ON ZEROS, BOUNDS, AND ASYMPTOTICS FOR ORTHOGONAL POLYNOMIALS ON THE UNIT CIRCLE

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Abstract. Let μ be a measure on the unit circle that is regular in the sense of Stahl Totik, and Ullmann. Let $\{\varphi_n\}$ be the orthonormal polynomials for μ and $\{z_{jn}\}$ their zeros. Let μ be absolutely continuous in an arc Γ of the unit circle, with μ' positive and continuous there. We show that uniform boundedness of the orthonormal polynomials in subarcs of Γ is equivalent to $n(1-|z_{jn}|)$ being bounded away from 0. If in addition as $n \to \infty$, $n(1-|z_{jn}|) \to \infty$, then $|\varphi_n|^2 \mu' \to 1$ uniformly.

Research supported by NSF grant DMS1800251

1. Main Results

Let μ be a finite positive Borel measure on $[-\pi, \pi)$ (or equivalently on the unit circle) with infinitely many points in its support. Then we may define orthonormal polynomials

$$\varphi_n(z) = \kappa_n z^n + ..., \kappa_n > 0,$$

$n = 0, 1, 2, ...$ satisfying the orthonormality conditions

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi_n(z) \overline{\varphi_m(z)} d\mu(\theta) = \delta_{mn},$$

where $z = e^{i\theta}$. We denote the zeros of φ_n by $\{z_{jn}\}_{j=1}^{n}$. They lie inside the unit circle, and may not be distinct.

We shall often assume that μ is *regular* in the sense of Stahl, Totik and Ullmann [14], so that

$$\lim_{n\to\infty} \kappa_n^{1/n} = 1.$$

This is true if for example $\mu' > 0$ a.e. in $[-\pi, \pi)$, but there are pure jump and pure singularly continuous measures that are regular.

Many aspects of the zeros $\{z_{jn}\}$ have been studied down the years: their asymptotics, their distribution (often when projected onto the unit circle), "clock spacing" of zeros of paraorthogonal polynomials, ... See

In a very interesting recent paper, Bessonov and Denisov [**?**] showed that the distance of the zeros to the unit circle is intimately related to asymptotics of orthogonal polynomials. The following is a reformulation of one of their results:

Theorem
Let μ be a measure on the unit circle satisfying the Szegő condition

$$\int_{-\pi}^{\pi} \log \mu'\left(e^{it}\right) dt > -\infty.$$

Date: December 12, 2020.

For almost every ζ with $|\zeta| = 1$, the following are equivalent:
(I)

$$\lim_{n\to\infty} |\varphi_n(\zeta)|^2 \mu'(\zeta) = 1.$$

(II)

$$\lim_{n\to\infty} \left(\inf_{1\le j\le n} |\zeta - z_{jn}| \right) = \infty.$$

We prove related equivalences for local bounds and asymptotics but for more general regular, rather than Szegő, measures:

Theorem 1.1
Let μ be a finite positive Borel measure on the unit circle that is regular in the sense of Stahl, Totik, and Ullmann. Let Δ be an arc of the unit circle in which μ is absolutely continuous, while μ' is positive and continuous there. The following are equivalent:
(I) In every proper subarc Γ of Δ,

$$\lim_{n\to\infty} \left(\inf \left\{ n(1 - |z_{jn}|) : z_{jn} \neq 0, \frac{z_{jn}}{|z_{jn}|} \in \Gamma \right\} \right) = \infty.$$

(II) In every proper subarc Γ of Δ, *as $n \to \infty$, uniformly for $\zeta \in \Gamma$,*

$$\lim_{n\to\infty} |\varphi_n(\zeta)|^2 \mu'(\zeta) = 1.$$

Theorem 1.2
Assume the hypotheses of Theorem 1.1. The following are equivalent:
(I) In every proper subarc Γ of Δ, *there exists $C_1 > 0$ such that for $n \ge 1$,*

$$\inf \left\{ n(1 - |z_{jn}|) : z_{jn} \neq 0, \frac{z_{jn}}{|z_{jn}|} \in \Gamma \right\} \ge C_1.$$

(II) In every proper subarc Γ of Δ, *there exists $C_2 > 0$ such that for $n \ge 1$,*

$$\|\varphi_n\|_{L_\infty(\Gamma)} \le C_2.$$

This paper is organized as follows: in Section 2, we present more background as well as more equivalences. ...

