Orthogonal Dirichlet Polynomials

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Abstract Let $\{\lambda_j\}_{j=1}^{\infty}$ be a sequence of distinct positive numbers. Let w be a non-negative function, integrable on the real line. One can form orthogonal Dirichlet polynomials $\{\phi_n\}$ from linear combinations of $\left\{\lambda_j^{-it}\right\}_{j=1}^{n}$, satisfying the orthogonality relation

$$\int_{-\infty}^{\infty} \phi_n(t)\, \overline{\phi_m(t)} w(t)\, dt = \delta_{mn}.$$

Weights that have been considered include the arctan density $w(t) = \frac{1}{\pi(1+t^2)}$; rational function choices of w; $w(t) = e^{-t}$; and $w(t)$ constant on an interval symmetric about 0. We survey these results and discuss possible future directions.

1 Introduction

Throughout, let

$$\{\lambda_j\}_{j=1}^{\infty} \text{ be a sequence of distinct positive numbers.} \tag{1.1}$$

Given $m \geq 1$, a *Dirichlet polynomial of degree* $\leq m$ [17, 23] associated with this sequence of exponents has the form

Research supported by NSF grant DMS1800251.

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N. J. Daras, Th. M. Rassias (eds.), *Approximation and Computation in Science and Engineering*, Springer Optimization and Its Applications 180,
https://doi.org/10.1007/978-3-030-84122-5_30

$$\sum_{n=1}^{m} a_n \lambda_n^{-it} = \sum_{n=1}^{m} a_n e^{-i(\log \lambda_n)t},$$

where $\{a_n\} \subset \mathbb{C}$. We denote the set of all such polynomials by $\mathcal{L}_m$.

The theory of almost-periodic functions [2, 3] is based on orthogonality in the mean:

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \lambda_j^{-it} \overline{\lambda_k^{-it}} dt = \delta_{jk}.$$

Thus in an asymptotic sense, the "monomials" $\left\{\lambda_j^{-it}\right\}_{j\geq 1}$ are orthonormal polynomials. In the hope that a more standard orthogonality relation might have some advantages, the author [6] introduced Dirichlet orthogonal polynomials associated with the arctan density.

In the general case, one can consider a non-negative function w, integrable on the real line, and positive on a set of positive measure. The corresponding orthonormal polynomials $\phi_n \in \mathcal{L}_n$ have positive leading coefficient and satisfy

$$\int_{-\infty}^{\infty} \phi_n(t) \overline{\phi_m(t)} w(t)\, dt = \delta_{mn}, \quad m, n \geq 1.$$

If we use as the inner product

$$(f, g) = \int_{-\infty}^{\infty} f(t) \overline{g(t)} w(t)\, dt$$

and assume $\int_{-\infty}^{\infty} w = 1$, then ϕ_n admits the representation

$$\phi_n(x) = \frac{(-1)^{n+1}}{\sqrt{A_{n-1}A_n}} \times \det \begin{bmatrix} \lambda_1^{-ix} & \lambda_2^{-ix} & \lambda_3^{-ix} & \cdots & \lambda_n^{-ix} \\ 1 & \left(\lambda_1^{-it}, \lambda_2^{-it}\right) & \left(\lambda_1^{-it}, \lambda_3^{-it}\right) & \cdots & \left(\lambda_1^{-it}, \lambda_n^{-it}\right) \\ \left(\lambda_2^{-it}, \lambda_1^{-it}\right) & 1 & \left(\lambda_2^{-it}, \lambda_3^{-it}\right) & \cdots & \left(\lambda_2^{-it}, \lambda_n^{-it}\right) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \left(\lambda_{n-1}^{-it}, \lambda_1^{-it}\right) & \left(\lambda_{n-1}^{-it}, \lambda_2^{-it}\right) & \left(\lambda_{n-1}^{-it}, \lambda_3^{-it}\right) & \cdots & \left(\lambda_{n-1}^{-it}, \lambda_n^{-it}\right) \end{bmatrix}, \quad (1.2)$$

where

$$A_n = \det\left[\left(\lambda_j^{-it}, \lambda_k^{-it}\right)\right]_{1\leq j,k\leq n}. \quad (1.3)$$

