VARIANCE OF REAL ZEROS OF RANDOM ORTHOGONAL POLYNOMIALS FOR VARYING AND EXPONENTIAL WEIGHTS

DORON S. LUBINSKY [1], IGOR E. PRITSKER [2]

ABSTRACT. We determine the asymptotics for the variance of the number of zeros of random linear combinations of orthogonal polynomials of degree $\leq n$ associated with varying weights. We deduce asymptotics of the variance for exponential weights. In particular, we show that very generally, the variance is asymptotic to Cn, where the constant C involves a universal constant and an equilibrium density associated with the weight.

1. INTRODUCTION AND MAIN RESULTS

Consider random linear combinations of polynomials of the form

$$G_n(x) = \sum_{j=0}^{n} a_j p_{n,j}(x), \quad n \geq 0, \tag{1.1}$$

where $\{a_j\}_{j=0}^{\infty}$ are standard Gaussian $\mathcal{N}(0,1)$ i.i.d. random variables, and $\{p_{n,j}\}$ are orthogonal polynomials with respect to some measure μ_n that depends on n.

The study of real zeros for random orthogonal polynomials of the form (1.1) is motivated to a large extent by classical results on random trigonometric polynomials. Random cosine polynomials $\sum_{j=0}^{n} a_j \cos(jx)$, $x \in [0, 2\pi]$, with $\mathcal{N}(0,1)$ i.i.d. coefficients were considered by Dunnage [9], who showed that the expected number of zeros in $[0, 2\pi]$, denoted by $\mathbb{E}N_n([0, 2\pi])$, is asymptotically equal to $2n/\sqrt{3}$. Qualls [20] studied trigonometric polynomials $\sum_{j=0}^{n} \xi_{j1} \cos(jx) + \xi_{j2} \sin(jx)$, $\quad x \in [0, 2\pi]$, and showed that $\mathbb{E}N_n([0, 2\pi])$ for this ensemble is also asymptotically equal to $2n/\sqrt{3}$.

The first result on random orthogonal polynomials is due to Das [5], who proved for random Legendre polynomials that $\mathbb{E}N_n([-1,1])$ is asymptotically equal to $n/\sqrt{3}$. Wilkins [22], [23] estimated the error term in this asymptotic relation. For more general random Jacobi polynomials, Das and Bhatt [6] established that $\mathbb{E}N_n([-1,1])$ is asymptotically equal to $n/\sqrt{3}$ too. The same asymptotic for the expected number of real zeros was shown to hold for very wide classes of random orthogonal polynomials by Lubinsky, Pritsker and Xie [16], [17]. Their work includes random orthogonal polynomials with i.i.d. normal coefficients spanned by orthonormal polynomials with respect to general measures supported compactly or on the whole real line. Do, O. Nguyen and Vu [8] recently extended the asymptotics $\mathbb{E}N_n(\mathbb{R})$ to the random orthogonal polynomials with general coefficients that possess finite moments of the order $(2+\varepsilon)$ via universality methods.

The asymptotics for the variance of real zeros are much more difficult to establish due to complexity of the corresponding Kac-Rice formula and numerous technical

Date: October 5, 2021.

difficulties associated with analysis. Bogomolny, Bohigas and Leboeuf [4] conjectured that $Var(N_n([0,2\pi]))$ is asymptotically equal to cn for random trigonometric polynomials, which was first verified by Granville and Wigman [12] for Qualls' ensemble, with an explicit formula for c (see also Azaïs and León [2]). The asymptotic variance for the trigonometric model of Dunnage was computed by Azaïs, Dalmao and León in [1].

In [18], the authors analyzed the variance for random linear combinations of orthogonal polynomials formed from a fixed measure with compact support. Similar techniques have recently been used by Gass to study the variance for random trigonometric polynomials, and to develop a general framework for finding the asymptotic variance results [11]. In this paper, we present analogous results for varying weights and consequently exponential weights on the real line. For any interval $[a,b] \subset \mathbb{R}$, let $N_n([a,b])$ denote the number of zeros of G_n lying in $[a,b]$. Our results involve some functions of the sinc kernel

$$S(u) = \frac{\sin \pi u}{\pi u}. \tag{1.2}$$

Let

$$F(u) = \det \begin{bmatrix} 1 & S(u) & 0 & S'(u) \\ S(u) & 1 & -S'(u) & 0 \\ 0 & -S'(u) & -S''(0) & -S''(u) \\ S'(u) & 0 & -S''(u) & -S''(0) \end{bmatrix}; \tag{1.3}$$

$$G(u) = \det \begin{bmatrix} 1 & S(u) & -S'(u) \\ S(u) & 1 & 0 \\ -S'(u) & 0 & -S''(0) \end{bmatrix}; \tag{1.4}$$

$$H(u) = \det \begin{bmatrix} 1 & S(u) & 0 \\ S(u) & 1 & -S'(u) \\ S'(u) & 0 & -S''(u) \end{bmatrix}; \tag{1.5}$$

$$\Xi(u) = \frac{1}{\pi^2} \left\{ \frac{\sqrt{F(u)}}{1 - S(u)^2} + \frac{1}{\left(1 - S(u)^2\right)^{3/2}} H(u) \arcsin\left(\frac{H(u)}{G(u)}\right) \right\} - \frac{1}{3}. \tag{1.6}$$

In [18], we proved that for fixed measures μ with support $[-1,1]$ and $(a,b) \subset (-1,1)$,

$$\lim_{n\to\infty} \frac{1}{n} \mathrm{Var}\left[N_n([a,b])\right] = \left(\int_a^b \omega(x)\,dx\right)\left(\int_{-\infty}^{\infty} \Xi(u)\,du + \frac{1}{\sqrt{3}}\right),$$

where ω is the equilibrium density, in the sense of potential theory, for the support of μ. The hypotheses on μ primarily involved assumptions on the orthonormal polynomials for μ, such as uniform boundedness in subintervals of the support. In this paper, our main hypotheses are:

Hypotheses on the Measures

For $n \geq 1$, let μ_n be a measure supported on I_n, where I_n is an interval that may be unbounded or unbounded, but contains $[-1,1]$. We assume that μ_n is absolutely continuous in $[-1,1]$, and in that interval

$$\mu_n'(x) = e^{-2nQ_n(x)},$$

and $Q_n'(x)$ exists there. We assume that for each $n \geq 1$, there are orthonormal polynomials $\{p_{n,m}(\mu_n, x)\}_{m=0}^{\infty}$ so that $p_{n,j}(x) = \gamma_{n,j}x + ..., \gamma_{n,j} > 0$, and

$$\int_{I_n} p_{n,j} p_{n,k} d\mu_n = \delta_{jk}.$$

We let

$$K_{n+1}(x,y) = K_{n+1}(\mu_n, x, y) = \sum_{j=0}^{n} p_{n,j}(x) p_{n,k}(y)$$

denote the $(n+1)$st reproducing kernel for μ_n. More generally, for non-negative integers r, s, we define the differentiated kernels

$$K_{n+1}^{(r,s)}(x,y) = \sum_{j=0}^{n} p_{n,j}^{(r)}(x) p_{n,k}^{(s)}(y) \tag{1.7}$$

and their normalized forms,

$$\tilde{K}_{n+1}^{(r,s)}(x,y) = K_{n+1}^{(r,s)}(x,y) \mu_n'(x)^{1/2} \mu_n'(y)^{1/2}. \tag{1.8}$$

We need a number of implicit hypotheses:

(I) Uniform Bounds on Orthogonal Polynomials and their Derivatives

For each $0 < \varepsilon < 1$, there exists $C > 0$ such that for $n \geq 1$, $k = n, n+1$, $j = 0, 1$, and $|x| \leq 1 - \varepsilon$,

$$\left| p_{n,k}^{(j)}(x) \right| \sqrt{\mu_n'(x)} \leq C n^j. \tag{1.9}$$

(II) Bounds on the Ratio of Leading Coefficients

There exists $C_1 > 1$ such that for $n \geq 1$,

$$C_1^{-1} \leq \frac{\gamma_{n,n}}{\gamma_{n,n+1}} \leq C_1. \tag{1.10}$$

(III) Bounds on the Reproducing Kernel

For each $0 < \varepsilon < 1$, there exists $C_2 > 1$ such that for $n \geq 1$ and $|x| \leq 1 - \varepsilon$,

$$C_2^{-1} \leq K_{n+1}(x,x) \mu_n'(x) / n \leq C_2. \tag{1.11}$$

(IV) Universality Limit

For each $0 < \varepsilon < 1$, we have uniformly for $|x| \leq 1 - \varepsilon$, and u, v in compact subsets of the plane,

$$\lim_{n\to\infty} \frac{K_{n+1}\left(x + \frac{u}{\tilde{K}_{n+1}(x,x)}, x + \frac{v}{\tilde{K}_{n+1}(x,x)}\right)}{K_{n+1}(x,x)} e^{-\frac{nQ_n'(x)}{\tilde{K}_{n+1}(x,x)}(u+v)} = S(v-u). \tag{1.12}$$

(V) Bounds on $\{Q_n'\}$

For each $0 < \varepsilon < 1$, there exists $C_3 > 0$ such that for $n \geq 1$ and $|x| \leq 1 - \varepsilon$, we have

$$|Q_n'(x)| \leq C_3. \tag{1.13}$$

Moreover, given $r > 0$, we assume that

$$\sup_{|x|\leq 1-\varepsilon} \sup_{|a|\leq r} \left| Q_n'(x) - Q_n'\left(x + \frac{a}{n}\right) \right| = o(1). \tag{1.14}$$

We prove:

Theorem 1.1
Assume the hypotheses (I) - (V) above. If $[a,b] \subset (-1,1)$, *then*
(1.15)

$$\lim_{n\to\infty} \left\{ \frac{1}{n} \mathrm{Var}\left[N_n\left([a,b]\right)\right] - \left(\int_a^b \frac{1}{n} \tilde{K}_n\left(x,x\right) dx \right) \left(\int_{-\infty}^{\infty} \Xi\left(u\right) du + \frac{1}{\sqrt{3}} \right) \right\} = 0.$$

Since the orthogonality measures μ_n are not necessarily related to one another for different values of n, one should not expect $\left\{\frac{1}{n} VarN_n\left([a,b]\right)\right\}_{n\geq 1}$ to converge in general. Indeed, one can construct examples of sequences of measures for which different subsequence have different limits. However, (1.11) and (1.15) show that $\left\{\frac{1}{n} VarN_n\left([a,b]\right)\right\}_{n\geq 1}$ is a bounded sequence.

In Section 2, we give two examples to which this theorem may be applied: varying exponential weights and fixed exponential weights on the real line. In both these cases, $\frac{1}{n}\tilde{K}_n\left(x,x\right)$ may be replaced by a more explicit term. The methods of proof follow those in [18]. However, there are substantial additional technical difficulties due to the varying weights.