We close this section with more notation. We let

$$\varphi_n^*(z) = z^n \overline{\varphi_n\left(\frac{1}{\bar{z}}\right)}.$$

The nth reproducing kernel for μ is

$$K_n(z,u) = \sum_{j=0}^{n-1} \varphi_j(z) \overline{\varphi_j(u)}. \tag{1.1}$$

The Christoffel-Darboux formula asserts that

$$K_n(z,u) = \frac{\overline{\varphi_n^*(u)}\varphi_n^*(z) - \overline{\varphi_n(u)}\varphi_n(z)}{1 - \bar{u}z}. \tag{1.2}$$

We let

$$R_n(z)=\sum_{j=1}^{n}\frac{1-|z_{jn}|^2}{|z-z_{jn}|^2} \tag{1.3}$$

and

$$g_n(z)=\frac{z\varphi_n'(z)}{n\varphi_n(z)}. \tag{1.4}$$

Throughout $C, C_1, C_2, \ldots$ denote positive constants independent of n, z, t and polynomials P of degree $\leq n$. The same symbol need not denote the same constant in different occurrences. For sequences $\{x_n\}, \{y_n\}$ of non-zero real numbers, we write

$$x_n \sim y_n$$

if there exists $C>1$ such that

$$C^{-1} \leq x_n/y_n \leq C \text{ for } n \geq 1.$$

2. Background and Further Results

Parts of the following theorem appear in Theorem 1.2 in [?], notably (b), (d), (e), while weaker forms of (a), (f) appear there.

Theorem 2.1
Let μ be a finite positive Borel measure on the unit circle that is regular in the sense of Stahl, Totik, and Ullmann. Let Δ be an arc of the unit circle in which μ is absolutely continuous, while μ' is positive and continuous there. Let Γ be a proper subarc of Δ. The following are equivalent: in every proper subarc Γ of Δ,
(a) Uniformly in Γ,

$$\lim_{n\to\infty}|\varphi_n(z)|^2\mu'(z)=1. \tag{2.1}$$

(b) Uniformly in Γ,

$$\lim_{n\to\infty}\frac{1}{n}R_n(z)=1. \tag{2.2}$$

(c) Uniformly in Γ,

$$\lim_{n\to\infty}\operatorname{Re}\left(\frac{z\varphi_n'(z)}{n\varphi_n(z)}\right)=1. \tag{2.3}$$

(d) Uniformly in Γ,

$$\lim_{n\to\infty}\frac{z\varphi_n'(z)}{n\varphi_n(z)}=1. \tag{2.4}$$

(e) Uniformly in Γ,

$$\lim_{n\to\infty}\frac{\varphi_n\left(ze^{i\pi/n}\right)}{\varphi_n(z)}=-1. \tag{2.5}$$

(f) Uniformly for $z\in\Gamma$ and u in compact subsets of $\mathbb{C}$,

$$\lim_{n\to\infty}\frac{\varphi_n\left(z\left(1+\frac{u}{n}\right)\right)}{\varphi_n(z)}=e^u. \tag{2.6}$$

(g)

$$\lim_{n\to\infty}\left(\inf\left\{n\left(1-\left|z_{jn}\right|\right): z_{jn}\neq 0, \frac{z_{jn}}{\left|z_{jn}\right|}\in\Gamma\right\}\right)=\infty. \tag{2.7}$$

(h) Uniformly for $z\in\Gamma$,

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{j=1}^{n}\frac{1}{\left|z-z_{jn}\right|^2}=0. \tag{2.8}$$

Theorem 2.2
Let μ satisfy the hypotheses of Theorem 2.1. The following are equivalent: in every proper subarc Γ of Δ,
(a)

$$\sup_{n\geq 1}\left\|\varphi_n\right\|_{L_\infty(\Gamma)}<\infty. \tag{2.9}$$

(b) There exist $n_0, C>0$ such that for $n\geq n_0$, and $z\in\Gamma$,

$$\frac{1}{n}R_n\left(z\right)\geq C. \tag{2.10}$$

(c) There exist $n_0, C>0$ such that for $n\geq n_0$, and $z\in\Gamma$,

$$\left|\operatorname{Re}\left(\frac{z\varphi_n'\left(z\right)}{n\varphi_n\left(z\right)}-\frac{1}{2}\right)\right|\geq C. \tag{2.11}$$

(d) There exist $n_0, C>0$ such that for $n\geq n_0$, and $z\in\Gamma$,

$$\left|\operatorname{Re}\left(\frac{\varphi_n\left(ze^{\pm i\pi/n}\right)}{\varphi_n\left(z\right)}\right)\right|\geq C. \tag{2.12}$$

(e) There exist $n_0, C>0$ such that for $n\geq n_0$, and $z\in\Gamma$,

$$\inf\left\{n\left(1-\left|z_{jn}\right|\right): z_{jn}\neq 0, \frac{z_{jn}}{\left|z_{jn}\right|}\in\Gamma\right\}\geq C. \tag{2.13}$$

(f) There exist $n_0, C>0$ such that for $n\geq n_0$, and $z\in\Gamma$

$$\sup_{\zeta\in\Gamma, n\geq 1}\frac{1}{n^2}\sum_{j=1}^{n}\frac{1}{\left|z-z_{jn}\right|^2}\leq C. \tag{2.14}$$

3. Preliminary Lemmas

Throughout, we assume the hypotheses of Theorem 1.1. We first recall some asymptotics for Christoffel functions and universality and local limits.

