1. Averages of Erdős-Szekeres Polynomials

We want to average the L_p norms of Erdős-Szekeres polynomials

$$P_n(\{s_j\};z) = \prod_{j=1}^{n} (1 - z^{s_j})$$

over all choices of $s_1, s_2, ..., s_n$ with $1 \leq s_1, s_2, ..., s_n \leq M$. For $0 < p < \infty$, the L_p norm of such a P_n is

$$\|P_n(\{s_j\};z)\|_p = \left(\frac{1}{2\pi} \int_0^{2\pi} \left| P_n\left(\{s_j\};e^{i\theta}\right)\right|^p d\theta \right)^{1/p}.$$

For $p = \infty$, we note that

$$\|P_n(\{s_j\};z)\|_\infty = \sup\left\{ \left| P_n\left(\{s_j\};e^{i\theta}\right)\right| : \theta \in [0, 2\pi] \right\}.$$

There are M^n choices for $(s_1, s_2, ..., s_n)$. So one natural average for $0 < p < \infty$ is

$$A_p(M,n) = \frac{1}{M^n} \sum_{\substack{(s_1,s_2,\ldots,s_n): \\ \text{all } 1 \leq s_j \leq M}} \left(\|P_n(\{s_j\};z)\|_p \right)^p.$$

There is a formula:

Lemma 1

$$A_p(M,n) = 2^{np} \frac{1}{\pi} \int_0^{\pi} \left(\frac{1}{M} \sum_{k=1}^{M} |\sin kt|^p \right)^n dt.$$

Proof
Now

$$\begin{aligned} \left|1 - e^{is_j\theta}\right| &= \left| \left(e^{is_j\theta/2}\right)\left(e^{-is_j\theta/2} - e^{is_j\theta/2}\right) \right| \\ &= \left| \left(e^{is_j\theta/2}\right)(2i \sin s_j\theta/2) \right| \\ &= 2 \left| \sin\left(\frac{s_j\theta}{2}\right) \right| \end{aligned}$$

and hence

$$\begin{aligned} \left(\|P_n(\{s_j\};z)\|_p \right)^p &= \frac{1}{2\pi} \int_0^{2\pi} \prod_{j=1}^{n} \left(2 \left| \sin\left(\frac{s_j\theta}{2}\right) \right| \right)^p d\theta \\ &= 2^{np} \frac{1}{2\pi} \int_0^{2\pi} \prod_{j=1}^{n} \left| \sin\left(\frac{s_j\theta}{2}\right) \right|^p d\theta \\ &= 2^{np} \frac{1}{\pi} \int_0^{\pi} \prod_{j=1}^{n} |\sin(s_j t)|^p \, dt \end{aligned}$$

by the substitution $t = \theta/2$. Then the average is

$$\begin{aligned} A_p(M,n) &= \frac{1}{M^n} \sum_{\substack{(s_1,s_2,\ldots,s_n): \\ \text{all } 1\le s_j \le M}} \left(\|P_n(\{s_j\};z)\|_p \right)^p \\ &= \frac{1}{M^n} \sum_{s_1=1}^{M} \sum_{s_2=1}^{M} \ldots \sum_{s_n=1}^{M} \left(2^{np} \frac{1}{\pi} \int_0^{\pi} \prod_{j=1}^{n} |\sin(s_j t)|^p \, dt \right) \\ &= \frac{2^{np}}{M^n} \frac{1}{\pi} \int_0^{\pi} \left[\sum_{s_1=1}^{M} \sum_{s_2=1}^{M} \ldots \sum_{s_n=1}^{M} \prod_{j=1}^{n} |\sin(s_j t)|^p \right] dt. \end{aligned}$$