The leading coefficient of $\phi_n(x)$ is

$$\gamma_n = \sqrt{\frac{A_{n-1}}{A_n}}.$$

In analyzing orthonormal polynomials, the reproducing kernels

$$K_n(x, y) = \sum_{j=1}^{n} \phi_j(x) \overline{\phi_j(y)}$$

are useful. The nth *Christoffel function* is

$$1/K_n(x, x) = 1/\sum_{j=1}^{n} \left|\phi_j(x)\right|^2.$$

The extremal property

$$K_n(x, x) = \sup_{P \in \mathcal{L}_n} \frac{|P(x)|^2}{\int_{-\infty}^{\infty} |P|^2 w}$$

facilitates estimation of $K_n(x, x)$ and the Christoffel function. The extremal property is an easy consequence of the Cauchy-Schwarz inequality.

Examples of weights w for which some analysis has been undertaken are the arctan density

$$w(t) = \frac{1}{\pi\left(1 + t^2\right)}, \quad t \in \mathbb{R};$$

rational functions of special form; $w(t) = e^{-t}, t \in [0, \infty)$ and $w(t) = 1$ on $[-T, T]$, $T > 0$. We shall survey some of the results in Sects. 2–5. It seems of some interest to develop also a theory for general weights.

One reason for studying Dirichlet orthogonal polynomials is that they might offer some insight into the behavior of general Dirichlet polynomials, just as classical orthogonal polynomials are useful in analyzing algebraic polynomials $P(x) = \sum_{j=0}^{n} c_j x^j$. There is of course a vast literature on Dirichlet polynomials, with connections to Turán's formulation of the Lindelöf hypothesis, Hilbert's inequality, the large sieve of number theory, the Montgomery-Vaughn theory, and higher dimensional results such as the Vinogradov Mean Value Theorem. We cannot hope to review these here, but present a few results relevant to our topic:

The classical conjecture of Lindelöf asserts that given $\varepsilon > 0$, the Riemann ζ function admits the bound

$$|\zeta(s + it)| \leq C(\varepsilon)(2 + |t|)^{\varepsilon}$$

provided $s \geq \frac{1}{2}$ and $s+it$ lies outside a small disk centered on 1. Using a very simple argument, Turán showed in a 1962 paper [22] that this conjecture is equivalent to the estimate on a specific Dirichlet polynomial: given $\varepsilon > 0$, we have for all real t and $n \geq 1$,

$$\left| \sum_{j=1}^{n} (-1)^j \, j^{-it} \right| < C(\varepsilon)\, n^{\frac{1}{2}+\varepsilon} (2+|t|)^{\varepsilon}.$$

Another classical connection, to Hilbert's inequality, involves the Montgomery-Vaughan refinement of the Mean Value Theorem. There are several versions, among them [13, 14], [15, p. 74, Corollary 2]

$$\int_0^T \left| \sum_{j=1}^{n} a_j \lambda_j^{-it} \right|^2 dt = T \sum_{j=1}^{n} \left| a_j^2 \right| + 3\pi\theta \sum_{j=1}^{n} \left| a_j^2 \right| \delta_j^{-1}. \tag{1.4}$$

Here $T > 0$, and (in the notation here):

$$\delta_j = \min \left\{ \left| \log \lambda_j - \log \lambda_k \right| : k \neq j, k \leq n \right\},$$

while $|\theta| \leq 1$.

A much more recent result is Weber's Mean Value Theorem [24] when there are non-negative coefficients:

$$\int_0^T \left| \sum_{j=1}^{n} a_j \lambda_j^{-it} \right|^{2q} dt \geq cT \left(\sum_{j=1}^{n} a_j^2 \right)^q.$$

Here we assume that q is a positive integer, all $a_j \geq 0$, while c is independent of $N, \{a_j\}, \{\lambda_j\}$.