Lemma 3.1
Let Γ be a proper subarc of Δ.
(a) Uniformly for $z\in\Gamma$,

$$\lim_{n\to\infty}\frac{1}{n}K_n\left(z,z\right)\mu'\left(z\right)=1. \tag{3.1}$$

(b) Uniformly for $z \in \Gamma$ and a, b in compact subsets of $\mathbb{C}$,

$$\lim_{n\to\infty} \frac{K_n\left(z\left(1+\frac{i2\pi a}{n}\right), z\left(1+\frac{i2\pi \bar{b}}{n}\right)\right)}{K_n(z,z)} = e^{i\pi(a-b)}\mathbb{S}(a-b). \tag{3.2}$$

(c) Let $\{\zeta_n\} \subset \Gamma$. Assume that

$$\sup_{n\geq 1} \frac{1}{n}\left|\sum_{j=1}^{n} \frac{1}{\zeta_n - z_{jn}}\right| < \infty \text{ and } \sup_{n\geq 1} \frac{1}{n^2}\sum_{j=1}^{n} \frac{1}{|\zeta_n - z_{jn}|^2} < \infty. \tag{3.3}$$

From every infinite sequence of positive integers, we can choose an infinite subsequence $\mathcal{S}$ such that uniformly for u in compact subsets of $\mathbb{C}$,

$$\lim_{n\to\infty, n\in\mathcal{S}} \frac{\varphi_n\left(\zeta_n\left(1+\frac{u}{n}\right)\right)}{\varphi_n(\zeta_n)} = e^u + C\left(e^u - 1\right), \tag{3.4}$$

where

$$C = \lim_{n\to\infty, n\in\mathcal{S}} \left(\frac{\zeta_n}{n}\frac{\varphi_n'(\zeta_n)}{\varphi_n(\zeta_n)} - 1\right), \tag{3.5}$$

Proof

(a) See for example [13, p. 123, Thm. 2.16.1].
(b) See for example [7, Thm. 6.3, p. 559].
(c) This follows immediatley from Theorem 1.3 in [**?**] as we have the universality limit ( ) We note that there was a mistake in Lemma 4.2(a) in [**?**] that was corrected in [ ]. However, the mistake did not affect Theorem 1.3 there. ■

Many of the assertions in the following lemma appear in the proof of Theorem 1.1 and 1.2 in [**?**], but we include proofs for the reader's convenience. We also note there was an error in Lemma 4.2(a) there, leading to an error in Lemma 4.3(d) and a gap in the proof of Theorems 1.1 and 1.2 of [**?**], but this was corrected in [ ]. Recall that R_n and g_n were defined by ( ), ( ).

Lemma 3.2
Let Γ be a proper subarc of Δ.
(a) For $|z| = 1$,

$$\frac{1}{n}R_n(z) = \operatorname{Re}\left[2g_n(z) - 1\right]. \tag{3.6}$$

(b) Uniformly for $z \in \Gamma$, and fixed real α,

$$\lim_{n\to\infty} \operatorname{Im}\left[e^{i\pi\alpha}\varphi_n(z)\overline{\varphi_n\left(ze^{2\pi i\alpha/n}\right)}\right]\mu'(z) = -\sin\pi\alpha. \tag{3.7}$$

(c) Uniformly for $z \in \Gamma$, and fixed real α,

$$\lim_{n\to\infty}\left\{\begin{array}{c} \operatorname{Re}\left[e^{\pi i\alpha}\varphi_n(z)\overline{\varphi_n\left(ze^{2\pi i\alpha/n}\right)}\right]\frac{1}{n}R_n\left(ze^{2\pi i\alpha/n}\right)\mu'(z) \\ -\left(2\sin\pi\alpha\right)\left(\operatorname{Im} g_n\left(ze^{2\pi i\alpha/n}\right)\right) \end{array}\right\} = \cos\pi\alpha. \tag{3.8}$$

(d) Uniformly for $z \in \Gamma$,

$$\lim_{n\to\infty} \frac{1}{n}R_n(z)\left|\varphi_n(z)\right|^2\mu'(z) = 1; \tag{3.9}$$

(e) Uniformly for $z \in \Gamma$, and fixed real α,

$$\Rightarrow \frac{\varphi_n\left(ze^{2\pi i\alpha/n}\right)}{\varphi_n(z)}\left(1+o(1)\right) + g_n(z)\,2ie^{-\pi i\alpha}\sin\pi\alpha = 1 + o(1). \tag{3.10}$$

Proof

(a) Elementary manipulation shows that

$$\frac{1-|z_{jn}|^2}{|z-z_{jn}|^2}=2\operatorname{Re}\left(\frac{z}{z-z_{jn}}\right)-1.$$

Dividing by $2n$ and adding for $j=1,2,...,n$ gives ( ).