This paper is organised as follows: In Section 3, we outline the proof of Theorem 1.1, deferring technical details to later. In Section 4, we present some auxiliary technical results. In Section 5, we handle the tail term. In Section 6, we handle the central term. In Section 7, we prove Theorem 2.1. In Section 8, we prove Theorem 2.3 and Corollary 2.4.

In the sequel, $C, C_1, C_2, ...$ denote constants independent of n, x, y. The same symbol may be different in different occurrences. We shall frequently need two versions of formulae that involve the reproducing kernels K_n or their normalized version $\tilde{K}_n$. If J is an expression involving terms such as $K_n^{(r,s)}$, we let $\tilde{J}$ denote the analogous expression where every $K_n^{(r,s)}$ is replaced by its normalization $\tilde{K}_n^{(r,s)}$. Thus, for example, if

$$\Delta(x,y) := K_{n+1}(x,x)K_{n+1}(y,y) - K_{n+1}^2(x,y)$$

then

$$\tilde{\Delta}(x,y) := \tilde{K}_{n+1}(x,x)\tilde{K}_{n+1}(y,y) - \tilde{K}_{n+1}^2(x,y).$$

If $\{\alpha_n\}, \{\beta_n\}$ are sequences of non-0 real numbers, then we write

$$\alpha_n \sim \beta_n$$

if there exists $C > 1$ such that for $n \geq 1$,

$$C^{-1} \leq \alpha_n/\beta_n \leq C.$$

Acknowledgements
The authors would like to thank Yen Do, Hoi Nguyen, and Oanh Nguyen for their perspectives on random polynomials and comments on this paper. This paper grew out of the AIM workshop on Zeros of Random Polynomials, held from 12-16 August 2019. D. Lubinsky was partially supported by NSF grant DMS1800251. I. Pritsker was partially supported by NSA grant H98230-21-1-0008.

2. Exponential Weights

We begin with varying exponential weights, as studied in [15]. The statement of the result involves equilibrium measures for external fields. We shall discuss that in more detail in section 7.

Theorem 2.1
For $n \geq 1$, let $I_n = (c_n, d_n)$, where $-\infty \leq c_n < d_n \leq \infty$. Assume that for some $r^* > 1$, $[-r^*, r^*] \subset I_n$, for all $n \geq 1$. Assume that

$$\mu_n'(x) = e^{-2nQ_n(x)}, x \in I_n, \tag{2.1}$$

where
(i) $Q_n(x) / \log(2+|x|)$ has limit ∞ as $x \to c_n+$ and $x \to d_n-$.
(ii) Q_n' is strictly increasing and continuous in I_n.
(iii) There exists $\alpha \in (0,1)$, $C > 0$ such that for $n \geq 1$ and $x, y \in [-r^*, r^*]$,

$$|Q_n'(x) - Q_n'(y)| \leq C |x-y|^{\alpha}. \tag{2.2}$$

(iv) There exists $\alpha_1 \in \left(\frac{1}{2}, 1\right)$, $C_1 > 0$, and an open neighborhood I_0 of 1 *and -1, such that for $n \geq 1$ and $x, y \in I_n \cap I_0$,*

$$|Q_n'(x) - Q_n'(y)| \leq C_1 |x-y|^{\alpha_1}. \tag{2.3}$$

(v) $[-1,1]$ is the support of the equilibrium distribution for the external field Q_n. Let $[a,b] \subset (-1,1)$. Then

$$\lim_{n\to\infty} \left\{ \frac{1}{n} \mathrm{Var}[N_n([a,b])] - \left(\int_a^b \sigma_{Q_n}(x)\, dx \right) \left(\int_{-\infty}^{\infty} \Xi(u)\, du + \frac{1}{\sqrt{3}} \right) \right\} = 0, \tag{2.4}$$

where for $x \in (-1,1)$,

$$\sigma_{Q_n}(x) = \frac{\sqrt{1-x^2}}{\pi^2} \int_{-1}^{1} \frac{Q_n'(s) - Q_n'(x)}{s-x} \frac{ds}{\sqrt{1-s^2}}. \tag{2.5}$$

Note that σ_{Q_n} is the Radon-Nikodym derivative of the equilibrium measure for the external field Q_n. We shall prove Theorem 2.1 in Section 7. Next we turn to fixed exponential weights. First we define a subclass of the weights presented in [13, Definition 1.1, p. 7]:

Definition 2.2
Let $W = e^{-Q}$, where $Q : \mathbb{R} \to [0, \infty)$ satisfies the following conditions:
(a) Q' is continuous in $\mathbb{R}$ and $Q(0) = 0$.
(b) Q'' exists and is positive in $\mathbb{R} \setminus \{0\}$;
(c)

$$\lim_{|t|\to\infty} Q(t) = \infty.$$

(d) The function

$$T(t) = \frac{tQ'(t)}{Q(t)}, \ t \neq 0,$$

is quasi-increasing in $(0, \infty)$, in the sense that for some $C > 0$,

$$0 < x < y \Rightarrow T(x) \leq CT(y).$$

We assume, with an analogous definition, that T is *quasi-decreasing in* $(-\infty, 0)$. *In addition, we assume that for some* $\Lambda > 1$,

$$T(t) \geq \Lambda \text{ in } \mathbb{R} \setminus \{0\}.$$

(e) There exists $C_1 > 0$ *such that*

$$\frac{Q''(x)}{|Q'(x)|} \leq C_1 \frac{Q'(x)}{Q(x)} \text{ a.e. } x \in \mathbb{R} \setminus \{0\}.$$

Then we write $W \in \mathcal{F}\left(C^2\right)$. *We also let*

$$\mu(x) = e^{-2Q(x)}, x \in \mathbb{R}.$$

Remarks

Examples of weights in this class are $W = \exp(-Q)$, where

$$Q(x) = \begin{cases} Ax^\alpha, & x \in [0, \infty) \\ B|x|^\beta, & x \in (-\infty, 0) \end{cases},$$

where $\alpha, \beta > 1$ and $A, B > 0$. More generally, if $\exp_k = \exp(\exp(... \exp()))$ denotes the kth iterated exponential, we may take

$$Q(x) = \begin{cases} \exp_k(Ax^\alpha) - \exp_k(0), & x \in [0, \infty), \\ \exp_\ell\left(B|x|^\beta\right) - \exp_\ell(0), & x \in (-\infty, 0), \end{cases}$$

where $k, \ell \geq 1$, $\alpha, \beta > 1$.

We shall need the Mhaskar-Rakhmanov-Saff numbers $a_{-n} < 0 < a_n$. These are defined for $n \geq 1$ by the equations

$$n = \frac{1}{\pi} \int_{a_{-n}}^{a_n} \frac{xQ'(x)}{\sqrt{(x - a_{-n})(a_n - x)}} dx; \; 0 = \frac{1}{\pi} \int_{a_{-n}}^{a_n} \frac{Q'(x)}{\sqrt{(x - a_{-n})(a_n - x)}} dx. \tag{2.6}$$

In the case where Q is even, $a_{-n} = -a_n$. We also define

$$\beta_n = \frac{1}{2}(a_n + a_{-n}) \text{ and } \delta_n = \frac{1}{2}(a_n + |a_{-n}|), \tag{2.7}$$

which are respectively the center, and half-length of the Mhaskar-Rakhmanov-Saff interval

$$\Delta_n = [a_{-n}, a_n]. \tag{2.8}$$

The linear transformation

$$L_n(x) = \frac{x - \beta_n}{\delta_n} \tag{2.9}$$

maps Δ_n onto $[-1, 1]$. Its inverse $L_n^{[-1]}(u) = \beta_n + u\delta_n$ maps $[-1, 1]$ onto Δ_n. For $0 < \varepsilon < 1$, we let

$$J_n(\varepsilon) = L_n^{[-1]}[-1 + \varepsilon, 1 - \varepsilon] = [a_{-n} + \varepsilon\delta_n, a_n - \varepsilon\delta_n]. \tag{2.10}$$

The equilibrium density on $[a_{-n}, a_n]$ is

$$\sigma_n(x) = \frac{\sqrt{(x - a_{-n})(a_n - x)}}{\pi^2} \int_{a_{-n}}^{a_n} \frac{Q'(x) - Q'(s)}{s - x} \frac{ds}{\sqrt{(s - a_{-n})(a_n - s)}}. \tag{2.11}$$

We also need the scaled density

$$\sigma_n^*(t) = \frac{\delta_n}{n} \sigma_n\left(L_n^{[-1]}(t)\right), t \in (-1, 1), \tag{2.12}$$

that satisfies

$$\int_{-1}^{1} \sigma_n^* = 1. \tag{2.13}$$

Let $\{p_j\}$ denote the orthonormal polynomials associated with the weight W^2, so that

$$\int_{-\infty}^{\infty} p_j p_k W^2 = \delta_{jk}.$$

Random linear combinations of these have the form

$$G_n(x) = \sum_{j=0}^{n} a_j p_j(x),$$

where the $\{a_j\}_{j=0}^n$ are standard Gaussian $\mathcal{N}(0,1)$ i.i.d. random variables. One expects that most zeros of these will lie in the Mhaskar-Rakhmanov-Saff interval, see [17]. It is hence convenient to scale this interval to $[-1,1]$. Accordingly, we consider

$$G_n^*(t) = G_n\left(L_n^{[-1]}(t)\right).$$

In particular, when Q is even,

$$G_n^*(t) = G_n(a_n t).$$

We let $N_n^*[a,b]$ denote the number of zeros of G_n^* in $[a,b]$, or equivalently of G_n in $L_n^{[-1]}([a,b])$. We prove:

Theorem 2.3
Let $W \in \mathcal{F}\left(C^2\right)$. Then for $[a,b] \subset (-1,1)$,

$$\lim_{n\to\infty} \left\{ \frac{1}{n} \mathrm{Var}\left[N_n^*([a,b])\right] - \left(\int_a^b \sigma_n^*(x)\,dx\right)\left(\int_{-\infty}^{\infty} \Xi(u)\,du + \frac{1}{\sqrt{3}}\right)\right\} = 0. \tag{2.14}$$

Under additional conditions, we can replace σ_n^* by a limiting distribution. For $\alpha > 0$, define the Nevai-Ullmann density

$$\sigma_\alpha(x) = \frac{2\sqrt{1-x^2}}{\pi^2 B_\alpha} \int_0^1 \frac{t^\alpha - x^\alpha}{t^2 - x^2} \frac{dt}{\sqrt{1-t^2}}, t \in (-1,1), \tag{2.15}$$

where

$$B_\alpha = \frac{2}{\pi} \int_0^1 \frac{t^\alpha}{\sqrt{1-t^2}} dt.$$

This is the equilibrium density for the Freud weight $\exp\left(-C|x|^\alpha\right)$ for appropriate C [21, Theorem 5.1, p. 240]. When $\alpha \to \infty$, this becomes the arcsine distribution

$$\sigma_\infty(x) = \frac{1}{\pi\sqrt{1-x^2}}, x \in (-1,1).$$

Corollary 2.4
Let $W \in \mathcal{F}\left(C^2\right)$ and assume in addition that W is even and for some $\alpha \in (1,\infty]$,

$$\lim_{x\to\infty} T(x) = \alpha. \tag{2.16}$$

Then for $[a,b] \subset (-1,1)$,

$$\lim_{n\to\infty} \frac{1}{n} \mathrm{Var}\left[N_n^*\left([a,b]\right)\right] = \left(\int_a^b \sigma_\alpha\left(x\right) dx\right)\left(\int_{-\infty}^{\infty} \Xi\left(u\right) du + \frac{1}{\sqrt{3}}\right). \tag{2.17}$$

3. The Proof of Theorem 1.1

We begin with the Kac-Rice formulas for the expectation and variance. These involve the reproducing kernels defined in (1.7).