Next, we use

$$\begin{aligned} & \sum_{s_1=1}^{M} \sum_{s_2=1}^{M} \ldots \sum_{s_n=1}^{M} \prod_{j=1}^{n} |\sin(s_j t)|^p \\ = & \sum_{s_1=1}^{M} \sum_{s_2=1}^{M} \ldots \sum_{s_n=1}^{M} |\sin s_1 t|^p \, |\sin s_2 t|^p \ldots |\sin s_n t|^p \\ = & \sum_{s_1=1}^{M} |\sin s_1 t|^p \sum_{s_2=1}^{M} |\sin s_2 t|^p \ldots \sum_{s_n=1}^{M} |\sin s_n t| \\ = & \left(\sum_{k=1}^{M} |\sin kt|^p \right)^n . \end{aligned}$$

So

$$\begin{aligned} A_p(M,n) &= \frac{2^{np}}{M^n} \frac{1}{\pi} \int_0^{\pi} \left(\sum_{k=1}^{M} |\sin kt|^p \right)^n dt \\ &= 2^{np} \frac{1}{\pi} \int_0^{\pi} \left(\frac{1}{M} \sum_{k=1}^{M} |\sin kt|^p \right)^n dt. \end{aligned}$$

■

Lemma 2
For $t \in [0,\pi]$ such that t/π is irrational,

$$\lim_{M\to\infty} \frac{1}{M} \sum_{k=1}^{M} |\sin kt|^p = \frac{1}{\pi} \int_0^{\pi} |\sin t|^p \, dt.$$

Proof
Let $\{x\}$ denote the fractional part of a real number, so that $\{x\} = x-$greatest integer $\le x$. For example, $\{7.34\} = 0.34$. Note that if α is irrational and $f : [0,1] \to \mathbb{R}$ is continuous, then

$$\lim_{M\to\infty} \frac{1}{M} \sum_{k=1}^{M} f(\{k\alpha\}) = \int_0^1 f(t) \, dt$$

This is a classic result in the theory of uniform distribution [3]. We apply this with $f(s) = |\sin \pi s|^p$, $s \in [0,1]$, and use the fact that f is periodic of period 1, that is,

$f(x+1) = f(x)$. Then for t/π irrational,

$$\begin{aligned}
\lim_{M\to\infty} \frac{1}{M} \sum_{k=1}^{M} |\sin kt|^p &= \lim_{M\to\infty} \frac{1}{M} \sum_{k=1}^{M} \left|\sin \pi \left(k\frac{t}{\pi}\right)\right|^p \\
&= \lim_{M\to\infty} \frac{1}{M} \sum_{k=1}^{M} \left|\sin \pi \left\{k\frac{t}{\pi}\right\}\right|^p \\
&= \int_0^1 |\sin \pi s|^p \, ds = \frac{1}{\pi} \int_0^\pi |\sin t|^p \, dt.
\end{aligned}$$

■

Lemma 3
Let $\{M_n\}$ be any sequence of positive integers with limit ∞. Then

$$\liminf_{n\to\infty} A_p(M_n, n)^{1/n} \geq \frac{2^p}{\pi} \int_0^\pi |\sin t|^p \, dt.$$

Remark
My guess is that the limit exists and equals the right-hand side, but it will be difficult to prove the corresponding upper bound.
Proof
From the previous lemma, for each $t \in [0,\pi]$ such that t/π is irrational,

$$\lim_{n\to\infty} \frac{1}{M_n} \sum_{k=1}^{M_n} |\sin kt|^p = \frac{1}{\pi} \int_0^\pi |\sin t|^p \, dt.$$

By Egorov's Theorem, a classic result of measure theory, given $\varepsilon \in (0,1)$, there exists a set $\mathcal{E}$ of Lebesgue measure ("length") $\leq \varepsilon$ such that the convergence above is uniform for $t \in [0,\pi] \setminus \mathcal{E}$. Then there exists K such that for $n \geq K$ and $t \in [0,\pi] \setminus \mathcal{E}$,

$$\left| \frac{1}{M_n} \sum_{k=1}^{M_n} |\sin kt|^p - \frac{1}{\pi} \int_0^\pi |\sin t|^p \, dt \right| < \varepsilon.$$