(b) Let $\zeta = ze^{2\pi i\alpha/n}$. The Christoffel-Darboux formula and universality limit ( ) (as well as the uniformity of that limit) give uniformly for α in compact subsets of $\mathbb{C}$,

$$\begin{aligned}&\lim_{n\to\infty}\frac{\overline{\varphi_n^*(z)}\varphi_n^*(\zeta)-\overline{\varphi_n(z)}\varphi_n(\zeta)}{[1-\bar{z}\zeta]K_n(z,z)}\\ =&\lim_{n\to\infty}\frac{K_n(\zeta,z)}{K_n(z,z)}\\ =&\lim_{n\to\infty}\frac{K_n\left(z\left(1+\frac{2\pi i\alpha}{n}[1+o(1)]\right),z\right)}{K_n(z,z)}=e^{i\pi\alpha}\mathbb{S}(\alpha).\end{aligned}$$

(3.11)

Here by (4.10),

$$\lim_{n\to\infty}[1-\bar{z}\zeta]K_n(z,z)=-2\pi i\alpha\mu'(z)^{-1}.$$

Thus

$$\begin{aligned}\lim_{n\to\infty}\left[\overline{\varphi_n^*(z)}\varphi_n^*(\zeta)-\overline{\varphi_n(z)}\varphi_n(\zeta)\right]\mu'(z)&=-2\pi i\alpha e^{i\pi\alpha}\mathbb{S}(\alpha)\\ &=-2ie^{\pi i\alpha}\sin\pi\alpha=1-e^{2\pi i\alpha}.\end{aligned}\tag{3.12}$$

Next, if α is real,

$$\overline{\varphi_n^*(z)}\varphi_n^*(\zeta)=e^{2\pi i\alpha}\varphi_n(z)\overline{\varphi_n(\zeta)}$$

so combining this and the last two limits gives

$$\lim_{n\to\infty}e^{\pi i\alpha}\left\{e^{i\pi\alpha}\varphi_n(z)\overline{\varphi_n(\zeta)}-e^{-\pi i\alpha}\overline{\varphi_n(z)}\varphi_n(\zeta)\right\}\mu'(z)=-2ie^{\pi i\alpha}\sin\pi\alpha.$$

(c) We go back to ( ), which holds uniformly for α in compact subsets of $\mathbb{C}$. This uniformity allows us to differentiate with respect to α: after cancelling a factor of $2\pi i$, we obtain

$$\lim_{n\to\infty}\left[\overline{\varphi_n^*(z)}\varphi_n^{*\prime}(\zeta)-\overline{\varphi_n(z)}\varphi_n'(\zeta)\right]\frac{\zeta}{n}\mu'(z)=-e^{2\pi i\alpha}.\tag{3.13}$$

Now we again specialize to real α, and use that for $|\zeta|=1$,

$$\varphi_n^{*\prime}(\zeta)=n\zeta^{n-1}\overline{\varphi_n(\zeta)}-\zeta^{n-2}\overline{\varphi_n'(\zeta)}$$

so that, recalling the definition of g_n,

$$\overline{\varphi_n^*(z)}\varphi_n^{*\prime}(\zeta)\frac{\zeta}{n}=e^{2\pi i\alpha}\varphi_n(z)\overline{\varphi_n(\zeta)}-e^{2\pi i\alpha}\varphi_n(z)\overline{\varphi_n(\zeta)g_n(\zeta)}.$$

Substituting in ( ), and cancelling a factor of $e^{\pi i\alpha}$,

(3.14)

$$\lim_{n\to\infty}\left[e^{\pi i\alpha}\varphi_n(z)\overline{\varphi_n(\zeta)}-e^{\pi i\alpha}\varphi_n(z)\overline{\varphi_n(\zeta)g_n(\zeta)}-e^{-\pi i\alpha}\overline{\varphi_n(z)}\varphi_n(\zeta)g_n(\zeta)\right]\mu'(z)=-e^{\pi i\alpha}$$

or

$$\lim_{n\to\infty}\left[e^{\pi i\alpha}\varphi_n(z)\overline{\varphi_n(\zeta)}-2\operatorname{Re}\left\{e^{\pi i\alpha}\varphi_n(z)\overline{\varphi_n(\zeta)g_n(\zeta)}\right\}\right]\mu'(z)=-e^{\pi i\alpha}.$$

Taking real parts,

$$\lim_{n\to\infty}\operatorname{Re}\left[e^{\pi i\alpha}\varphi_n(z)\overline{\varphi_n(\zeta)}\left\{1-2\overline{g_n(\zeta)}\right\}\right]=-\cos\pi\alpha.$$

Then using (b),

$$\lim_{n\to\infty}\left\{\begin{array}{c}\operatorname{Re}\left[e^{\pi i\alpha}\varphi_n(z)\overline{\varphi_n(\zeta)}\right]\operatorname{Re}\left[1-2g_n(\zeta)\right]\\ +2\sin\pi\alpha\operatorname{Im}g_n(\zeta)\end{array}\right\}\mu'(z)=-\cos\pi\alpha.$$

Finally apply (a).
(d) Here we set $\alpha=0$, then $\alpha=\frac{1}{2}$, then $\alpha=1$ in (c).
(e) From (a),

$$\overline{g_n(\zeta)}=2\operatorname{Re}g_n(\zeta)-g_n(\zeta)=\frac{1}{n}R_n(\zeta)+1-g_n(\zeta).$$

