_2(x) = x & \eta_3(x) = 1 & \eta_4(x) = 1 \\
\eta_5(x) = x & \eta_6(x) = 1 & \eta_7(x) = 1 & \eta_8(x) = x
\end{array}$$

The second choice, referred to as choice B, is given by the functions:

$$\begin{array}{llll}
\eta_1(x) = \sqrt{|x|} & \eta_2(x) = \sqrt{|x|}\operatorname{sgn}(x) & \eta_3(x) = \sqrt[3]{|x|} & \eta_4(x) = \sqrt[3]{|x|} \\
\eta_5(x) = \sqrt[3]{|x|}\operatorname{sgn}(x) & \eta_6(x) = \sqrt[3]{|x|} & \eta_7(x) = \sqrt[3]{|x|} & \eta_8(x) = \sqrt[3]{|x|}\operatorname{sgn}(x)
\end{array}$$

The simulation results are shown in Fig.(2). We used a initial condition $\boldsymbol{x}_0 = (1, 1, 1, 1, 1)$, interval length $T = 0.05$ and frequency $j = 40\pi$. We clearly see the advantage of choice B, which shows the asymptotic convergence of the trajectories to the origin, whereas choice A suffers from persistent oscillations.

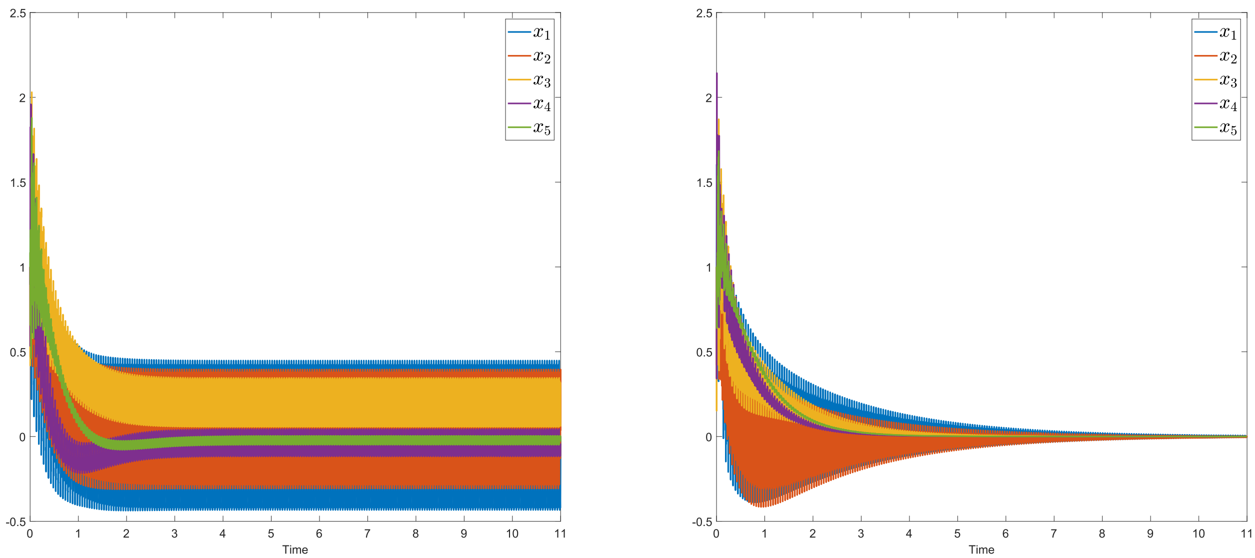
Choice B turns out to be always possible no matter the order of the Lie brackets involved in the extended system. The control laws produced by the algorithm in [19] will contain $|V|$ sets of functions $\big\{\{\eta_i^v\}_{i\in\mathrm{I}}\big\}_{v\in V}$, each set being associated to one of the extended inputs $v \in V$. All of the functions need to be absolutely continuous, and they must satisfy the condition that

$$\prod_{i\in\mathrm{I}} \eta_i^v(x) = x,\ \forall v \in V$$

It is now clear that we can always pick all the functions associated to an extended input v to be $\eta_i^v(x) = \sqrt[|\mathrm{I}|]{|x|}$, except for one function which we pick to be $\eta_i^v(x) = \sqrt[|\mathrm{I}|]{|x|}\operatorname{sgn}(x)$. This choice satisfies the requirement on the functions, and so the convergence properties of the trajectories are not affected. However, the qualitative behavior as the system approaches the origin will differ. The functions η_i^v appear as the amplitude of a high frequency high amplitude oscillations in the ordinary input. When these functions do not vanish at the origin, amplitude of the oscillations remain finite and so the oscillations in the input will persist. However, if the functions are made to vanish at the origin, then the amplitude of the oscillations will decrease asymptotically, and the system will not suffer from the persistent oscillations.