This paper is organized as follows: in Sect. 2, we review results for the arctangent density. In Sect. 3, we consider the exponential weight and the connection to Müntz orthogonal polynomials. In Sect. 4, we look at rational weights, and in Sect. 5, we look at constant weights on $[-T, T]$.

2 The Arctangent Density

Let

$$w(t) = \frac{1}{\pi \left(1+t^2\right)}, \quad t \in \mathbb{R}.$$

We also assume that

$$1 = \lambda_1 < \lambda_2 < \lambda_3 < \cdots$$

It was shown in [6] that $\phi_1 = 1$ and for $n \geq 2$,

$$\phi_n(t) = \frac{\lambda_n^{1-it} - \lambda_{n-1}^{1-it}}{\sqrt{\lambda_n^2 - \lambda_{n-1}^2}}.$$

Here it is essential that the $\{\lambda_j\}$ are increasing, while it is intriguing that ϕ_n involves only the last two powers. The proof of course is elementary, and based on the following integral (itself a simple consequence of the residue theorem):

$$\int_{-\infty}^{\infty} \frac{e^{i\mu t}}{\pi\left(1+t^2\right)} dt = e^{-|\mu|}.$$

The nth reproducing kernel along the diagonal is given for real x by [6, p. 46]

$$K_n(x,x) = 1 + \sum_{n=1}^{m} \frac{1}{\lambda_n^2 - \lambda_{n-1}^2}\left[(\lambda_n - \lambda_{n-1})^2 + 4\lambda_{n-1}\lambda_n \sin^2\left(\frac{x}{2}\log\frac{\lambda_n}{\lambda_{n-1}}\right)\right].$$

Because of the simple explicit form, it is easy to do analysis. Thus one can check that

$$\sup_{t\in\mathbb{R}} |\phi_n(t)| = \sqrt{\frac{\lambda_n + \lambda_{n-1}}{\lambda_n - \lambda_{n-1}}}$$

while

$$\sup_{t\in\mathbb{R}} \left|\phi_n'(t)\right| = \frac{\lambda_n \log\lambda_n + \lambda_{n-1}\log\lambda_{n-1}}{\sqrt{\lambda_n^2 - \lambda_{n-1}^2}}.$$

The zeros of ϕ_n have the form $-i + \frac{2k\pi}{\log(\lambda_n/\lambda_{n-1})}$, $k \in \mathbb{Z}$.

If $\lambda_m \to \infty$, as $m \to \infty$, the reproducing kernel admits the asymptotic

$$\lim_{m\to\infty} \frac{1}{\log\lambda_m} K_m(x,x) = \frac{1+x^2}{2},$$

uniformly for x in compact subsets of the real line. The universality limit takes the form

$$\lim_{m\to\infty}\frac{1}{\log\lambda_m}K_m\left(x+\frac{\alpha}{\log\lambda_m},x+\frac{\beta}{\log\lambda_m}\right)=\frac{1+x^2}{2}e^{i(\beta-\alpha)/2}\mathbb{S}\left(\frac{\alpha-\beta}{2}\right),$$

where

$$\mathbb{S}(t)=\frac{\sin t}{t}$$

is the usual sinc kernel. The limit holds uniformly for x in compact subsets of $\mathbb{R}$ and α,β in compact subsets of $\mathbb{C}$. Markov-Bernstein inequalities for derivatives of Dirichlet polynomials were also established in [6].