We substitute this in ( ):

$$\lim_{n\to\infty}\left[-e^{\pi i\alpha}\varphi_n(z)\overline{\varphi_n(\zeta)}\frac{1}{n}R_n(\zeta)+g_n(\zeta)\left\{e^{\pi i\alpha}\varphi_n(z)\overline{\varphi_n(\zeta)}-e^{-\pi i\alpha}\overline{\varphi_n(z)}\varphi_n(\zeta)\right\}\right]\mu'(z)=-e^{\pi i\alpha}$$

Using (d) and (b), we obtain

$$\lim_{n\to\infty}\left[-e^{\pi i\alpha}\frac{\varphi_n(z)}{\varphi_n(\zeta)}(1+o(1))+g_n(\zeta)\,2i(-\sin\pi\alpha)\right]=-e^{\pi i\alpha}$$

$$\Rightarrow\lim_{n\to\infty}\left[\frac{\varphi_n(z)}{\varphi_n(\zeta)}(1+o(1))+g_n(\zeta)\,2ie^{-\pi i\alpha}\sin\pi\alpha\right]=1.$$

Because of the uniformity, we can substitute $ze^{-2\pi i\alpha}$ for z so that $\zeta=z$. ■

We now prove parts of Theorems 2.1, 2.2:

Lemma 3.3
(a) The following are equivalent:
(i)

$$\inf\left\{n(1-|z_{jn}|):\tau_{jn}\in J,n\geq 1\right\}\geq C>0.$$

(ii)

$$\sup_{\zeta\in\Gamma,n\geq 1}\frac{1}{n^2}\sum_{j=1}^{n}\frac{1}{|\zeta_n-z_{jn}|^2}<\infty.$$

(b) The following are equivalent:
(i) As $n\to\infty$,

$$\inf\left\{n(1-|z_{jn}|):\tau_{jn}\in J\right\}\to\infty$$

(ii)

$$\sup_{\zeta\in I}\frac{1}{n^2}\sum_{j=1}^{n}\frac{1}{|\zeta_n-z_{jn}|^2}=o(1).$$

Proof
(a) (i)⇒(ii)
We have

$$\begin{aligned}\frac{1}{n^2}\sum_{\tau_{jn}\in J}\frac{1}{|\zeta-z_{jn}|^2}&\leq\frac{C^{-1}}{n}\sum_{\tau_{jn}\in J}\frac{1-|z_{jn}|^2}{|\zeta-z_{jn}|^2}\\&\leq\frac{1}{Cn}R_n(\zeta)\leq C_1|\varphi_n(\zeta)|^{-2},\end{aligned}$$

by the old (4.8). Now we know from (4.7) that

$$\left|\varphi_n(\zeta)\right|\left|\varphi_n\left(\zeta e^{i\pi/n}\right)\right| \geq 1+o(1)$$

so it follows that either

$$\frac{1}{n^2}\sum_{\tau_{jn}\in J}\frac{1}{\left|\zeta-z_{jn}\right|^2}\leq C \text{ or } \frac{1}{n^2}\sum_{\tau_{jn}\in J}\frac{1}{\left|\zeta e^{i\pi/n}-z_{jn}\right|^2}\leq C$$

(or possibly both). But because of our hypothesis, for $\tau_{jn},\zeta\in J$,

$$\begin{aligned}\left|\frac{\zeta-z_{jn}}{\zeta e^{i\pi/n}-z_{jn}}\right| &= \left|1+\frac{\zeta\left(1-e^{i\pi/n}\right)}{\zeta e^{i\pi/n}-z_{jn}}\right| \\ &\leq 1+\frac{2\sin(\pi/2n)}{1-|z_{jn}|}\leq C\end{aligned}$$

while a similar bound holds for the reciprocal. So for $\zeta\in J$,

$$\frac{1}{n^2}\sum_{\tau_{jn}\in J}\frac{1}{\left|\zeta-z_{jn}\right|^2}\leq C.$$

Then also for the remaining terms,

$$\frac{1}{n^2}\sum_{\tau_{jn}\notin J}\frac{1}{\left|\zeta-z_{jn}\right|^2}\leq\frac{Cn}{n^2}=o(1).$$

(OK one might need to worry near the boundary).
(ii)⇒(i)
Choosing $\zeta=\tau_{jn}\in J$ gives

$$\frac{1}{n^2\left(1-|z_{jn}|\right)^2}=\frac{1}{n^2\left|\zeta-z_{jn}\right|^2}\leq C,$$

by our hypothesis.
(b) (i)⇒(ii)
We have

$$\begin{aligned}\frac{1}{n^2}\sum_{\tau_{jn}\in J}\frac{1}{\left|\zeta-z_{jn}\right|^2} &\leq \frac{1}{n}\sum_{\tau_{jn}\in J}\frac{1-|z_{jn}|^2}{\left|\zeta-z_{jn}\right|^2}\frac{1}{\inf_{\tau_{jn}\in J} n\left(1-|z_{jn}|^2\right)} \\ &= o\left(\frac{1}{n}R_n(\zeta)\right).\end{aligned}$$

We can now proceed as in (a).
(b) (ii)⇒(i)
Again, proceed much as in (a). ■

4. Proof of Theorem 2.1

Proof of Theorem 2.1
(a)⇔(b)
This is immediate from Lemma 3.2(d).
(b)⇔(c)
This is immediate from Lemma 3.2(a).