Then

$$\begin{aligned}
A_p(M_n, n) &\geq 2^{np} \frac{1}{\pi} \int_{[0,1]\setminus\mathcal{E}} \left(\frac{1}{M_n} \sum_{k=1}^{M_n} |\sin kt|^p \right)^n dt \\
&\geq 2^{np} \frac{1}{\pi} \int_{[0,1]\setminus\mathcal{E}} \left(\frac{1}{\pi} \int_0^\pi |\sin s|^p \, ds - \varepsilon \right)^n dt \\
&= 2^{np} \frac{1}{\pi} meas\left([0,1] \setminus \mathcal{E}\right) \left(\frac{1}{\pi} \int_0^\pi |\sin s|^p \, ds - \varepsilon \right)^n
\end{aligned}$$

so

$$A_p(M_n, n)^{1/n} \geq 2^p \left[\frac{1}{\pi} meas\left([0,1] \setminus \mathcal{E}\right) \right]^{1/n} \left(\frac{1}{\pi} \int_0^\pi |\sin s|^p \, ds - \varepsilon \right).$$

Letting $n \to \infty$,

$$\liminf_{n\to\infty} A(M_n, n)^{1/n} \geq 2^p \left(\frac{1}{\pi} \int_0^\pi |\sin s|^p \, ds - \varepsilon \right).$$

As $\varepsilon > 0$ is arbitrary, we obtain the lower bound. ■

It might be difficult to establish a corresponding upper bound. On the other hand, for fixed n and $M \to \infty$, it is doable:

Lemma 4
Fix $n \geq 1$. Then

$$\lim_{M \to \infty} A_p(M, n) = \left(\frac{2^p}{\pi} \int_0^\pi |\sin t|^p \, dt \right)^n.$$

Proof
We have for t/π irrational, and hence for almost every t, that

$$\lim_{M \to \infty} \left(\frac{1}{M} \sum_{k=1}^M |\sin kt|^p \right)^n = \left(\frac{1}{\pi} \int_0^\pi |\sin t|^p \, dt \right)^n.$$

Also, for all $t \in [0, 1]$,

$$\left(\frac{1}{M} \sum_{k=1}^M |\sin kt|^p \right)^n \leq 1.$$

Lebesgue's Dominated Convergence Theorem then shows that the stated limit is true. ■

Lemma 5
(a)

$$\int_0^\pi |\sin t|^p \, dt = 2^p \frac{\Gamma\left(\frac{p+1}{2}\right)^2}{\Gamma(p+1)}.$$

(b) For $p \geq 1$,

$$\frac{2^p}{\pi} \int_0^\pi |\sin t|^p \, dt \geq \frac{4}{\pi} > 1.$$

Proof
(a) From 3.621.1 in [2], with $\mu - 1 = p$ there,

$$\begin{aligned} \int_0^{\pi/2} (\sin x)^p \, dx &= 2^{p-1} B\left(\frac{p+1}{2}, \frac{p+1}{2} \right) \\ &= 2^{p-1} \frac{\Gamma\left(\frac{p+1}{2}\right)^2}{\Gamma(p+1)} \end{aligned}$$

so the identity follows. Next,

$$B\left(\frac{p+1}{2}, \frac{p+1}{2} \right) = \int_0^1 t^{\frac{p-1}{2}} (1-t)^{\frac{p-1}{2}} \, dt$$

(b) We use a well known inequality: for $x, y \geq 1$,

$$B(x, y) \geq \frac{1}{xy}.$$

See for example [1]. If $p \geq 1$, then $\frac{p+1}{2} \geq 1$, so

$$B\left(\frac{p+1}{2}, \frac{p+1}{2} \right) \geq \left(\frac{2}{p+1} \right)^2$$

and then

$$\frac{2^p}{\pi} \int_0^\pi |\sin t|^p \, dt \geq \frac{1}{\pi} \left(\frac{2^{p+1}}{p+1} \right)^2$$