C. Example 3

In the third example, we consider the three DOF model illustrating the LIBRA mechanism which was introduced in the recent article [16]. The model represents an airplane in a wind-tunnel with three degrees of freedom: pitching, rolling, and heaving. Near stall, the sensitivity to aileron deflections becomes zero and even changes sign. This notorious



(a) Choice A **(b) Choice B**

Fig. 2 Simulation Results for Example 2

situation leads to the failure of linear techniques in recovering from stall. The situation is exacerbated by the fact the dynamics of rolling motion is inherently unstable. The model in [16] can be put on the form:

$$\dot{z} = W \tag{8}$$

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 1 & \tan(\theta)\sin(\phi) \\ 0 & \cos(\phi) \end{bmatrix} \begin{bmatrix} P \\ Q \end{bmatrix} \tag{9}$$

$$\dot{W} = g_0 + g_1 \delta_e \tag{10}$$

$$\begin{bmatrix} \dot{P} \\ \dot{Q} \end{bmatrix} = \boldsymbol{f}_0 + \boldsymbol{f}_1\, \delta_e + \boldsymbol{f}_2\, \delta_a + [\boldsymbol{f}_1, \boldsymbol{f}_2]\, \delta_{e,a} \tag{11}$$

where the vector fields $\boldsymbol{f}_i$ can be found in [16]. It is shown in [15–17] that the Lie Bracket vector field $[\boldsymbol{f}_1, \boldsymbol{f}_2]$ recovers controllability near stall; it provides rolling control authority even when the vector field $\boldsymbol{f}_2 = 0$. Therefore, the extended system given by the equations:

$$\dot{z} = W \tag{12}$$

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 1 & \tan(\theta)\sin(\phi) \\ 0 & \cos(\phi) \end{bmatrix} \begin{bmatrix} P \\ Q \end{bmatrix} \tag{13}$$

$$\dot{W} = g_0 + g_1 \delta_e \tag{14}$$

$$\begin{bmatrix} \dot{P} \\ \dot{Q} \end{bmatrix} = \boldsymbol{f}_0 + \boldsymbol{f}_1\, \delta_e + \boldsymbol{f}_2\, \delta_a + [\boldsymbol{f}_1, \boldsymbol{f}_2]\, \delta_{e,a} \tag{15}$$

which does not suffer from the loss of controllability problem near stall. Consequently, we may apply the techniques introduced in the previous subsection to design a controller that provides control authority near-stall. We begin by designing a control law for the extended system. To that end, we design a controller for the extended system using a backstepping approach. First, we design the desired rates P_d and Q_d to track a step command ϕ_d and θ_d:

$$\boldsymbol{u_x} = \begin{bmatrix} P_d \\ Q_d \end{bmatrix} = \begin{bmatrix} 1 & \tan(\theta)\sin(\phi) \\ 0 & \cos(\phi) \end{bmatrix}^{-1} \begin{bmatrix} \phi_d - \phi \\ \theta_d - \theta \end{bmatrix} \tag{16}$$