(c)⇔(d)
It is immediate that (d)⇒(c). Now assume (c) holds. From Lemma 3.2(c) with $\alpha = \frac{1}{2}$, and Lemma 3.2(d),

$$\lim_{n\to\infty} \left\{ \operatorname{Im} \left[\frac{\varphi_n(z)\overline{\varphi_n\left(ze^{i\pi/n}\right)}}{|\varphi_n(z)|\left|\varphi_n\left(ze^{i\pi/n}\right)\right|} \right] - 2\left(\operatorname{Im} g_n\left(ze^{\pi i/n}\right)\right) \right\} = 0. \tag{4.1}$$

Next, using (a) for $\zeta = z, ze^{i\pi/n}$

$$\lim_{n\to\infty} |\varphi_n(\zeta)|^2 \mu'(\zeta) = 1,$$

while from 3.2(d) with $\alpha = \frac{1}{2}$,

$$\lim_{n\to\infty} \operatorname{Re}\left[\varphi_n(z)\overline{\varphi_n\left(ze^{i\pi/n}\right)}\right]\mu'(z) = -1$$

so that

$$\lim_{n\to\infty} \operatorname{Re}\left[\frac{\varphi_n(z)\overline{\varphi_n\left(ze^{i\pi/n}\right)}}{|\varphi_n(z)|\left|\varphi_n\left(ze^{i\pi/n}\right)\right|}\right] = -1$$

and hence

$$\lim_{n\to\infty} \operatorname{Im}\left[\frac{\varphi_n(z)\overline{\varphi_n\left(ze^{i\pi/n}\right)}}{|\varphi_n(z)|\left|\varphi_n\left(ze^{i\pi/n}\right)\right|}\right] = 0.$$

Then

$$\lim_{n\to\infty} \operatorname{Im} g_n\left(ze^{\pi i/n}\right) = 0. \tag{4.2}$$

Because of the unformity in z, we may replace $ze^{i\pi/n}$ by z. So indeed (c)⇒(d).
(d) ⇔(e)
From Lemma 3.2(a) with $\alpha = \frac{1}{2}$,

$$\frac{\varphi_n\left(ze^{\pi i/n}\right)}{\varphi_n(z)}(1 + o(1)) + 2g_n(z) = 1 + o(1)$$

and so ( ) holds iff $g_n \to 1$.
(a)⇔(f)
Let Γ_1 be a proper subarc of Δ containing Γ. Assume first the conclusion ( ) of (a) holds. We apply Lemma 3.1(c), so must verify there. The first condition in ( ) follows immediately from (d). For the second, observe first from Lemma 3.2(d) and our hypothesis, that

$$\frac{1}{n^2}\sum_{z_{jn}\in\Gamma_1}\frac{1}{|z - z_{jn}|^2} \le \frac{C}{n}\sum_{z_{jn}\in\Gamma_1}\frac{1 - |z_{jn}|^2}{|z - z_{jn}|^2} \le \frac{C}{n}R_n(z) \le C.$$

Next, for $z \in \Gamma$, if d is the distance from Γ to Γ_1,

$$\frac{1}{n^2}\sum_{z_{jn}\notin\Gamma_1}\frac{1}{|z - z_{jn}|^2} \le \frac{n}{n^2 d} = o(1).$$

Thus

$$\frac{1}{n^2}\sum_{j=1}^{n}\frac{1}{|z - z_{jn}|^2} \le C$$

and we have the second condition in ( ). From Lemma 3.4, we obtain that every subsequence of positive integers contains a further subsequence $\mathcal{S}$ such that uniformly for u in compact subsets of $\mathbb{C}$, we have ( ), where C is given by ( ). But from (d), $C = 0$, so the limit is independent of the subsequence, and we have ( ).

Now conversely assume we have the local limit ( ). Then setting $u = i\pi/n$ and using the uniformity,

$$\lim_{n\to\infty} \frac{\varphi_n(ze^{i\pi/n})}{\varphi_n(z)} = \lim_{n\to\infty} \frac{\varphi_n(z\left(1+\frac{i\pi}{n}\left[1+o(1)\right]\right))}{\varphi_n(z)} = e^{i\pi} = -1,$$

so we have ( ) and hence the result from (x?).