Now $\frac{2^x}{x}$ is an increasing function of x for $x \geq 2$ (calculus exercise) so the smallest is when $p = 1$ and then

$$\frac{2^p}{\pi} \int_0^{\pi} |\sin t|^p \, dt \geq \frac{4}{\pi}.$$

■

2. Variance of Erdős-Szekeres Polynomials

If X is a random variable with mean or average $E(X)$, then the variance is

$$\sqrt{E\left(X - E(X)^2\right)}.$$

It simplifies after squaring out terms, to

$$\sqrt{E(X^2) - E(X)^2}$$

In our case $X = \|P_n(\{s_j\}; z)\|_p$ and $E(X) = A_p(M, n)$. The square of the variance is

$$V_p(m, n) = \frac{1}{M^n} \sum_{\substack{(s_1, s_2, \ldots, s_n): \\ \text{all } 1 \leq s_j \leq M}} \left(\|P_n(\{s_j\}; z)\|_p\right)^{2p} - A_p^2(M, n).$$

Let us call the first term $B_p(M, n)$:

$$\begin{aligned} B_p(M, n) &= \frac{1}{M^n} \sum_{\substack{(s_1, s_2, \ldots, s_n): \\ \text{all } 1 \leq s_j \leq M}} \left(\|P_n(\{s_j\}; z)\|_p\right)^{2p} \\ &= \frac{1}{M^n} \sum_{\substack{(s_1, s_2, \ldots, s_n): \\ \text{all } 1 \leq s_j \leq M}} \left(\frac{1}{2\pi} \int_0^{2\pi} \left|P_n\left(\{s_j\}; e^{i\theta}\right)\right|^p d\theta\right)^2 \end{aligned}$$

I want you to try find a formula for $B_p(M, n)$. Show that we can write

$$\begin{aligned} & B_p(M, n) \\ = & \frac{1}{M^n} \sum_{\substack{(s_1, s_2, \ldots, s_n): \\ \text{all } 1 \leq s_j \leq M}} \left(\frac{1}{2\pi}\right)^2 \int_0^{2\pi} \int_0^{2\pi} \prod_{j=1}^n \left(2 \left|\sin\left(\frac{s_j \theta}{2}\right)\right|\right)^p \prod_{j=1}^n \left(2 \left|\sin\left(\frac{s_j \phi}{2}\right)\right|\right)^p d\theta d\phi. \end{aligned}$$

Now follow similar steps to that we did for $A_p(M, n)$.

3. The Case $p = 2$

Lemma 3.1

$$h_M(t) = \frac{1}{M} \sum_{k=1}^{M} \sin^2 kt = \frac{1}{2}\left(1 + \frac{1}{2M} - \frac{\sin(2M+1)t}{2M \sin t}\right).$$

Proof

$$\begin{aligned}
\sum_{k=1}^{M} \sin^2 kt &= \sum_{k=1}^{M} \frac{1}{2}(1-\cos 2kt) \\
&= \frac{M}{2} - \frac{1}{2}\sum_{k=1}^{M} \cos 2kt \\
&= \frac{M}{2} - \frac{1}{2}\sum_{k=1}^{M} \frac{\sin((2k+1)t) - \sin((2k-1)t)}{2\sin t} \\
&= \frac{M}{2} - \frac{1}{4\sin t}(\sin(2M+1)t - \sin t) \\
&= \frac{M+1/2}{2} - \frac{\sin(2M+1)t}{4\sin t}
\end{aligned}$$

so

$$\frac{1}{M}\sum_{k=1}^{M} \sin^2 kt = \frac{1}{2}\left(1+\frac{1}{2M} - \frac{\sin(2M+1)t}{2M\sin t}\right).$$