Lemma 3.1
Let $[a,b] \subset \mathbb{R}$. *Then the expected number of real zeros for* G_n *is*

$$\mathbb{E}\left[N_n\left([a,b]\right)\right] = \frac{1}{\pi}\int_a^b \rho_1\left(x\right)\ dx, \tag{3.1}$$

where

$$\rho_1\left(x\right) = \frac{1}{\pi}\sqrt{\frac{K_{n+1}^{(1,1)}\left(x,x\right)}{K_{n+1}\left(x,x\right)} - \left(\frac{K_{n+1}^{(0,1)}\left(x,x\right)}{K_{n+1}\left(x,x\right)}\right)^2}. \tag{3.2}$$

Moreover,

$$\rho_1\left(x\right) = \tilde{\rho}_1\left(x\right). \tag{3.3}$$

Proof
See [16]. Note that $\frac{\tilde{K}_{n+1}^{(1,1)}(x,x)}{\tilde{K}_{n+1}(x,x)} = \frac{K_{n+1}^{(1,1)}(x,x)}{K_{n+1}(x,x)}$ and so on. ■

Recall that $\tilde{\rho}_1$ is the expression defined by the same formula as ρ_1 but with every occurrence of $K_n^{(r,s)}$ replaced by $\tilde{K}_n^{(r,s)}$. Note that ρ_1 depends on n, but we omit this dependence to simplify the notation. The same applies to ρ_2 below. We also need

$$\Sigma = \begin{bmatrix} K_{n+1}\left(x,x\right) & K_{n+1}\left(x,y\right) & K_{n+1}^{(0,1)}\left(x,x\right) & K_{n+1}^{(0,1)}\left(x,y\right) \\ K_{n+1}\left(x,y\right) & K_{n+1}\left(y,y\right) & K_{n+1}^{(0,1)}\left(y,x\right) & K_{n+1}^{(0,1)}\left(y,y\right) \\ K_{n+1}^{(0,1)}\left(x,x\right) & K_{n+1}^{(0,1)}\left(y,x\right) & K_{n+1}^{(1,1)}\left(x,x\right) & K_{n+1}^{(1,1)}\left(x,y\right) \\ K_{n+1}^{(0,1)}\left(x,y\right) & K_{n+1}^{(0,1)}\left(y,y\right) & K_{n+1}^{(1,1)}\left(x,y\right) & K_{n+1}^{(1,1)}\left(y,y\right) \end{bmatrix}. \tag{3.4}$$

The variance of real zeros of G_n is found from the following formula, which was derived in [24] by using the method of [12].

Lemma 3.2
Let $[a,b] \subset \mathbb{R}$, *and let* G_n *be defined by (1.1).*

$$\mathrm{Var}\left[N_n\left([a,b]\right)\right] = \int_a^b \int_a^b \left\{\rho_2\left(x,y\right) - \rho_1\left(x\right)\rho_1\left(y\right)\right\} dxdy + \int_a^b \rho_1\left(x\right) dx, \tag{3.5}$$

where

$$\rho_2(x,y) = \frac{1}{\pi^2\sqrt{\Delta}}\left(\sqrt{\Omega_{11}\Omega_{22} - \Omega_{12}^2} + \Omega_{12}\arcsin\left(\frac{\Omega_{12}}{\sqrt{\Omega_{11}\Omega_{22}}}\right)\right) = \tilde{\rho}_2\left(x,y\right) \tag{3.6}$$

Here

$$\Delta(x,y) = K_{n+1}(x,x)K_{n+1}(y,y) - K_{n+1}^2(x,y); \tag{3.7}$$

(3.8)
$$\Delta\Omega_{11} = \det\begin{bmatrix} K_{n+1}(y,y) & K_{n+1}(y,x) & K_{n+1}^{(0,1)}(y,x) \\ K_{n+1}(x,y) & K_{n+1}(x,x) & K_{n+1}^{(0,1)}(x,x) \\ K_{n+1}^{(1,0)}(x,y) & K_{n+1}^{(0,1)}(x,x) & K_{n+1}^{(1,1)}(x,x) \end{bmatrix};$$

(3.9)
$$\Delta\Omega_{22} = \det\begin{bmatrix} K_{n+1}(x,x) & K_{n+1}(x,y) & K_{n+1}^{(0,1)}(x,y) \\ K_{n+1}(y,x) & K_{n+1}(y,y) & K_{n+1}^{(0,1)}(y,y) \\ K_{n+1}^{(1,0)}(y,x) & K_{n+1}^{(1,0)}(y,y) & K_{n+1}^{(1,1)}(y,y) \end{bmatrix};$$

(3.10)
$$\Delta\Omega_{12} = \det\begin{bmatrix} K_{n+1}(x,x) & K_{n+1}(x,y) & K_{n+1}^{(0,1)}(x,x) \\ K_{n+1}(y,x) & K_{n+1}(y,y) & K_{n+1}^{(0,1)}(y,x) \\ K_{n+1}^{(1,0)}(y,x) & K_{n+1}^{(0,1)}(y,y) & K_{n+1}^{(1,1)}(y,x) \end{bmatrix}.$$

Moreover,

(3.11)
$$\det(\Sigma) = \Delta\left(\Omega_{22}\Omega_{11} - \Omega_{12}^2\right).$$

The formulae avove also hold for $\tilde{\Delta}, \tilde{\Omega}_{11}, \tilde{\Omega}_{12}, \tilde{\Omega}_{22}$ when every $K_n^{(r,s)}$ term is replaced by $\tilde{K}_n^{(r,s)}$.

Proof
See Lemma 2.2 and 3.1 in [18]. For those involving $\tilde{\rho}_2, \tilde{\Delta}, \tilde{\Omega}_{11}, \tilde{\Omega}_{12}, \tilde{\Omega}_{22}$, one can check that the requisite powers of $\mu_n'(x)$ and $\mu_n'(y)$ on both sides match. ■

To prove Theorem 1.1, we split the first integral in (3.5) into a central term that provides the main contribution, and a tail term: for some large enough Λ, write

$$\begin{aligned} & \int_a^b \int_a^b \{\rho_2(x,y) - \rho_1(x)\rho_1(y)\}\, dx\, dy \\ = & \left[\int\int_{\left\{(x,y):x,y\in[a,b],|x-y|\geq\Lambda/\tilde{K}_n(x,x)\right\}} + \int\int_{\left\{(x,y):x,y\in[a,b],|x-y|<\Lambda/\tilde{K}_n(x,x)\right\}}\right] \\ & \{\rho_2(x,y) - \rho_1(x)\rho_1(y)\}\, dx\, dy \\ = & \text{ Tail } + \text{Central}. \end{aligned}$$

We handle the tail term by proving the following estimate and a simple consequence:

Lemma 3.3
(a) There exist C_1, n_0, and Λ_0 such that for $n \geq n_0$ and $|x-y| \geq \frac{\Lambda_0}{n}$,

(3.12)
$$|\rho_2(x,y) - \rho_1(x)\rho_1(y)| \leq \frac{C_1}{|x-y|^2}.$$

(b) There exist C_2, n_0, and Λ_0 such that for $n \geq n_0$ and $\Lambda \geq \Lambda_0$,

(3.13)
$$\int\int_{\{(x,y):x,y\in[a,b],|x-y|\geq\Lambda/n\}} |\rho_2(x,y) - \rho_1(x)\rho_1(y)|\, dx\, dy \leq C_2\frac{n}{\Lambda}.$$

Proof
See Section 5. ■

Recall that Ξ is defined by (1.6). For the central term we will prove:

Lemma 3.4

(a) Uniformly for u in compact subsets of $\mathbb{C}\backslash\{0\}$, for $|x| \leq 1-\varepsilon$, and $y = x + \frac{u}{\tilde{K}_n(x,x)}$,

$$\frac{1}{\tilde{K}_n(x,x)^2}\{\rho_2(x,y)-\rho_1(x)\rho_1(y)\} = \Xi(u)+o(1). \tag{3.14}$$

(b) Let $\eta > 0$. There exists C such that for $|x| \leq 1-\varepsilon$ and $y = x + \frac{u}{\tilde{K}_n(x,x)}$, $u \in [-\eta, \eta]$,

$$|\rho_2(x,y)-\rho_1(x)\rho_1(y)| \leq Cn^2.$$

(c) For any $[a,b] \subset [-1+\varepsilon, 1-\varepsilon]$,

$$\frac{1}{n}\int_a^b \rho_1(x)\,dx - \frac{1}{\sqrt{3}}\int_a^b \frac{1}{n}\tilde{K}_n(x,x)\,dx = o(1). \tag{3.15}$$

Proof
See Section 6. ■

Proof of Theorem 1.1
We fix $\Lambda > \eta > 0$ and split

$$\begin{aligned} &\int_a^b\int_a^b \{\rho_2(x,y)-\rho_1(x)\rho_1(y)\}\,dy\;dx \\ = &\int_a^b\left[\int_I+\int_J+\int_K\right]\{\rho_2(x,y)-\rho_1(x)\rho_1(y)\}\,dy\;dx, \end{aligned} \tag{3.16}$$

where for a given x,

$$\begin{aligned} I &= \left\{y\in[a,b]:|y-x|\geq \Lambda/\tilde{K}_n(x,x)\right\}; \\ J &= \left\{y\in[a,b]:\eta/\tilde{K}_n(x,x)\leq|y-x|<\Lambda/\tilde{K}_n(x,x)\right\}; \\ K &= \left\{y\in[a,b]:|y-x|<\eta/\tilde{K}_n(x,x)\right\}. \end{aligned}$$

Recall from (1.11) that $\tilde{K}_n(x,x) \sim n$ uniformly for $n \geq 1$ and $|x| \leq 1-\varepsilon$. If A is a uniform upper bound for $\frac{1}{n}\tilde{K}_n(x,x)$ in $[a,b]$ for $n \geq 1$,

$$\begin{aligned} &\left|\int_a^b\int_I \{\rho_2(x,y)-\rho_1(x)\rho_1(y)\}\,dy\;dx\right| \\ \leq &\int\int_{\{(x,y):x,y\in[a,b],|x-y|\geq\Lambda/(nA)\}} |\rho_2(x,y)-\rho_1(x)\rho_1(y)|\,dy\;dx \\ \leq &\; C_1\frac{nA}{\Lambda}, \end{aligned}$$