(f)⇒(g)

This is a consequence of the fact that e^u has no zeros. Indeed, if there were a subsequence of zeros z_{jn}, $n \in \mathcal{S}$, $j = j(n)$, then writing $z_{jn} = \zeta_n(1 + i\alpha_n/n)$, we have $\alpha_n = O(1)$, and by the local limit,

$$0 = \frac{\varphi_n(\zeta_n(1 + i\alpha_n/n))}{\varphi_n(\zeta_n)} = e^{\pi i\alpha_n} + o(1).$$

leading to a contradiction.

(g)⇔(h)

This is Lemma 3.3.

(h)⇒(f)

Now

$$\begin{aligned} \frac{1}{n}g_n'(z) &= \frac{1}{n^2}\frac{d}{dz}\left(\sum_{j=1}^n \left[1 + \frac{z_{jn}}{z - z_{jn}}\right]\right) \\ &= -\frac{1}{n^2}\sum_{j=1}^n \frac{z_{jn}}{(z - z_{jn})^2}. \end{aligned}$$

Our hypothesis gives uniformly for $z \in \Gamma$,

$$\frac{1}{n}g_n'(z) = o(1)$$

and hence for $\zeta, z \in \Gamma$ with $|\zeta - z| \leq A/n$,

$$|g_n(z) - g_n(\zeta)| = o(1).$$

Then from Lemma 3.2(e), with $\zeta = ze^{-i\pi/n}$,

$$\left| \frac{\varphi_n\left(ze^{i\pi/n}\right)}{\varphi_n(z)} - \frac{\varphi_n(z)}{\varphi_n\left(ze^{i\pi/n}\right)} \right| = o(1)$$

so that in view of ( ),

$$\lim_{n\to\infty} \frac{\varphi_n\left(ze^{i\pi/n}\right)}{\varphi_n(z)} = -1.$$

Then we have the conclusion of (e) and ( ) gives the result. ■

5. Proof of Theorem 2.2

Lemma 5.1

Assume

$$\sup_{n\leq 1} \|\varphi_n\|_{L_\infty(\Gamma)} < \infty,$$

then there exists $C > 0$ such that for $n \geq 1$ and $z_{jn} \neq 0$, $\frac{z_{jn}}{|z_{jn}|} \in \Gamma$

$$n(1 - |z_{jn}|) \geq C.$$

Proof

Suppose the conclusion is false. Then we can can choose an infinite subsequence $\mathcal{S}$ of integers, and for $j = j(n) \in \mathcal{S}$,

$$n(1 - |z_{jn}|) \to 0.$$

Write

$$z_{jn} = \tau_{jn}\left(1 + 2\pi i \frac{\alpha_n}{n}\right); u = \tau_{jn}\left(1 + 2\pi i \frac{\bar{v}}{n}\right)$$

where $v = v(n)$ and $\alpha_n \to 0$ as $n \to \infty$. Then from the university limit, uniformly for v in compact sets,

$$\begin{aligned}\frac{K_n(z_{jn}, u)}{K_n(\tau_{jn}, \tau_{jn})} &= e^{i\pi(v-\alpha_n)}\mathbb{S}(v - \alpha_n) + o(1) \\ &= e^{i\pi v}\mathbb{S}(v) + o(1).\end{aligned}$$

Next from the Christoffel-Darboux formula,

$$\overline{\varphi_n^*(z_{jn})}\varphi_n(u) = \{K_n(\tau_{jn}, \tau_{jn})(1 - \bar{u}z_{jn})\}\frac{K_n(z_{jn}, u)}{K_n(\tau_{jn}, \tau_{jn})} \tag{5.1}$$

and setting $u = \tau_{jn}$ so that $v = 0$ in both formulas, gives

$$\begin{aligned}\overline{\varphi_n^*(z_{jn})}\varphi_n(\tau_{jn}) &= K_n(\tau_{jn}, \tau_{jn})(1 - |z_{jn}|)\frac{K_n(z_{jn}, \tau_{jn})}{K_n(\tau_{jn}, \tau_{jn})} \\ &= o(1)(1 + o(1)) = o(1). \qquad (5.2)\end{aligned}$$

Now apply the above with $u = \tau_{jn}e^{i\pi/n}$, so that $v = -\frac{1}{2} + o(1)$,

$$\frac{K_n\left(z_{jn}, \tau_{jn}e^{i\pi/n}\right)}{K_n(\tau_{jn}, \tau_{jn})} = e^{-i\pi/2}\mathbb{S}\left(\frac{1}{2}\right) + o(1),$$

while

$$\begin{aligned}&\overline{\varphi_n^*(z_{jn})}\varphi_n\left(\tau_{jn}e^{i\pi/n}\right) \\ =\ &\left\{K_n(\tau_{jn}, \tau_{jn})\left(1 - e^{-i\pi/n}|z_{jn}|\right)\right\}\frac{K_n(z_{jn}, u)}{K_n(\tau_{jn}, \tau_{jn})} \\ =\ &\left\{K_n(\tau_{jn}, \tau_{jn})\left(1 - e^{-i\pi/n} + o\left(\frac{1}{n}\right)\right)\right\}\left\{e^{-i\pi/2}\mathbb{S}\left(\frac{1}{2}\right) + o(1)\right\}\end{aligned}$$

so that

$$\left|\overline{\varphi_n^*(z_{jn})}\varphi_n\left(\tau_{jn}e^{i\pi/n}\right)\right| \sim 1.$$