Orthonormal expansions in the $\{\phi_n\}$ were also considered there, and in the follow up paper [7]. For example, it was shown using such orthonormal expansions that if

$$f(t)=\sum_{n=1}^{\infty}a_n\lambda_n^{-it}$$

where the coefficients are complex numbers, and $r>0$, then

$$\int_{-\infty}^{\infty}|f(rt)|^2\frac{dt}{\pi\left(1+t^2\right)}=\sum_{k=1}^{\infty}\left(\lambda_k^{2r}-\lambda_{k-1}^{2r}\right)\left|\sum_{n=k}^{\infty}\frac{a_n}{\lambda_n^r}\right|^2,$$

provided the series on the right-hand side converges. This was used to establish a number of inequalities of Hilbert/mean value type. If for example, $r>0$ and $\{a_k\}$ are non-negative numbers with $\left\{a_k/\lambda_k^r\right\}$ decreasing, then

$$F(t)=\sum_{n=1}^{\infty}(-1)^{n-1}a_n\lambda_n^{-it}$$

satisfies

$$\int_{-\infty}^{\infty}|F(rt)|^2\frac{dt}{\pi\left(1+t^2\right)}\leq\sum_{n=1}^{\infty}a_n^2.$$

M. Weber used the orthonormal expansions above in studying Cauchy means of Dirichlet polynomials and series, with a more definitive version of the limits for orthonormal expansions than given in [6, 7]. For example, he proved that if q is a positive integer, and $\{a_n\}$ are complex, [26, p. 65, Proposition 1.4]

$$\lim_{s\to\infty}\int_{-\infty}^{\infty}\left|\sum_{j=1}^{\infty}a_j j^{-ist}\right|^{2q}\frac{dt}{\pi\left(1+t^2\right)}=\lim_{s\to\infty}\frac{1}{2s}\int_{-\infty}^{\infty}\left|\sum_{j=1}^{\infty}a_j j^{-it}\right|^{2q}dt,$$

provided the limit on the right exists. He established estimates such as [26, p. 65, Proposition 1.5]

$$\frac{1}{S}\int_0^S \left|\sum_{j=1}^{n} a_j j^{-it}\right|^{2q} dt \le \frac{2\pi}{\log 2}\sup_{S\le s\le 2S}\int_{-\infty}^{\infty}\left|\sum_{j=1}^{N} a_j j^{-ist}\right|^{2q}\frac{dt}{\pi\left(1+t^2\right)}.$$

Another application has been given by D. Dimitrov and W.D. Oliviera [5], to finding the Dirichlet polynomials that minimize

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|P\left(\frac{1}{p}+it\right)\right|^2\frac{dt}{\frac{1}{p}+t^2}$$

among all Dirichlet polynomials of degree $\le n$ satisfying the interpolation conditions $P\left(\frac{1}{p}+it_j\right)=1$, at m distinct points $\left\{t_j\right\}_{j=1}^{m}$. See also [16].

3 Laguerre Weight

Let

$$w(t)=e^{-t},\quad t\in[0,\infty),$$

so that our orthogonality relation becomes

$$\int_0^{\infty}\phi_n(t)\,\overline{\phi_m(t)}e^{-t}dt=\delta_{mn}. \tag{3.1}$$

In [8], it was shown that

$$\phi_n(t)=\frac{\Delta_n}{2\pi i}\int_{\Gamma}e^{-tz}R_n(t)\,dt,$$

where Γ is a simple closed positively oriented curve in the half plane Re $z>-1$ that encloses $i\log\lambda_j$, $1\le j\le n$, while

$$R_n(z)=\frac{1}{z-i\log\lambda_n}\prod_{j=1}^{n-1}\left(1+\frac{1}{z-i\log\lambda_j}\right);$$

$$\Delta_n=\frac{D_n}{|D_n|};$$

and

$$D_n = \prod_{j=1}^{n-1} \left(1 + \left[i \log \frac{\lambda_j}{\lambda_n} \right]^{-1} \right).$$

For $x \in (0, \infty)$, there is the simplified form

$$\phi_n (x) = -\Delta_n \frac{e^{\alpha x}}{2\pi} \int_{-\infty}^{\infty} e^{-ixs} R_n (-\alpha + is)\, ds.$$