(3.17)

by Lemma 3.3(b), provided $\Lambda/A \geq \Lambda_0$. Next,

$$\begin{aligned}
&\frac{1}{n}\int_a^b \int_J \{\rho_2(x,y) - \rho_1(x)\rho_1(y)\}\, dy\ dx \\
= \quad &\int_a^b \frac{\tilde{K}_n(x,x)}{n} \int_{\substack{\eta\leq|u|\leq\Lambda, \\ x+\frac{u}{\tilde{K}_n(x,x)}\in[a,b]}} \left\{\rho_2\left(x, x+\frac{u}{\tilde{K}_n(x,x)}\right) - \rho_1(x)\rho_1\left(x+\frac{u}{\tilde{K}_n(x,x)}\right)\right\} \\
&\frac{1}{\tilde{K}_n(x,x)^2} du\ dx.
\end{aligned}$$

Note that if $\eta \leq |u| \leq \Lambda$ and $x \in [a,b]$ but $x + \frac{u}{\tilde{K}_n(x,x)} \notin [a,b]$, then x is at a distance of $O\left(\frac{\Lambda}{n}\right)$ to a or b, and in view of Lemma 3.4(b) and (1.11), the integral over such (x,u) is $O\left(\frac{1}{n}\right)$. Using Lemma 3.4(a) and (1.11), we deduce that

$$\begin{aligned}
&\int_a^b \frac{\tilde{K}_n(x,x)}{n} \int_J \{\rho_2(x,y) - \rho_1(x)\rho_1(y)\}\, dydx \\
= \quad &\left(\int_a^b \frac{\tilde{K}_n(x,x)}{n} dx\right) \left(\int_{\eta\leq|u|\leq\Lambda} \Xi(u)\, du\right) + o(1).
\end{aligned} \tag{3.18}$$

Finally, from Lemma 3.4(b) and (1.11), (but with a different fixed η there),

$$\frac{1}{n}\left|\int_a^b \int_K \{\rho_2(x,y) - \rho_1(x)\rho_1(y)\}\, dydx\right| \leq C\eta, \tag{3.19}$$

where C is independent of n, η. Combining the three estimates (3.17)-(3.19), over I, J, K with (3.5), (3.15) and (3.16), we obtain

$$\begin{aligned}
&\limsup_{n\to\infty}\left|\frac{1}{n}Var[N_n(a,b)] - \left(\int_a^b \frac{\tilde{K}_n(x,x)}{n} dx\right) \left(\int_{\eta\leq|u|\leq\Lambda} \Xi(u)\, du + \frac{1}{\sqrt{3}}\right)\right| \\
\leq \quad &C\left(\frac{1}{\Lambda} + \eta\right).
\end{aligned}$$

Here C is independent of Λ and η. In [18, Proof of Theorem 1.2] it was shown that $\int_{-\infty}^{\infty} \Xi(u)\, du$ converges. We can let $\Lambda \to \infty$ and $\eta \to 0$ to deduce the result. ■

4. Auxiliary Results

We first record some universality limits. Recall that S is defined by (1.2). We also introduce some auxiliary parameters that will simplify notation and will be used throughout the sequel. For a given n and x, we set

$$\kappa = \tilde{K}_{n+1}(x,x) \tag{4.1}$$

and

$$\tau = \frac{nQ_n'(x)}{\tilde{K}_{n+1}(x,x)}. \tag{4.2}$$

We do not display this dependence on n and x. From (1.11) and (1.13), uniformly in $[-1+\varepsilon, 1-\varepsilon]$, $n \geq 1$,

$$|\tau| \leq C. \tag{4.3}$$

We use both κ and $K_{n+1}(x,x)$ in the same formulae where convenient.

Lemma 4.1
Let $\varepsilon \in (0,1)$. Let r, s be non-negative integers. Then
(a) Uniformly for $|x| \leq 1-\varepsilon$ and u, v in compact subsets of $\mathbb{C}$,

$$\lim_{n\to\infty}\left\{\frac{K_{n+1}^{(1,0)}\left(x+\frac{u}{\kappa}, x+\frac{v}{\kappa}\right)}{K_{n+1}(x,x)}\frac{e^{-\tau(u+v)}}{\kappa}-\tau S(v-u)\right\}=-S'(v-u). \tag{4.4}$$

$$\lim_{n\to\infty}\left\{\frac{K_{n+1}^{(0,1)}\left(x+\frac{u}{\kappa}, x+\frac{v}{\kappa}\right)}{K_{n+1}(x,x)}\frac{e^{-\tau(u+v)}}{\kappa}-\tau S(v-u)\right\}=S'(v-u). \tag{4.5}$$

(b)

$$\lim_{n\to\infty}\left\{\frac{K_{n+1}^{(1,1)}\left(x+\frac{u}{\kappa}, x+\frac{v}{\kappa}\right)}{K_{n+1}(x,x)}\frac{e^{-\tau(u+v)}}{\kappa^2}-\tau^2 S(v-u)\right\}=-S''(v-u.) \tag{4.6}$$

(c) In particular, uniformly for $|x| \leq 1-\varepsilon$,

$$\lim_{n\to\infty}\left\{\frac{K_{n+1}^{(1,0)}(x,x)}{K_{n+1}(x,x)\,\kappa}-\tau\right\}=0. \tag{4.7}$$

and

$$\lim_{n\to\infty}\left\{\frac{K_{n+1}^{(1,1)}(x,x)}{K_{n+1}(x,x)\,\kappa^2}-\tau^2\right\}=\frac{\pi^2}{3}. \tag{4.8}$$

(d) Uniformly for $|x| \leq 1-\varepsilon$,

$$\lim_{n\to\infty}\frac{\tilde{K}_{n+1}^{(1,1)}(x,x)\,\tilde{K}_{n+1}(x,x)-\tilde{K}_{n+1}^{(0,1)}(x,x)^2}{\kappa^4}=\frac{\pi^2}{3}. \tag{4.9}$$

(e) Uniformly for $|x| \leq 1-\varepsilon$, and $r = 0, 1$,

$$\tilde{K}_n^{(r,r)}(x) \sim n^{2r+1}. \tag{4.10}$$

Proof
(a) We start with our hypothesis (1.12) that uniformly for $x \in [a,b]$ and u, v in compact subsets of $\mathbb{C}$,

$$\lim_{n\to\infty}\frac{K_{n+1}\left(x+\frac{u}{\kappa}, x+\frac{v}{\kappa}\right)}{K_{n+1}(x,x)}e^{-\tau(u+v)}=S(v-u).$$

Because this holds uniformly for u, v in compact subsets of the plane, we can differentiate this relation w.r.t. u, v. Differentiating once w.r.t. u gives

$$\lim_{n\to\infty}\left\{\frac{K_{n+1}^{(1,0)}\left(x+\frac{u}{\kappa}, x+\frac{v}{\kappa}\right)}{K_{n+1}(x,x)}\frac{e^{-\tau(u+v)}}{\kappa}-\tau\frac{K_{n+1}\left(x+\frac{u}{\kappa}, x+\frac{v}{\kappa}\right)}{K_{n+1}(x,x)}e^{-\tau(u+v)}\right\}=-S'(v-u).$$

Hence

$$\lim_{n\to\infty}\left\{\frac{K_{n+1}^{(1,0)}\left(x+\frac{u}{\kappa}, x+\frac{v}{\kappa}\right)}{K_{n+1}(x,x)}\frac{e^{-\tau(u+v)}}{\kappa}-\tau S(v-u)\right\}=-S'(v-u).$$

So we obtain (4.4). Similarly we obtain (4.5).
(b) Differentiating (4.4) w.r.t. v gives

$$\lim_{n\to\infty}\left\{\begin{array}{c}\frac{K_{n+1}^{(1,1)}\left(x+\frac{u}{\kappa},x+\frac{v}{\kappa}\right)}{K_{n+1}(x,x)}\frac{e^{-\tau(u+v)}}{\kappa^2}-\frac{K_{n+1}^{(1,0)}\left(x+\frac{u}{\kappa},x+\frac{v}{\kappa}\right)}{K_{n+1}(x,x)}\frac{e^{-\tau(u+v)}\tau}{\kappa}\\ -\tau S'(v-u)\end{array}\right\}$$
$$= \quad -S''(v-u).$$

and then

$$\lim_{n\to\infty}\left\{\begin{array}{c}\frac{K_{n+1}^{(1,1)}\left(x+\frac{u}{\kappa},x+\frac{v}{\kappa}\right)}{K_{n+1}(x,x)}\frac{e^{-\tau(u+v)}}{\kappa^2}-\tau\left[\tau S(v-u)-S'(v-u)\right]\\ -\tau S'(v-u)\end{array}\right\}$$
$$= \quad -S''(v-u).$$

$$-\tau S'(v-u)\} = -S''(v-u).$$

This simplifies to (4.6).
(c) Since $S(0)=1$; $S'(0)=0$ and $S''(0)=-\frac{\pi^2}{3}$ [18, p. 13, (3.15)] we obtain also the results for $u=v=0$.
(d) From (c),

$$\begin{aligned}&\frac{\tilde{K}_{n+1}^{(1,1)}(x,x)\,\tilde{K}_{n+1}(x,x)-\tilde{K}_{n+1}^{(0,1)}(x,x)^2}{\kappa^4}\\ =\;&\frac{K_{n+1}^{(1,1)}(x,x)}{K_{n+1}(x,x)\,\kappa^2}-\left(\frac{K_{n+1}^{(0,1)}(x,x)}{\kappa K_{n+1}(x,x)}\right)^2\\ =\;&\left(\tau^2+\frac{\pi^2}{3}+o(1)\right)-(\tau+o(1))^2\\ =\;&\frac{\pi^2}{3}+o(\tau)+o(1)=\frac{\pi^2}{3}+o(1),\end{aligned}$$

recall (4.3).
(e) For $r=0$, this is our hypothesis (1.11). For $r=1$, from (4.8) and (4.3), uniformly for $|x|\leq 1-\varepsilon$,

$$\frac{\tilde{K}_{n+1}^{(1,1)}(x,x)}{\kappa^3}=\tau^2+\frac{\pi^2}{3}+o(1)\sim 1.$$

Since $\kappa\sim n$ as follows from (1.11), we obtain the result for $r=1$. ■

Lemma 4.2
Let $\varepsilon\in(0,1)$. Then for $r,s=0,1$, and for all $n\geq 1$ and $x,y\in[-1+\varepsilon,1-\varepsilon]$,

$$\left|\tilde{K}_{n+1}^{(r,s)}(x,y)\right|\leq\frac{C_4 n^{r+s}}{|x-y|+\frac{1}{n}}. \tag{4.11}$$