Dividing (5.2) by this, gives

$$\left|\frac{\varphi_n(\tau_{jn})}{\varphi_n\left(\tau_{jn}e^{i\pi/n}\right)}\right| = o(1).$$

But from the old (4.7),

$$\left|\varphi_n\left(\tau_{jn}e^{i\pi/n}\right)\varphi_n(\tau_{jn})\right| \geq 1 + o(1),$$

so

$$\begin{aligned}\left|\varphi_n\left(\tau_{jn}e^{i\pi/n}\right)\right|^2 &= \left|\frac{\varphi_n(\tau_{jn})}{\varphi_n\left(\tau_{jn}e^{i\pi/n}\right)}\right|^{-1}\left|\varphi_n\left(\tau_{jn}e^{i\pi/n}\right)\varphi_n(\tau_{jn})\right| \\ &\to \infty \text{ as } n \to \infty,\end{aligned}$$

contradicting boundedness. ■

Proof of Theorem 2.1

(a)⇔(b)

This is immediate from Lemma 3.2(d).

(b)⇔(c)

This is immediate from Lemma 3.2(a).

(c)⇔(d)

This follows from Lemma 3.2(e).

(a)⇒(e)

This follows from Lemma 5.1.

(e)⇔(f)

This was proved in Lemma 3.3.

(f)⇒(a)

Assume the result is false. Then can choose a sequence $\mathcal{S}$ and for $n \in \mathcal{S}$, ζ_n such that

$$\left|\varphi_n\left(\zeta_n\right)\right| \rightarrow \infty.$$

Then from ( )

$$\frac{1}{n} R_n\left(\zeta_n\right) \rightarrow 0.$$

But then for $\left|\zeta_n - z_n\right| \leq \frac{C}{n}$,

$$\frac{1}{n} R_n\left(z_n\right) \rightarrow 0.$$

Then also

$$\frac{1}{n^2} \sum_{\tau_{jn} \in J} \frac{1}{\left|z_n - z_{jn}\right|^2} \leq \frac{C}{n} \sum_{\tau_{jn} \in J} \frac{1 - \left|z_{jn}\right|^2}{\left|z_n - z_{jn}\right|^2} = o(1)$$

while the tail sum is smaller, so

$$\frac{1}{n^2} \sum_{j=1}^{n} \frac{1}{\left|z - z_{jn}\right|^2} = o(1).$$

Then

$$\begin{aligned}
\frac{1}{n} \frac{d}{dz}\left(\frac{z\varphi_n'(z)}{n\varphi_n(z)}\right) &= \frac{1}{n^2} \frac{d}{dz}\left(\sum_{j=1}^{n} \frac{z}{z - z_{jn}}\right) \\
&= \frac{1}{n^2} \frac{d}{dz}\left(\sum_{j=1}^{n}\left[1 + \frac{z_{jn}}{z - z_{jn}}\right]\right) \\
&= -\frac{1}{n^2} \sum_{j=1}^{n} \frac{z_{jn}}{\left(z - z_{jn}\right)^2} = o(1).
\end{aligned}$$

It follows that if $\left|z_n - \zeta_n\right| \leq C/n$,

$$\frac{z_n\varphi_n'\left(z_n\right)}{n\varphi_n\left(z_n\right)} - \frac{\zeta_n\varphi_n'\left(\zeta_n\right)}{n\varphi_n\left(\zeta_n\right)} = o(1).$$

From the old (4.12),

$$\left|\frac{\varphi_n\left(\zeta_n e^{i\pi//n}\right)}{\varphi_n\left(\zeta_n\right)} - \frac{\varphi_n\left(\zeta_n\right)}{\varphi_n\left(\zeta_n e^{i\pi//n}\right)}\right| = o(1)$$

which forces
$$\frac{\varphi_n\left(\zeta_n e^{i\pi//n}\right)}{\varphi_n\left(\zeta_n\right)} = 1 + o\left(1\right)$$
which forces
$$\frac{\zeta_n \varphi_n'\left(\zeta_n\right)}{n\varphi_n\left(\zeta_n\right)} = 1 + o\left(1\right)$$
which forces from the old (4.9) that
$$\left|\varphi_n\left(\zeta_n\right)\right|^2 \mu'\left(\zeta_n\right) = 1 + o\left(1\right)$$
which forces
$$\frac{1}{n} R_n\left(\zeta_n\right) = 1 + o\left(1\right).$$
This contradicts our hypothesis. So uniformly in n and $\zeta \in J$,
$$\frac{1}{n} R_n\left(\zeta\right) \geq C$$
which forces
$$\left\|\varphi_n\right\|_{L_\infty(J)} \leq C.$$
■

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