Here $\alpha \in (0, 1)$. It was shown there that

$$\phi_n (x) = \sum_{j=1}^{n} B_{nj} \lambda_j^{-ix}, \tag{3.2}$$

where

$$B_{nj} = \frac{\Delta_n}{i \log \frac{\lambda_j}{\lambda_n}} \prod_{k=1, k \neq j}^{n-1} \left(1 + \frac{1}{i \log \frac{\lambda_j}{\lambda_k}} \right).$$

In addition, formulae were given for ϕ_n' and Markov-Bernstein inequalities were established. Among the more interesting inequalities established are the bounds

$$e^{-x} \sum_{j=1}^{n} \left| \phi_j (x) \right|^2 \leq \sum_{j=1}^{n} \left| \phi_j (0) \right|^2 = n.$$

Moreover, the left-hand side is a decreasing function of $x \in [0, \infty)$. Similarly,

$$\begin{aligned} & e^{-x} \sum_{j=1}^{n} \left| \phi_j' (x) \right|^2 \\ & \leq \sum_{j=1}^{n} \left| \phi_j' (0) \right|^2 = \frac{n (n-1) (2n-1)}{6} + \sum_{j=1}^{n} \left(\log \lambda_j \right)^2, \end{aligned}$$

and the left-hand side is also a decreasing function of x.

As it turns out, many of the above results were not new, and subsumed by existing results on Müntz orthogonal polynomials. Suppose we make the substitution $x = e^{-t}$ in (3.1). We obtain

$$\int_0^1 \phi_n \left(\log \frac{1}{x} \right) \overline{\phi_m \left(\log \frac{1}{x} \right)} dx = \delta_{mn},$$

and

$$\phi_n\left(\log\frac{1}{x}\right)=\sum_{j=1}^{n}B_{nj}x^{-i\lambda_j}.$$

These are Müntz orthogonal polynomials that were explored in the Russian literature as far back as 1955—see [1, 21]. An excellent reference is the beautiful book of Borwein and Erdelyi [4, p. 125 ff.]. As the author knows much of that book well, he ought to have noticed the connection.

The treatment in [4] allows complex λ_j, so let us change notation: given complex ρ_j with Re $\rho_j>-\frac{1}{2}$, $j\geq 0$, define the nth Müntz-Legendre polynomial

$$L_n(x)=\frac{1}{2\pi i}\int_{\Gamma}\prod_{k=0}^{n-1}\frac{t+\overline{\varrho_k}+1}{t-\rho_k}\frac{x^t}{t-\rho_n}dt.$$

Here Γ is a simple closed positively oriented curve enclosing all the $\{\rho_j\}$. It can be shown that L_n is a linear combination of $\{x^{\rho_j}\}_{j=0}^{n}$ admitting the orthogonality relation

$$\int_0^1 L_n(x)\overline{L_m(x)}dx=\delta_{mn}\frac{1}{1+2\text{Re }\rho_n}.$$

Müntz orthogonal polynomials have been used in numerical quadrature [11, 12]. A thorough study of their asymptotics was undertaken by Ulfar Stefansson. See for example [18, 19].

4 Rational Weights

Since the formulae for the arctan density are so simple, it is natural to try generalize them to linear combinations of scaled arctan densities. Let

$$w(t)=\sum_{m=1}^{L}\frac{c_m}{\pi\left(1+(b_m t)^2\right)},\tag{4.1}$$

where $L\geq 2$, the $\{c_j\}$ are real, and

$$1=b_1<b_2<\cdots<b_m.\tag{4.2}$$

One would also hope to preserve the simple structure for the arctan density. Some guidance is provided by expressing ϕ_n of Sect. 2, in the determinant form (1.2):

$$\phi_n(t) = \frac{\lambda_n^{1-it} - \lambda_{n-1}^{1-it}}{\sqrt{\lambda_n^2 - \lambda_{n-1}^2}} = -\frac{1}{\sqrt{\lambda_n^2 - \lambda_{n-1}^2}} \det \begin{bmatrix} \lambda_{n-1}^{-it} & \lambda_n^{-it} \\ \lambda_{n-1}^{-1} & \lambda_n^{-1} \end{bmatrix}.$$