Proof
The Christoffel-Darboux formula asserts that

$$K_{n+1}(x,y)=\frac{\gamma_{n,n}}{\gamma_{n,n+1}}\frac{p_{n,n+1}(x)\,p_{n,n}(y)-p_{n,n}(x)\,p_{n,n+1}(y)}{x-y}$$

so that using our bounds (1.9), (1.10),

$$\left|\tilde{K}_{n+1}(x,y)\right|\leq\frac{2C_1C^2}{|x-y|}.$$

Moreover, by Cauchy-Schwartz, and our bound (1.11) on $\tilde{K}_n$,

$$\left|\tilde{K}_{n+1}(x,y)\right| \leq \tilde{K}_{n+1}(x,x)^{1/2}\,\tilde{K}_{n+1}(y,y)^{1/2} \leq C_2^2 n.$$

Combining the last two inequalities yields

$$\left|\tilde{K}_{n+1}(x,y)\right| \leq C_3 \min\left\{\frac{1}{|x-y|}, n\right\},$$

giving (4.11) for $r = s = 0$. Next,

$$\begin{aligned} & K_{n+1}^{(1,0)}(x,y) \\ = \quad & \frac{\gamma_{n,n}}{\gamma_{n,n+1}}\left(\frac{p'_{n,n+1}(x)\,p_{n,n}(y) - p'_{n,n}(x)\,p_{n,n+1}(y)}{x-y} + \frac{p_{n,n+1}(x)\,p_{n,n}(y) - p_{n,n}(y)\,p_{n,n+1}(x)}{(x-y)^2}\right). \end{aligned}$$

(4.12)

Using our bounds on the orthogonal polynomials and their derivatives,

$$\left|\tilde{K}_{n+1}^{(1,0)}(x,y)\right| \leq C_5\left\{\frac{n}{|x-y|} + \frac{1}{|x-y|^2}\right\}.$$

Next, by Cauchy-Schwartz, and the bound (4.10) on $\tilde{K}_n^{(1,1)}$

$$\left|\tilde{K}_{n+1}^{(1,0)}(x,y)\right| \leq \tilde{K}_{n+1}^{(1,1)}(x,x)^{1/2}\,\tilde{K}_{n+1}(x,x)^{1/2} \leq C_6 n^2.$$

Thus

$$\left|\tilde{K}_{n+1}^{(1,0)}(x,y)\right| \leq C_7 \min\left\{\frac{n}{|x-y|} + \frac{1}{|x-y|^2}, n^2\right\}.$$

This yields (4.11) for $r = 1, s = 0$. Of course $r = 0, s = 1$ follows by symmetry. Finally,

$$\begin{aligned} K_{n+1}^{(1,1)}(x,y) \quad = \quad & \frac{\gamma_{n,n}}{\gamma_{n,n+1}}\Big(\frac{p'_{n,n+1}(x)\,p'_{n,n}(y) - p'_{n,n}(x)\,p'_{n,n+1}(y)}{x-y} \\ & + \frac{p'_{n,n+1}(x)\,p_{n,n}(y) - p'_{n,n}(x)\,p_{n,n+1}(y)}{(x-y)^2} \\ & + \frac{p_{n,n}(x)\,p'_{n,n+1}(y) - p'_{n,n}(y)\,p_{n,n+1}(x)}{(x-y)^2} \\ & + 2\frac{p_{n,n}(x)\,p_{n,n+1}(y) - p_{n,n}(y)\,p_{n.n+1}(x)}{(x-y)^3}\Big). \end{aligned}$$

Thus using our bounds on $\left\{p_k^{(j)}\right\}$, $j = 0, 1, 2$, $k = n, n+1$, gives for $x, y \in [a, b]$,

$$\left|\tilde{K}_{n+1}^{(1,1)}(x,y)\right| \leq C_8\left\{\frac{n^2}{|x-y|} + \frac{n}{|x-y|^2} + \frac{1}{|x-y|^3}\right\}$$

and again Cauchy-Schwartz gives

$$\left|\tilde{K}_n^{(1,1)}(x,y)\right| \leq \tilde{K}_{n+1}^{(1,1)}(x,x)^{1/2}\,\tilde{K}_{n+1}^{(1,1)}(y,y)^{1/2} \leq C_9 n^3.$$

This and the previous inequality give (4.11) for $r = s = 1$. ■

5. The Tail Term - Lemma 3.3

Recall that ρ_1, ρ_2 are defined by (3.3) and (3.6). We shall consistently use the $\sim$ versions of expressions and formulae in this section. First write

$$\tilde{\rho}_1(x) = \frac{1}{\pi \tilde{K}_{n+1}(x,x)} \sqrt{\tilde{\Psi}(x)} \tag{5.1}$$

where

$$\tilde{\Psi}(x) = \tilde{K}_{n+1}^{(1,1)}(x,x)\,\tilde{K}_{n+1}(x,x) - \tilde{K}_{n+1}^{(0,1)}(x,x)^2. \tag{5.2}$$

Next, recall $\rho_j = \tilde{\rho}_j$ for $j = 1, 2$ and write

$$\tilde{\rho}_2(x,y) - \tilde{\rho}_1(x)\,\tilde{\rho}_1(y) = \tilde{T}_1 + \tilde{T}_2 + \tilde{T}_3, \tag{5.3}$$

where

$$\begin{aligned} \tilde{T}_1 &= \frac{1}{\pi^2 \tilde{\Delta}} \left(\sqrt{\left(\tilde{\Omega}_{11}\tilde{\Omega}_{22} - \tilde{\Omega}_{12}^2\right)\tilde{\Delta}} - \sqrt{\tilde{\Psi}(x)\,\tilde{\Psi}(y)} \right); \\ \tilde{T}_2 &= \frac{1}{\pi^2 \sqrt{\tilde{\Delta}}} \left|\tilde{\Omega}_{12}\right| \arcsin\left(\frac{\left|\tilde{\Omega}_{12}\right|}{\sqrt{\tilde{\Omega}_{11}\tilde{\Omega}_{22}}} \right); \\ \tilde{T}_3 &= \frac{1}{\pi^2}\left(\frac{1}{\tilde{\Delta}} - \frac{1}{\tilde{K}_{n+1}(x,x)\,\tilde{K}_{n+1}(y,y)} \right) \sqrt{\tilde{\Psi}(x)\,\tilde{\Psi}(y)}. \end{aligned} \tag{5.4}$$

We estimate each $\tilde{T}$ term separately.

Lemma 5.1
There exists $\Lambda_0 > 0$ such that for all $x, y \in [-1+\varepsilon, 1-\varepsilon]$, with $|x-y| \geq \Lambda_0/n$,

$$\left|\tilde{T}_1\right| \leq \frac{C}{\left(|x-y| + \frac{1}{n}\right)^2}. \tag{5.5}$$

Proof
Write

$$\tilde{T}_1 = \frac{\left(\tilde{\Omega}_{11}\tilde{\Omega}_{22} - \tilde{\Omega}_{12}^2\right)\tilde{\Delta} - \tilde{\Psi}(x)\,\tilde{\Psi}(y)}{\pi^2 \tilde{\Delta}\left[\sqrt{\left(\tilde{\Omega}_{11}\tilde{\Omega}_{22} - \tilde{\Omega}_{12}^2\right)\tilde{\Delta}} + \sqrt{\tilde{\Psi}(x)\,\tilde{\Psi}(y)}\right]} = \frac{\text{Num}}{\text{Denom}}.$$

The numerator is (recall (3.11))

$$\begin{aligned} \text{Num} &= \left(\tilde{\Omega}_{11}\tilde{\Omega}_{22} - \tilde{\Omega}_{12}^2\right)\tilde{\Delta} - \tilde{\Psi}(x)\,\tilde{\Psi}(y) \\ &= \det\left(\tilde{\Sigma}\right) - \tilde{\Psi}(x)\,\tilde{\Psi}(y) \end{aligned}$$

$$\begin{aligned} = \ & \det \begin{bmatrix} \tilde{K}_{n+1}(x,x) & \tilde{K}_{n+1}(x,y) & \tilde{K}_{n+1}^{(0,1)}(x,x) & \tilde{K}_{n+1}^{(0,1)}(x,y) \\ \tilde{K}_{n+1}(x,y) & \tilde{K}_{n+1}(y,y) & \tilde{K}_{n+1}^{(0,1)}(y,x) & \tilde{K}_{n+1}^{(0,1)}(y,y) \\ \tilde{K}_{n+1}^{(0,1)}(x,x) & \tilde{K}_{n+1}^{(0,1)}(y,x) & \tilde{K}_{n+1}^{(1,1)}(x,x) & \tilde{K}_{n+1}^{(1,1)}(x,y) \\ \tilde{K}_{n+1}^{(0,1)}(x,y) & \tilde{K}_{n+1}^{(0,1)}(y,y) & \tilde{K}_{n+1}^{(1,1)}(x,y) & \tilde{K}_{n+1}^{(1,1)}(y,y) \end{bmatrix} \\ & - \det \begin{bmatrix} \tilde{K}_{n+1}(x,x) & \tilde{K}_{n+1}^{(0,1)}(x,x) \\ \tilde{K}_{n+1}^{(0,1)}(x,x) & \tilde{K}_{n+1}^{(1,1)}(x,x) \end{bmatrix} \det \begin{bmatrix} \tilde{K}_{n+1}(y,y) & \tilde{K}_{n+1}^{(0,1)}(y,y) \\ \tilde{K}_{n+1}^{(0,1)}(y,y) & \tilde{K}_{n+1}^{(1,1)}(y,y) \end{bmatrix}. \end{aligned}$$