■

Lemma 3.2

$$\begin{aligned}
A_2(M,n) &= 2^n \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \left(1+\frac{1}{2M} - \frac{\sin(2M+1)t}{2M\sin t}\right)^n dt \\
&= 2^{2n}\frac{2}{\pi}\int_0^{\frac{\pi}{2}} h_M(t)^n\, dt
\end{aligned}$$

Proof
Since $\sin(\pi - t) = \sin t$, and $\sin(2M+1)(\pi - t) = \sin(2M+1)t$, we can replace $[0,\pi]$ by $\left[0,\frac{\pi}{2}\right]$, and

$$\begin{aligned}
A_2(M,n) &= 2^{2n}\frac{2}{\pi}\int_0^{\frac{\pi}{2}} \left(\frac{1}{M}\sum_{k=1}^{M} |\sin kt|^2\right)^n dt \\
&= 2^{2n}\frac{2}{\pi}\int_0^{\frac{\pi}{2}} \left(\frac{1}{2}\left(1+\frac{1}{2M} - \frac{\sin(2M+1)t}{2M\sin t}\right)\right)^n dt \\
&= 2^{n}\frac{2}{\pi}\int_0^{\frac{\pi}{2}} \left(1+\frac{1}{2M} - \frac{\sin(2M+1)t}{2M\sin t}\right)^n dt \\
&= 2^{2n}\frac{2}{\pi}\int_0^{\frac{\pi}{2}} (h_M(t))^n\, dt.
\end{aligned}$$

■

Lemma 3.3
(a) h_M has a unique maximim $t_M \in \left(\frac{\pi}{2M+1}, \frac{3}{2}\frac{\pi}{2M+1}\right)$ and

$$t_M = \frac{s_0}{2M+1}(1+o(1)),$$

where s_0 is the unique root of the equation $\tan s_0 = s_0$ with $s_0 \in \left(\pi, \frac{3}{2}\pi\right)$.
(b)

$$\left|h_M(t) - h_M(s)\right| \le \frac{M+1}{2}\left|t-s\right|.$$

Proof
(a) First note that if $t \in \left[0, \frac{\pi}{2M+1}\right]$, so $\sin(2M+1)t \ge 0$, then

$$0 \le h_M(t) \le \frac{1}{2}\left(1 + \frac{1}{2M}\right).$$

If $t \in (\frac{3}{2}\frac{\pi}{2M+1}, \frac{\pi}{2}]$, then

$$\begin{aligned}
0 &\le h_M(t) \\
&= \frac{1}{2}\left(1 + \frac{1}{2M} - \frac{\sin(2M+1)t}{2M\sin t}\right) \\
&\le \frac{1}{2}\left(1 + \frac{1}{2M} + \frac{1}{2M\sin t}\right) \\
&< \frac{1}{2}\left(1 + \frac{1}{2M} + \frac{1}{2M\sin\frac{3}{2}\frac{\pi}{2M+1}}\right) \\
&= h_M\left(\frac{3}{2}\frac{\pi}{2M+1}\right).
\end{aligned}$$

So h_M must have a maximum $t_M \in \left(\frac{\pi}{2M+1}, \frac{3}{2}\frac{\pi}{2M+1}\right)$, possibly not unique. Write

$$t_M = \frac{s_m}{2M+1},\ s_M \in \left(\pi, \frac{3}{2}\pi\right).$$

Then

$$\begin{aligned}
h_M(t_M) &= \frac{1}{2}\left(1 + \frac{1}{2M} - \frac{\sin(2M+1)t_M}{2M\sin t_M}\right) \\
&= \frac{1}{2}\left(1 + \frac{1}{2M} - \frac{\sin s_M}{2M\sin\left(\frac{s_m}{2M+1}\right)}\right) \\
&= \frac{1}{2}\left(1 + \frac{1}{2M} - \frac{\sin s_M}{2M\frac{s_M}{2M+1}\left(1+O\left(\frac{1}{M}\right)\right)}\right) \\
&= \frac{1}{2}\left(1 + \frac{1}{2M} - \frac{\sin s_M}{s_M\left(1+O\left(\frac{1}{M}\right)\right)}\right) \\
&= \frac{1}{2}\left(1 - \frac{\sin s_M}{s_M} + O\left(\frac{1}{M}\right)\right).
\end{aligned}$$