By analogy, define for $n \geq L$,

$$\psi_n(t) = \det \begin{bmatrix} \lambda_{n-L}^{it} & \lambda_{n-L+1}^{it} & \cdots & \lambda_{n-1}^{it} & \lambda_n^{it} \\ \lambda_{n-L}^{-1/b_1} & \lambda_{n-L+1}^{-1/b_1} & \cdots & \lambda_{n-1}^{-1/b_1} & \lambda_n^{-1/b_1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \lambda_{n-L}^{-1/b_{L-1}} & \lambda_{n-L+1}^{-1/b_{L-1}} & \cdots & \lambda_{n-1}^{-1/b_{L-1}} & \lambda_n^{-1/b_{L-1}} \\ \lambda_{n-L}^{-1/b_L} & \lambda_{n-L+1}^{-1/b_L} & \cdots & \lambda_{n-1}^{-1/b_L} & \lambda_n^{-1/b_L} \end{bmatrix}. \tag{4.3}$$

Observe that $\psi_n(t)$ is a linear combination of only $\left\{\lambda_j^{-it}\right\}_{n-L \leq j \leq n}$. Also define for a given fixed n, and $j \geq 1$, $1 \leq m \leq L$,

$$d_{jm} = \int_{-\infty}^{\infty} \psi_n(t) \frac{\lambda_j^{it}}{\pi\left(1 + (b_m t)^2\right)} dt \tag{4.4}$$

and let B be the $(L-1) \times L$ matrix

$$B = \begin{bmatrix} d_{n-L+1,1} & d_{n-L+1,2} & \cdots & d_{n-L+1,L} \\ d_{n-L+2,1} & d_{n-L+2,2} & \cdots & d_{n-L+2,L} \\ \vdots & \vdots & \ddots & \vdots \\ d_{n-1,1} & d_{n-1,2} & \cdots & d_{n-1,L} \end{bmatrix} \tag{4.5}$$

and

$$D = \det \begin{bmatrix} d_{n-L+1,1} & d_{n-L+1,2} & \cdots & d_{n-L+1,L} \\ d_{n-L+2,1} & d_{n-L+2,2} & \cdots & d_{n-L+2,L} \\ \vdots & \vdots & \ddots & \vdots \\ d_{n-1,1} & d_{n-1,2} & \cdots & d_{n-1,L} \\ d_{n,1} & d_{n,2} & \cdots & d_{n,L} \end{bmatrix}. \tag{4.6}$$

In [9] we proved:

Proposition 4.1 *Let* $\mathbf{c} = [c_1\ c_2 \ldots c_L]^T$ *be taken as any non-trivial solution of* $B\mathbf{c} = \mathbf{0}$. *Let* w *be as in (4.1). Then for* $1 \leq j \leq n-1$,

$$\int_{-\infty}^{\infty} \psi_n(t)\, \lambda_j^{it} w(t)\, dt = 0. \tag{4.7}$$

*If D defined by (4.6) is non-*0*, then we can take*

$$w(t) = A\det\begin{bmatrix} d_{n-L+1,1} & d_{n-L+1,2} & \cdots & d_{n-L+1,L} \\ d_{n-L+2,1} & d_{n-L+2,2} & \cdots & d_{n-L+2,L} \\ \vdots & \vdots & \ddots & \vdots \\ d_{n-1,1} & d_{n-1,2} & \cdots & d_{n-1,L} \\ \frac{1}{\pi\left(1+(b_1 t)^2\right)} & \frac{1}{\pi\left(1+(b_2 t)^2\right)} & \cdots & \frac{1}{\pi\left(1+(b_L t)^2\right)} \end{bmatrix}, \tag{4.8}$$

for any $A \neq 0$*, while*

$$\int_{-\infty}^{\infty} \psi_n(t)\,\lambda_n^{it} w(t)\,dt = AD.$$

Only in the case $L = 2$, could we prove positivity of the weight, with appropriately chosen $0 < c_1 < c_2$. It seems a worthwhile project to investigate if for $L \geq 3$ that the weight can be chosen to be of one sign.