Using Laplace's determinant expansion exactly as in the proof of Lemma 4.1 in [18], we continue this as

$$= -\det\begin{bmatrix} \tilde{K}_{n+1}(x,x) & \tilde{K}_{n+1}(x,y) \\ \tilde{K}_{n+1}^{(0,1)}(x,x) & \tilde{K}_{n+1}^{(0,1)}(y,x) \end{bmatrix} \det\begin{bmatrix} \tilde{K}_{n+1}^{(0,1)}(y,x) & \tilde{K}_{n+1}^{(0,1)}(y,y) \\ \tilde{K}_{n+1}^{(1,1)}(x,y) & \tilde{K}_{n+1}^{(1,1)}(y,y) \end{bmatrix}$$

$$-\det\begin{bmatrix} \tilde{K}_{n+1}(x,x) & \tilde{K}_{n+1}^{(0,1)}(x,y) \\ \tilde{K}_{n+1}^{(0,1)}(x,x) & \tilde{K}_{n+1}^{(1,1)}(x,y) \end{bmatrix} \det\begin{bmatrix} \tilde{K}_{n+1}(y,y) & \tilde{K}_{n+1}^{(0,1)}(y,x) \\ \tilde{K}_{n+1}^{(0,1)}(y,y) & \tilde{K}_{n+1}^{(1,1)}(x,y) \end{bmatrix}$$

$$-\det\begin{bmatrix} \tilde{K}_{n+1}(x,y) & \tilde{K}_{n+1}^{(0,1)}(x,x) \\ \tilde{K}_{n+1}^{(0,1)}(y,x) & \tilde{K}_{n+1}^{(1,1)}(x,x) \end{bmatrix} \det\begin{bmatrix} \tilde{K}_{n+1}(x,y) & \tilde{K}_{n+1}^{(0,1)}(y,y) \\ \tilde{K}_{n+1}^{(0,1)}(x,y) & \tilde{K}_{n+1}^{(1,1)}(y,y) \end{bmatrix}$$

$$+\det\begin{bmatrix} \tilde{K}_{n+1}(x,y) & \tilde{K}_{n+1}^{(0,1)}(x,y) \\ \tilde{K}_{n+1}^{(0,1)}(y,x) & \tilde{K}_{n+1}^{(1,1)}(x,y) \end{bmatrix} \det\begin{bmatrix} \tilde{K}_{n+1}(x,y) & \tilde{K}_{n+1}^{(0,1)}(y,x) \\ \tilde{K}_{n+1}^{(0,1)}(x,y) & \tilde{K}_{n+1}^{(1,1)}(x,y) \end{bmatrix}$$

$$-\det\begin{bmatrix} \tilde{K}_{n+1}^{(0,1)}(x,x) & \tilde{K}_{n+1}^{(0,1)}(x,y) \\ \tilde{K}_{n+1}^{(1,1)}(x,x) & \tilde{K}_{n+1}^{(1,1)}(x,y) \end{bmatrix} \det\begin{bmatrix} \tilde{K}_{n+1}(x,y) & \tilde{K}_{n+1}(y,y) \\ \tilde{K}_{n+1}^{(0,1)}(x,y) & \tilde{K}_{n+1}^{(0,1)}(y,y) \end{bmatrix}.$$

We now use the estimate (4.11) and that $\left(|x-y|+\frac{1}{n}\right)^{-1} \leq n$, on each of the terms in these deteminants. We obtain, exactly as in the proof of Lemma 4.1 in [18] that this is $O\left(\frac{n^6}{\left(|x-y|+\frac{1}{n}\right)^2}\right)$. Thus

$$\text{Num} = O\left(\frac{n^6}{\left(|x-y|+\frac{1}{n}\right)^2}\right). \tag{5.6}$$

Also

$$\begin{aligned} \text{Denom} &= \pi^2\tilde{\Delta}\left[\sqrt{\left(\tilde{\Omega}_{11}\tilde{\Omega}_{22}-\tilde{\Omega}_{12}^2\right)\tilde{\Delta}}+\sqrt{\tilde{\Psi}(x)\,\tilde{\Psi}(y)}\right] \\ &\geq \pi^2\tilde{\Delta}\sqrt{\tilde{\Psi}(x)\,\tilde{\Psi}(y)}. \end{aligned}$$

Here from Lemma 4.1(d) and (1.11),

$$\tilde{\Psi}(x) = \tilde{K}_{n+1}^{(1,1)}(x,x)\,\tilde{K}_{n+1}(x,x) - \tilde{K}_{n+1}^{(0,1)}(x,x)^2 \geq \frac{\pi^2}{3}\tilde{K}_{n+1}(x,x)^4(1+o(1)) \geq Cn^4.$$

Also from (1.11) and (4.11),

$$\begin{aligned} 1-\frac{\tilde{\Delta}}{\tilde{K}_{n+1}(x,x)\,\tilde{K}_{n+1}(x,x)} &= \frac{\tilde{K}_{n+1}^2(x,y)}{\tilde{K}_{n+1}(x,x)\,\tilde{K}_{n+1}(y,y)} \\ &\leq \frac{C}{(n|x-y|+1)^2} \leq \frac{1}{2}, \end{aligned}$$

if $|x-y| \geq \Lambda_0/n$ with Λ_0 large enough. Then

$$\tilde{\Delta} \geq \frac{1}{2}\tilde{K}_{n+1}(x,x)\,\tilde{K}_{n+1}(y,y) \geq Cn^2 \tag{5.7}$$

and

$$\text{Denom} \geq Cn^6. \tag{5.8}$$

Combined with (5.6), this yields

$$\left|\tilde{T}_1\right| = \left|\frac{\text{Num}}{\text{Denom}}\right| \leq \frac{C}{\left(|x-y|+\frac{1}{n}\right)^2}.$$

■

Next, let us deal with T_2 :

Lemma 5.2
There exist Λ_0 such that for all $x, y \in [-1+\varepsilon, 1-\varepsilon]$, with $|x-y| \geq \Lambda_0/n$,

$$\left|\tilde{T}_2\right| \leq \frac{C}{\left(|x-y|+\frac{1}{n}\right)^2}. \tag{5.9}$$

Proof
Recall that

$$\left|\tilde{T}_2\right| = \tilde{T}_2 = \frac{1}{\pi^2\sqrt{\tilde{\Delta}}}\left|\tilde{\Omega}_{12}\right| \arcsin\left(\frac{\left|\tilde{\Omega}_{12}\right|}{\sqrt{\tilde{\Omega}_{11}\tilde{\Omega}_{22}}}\right).$$

Using $|\arcsin v| \leq \frac{\pi}{2}|v|$, $|v| \leq 1$, we obtain

$$\left|\tilde{T}_2\right| \leq \frac{1}{2\pi\tilde{\Delta}^{3/2}} \frac{\left|\tilde{\Omega}_{12}\tilde{\Delta}\right|^2}{\sqrt{\tilde{\Omega}_{11}\tilde{\Omega}_{22}\tilde{\Delta}^2}}. \tag{5.10}$$

Here from (3.10) and (4.11), and expanding by the first row,

$$\tilde{\Omega}_{12}\tilde{\Delta} = \det\begin{bmatrix} \tilde{K}_{n+1}(x,x) & \tilde{K}_{n+1}(x,y) & \tilde{K}_{n+1}^{(0,1)}(x,x) \\ \tilde{K}_{n+1}(y,x) & \tilde{K}_{n+1}(y,y) & \tilde{K}_{n+1}^{(0,1)}(y,x) \\ \tilde{K}_{n+1}^{(1,0)}(y,x) & \tilde{K}_{n+1}^{(0,1)}(y,y) & \tilde{K}_{n+1}^{(1,1)}(y,x) \end{bmatrix} = O\left(\frac{n^4}{|x-y|+\frac{1}{n}}\right).$$

(5.11)

Next, we examine $\tilde{\Omega}_{11}$ and $\tilde{\Omega}_{22}$. From (3.8) and (4.11), and expanding by the first row,

$$\tilde{\Omega}_{11}\tilde{\Delta} = \det\begin{bmatrix} \tilde{K}_{n+1}(y,y) & \tilde{K}_{n+1}(y,x) & \tilde{K}_{n+1}^{(0,1)}(y,x) \\ \tilde{K}_{n+1}(x,y) & \tilde{K}_{n+1}(x,x) & \tilde{K}_{n+1}^{(0,1)}(x,x) \\ \tilde{K}_{n+1}^{(1,0)}(x,y) & \tilde{K}_{n+1}^{(0,1)}(x,x) & \tilde{K}_{n+1}^{(1,1)}(x,x) \end{bmatrix}$$

$$= \tilde{K}_{n+1}(y,y)\left\{\tilde{K}_{n+1}(x,x)\tilde{K}_{n+1}^{(1,1)}(x,x) - \tilde{K}_{n+1}^{(0,1)}(x,x)^2\right\} + O\left(\frac{n^3}{\left(|x-y|+\frac{1}{n}\right)^2}\right)$$

so if $|x-y| \geq \Lambda_0/n$, and $\Lambda_0 \geq 1$,

$$\begin{aligned} \tilde{\Omega}_{11}\tilde{\Delta} &= \tilde{K}_{n+1}(y,y)\left\{\tilde{K}_{n+1}(x,x)\tilde{K}_{n+1}^{(1,1)}(x,x) - \tilde{K}_{n+1}^{(0,1)}(x,x)^2\right\} + O\left(\frac{n^5}{\Lambda_0^2}\right) \\ &\geq Cn^5 + O\left(\frac{n^5}{\Lambda_0^2}\right) \geq C_1 n^5, \end{aligned}$$

(5.12)

by (4.9), if Λ_0 and n are large enough. In much the same way,

$$\begin{aligned}
\tilde{\Omega}_{22}\tilde{\Delta} &= \det\begin{bmatrix} \tilde{K}_{n+1}(x,x) & \tilde{K}_{n+1}(x,y) & \tilde{K}_{n+1}^{(0,1)}(x,y) \\ \tilde{K}_{n+1}(y,x) & \tilde{K}_{n+1}(y,y) & \tilde{K}_{n+1}^{(0,1)}(y,y) \\ \tilde{K}_{n+1}^{(1,0)}(y,x) & \tilde{K}_{n+1}^{(1,0)}(y,y) & \tilde{K}_{n+1}^{(1,1)}(y,y) \end{bmatrix} \\
&= \tilde{K}_{n+1}(x,x)\left\{\tilde{K}_{n+1}(y,y)\tilde{K}_{n+1}^{(1,1)}(y,y) - \tilde{K}_{n+1}^{(0,1)}(y,y)^2\right\} + O\left(\frac{n^5}{\Lambda_0^2}\right) \\
&\geq C_1 n^5.
\end{aligned}$$

(5.13)

Then combining (5.10-5.13), followed by (5.7),

$$\tilde{T}_2 \leq C\left(\frac{n^4}{|x-y|+\frac{1}{n}}\right)^2 \frac{1}{\Delta^{3/2}}\frac{1}{n^5} \leq C\left(\frac{1}{|x-y|+\frac{1}{n}}\right)^2.$$

■

Next, we handle $\tilde{T}_3$:

Lemma 5.3
There exists Λ_0 such that for all $x, y \in [-1+\varepsilon, 1-\varepsilon]$, with $|x-y| \geq \Lambda_0/n$,

(5.14) $$\left|\tilde{T}_3\right| \leq \frac{C}{\left(|x-y|+\frac{1}{n}\right)^2}.$$

Proof
From (5.4), with Ψ given by (5.2),

$$\tilde{T}_3 = \frac{1}{\pi^2}\frac{\tilde{K}_{n+1}^2(x,y)}{\tilde{\Delta}\tilde{K}_{n+1}(x,x)\tilde{K}_{n+1}(y,y)}\sqrt{\tilde{\Psi}(x)\tilde{\Psi}(y)}.$$

Here from (4.9) and (1.11),

$$\left|\tilde{\Psi}(x)\right|, \left|\tilde{\Psi}(y)\right| \leq Cn^4.$$

Then

$$\tilde{T}_3 \leq \frac{C}{\left(|x-y|+\frac{1}{n}\right)^2},$$

by (4.11) and (5.7). Note too that $\tilde{T}_3 \geq 0$. ■

Proof of Lemma 3.3(a)
Just combine the estimates for $\tilde{T}_1, \tilde{T}_2, \tilde{T}_3$ from Lemmas 5.1, 5.2, 5.3 and recall (5.3).
■