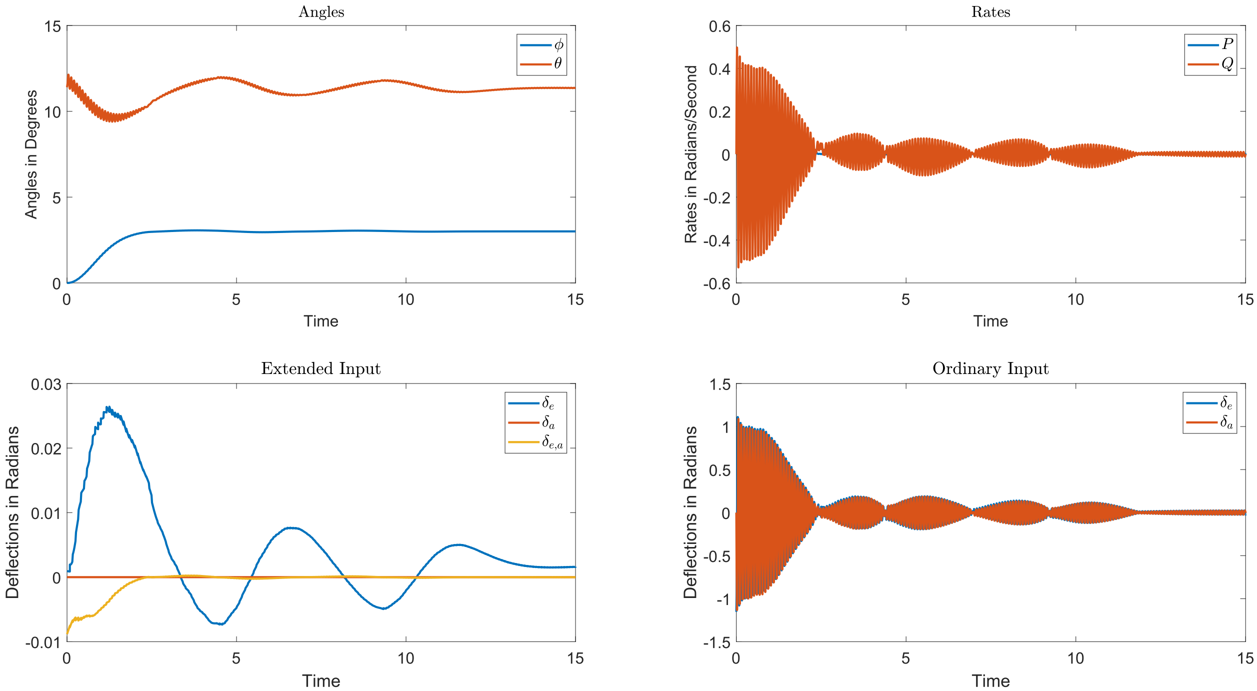


Fig. 3 Simulation Results for Example 3

We observe that:

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 1 & \tan(\theta)\sin(\phi) \\ 0 & \cos(\phi) \end{bmatrix} \boldsymbol{u_x} = \begin{bmatrix} \phi_d - \phi \\ \theta_d - \theta \end{bmatrix} \tag{17}$$

which is an exponentially stable system with the Lyapunov function:

$$V_{\boldsymbol{x}} = \frac{1}{2}(\phi - \phi_d)^2 + \frac{1}{2}(\theta - \theta_d)^2 \tag{18}$$

Applying the back stepping technique, we obtain the following control law for the extended system:

$$\begin{bmatrix} \delta_e \\ \delta_a \\ \delta_{e,a} \end{bmatrix} = \left[\boldsymbol{f}_1,\, \boldsymbol{f}_2,\, [\boldsymbol{f}_1, \boldsymbol{f}_2]\right]^{\dagger} \left(\dot{\boldsymbol{u}}_{\boldsymbol{x}} - \begin{bmatrix} 1 & \tan(\theta)\sin(\phi) \\ 0 & \cos(\phi) \end{bmatrix}^{\intercal} \nabla V_{\boldsymbol{x}} - \boldsymbol{f}_0 - \boldsymbol{e} \right) \tag{19}$$

where $\boldsymbol{e} = \left[P,\, Q\right]^{\intercal} - \boldsymbol{u_x}$, and † denotes the Moore-Penrose inverse. Observe that, for the sake of demonstration, we are effectively neglecting the dynamics of heaving $\dot{W}$ in the design and are only interested in regulating the angles θ and ϕ. The above control law stabilizes the extended system around a step input defined by θ_d and ϕ_d. Using the techniques introduced in the previous subsection, we design the ordinary inputs of the original system to perform a rolling maneuver near stall for the original system. The model is required to track a step input roll angle $\phi_d = 3°$ and maintain $\theta_d = 11.4°$ at stall. The initial conditions are taken to be $\alpha = \theta = 11.4°$, $\phi = W = P = Q = 0$. The simulation results are shown in Fig.3. We observe how the system is able to track the step input despite the fact that there is a complete loss of aileron sensitivity at this initial condition. This fact can be observed in the extended inputs time-history where only the elevator input δ_e and the extended input $\delta_{e,a}$ are non-zero, whereas the aileron deflection input δ_a is zero.