If one can prove positivity of w for arbitrary L, there is the hope that one can use such rational weights to approximate general weights in much the same way as Bernstein-Szegő weights are used in the theory of “algebraic” orthogonal polynomials [20]. However, this might be quite a reach, as there is at present no indication that even if we could prove positivity, that there is the wealth of detail and formulae that make Bernstein-Szegő weights such a valuable tool.

5 Legendre Weight

A natural choice for the weight is the Legendre weight $w =$ Constant on some interval or subset of the real line. In [10], we considered the normalized Legendre weight $w = \frac{1}{2T}$ on $[-T, T]$ for $T > 0$. To emphasize the dependence on $T > 0$, we denote the Dirichlet orthogonal polynomial by $\phi_{n,T}$, with positive leading coefficient $\gamma_{n,T}$, such that

$$\left(\phi_{n,T}, \phi_{m,T}\right)_T = \frac{1}{2T}\int_{-T}^{T} \phi_{n,T}(t)\,\overline{\phi_{m,T}(t)}dt = \delta_{mn}.$$

The nth reproducing kernel is

$$K_{n,T}(u, v) = \sum_{j=1}^{n} \phi_{j,T}(u)\,\overline{\phi_{j,T}(v)}.$$

Let, as above,

$$\mathbb{S}(u) = \frac{\sin u}{u}$$

denote the sinc kernel. From

$$\frac{1}{2T}\int_{-T}^{T} \left(\lambda_j/\lambda_k\right)^{-it} dt = \mathbb{S}\left(T \log\left(\lambda_j/\lambda_k\right)\right),$$

the determinantal representation (1.2) becomes

$$\phi_{n,T}(x) = \frac{(-1)^{n+1}}{\sqrt{A_{n-1,T}A_{n,T}}}$$
$$\times \det \begin{bmatrix} \lambda_1^{-ix} & \lambda_2^{-ix} & \lambda_3^{-ix} & \cdots & \lambda_n^{-ix} \\ 1 & \mathbb{S}(T\log\lambda_1/\lambda_2) & \mathbb{S}(T\log\lambda_1/\lambda_3) & \cdots & \mathbb{S}(T\log\lambda_1/\lambda_n) \\ \mathbb{S}(T\log\lambda_2/\lambda_1) & 1 & \mathbb{S}(T\log\lambda_2/\lambda_3) & \cdots & \mathbb{S}(T\log\lambda_2/\lambda_n) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbb{S}(T\log\lambda_{n-1}/\lambda_1) & \mathbb{S}(T\log\lambda_{n-1}/\lambda_2) & \mathbb{S}(T\log\lambda_{n-1}/\lambda_3) & \cdots & \mathbb{S}(T\log\lambda_{n-1}/\lambda_n) \end{bmatrix}.$$

The leading coefficient of $\phi_{n,T}(x)$ is $\gamma_{n,T} = \sqrt{\frac{A_{n-1,T}}{A_{n,T}}}$, where

$$A_{n,T} = \det\left[\mathbb{S}\left(T\log\lambda_j/\lambda_k\right)\right]_{1\le j,k\le n}. \tag{5.1}$$