Here $\frac{\sin s}{s}$, $s \in \left(\pi, \frac{3}{2}\pi\right)$ has a minimum at s_0 where

$$(\cos s_0)s_0 - \sin s_0 = 0 \Leftrightarrow s_0 = \tan s_0.$$

(b) Now

$$\begin{aligned} \left|h_M(t)-h_M(s)\right| &= \frac{1}{M}\left|\sum_{k=1}^{M}\left(\sin^2 kt-\sin^2 ks\right)\right| \\ &\le \frac{1}{M}\sum_{k=1}^{M}\left|\sin kt-\sin ks\right| \\ &\le \frac{|t-s|}{M}\sum_{k=1}^{M}k=\frac{M+1}{2}\left|t-s\right|. \end{aligned}$$

■

Lemma 3.4
Assume $M=M(n)$ is such that $M^{1/n}\to 1$ as $n\to\infty$. Let s_0 be as above. Then

$$\lim_{n\to\infty}A_2(M,n)^{1/n}=2\left(1-\frac{\sin s_0}{s_0}\right).$$

Proof

$$A_2(M,n)\le 2^{2n}\frac{2}{\pi}\frac{\pi}{2}h_M(t_M)^n$$

so that

$$\begin{aligned} \limsup_{M\to\infty}A_2(M,n)^{1/M} &\le \limsup_{M\to\infty}2^2h_M(t_M) \\ &= \limsup_{M\to\infty}2\left(1+\frac{1}{2M}-\frac{\sin(2M+1)t_M}{2M\sin t_M}\right) \\ &= \limsup_{M\to\infty}2\left(1+\frac{1}{2M}-\frac{\sin s_M}{2M\sin\frac{s_M}{2M+1}}\right) \\ &= 2\left(1-\frac{\sin s_0}{s_0}\right). \end{aligned}$$

In the other direction, let $\varepsilon\in(0,1)$. Then by the lemmas above,

$$\begin{aligned} A_2(M,n) &> 2^{2n}\frac{2}{\pi}\int_{t_M-\frac{\varepsilon}{2M+1}}^{t_M+\frac{\varepsilon}{2M+1}}\left(h_M(t)\right)^n dt \\ &\ge 2^{2n}\frac{2}{\pi}\frac{2\varepsilon}{2M+1}\left(h_M(t_M)-\varepsilon\right)^n \end{aligned}$$

so that as $M^{1/n}\to 1$,

$$\begin{aligned} \liminf_{n\to\infty}A_2(M,n)^{1/n} &\ge \liminf_{n\to\infty}2^2\left(\frac{2}{\pi}\frac{2\varepsilon}{2M+1}\right)^{1/N}\left(h_M(t_M)-\varepsilon\right) \\ &\ge 2\left(1-\frac{\sin s_0}{s_0}\right)-4\varepsilon, \end{aligned}$$

as above. Since $\varepsilon>0$ is arbitrary,

$$\lim_{n\to\infty}A_2(M,n)^{1/n}=2\left(1-\frac{\sin s_0}{s_0}\right).$$

■

Lemma 3.5
Assume that $M, n \to \infty$ in such a way that

$$\lim_{n\to\infty} M^{1/n} = \rho \geq 1.$$

Then

$$\lim_{n\to\infty} A_2 (M, n)^{1/n} = 2 \max\{1, \frac{1}{\rho}\left(1 - \frac{\sin s_0}{s_0}\right)\}.$$