Proof of Lemma 3.3(b)

From Lemma 3.3(a), for $y \in [-1+\varepsilon, 1-\varepsilon]$,

$$\begin{aligned}
&\int_{\{x\in[a,b],|x-y|\geq\Lambda/n\}} |\tilde{\rho}_2(x,y) - \tilde{\rho}_1(x)\tilde{\rho}_1(y)|\,dx \\
\leq\ &\int_{\{x\in[a,b],|x-y|\geq\Lambda/n\}} \frac{C}{|x-y|^2}dx \\
\leq\ &\int_{\{x\in[a,b],|x-y|\geq\Lambda/n\}} \frac{2C}{|x-y|^2+\left(\frac{\Lambda}{n}\right)^2}dx \\
\leq\ &\int_{-\infty}^{\infty} \frac{2C}{|x-y|^2+\left(\frac{\Lambda}{n}\right)^2}dx.
\end{aligned}$$

We make the substitution $x - y = \frac{\Lambda}{n}t$ in the latter integral:

$$= \frac{n}{\Lambda}\int_{\mathbb{R}\setminus[-1,1]} \frac{2C}{t^2+1}dt.$$

Then (3.13) follows. ■

6. The Central Term - Lemma 3.4

Recall that $\Delta, \Omega_{11}, \Omega_{22}, \Omega_{12}$ were defined in (3.7-3.10), while S, F, G, H were defined in (1.2-1.5). In this section, we use the non-normalized versions of our formulae. Recall that we defined κ and τ by (4.1) and (4.2) respectively.

Lemma 6.1
Uniformly for u in compact subsets of the plane, and uniformly for $x \in [-1+\varepsilon, 1-\varepsilon]$ and $y = x + \frac{u}{\tilde{K}_{n+1}(x,x)}$,
(a)

$$\frac{\left(\Omega_{11}\Omega_{22} - \Omega_{12}^2\right)\Delta}{K_{n+1}(x,x)^4}\left(\frac{e^{-\tau u}}{\kappa}\right)^4 = F(u) + o(1); \tag{6.1}$$

(b)

$$\frac{\Delta}{K_{n+1}(x,x)^2}e^{-2\tau u} = 1 - S(u)^2 + o(1); \tag{6.2}$$

(c)

$$\frac{\Delta\Omega_{11}}{K_{n+1}(x,x)^3}\frac{e^{-2\tau u}}{\kappa^2} = G(u) + o(1); \tag{6.3}$$

(d)

$$\frac{\Delta\Omega_{22}}{K_{n+1}(x,x)^3}\frac{e^{-4\tau u}}{\kappa^2} = G(u) + o(1); \tag{6.4}$$

(e)

$$\frac{\Omega_{12}\Delta}{K_{n+1}(x,x)^3}\frac{e^{-3\tau u}}{\kappa^2} = H(u) + o(1). \tag{6.5}$$

Proof
From (1.12) and the limits in Lemma 4.1, uniformly for u in compact subsets of the plane,

$$\lim_{n\to\infty}\frac{K_{n+1}(x,y)}{K_{n+1}(x,x)}e^{-\tau u}=S(u);$$
$$\lim_{n\to\infty}\left\{\frac{K^{(1,0)}_{n+1}(x,y)}{K_{n+1}(x,x)}\frac{e^{-\tau u}}{\kappa}-\tau S(u)\right\}=-S'(u);$$
$$\lim_{n\to\infty}\left\{\frac{K^{(0,1)}_{n+1}(x,y)}{K_{n+1}(x,x)}\frac{e^{-\tau u}}{\kappa}-\tau S(u)\right\}=S'(u);$$
$$\lim_{n\to\infty}\left\{\frac{K^{(1,1)}_{n+1}(x,y)}{K_{n+1}(x,x)}\frac{e^{-\tau u}}{\kappa^2}-\tau^2 S(u)\right\}=-S''(u);$$
$$\lim_{n\to\infty}\frac{K_{n+1}(y,y)}{K_{n+1}(x,x)}e^{-2\tau u}=1;$$
$$\lim_{n\to\infty}\left\{\frac{K^{(1,0)}_{n+1}(y,y)}{K_{n+1}(x,x)}\frac{e^{-2\tau u}}{\kappa}-\tau\right\}=0;$$
$$\lim_{n\to\infty}\left\{\frac{K^{(1,1)}_{n+1}(y,y)}{K_{n+1}(x,x)}\frac{e^{-2\tau u}}{\kappa^2}-\tau^2\right\}=-S''(0)=\frac{\pi^2}{3}. \tag{6.6}$$

We shall repeatedly refer to these limits using this single equation number.
(a) Recall that Σ was defined by (3.4). Then (3.11) gives

$$\frac{\left[\left(\Omega_{11}\Omega_{22}-\Omega_{12}^2\right)\Delta\right]}{K_{n+1}(x,x)^4}\left(\frac{e^{-\tau u}}{\kappa}\right)^4=\frac{\det\Sigma}{K_{n+1}(x,x)^4}\left(\frac{e^{-\tau u}}{\kappa}\right)^4$$

$$=\det\begin{bmatrix}1 & \frac{K_{n+1}(x,y)}{K_{n+1}(x,x)}e^{-\tau u} & \frac{K^{(0,1)}_{n+1}(x,x)}{K_{n+1}(x,x)}\frac{1}{\kappa} & \frac{K^{(0,1)}_{n+1}(x,y)}{K_{n+1}(x,x)}\frac{e^{-\tau u}}{\kappa}\\ \frac{K_{n+1}(x,y)}{K_{n+1}(x,x)}e^{-\tau u} & \frac{K_{n+1}(y,y)}{K_{n+1}(x,x)}e^{-2\tau u} & \frac{K^{(0,1)}_{n+1}(y,x)}{K_{n+1}(x,x)}\frac{e^{-\tau u}}{\kappa} & \frac{K^{(0,1)}_{n+1}(y,y)}{K_{n+1}(x,x)}\frac{e^{-2\tau u}}{\kappa}\\ \frac{K^{(0,1)}_{n+1}(x,x)}{K_{n+1}(x,x)}\frac{1}{\kappa} & \frac{K^{(0,1)}_{n+1}(y,x)}{K_{n+1}(x,x)}\frac{e^{-\tau u}}{\kappa} & \frac{K^{(1,1)}_{n+1}(x,x)}{K_{n+1}(x,x)}\frac{1}{\kappa^2} & \frac{K^{(1,1)}_{n+1}(x,y)}{K_{n+1}(x,x)}\frac{e^{-\tau u}}{\kappa^2}\\ \frac{K^{(0,1)}_{n+1}(x,y)}{K_{n+1}(x,x)}\frac{e^{-\tau u}}{\kappa} & \frac{K^{(0,1)}_{n+1}(y,y)}{K_{n+1}(x,x)}\frac{e^{-2\tau u}}{\kappa} & \frac{K^{(1,1)}_{n+1}(x,y)e^{-\tau u}}{K_{n+1}(x,x)}\frac{1}{\kappa^2} & \frac{K^{(1,1)}_{n+1}(y,y)}{K_{n+1}(x,x)}\left(\frac{e^{-\tau u}}{\kappa}\right)^2\end{bmatrix}.$$

Here we have factored in $\frac{1}{\kappa}$ into the 3rd and 4th rows and columns. In addition, we have factored in $e^{-\tau u}$ into the second and fourth rows and columns. Using the limits in (6.6) and that $S(0)=1, S'(0)=0$, while $S(-u)=S(u)$, we continue this as

$$=\det\begin{bmatrix}1 & S(u) & \tau & \tau S(u)+S'(u)\\ S(u) & 1 & \tau S(u)-S'(u) & \tau\\ \tau & \tau S(u)-S'(u) & \tau^2-S''(0) & \tau^2 S(u)-S''(u)\\ \tau S(u)+S'(u) & \tau & \tau^2 S(u)-S''(u) & \tau^2-S''(0)\end{bmatrix}+o(1)$$

Now subtract $\tau\times$Row 2 from Row 4:

$$=\det\begin{bmatrix}1 & S(u) & \tau & \tau S(u)+S'(u)\\ S(u) & 1 & \tau S(u)-S'(u) & \tau\\ \tau & \tau S(u)-S'(u) & \tau^2-S''(0) & \tau^2 S(u)-S''(u)\\ S'(u) & 0 & \tau S'(u)-S''(u) & -S''(0)\end{bmatrix}+o(1)$$

Next, subtract $\tau\times$Column 1 from Column 3

$$= \det \begin{bmatrix} 1 & S(u) & 0 & \tau S(u) + S'(u) \\ S(u) & 1 & -S'(u) & \tau \\ \tau & \tau S(u) - S'(u) & -S''(0) & \tau^2 S(u) - S''(u) \\ S'(u) & 0 & -S''(u) & -S''(0) \end{bmatrix} + o(1)$$

Next subtract $\tau\times$Row 1 from Row 3

$$= \det \begin{bmatrix} 1 & S(u) & 0 & \tau S(u) + S'(u) \\ S(u) & 1 & -S'(u) & \tau \\ 0 & -S'(u) & -S''(0) & -\tau S'(u) - S''(u) \\ S'(u) & 0 & -S''(u) & -S''(0) \end{bmatrix} + o(1)$$

Finally subtract $\tau\times$Column 2 from Column 4

$$= \det \begin{bmatrix} 1 & S(u) & 0 & S'(u) \\ S(u) & 1 & -S'(u) & 0 \\ 0 & -S'(u) & -S''(0) & -S''(u) \\ S'(u) & 0 & -S''(u) & -S''(0) \end{bmatrix} + o(1) = F(u) + o(1).$$

(b) From (3.7) and (6.6),

$$\begin{aligned} \frac{\Delta}{K_{n+1}(x,x)^2} e^{-2\tau u} &= \det \begin{bmatrix} 1 & \frac{K_{n+1}(x,y)}{K_{n+1}(x,x)} e^{-\tau u} \\ \frac{K_{n+1}(x,y)}{K_{n+1}(x,x)} e^{-\tau u} & \frac{K_{n+1}(y,y)}{K_{n+1}(x,x)} e^{-2\tau u} \end{bmatrix} \\ &= \det \begin{bmatrix} 1 & S(u) \\ S(u) & 1 \end{bmatrix} + o(1). \end{aligned}$$