It follows from the determinantal expression and the limit $\lim_{x\to\infty}\mathbb{S}(x) = 0$ that

$$\lim_{T\to\infty}\phi_{n,T}(x) = \lambda_n^{-ix}.$$

One motivation for considering the Legendre weight is the Montgomery-Vaughan mean value relation (1.4). It is to be hoped that a theory of orthogonal Dirichlet polynomials might contribute to this circle of ideas and to estimates involving Dirichlet polynomials. In this vein, write for $j \ge 1$, $T > 0$,

$$\lambda_j^{-it} = \sum_{k=1}^{j} c_{T,j,k}\phi_{k,T}(t).$$

Let

$$C_{T,n} = \begin{bmatrix} c_{T,1,1} & c_{T,2,1} & c_{T,3,1} & \cdots & c_{T,n,1} \\ 0 & c_{T,2,2} & c_{T,3,2} & \cdots & c_{T,n,2} \\ 0 & 0 & c_{T,3,3} & \cdots & c_{T,n,3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & c_{T,n,n} \end{bmatrix}.$$

In [10], there is the simple observation that

$$\sup_{\{a_j\}} \frac{1}{2T} \int_{-T}^{T} \left| \sum_{j=1}^{n} a_j \lambda_j^{-it} \right|^2 dt / \sum_{j=1}^{n} \left| a_j \right|^2 = \left\| C_{T,n} \right\|^2,$$

where the norm is the usual matrix norm induced by the Euclidean norm on $\mathbb{C}^n$. The Montgomery-Vaughan inequality shows that

$$\left\| C_{T,n} \right\|^2 = T + 3\pi\theta_0 / \min_{j \neq k} \left| \log \lambda_j - \log \lambda_k \right|,$$

where $|\theta_0| \leq 1$, but it would be of interest to use $\left\| C_{T,n} \right\|$ to study refinements in the other direction as $T \to \infty$. Of course this would require understanding how $\phi_{n,T}$ changes as T does. Some initial estimates were obtained in [10]:

Proposition 5.1 *Let $S > T$.*
(a)

$$\frac{1}{2T} \int_{-T}^{T} \left| \psi_{n,S}(t) - \frac{\gamma_{n,S}}{\gamma_{n,T}} \psi_{n,T}(t) \right|^2 dt \leq \frac{S}{T} - \left(\frac{\gamma_{n,S}}{\gamma_{n,T}} \right)^2.$$

(b)

$$\frac{\gamma_{n,S}}{\gamma_{n,T}} \leq \left(\frac{S}{T} \right)^{1/2}.$$

(c)

$$K_{n,T}(x,x) + \left(\frac{S}{T} - 2 \right) K_{n,S}(x,x) \geq 0. \tag{5.2}$$

(d)

$$\frac{\partial}{\partial T} K_{n,T}(x,x) = \frac{1}{T} K_{n,T}(x,x) - \frac{1}{2T} \left(|K_n(x,T)|^2 + |K_n(x,-T)|^2 \right). \tag{5.3}$$

(e)

$$\frac{\partial \left(\ln \gamma_{n,T} \right)}{\partial T} = \frac{1}{2T} (1 - \left| \psi_{n,T}(T) \right|^2).$$

(f)

$$\frac{\partial}{\partial T} \ln A_{n,T} = -\frac{1}{T} \left(n - K_{n,T}(T,T) \right).$$

(g)

$$\frac{\partial}{\partial T}c_{T,j,k}+\frac{1}{T}c_{T,j,k}=\frac{1}{2T}\left[\lambda_j^{-iT}\overline{\phi_{k,T}(T)}+\lambda_j^{iT}\phi_{k,T}(T)\right]$$
$$+\frac{1}{2T}\int_{-T}^{T}\lambda_j^{-it}\frac{\partial}{\partial T}\overline{\phi_{k,T}(t)}dt.$$

6 Conclusions

The hope in studying Dirichlet orthogonal polynomials is that they might give new insights into estimates for Dirichlet polynomials such as mean value theorems. At this preliminary stage, this is little more than a hope. However, it seems of intrinsic interest to develop analogues of the analysis for ordinary orthogonal polynomials: estimates and asymptotics for the Christoffel functions, orthogonal polynomials, and reproducing kernels for general weights. The first step in such a direction would be explicit formulae for a significant set of special weights that can approximate others—perhaps something like the Bernstein-Szegő weight. As is clear from the above, even a more basic theory for special weights is incomplete.

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