Proof
We have

$$\begin{aligned}
& \frac{2}{\pi}\int_{A/(2M+1)}^{\frac{\pi}{2}} \left(1 + \frac{1}{2M} - \frac{\sin(2M+1)\, t}{2M \sin t}\right)^n dt \\
\leq\ & \frac{2}{\pi}\int_{A/(2M+1)}^{\frac{\pi}{2}} \exp\left(n\left[\frac{1}{2M} - \frac{\sin(2M+1)\, t}{2M\sin t}\right]\right) dt \\
\leq\ & \exp\left(\frac{n}{2M}\right)\frac{2}{\pi}\int_{A/(2M+1)}^{\frac{\pi}{2}} \exp\left(\frac{n}{2M\sin t}\right) dt \\
\leq\ & \exp\left(\frac{n}{2M}\right)\frac{2}{\pi}\frac{\pi}{2}\exp\left(\frac{n}{2M\sin\frac{A}{2M+1}}\right) \\
=\ & \exp\left(\frac{n}{2M}\right)\exp\left(\frac{n}{2M\frac{2}{\pi}\frac{A}{2M+1}}\right) \\
=\ & \exp\left(\frac{n}{2M}\right)\left(\exp\frac{2M+1}{2M}\frac{\pi n}{2A}\right).
\end{aligned}$$

Also,

$$\begin{aligned}
& \frac{2}{\pi}\int_0^{A/(2M+1)} \left(1 + \frac{1}{2M} - \frac{\sin(2M+1)\, t}{2M\sin t}\right)^n dt \\
=\ & \frac{2}{\pi}\int_0^{A/(2M+1)} \left(\frac{1}{2}h_m(t)\right)^n dt \\
\leq\ & \frac{2}{\pi}\frac{A}{2M+1}\left(1 - \frac{\sin s_0}{s_0} + o(1)\right)^n.
\end{aligned}$$

Then

$$\begin{aligned}
& \frac{2}{\pi}\int_0^{A/(2M+1)} \left(1 + \frac{1}{2M} - \frac{\sin(2M+1)\, t}{2M\sin t}\right)^n dt \\
\leq\ & \exp\left(\frac{n}{2M}\right)\left(\exp\frac{2M+1}{2M}\frac{\pi n}{2A}\right) + \frac{2}{\pi}\frac{A}{2M+1}\left(1 - \frac{\sin s_0}{s_0} + o(1)\right)^n
\end{aligned}$$

so

$$\begin{aligned}
& \left(\frac{2}{\pi}\int_0^{A/(2M+1)} \left(1 + \frac{1}{2M} - \frac{\sin(2M+1)\, t}{2M\sin t}\right)^n dt\right)^{1/n} \\
\leq\ & (1 + o(1)) \max\left\{\exp\left(\frac{\pi}{2A}\right), \frac{1}{M^{1/n}}\left(1 - \frac{\sin s_0}{s_0}\right)\right\}.
\end{aligned}$$

Thus

$$\begin{aligned}&\limsup_{M\to\infty} A_2\left(M,n\right)^{1/n}\\ \leq\quad & 2\max\left\{\exp\left(\frac{\pi}{2A}\right),\frac{1}{\rho}\left(1-\frac{\sin s_0}{s_0}\right)\right\}.\end{aligned}$$

As we can take A arbitrarily large,

$$\limsup_{M\to\infty} A_2\left(M,n\right)^{1/n}\leq 2\max\{1,\frac{1}{\rho}\left(1-\frac{\sin s_0}{s_0}\right)\}.$$

Also as in the proof of the lemma above,

$$\begin{aligned}A_2\left(M,n\right) &> 2^{2n}\frac{2}{\pi}\int_{t_M-\frac{\varepsilon}{2M+1}}^{t_M+\frac{\varepsilon}{2M+1}}\left(h_M\left(t\right)\right)^n dt\\ &\geq 2^{2n}\frac{2}{\pi}\frac{2\varepsilon}{2M+1}\left(h_M\left(t_M\right)-\varepsilon\right)^n\end{aligned}$$

so

$$\liminf_{n\to\infty} A_2(M,n)^{1/n}\geq\frac{2}{\rho}\left(1-\frac{\sin s_0}{s_0}\right).$$