VI. Conclusion

In this manuscript we utilized trajectory tracking technique to design a time-varying control scheme that provides practical stability for control-affine systems with drift when the **Strong LARC** is satisfied. We provided several applications of the results here, including the problem of airplane dynamics in the case of Loss of Controllability during stall using a minimal model that illustrates the recently introduced LIBRA mechanism. Future work will consider a better control design technique for the extended system and a comparison with the traditional methods employed in the case of loss of controllability.

Appendix: Proof Sketch and Remarks

In this appendix, we provide a sketch of the proof of practical stability for the proposed control scheme. We begin by recalling an important proposition from [19]

Proposition 1 *Let $\mathcal{F} = \{\boldsymbol{f}_0, \boldsymbol{f}_1, \ldots, \boldsymbol{f}_m\}$ be a family of smooth time-varying vector fields on $\mathbb{R}^n$ that are jointly differentiable in $(\boldsymbol{x}, t)$. Assume that the solution $\boldsymbol{x}$ generated by $\boldsymbol{v}$ and the extended system for $\mathcal{F}$ with initial condition $\boldsymbol{x}(0) = \boldsymbol{x}_0 \in \mathbb{R}^n$ is defined on $[0, T]$, $T > 0$. Then there exists a constant K such that the solutions $\boldsymbol{x}^j$ generated by $\boldsymbol{u}_j(\cdot) = Liu(\boldsymbol{v}, \cdot, j)$ and $\mathcal{F}$ and initial condition $\boldsymbol{x}(0) = \boldsymbol{x}_0$ are defined on $[0, T]$ for j large enough and*

$$\sup_{t\in[0,T]} \left\|\boldsymbol{x}^j(t) - \boldsymbol{x}(t)\right\| \le K\, j^{-\frac{1}{r}}$$

where r is the highest order of Lie brackets in the extended system. Let $\Delta \subset \mathbb{R}^n$ be a compact set that contains the trajectory $\boldsymbol{x}$ in its interior. Then the constant K depends on $\mathcal{F}$ restricted on Δ, for a fixed choice of the functions $\{\eta_i\}$.

This proposition allows us to choose the frequency j independently from the interval in the proposed algorithm when all the trajectories of the extended system lie inside a compact set. Such is the case when the extended system is asymptotically stable under a smooth feedback control law.

Fix $T > 0$, and consider the control affine system in Eq.(3) under the proposed control scheme with interval length T. Let $R > 0$, and consider initial conditions in the compact ball around the origin:

$$\mathbb{B}_R(\mathbf{0}) = \{\boldsymbol{x} \in \mathbb{R}^n \mid \|\boldsymbol{x}\| < R\} \subset \mathbb{R}^n$$

Any trajectory of the closed loop extended system with initial condition $\boldsymbol{x}_0 \in \mathbb{B}_R(\mathbf{0})$ will remain inside $\mathbb{B}_R(\mathbf{0})$. The norm of the trajectory of the extended system in the first interval is given by:

$$\|\boldsymbol{x}(t)\| \le \|\boldsymbol{x}(0)\| e^{-\lambda t} \le R e^{-\lambda t}, \quad t \in [0, T]$$

So, in particular $\boldsymbol{x}(t) \in \mathbb{B}_R(\mathbf{0})$, $\forall t \in [0, T]$. Let $\delta \in (0, 1), \varepsilon \in (0, (1-\delta)R)$, and let j_1 be such that $\forall j > j_1$, the trajectories of the ordinary system exist and satisfy:

$$\sup_{t\in[0,T]} \left\|\boldsymbol{x}^j(t) - \boldsymbol{x}(t)\right\| \le K\, j^{-\frac{1}{r}}$$

where $K > 0$ is as in Proposition 1 with a fixed choice for the functions $\{\eta_i\}$ and the compact subset $\mathbb{B}_{R+\varepsilon}(\mathbf{0})$. Let $\lambda^* = \frac{1}{T} \log\left(\frac{R}{(1-\delta)R-\varepsilon}\right)$. Then, we see that:

$$R e^{-\lambda T} \le R e^{-\alpha T} - \varepsilon, \quad \forall \lambda \in (\lambda^*, \infty)$$

where $\alpha = \frac{1}{T}\log\left(\frac{R}{\varepsilon + R e^{-\lambda T}}\right) > 0$. Fix $\lambda \in (\lambda^*, \infty)$. Then, the trajectory of the ordinary system satisfies:

$$\|\boldsymbol{x}^j(t)\| = \|\boldsymbol{x}^j(t) - \boldsymbol{x}(t) + \boldsymbol{x}(t)\| \le \|\boldsymbol{x}^j(t) - \boldsymbol{x}(t)\| + \|\boldsymbol{x}(t)\| \le K j^{-\frac{1}{r}} + \|\boldsymbol{x}(0)\| e^{-\lambda t}, \quad t \in [0, T]$$

Let $j_2 > \left(\frac{K}{\varepsilon}\right)^r$. Then, $\forall j \in (\max(j_1, j_2), \infty)$, we have:

$$\|\boldsymbol{x}^j(t)\| \le \varepsilon + \|\boldsymbol{x}(0)\| e^{-\lambda t} \le \varepsilon + R e^{-\lambda t}, \quad t \in [0, T]$$

Thus, the trajectory of the ordinary system satisfies:

$$\|\boldsymbol{x}^j(t)\| \le R e^{-\alpha t}, \quad t \in [0, T]$$

In particular, we have that $\|\boldsymbol{x}^j(T)\| \leq Re^{-\alpha T}$, which implies that $\boldsymbol{x}^j(T) \in \mathbb{B}_R(\mathbf{0})$, and so the initial condition for the extended system in the next interval is again inside $\mathbb{B}_R(\mathbf{0})$. Consequently, the same choice made for j holds for the second interval, and all subsequent intervals as well. Hence, we have:

$$\begin{aligned}
\|\boldsymbol{x}^j(t)\| &\leq \varepsilon + \|\boldsymbol{x}(0)\| e^{-\alpha t}, && t \in [0, T] \\
\|\boldsymbol{x}^j(t)\| &\leq \varepsilon + \|\boldsymbol{x}(T)\| e^{-\alpha (t-T)}, && t \in [T, 2T] \\
&\vdots \\
\|\boldsymbol{x}^j(t)\| &\leq \varepsilon + \|\boldsymbol{x}(kT)\| e^{-\alpha (t-kT)}, && t \in [kT, (k+1)T]
\end{aligned}$$

All these inequalities can be combined to obtain:

$$\|\boldsymbol{x}^j(t)\| \leq \varepsilon \Big(\sum_{i=0}^{k} \left(e^{-\alpha T}\right)^i \Big) + \|\boldsymbol{x}^j(0)\| e^{-\alpha t}, \quad t \in [0, (k+1)T]$$

Taking the limit as $k \to \infty$, we obtain the relation:

$$\|\boldsymbol{x}^j(t)\| \leq \frac{\varepsilon}{1 - e^{-\alpha T}} + \|\boldsymbol{x}^j(0)\| e^{-\alpha t} \leq \frac{\varepsilon}{\delta} + \|\boldsymbol{x}^j(0)\| e^{-\alpha t}, \quad t \in [0, \infty)$$

We see that starting from a fixed value of ε, we were able to find a parameter j large enough such that the trajectories of the ordinary system satisfy a relationship of the form:

$$\|\boldsymbol{x}^j(t)\| \leq \|\boldsymbol{x}^j(0)\| e^{-\alpha t} + C\varepsilon, \quad t \in [0, \infty)$$

where $C > 0$ is a constant. This concludes the sketch of the proof.

A few remarks concerning the required frequency of oscillation are due. First, notice that the sufficiently high frequency grows exponentially with the order of the Lie bracket used. It also depends on the size of the compact set in which we would like to stabilize the system, and the small neighborhood around the equilibrium point that the system will eventually converge to. Thus, the frequency can be prohibitively high for normal operation. However, in the case of Loss of Controllability, traditional methods fail to provide stabilizing control laws. Hence, the merit of the proposed scheme.

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