Next, we showed above that

$$\liminf_{n\to\infty} A_2(M,n)^{1/n}\geq\frac{2^2}{\pi}\int_0^\pi \sin^2 dt=2$$

Thus,

$$\liminf_{n\to\infty} A_2(M,n)^{1/n}\geq 2\max\{1,\frac{1}{\rho}\left(1-\frac{\sin s_0}{s_0}\right)\}.$$

■

3.1. **The case p an even integer.** We use the formula [2, p. 25, 1.320.1]

$$\begin{aligned}\sin^{2r} x &= \frac{1}{2^{2r}}\left\{2\sum_{k=0}^{r-1}\left(-1\right)^{r-k}\binom{2r}{k}\cos\left(2\left(r-k\right)x\right)+\binom{2r}{r}\}\right\}\\ &= \frac{1}{2^{2r}}\left\{2\sum_{j=1}^{r}\left(-1\right)^{j}\binom{2r}{r-j}\cos\left(2jx\right)+\binom{2r}{r}\}\right\}.\end{aligned}$$

Lemma 4.1
Let $r\geq 1$ and

$$h_M\left(t\right)=\frac{1}{M}\sum_{k=1}^{M}\sin^{2r} kt.$$

Then

$$\begin{aligned}h_M\left(t\right) &= \frac{1}{2^{2r}}\binom{2r}{r}-\frac{1}{2^{2r}}\frac{2}{M}\sum_{j=1}^{r}\left(-1\right)^{j}\binom{2r}{r-j}\\ &\quad+\frac{1}{2^{2r}}\frac{2}{M}\sum_{j=1}^{r}\left(-1\right)^{j}\binom{2r}{r-j}\frac{\sin\left(j\left(2M+1\right)t\right)}{\sin jt}.\end{aligned}$$

Proof

$$\begin{aligned}
h_M\left(t\right) &= \frac{1}{M}\sum_{k=1}^{M}\frac{1}{2^{2r}}\left\{2\sum_{j=1}^{r}\left(-1\right)^j\binom{2r}{r-j}\cos\left(2jkt\right)+\binom{2r}{r}\}\right\} \\
&= \frac{1}{2^{2r}}\binom{2r}{r}+\frac{1}{2^{2r}}\frac{2}{M}\sum_{j=1}^{r}\left(-1\right)^j\binom{2r}{r-j}\sum_{k=1}^{M}\cos\left(2jkt\right) \\
&= \frac{1}{2^{2r}}\binom{2r}{r}+\frac{1}{2^{2r}}\frac{2}{M}\sum_{j=1}^{r}\left(-1\right)^j\binom{2r}{r-j}\sum_{k=1}^{M}\frac{\sin\left(j\left(2k+1\right)t\right)-\sin\left(j\left(2k-1\right)t\right)}{\sin jt} \\
&= \frac{1}{2^{2r}}\binom{2r}{r}+\frac{1}{2^{2r}}\frac{2}{M}\sum_{j=1}^{r}\left(-1\right)^j\binom{2r}{r-j}\left\{\frac{\sin\left(j\left(2M+1\right)t\right)-\sin jt}{\sin jt}\right\} \\
&= \frac{1}{2^{2r}}\binom{2r}{r}-\frac{1}{2^{2r}}\frac{2}{M}\sum_{j=1}^{r}\left(-1\right)^j\binom{2r}{r-j}+\frac{1}{2^{2r}}\frac{2}{M}\sum_{j=1}^{r}\left(-1\right)^j\binom{2r}{r-j}\frac{\sin\left(j\left(2M+1\right)t\right)}{\sin jt}
\end{aligned}$$

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