(c) From (3.8), and then factoring $e^{-\tau u}$ into the first row and first column and $\frac{1}{\kappa}$ into the third row and third column, and then using (6.6) as well as $S(0) = 1, S'(0) = 0$,

$$\begin{aligned} & \frac{\Delta \Omega_{11}}{K_{n+1}(x,x)^3} \left(\frac{e^{-\tau u}}{\kappa} \right)^2 \\ = & \det \begin{bmatrix} \frac{K_{n+1}(y,y)}{K_{n+1}(x,x)} e^{-2\tau u} & \frac{K_{n+1}(y,x)}{K_{n+1}(x,x)} e^{-\tau u} & \frac{K_{n+1}^{(0,1)}(y,x)}{K_{n+1}(x,x)} \frac{e^{-\tau u}}{\kappa} \\ \frac{K_{n+1}(x,y)}{K_{n+1}(x,x)} e^{-\tau u} & 1 & \frac{K_{n+1}^{(0,1)}(x,x)}{K_{n+1}(x,x)} \frac{1}{\kappa} \\ \frac{K_{n+1}^{(1,0)}(x,y)}{K_{n+1}(x,x)} \frac{e^{-\tau u}}{\kappa} & \frac{K_{n+1}^{(0,1)}(x,x)}{K_{n+1}(x,x)} \frac{1}{\kappa} & \frac{K_{n+1}^{(1,1)}(x,x)}{K_{n+1}(x,x)} \frac{1}{\kappa^2} \end{bmatrix} \\ = & \det \begin{bmatrix} 1 & S(u) & \tau S(u) - S'(u) \\ S(u) & 1 & \tau \\ \tau S(u) - S'(u) & \tau & \tau^2 - S''(0) \end{bmatrix} + o(1) \end{aligned}$$

Subtract $\tau\times$Row2 from Row 3

$$= \det \begin{bmatrix} 1 & S(u) & \tau S(u) - S'(u) \\ S(u) & 1 & \tau \\ S'(-u) & 0 & -S''(0) \end{bmatrix}$$

Subtract $\tau\times$Column 2 from Column 3:

$$= \det \begin{bmatrix} 1 & S(u) & -S'(u) \\ S(u) & 1 & 0 \\ -S'(u) & 0 & -S''(0) \end{bmatrix} + o(1) = G(u) + o(1),$$

recall (1.4).
(d) From (3.9), and factoring $e^{-\tau u}$ into the 2nd and 3rd rows and columns and $\frac{1}{\kappa}$ into the 3rd row and column,

$$\begin{aligned}
&\frac{\Delta\Omega_{22}}{K_{n+1}(x,x)^3}\frac{e^{-4\tau u}}{\kappa^2} \\
=&\ \det\begin{bmatrix} 1 & \frac{K_{n+1}(x,y)}{K_{n+1}(x,x)}e^{-\tau u} & \frac{K_{n+1}^{(0,1)}(x,y)}{K_{n+1}(x,x)}\frac{e^{-\tau u}}{\kappa} \\ \frac{K_{n+1}(y,x)}{K_{n+1}(x,x)}e^{-\tau u} & \frac{K_{n+1}(y,y)}{K_{n+1}(x,x)}e^{-2\tau u} & \frac{K_{n+1}^{(0,1)}(y,y)}{K_{n+1}(x,x)}\frac{e^{-2\tau u}}{\kappa} \\ \frac{K_{n+1}^{(1,0)}(y,x)}{K_{n+1}(x,x)}\frac{e^{-\tau u}}{\kappa} & \frac{K_{n+1}^{(1,0)}(y,y)}{K_{n+1}(x,x)}\frac{e^{-2\tau u}}{\kappa} & \frac{K_{n+1}^{(1,1)}(y,y)}{K_{n+1}(x,x)}\left(\frac{e^{-2\tau u}}{\kappa}\right)^2 \end{bmatrix} \\
=&\ \det\begin{bmatrix} 1 & S(-u) & \tau S(u)+S'(u) \\ S(u) & 1 & \tau \\ \tau S(u)+S'(u) & \tau & \tau^2-S''(0) \end{bmatrix} + o(1).
\end{aligned}$$

Subtract $\tau\times$Row 2 from Row 3:

$$= \det\begin{bmatrix} 1 & S(-u) & \tau S(u)+S'(u) \\ S(u) & 1 & \tau \\ S'(u) & 0 & -S''(0) \end{bmatrix} + o(1)$$

Subtract $\tau\times$Column 2 from Column 3:

$$= \det\begin{bmatrix} 1 & S(u) & S'(u) \\ S(u) & 1 & 0 \\ S'(u) & 0 & -S''(0) \end{bmatrix} + o(1) = G(u)+o(1).$$

Here we have multiplied the 3rd row and 3rd column in G in (1.4) by -1.
(e) From (3.10), and factoring $e^{-\tau u}$ into the 2nd and 3rd rows and the 2nd column, and $\frac{1}{\kappa}$ into the 3rd row and 3rd column,

$$\begin{aligned}
&\frac{\Omega_{12}\Delta}{K_{n+1}(x,x)^3}\frac{e^{3\tau u}}{\kappa^2} \\
=&\ \det\begin{bmatrix} 1 & \frac{K_{n+1}(x,y)}{K_{n+1}(x,x)}e^{-\tau u} & \frac{K_{n+1}^{(0,1)}(x,x)}{K_{n+1}(x,x)}\frac{1}{\kappa} \\ \frac{K_{n+1}(y,x)}{K_{n+1}(x,x)}e^{-\tau u} & \frac{K_{n+1}(y,y)}{K_{n+1}(x,x)}e^{-2\tau u} & \frac{K_{n+1}^{(0,1)}(y,x)}{K_{n+1}(x,x)}\frac{e^{-\tau u}}{\kappa} \\ \frac{K_{n+1}^{(1,0)}(y,x)}{K_{n+1}(x,x)}\frac{e^{-\tau u}}{\kappa} & \frac{K_{n+1}^{(0,1)}(y,y)}{K_{n+1}(x,x)}\frac{e^{-2\tau u}}{\kappa} & \frac{K_{n+1}^{(1,1)}(y,x)}{K_{n+1}(x,x)}e^{-\tau u}\frac{1}{\kappa^2} \end{bmatrix} \\
=&\ \det\begin{bmatrix} 1 & S(-u) & \tau \\ S(u) & 1 & \tau S(u)-S'(u) \\ \tau S(u)+S'(u) & \tau & \tau^2 S(u)-S''(u) \end{bmatrix} + o(1).
\end{aligned}$$

Subtract $\tau\times$Row 2 from Row 3:

$$= \det\begin{bmatrix} 1 & S(-u) & \tau \\ S(u) & 1 & \tau S(u)-S'(u) \\ S'(u) & 0 & \tau S'(u)-S''(u) \end{bmatrix} + o(1).$$

Subtract $\tau\times$Column 1 from Column 3:

$$= \det\begin{bmatrix} 1 & S(-u) & 0 \\ S(u) & 1 & -S'(u) \\ S'(u) & 0 & -S''(u) \end{bmatrix} + o(1) = H(u)+o(1),$$

recall (1.5). ■

Now we can obtain the asymptotics for $\rho_2(x,y) - \rho_1(x)\rho_1(y)$ stated in (3.14):

Proof of Lemma 3.4(a)
Recall as in (5.3), that

$$\rho_2(x,y) - \rho_1(x)\rho_1(y) = T_1 + T_2 + T_3. \tag{6.7}$$

We handle the terms $T_j, j = 1,2,3$ one by one:
Step 1: T_1
Firstly from Lemma 4.1(d), and (5.2),

$$\frac{\Psi(x)}{K_{n+1}(x,x)^2\kappa^2} = \frac{\pi^2}{3} + o(1). \tag{6.8}$$

Then

$$\begin{aligned}\frac{\Psi(y)}{K_{n+1}(x,x)^2}\frac{e^{-4\tau u}}{\kappa^2} &= \left[\frac{\Psi(y)}{K_{n+1}(y,y)^2}\frac{1}{\tilde{K}_{n+1}(y,y)^2}\right]\left[\frac{K_{n+1}(y,y)e^{-2\tau u}}{K_{n+1}(x,x)}\right]^2\left[\frac{\tilde{K}_{n+1}(y,y)}{\tilde{K}_{n+1}(x,x)}\right]^2 \\ &= \left[\frac{\pi^2}{3} + o(1)\right][1+o(1)][1+o(1)] = \frac{\pi^2}{3} + o(1).\end{aligned}$$

(6.9)

Here we are using (6.6) and also that

$$\frac{\mu_n'(y)}{\mu_n'(x)} = e^{2n[Q_n(x)-Q_n(y)]} = e^{-2nQ_n'(x)(y-x)+o(n(y-x))} = e^{-2\tau u + o(1)},$$

by (1.14). Then using (6.2),

$$\begin{aligned}&\frac{1}{\pi^2\Delta}\sqrt{\Psi(x)\Psi(y)}\frac{1}{\kappa^2} \\ =\ &\frac{1}{\pi^2}\left[\frac{K_{n+1}(x,x)^2}{\Delta e^{-2\tau u}}\right]\sqrt{\frac{\Psi(x)}{K_{n+1}(x,x)^2}\frac{1}{\kappa^2}\frac{\Psi(y)}{K_{n+1}(x,x)^2}\frac{e^{-4\tau u}}{\kappa^2}} \\ =\ &\frac{1}{\pi^2}\frac{1}{1-S(u)^2}\left(\frac{\pi^2}{3} + o(1)\right).\end{aligned}$$

Then from (6.1) and (6.8), and recalling the definition of T_1 at (5.4),

$$\begin{aligned}&\frac{T_1}{\kappa^2} \\ =\ &\frac{1}{\pi^2}\left[\frac{K_{n+1}(x,x)^2}{\Delta e^{-2\tau u}}\right]\sqrt{\frac{\left(\Omega_{11}\Omega_{22} - \Omega_{12}^2\right)\Delta}{K_{n+1}(x,x)^4}\left(\frac{e^{-\tau u}}{\kappa}\right)^4} - \frac{1}{\pi^2}\frac{1}{1-S(u)^2}\left(\frac{\pi^2}{3} + o(1)\right) \\ =\ &\frac{1}{\pi^2\left(1-S(u)^2\right)}\left(\sqrt{F(u)} - \frac{\pi^2}{3}\right) + o(1),\end{aligned}$$

by (6.1) and (6.2).
Step 2: T_2

From (5.4),

$$\begin{aligned}
&\frac{T_2}{\kappa^2}\\
=\ &\frac{1}{\pi^2\Delta^{3/2}}\left|\Omega_{12}\Delta\right|\arcsin\left(\frac{\left|\Omega_{12}\Delta\right|}{\sqrt{\left|\Omega_{11}\Delta\right|\left|\Omega_{22}\Delta\right|}}\right)\frac{1}{\kappa^2}\\
=\ &\frac{1}{\pi^2}\left[\frac{K_{n+1}(x,x)^2}{\Delta e^{-2\tau u}}\right]^{3/2}\left|\frac{\Omega_{12}\Delta}{K_{n+1}(x,x)^3}\frac{e^{-3\tau u}}{\kappa^2}\right|\arcsin\left(\frac{\left|\Omega_{12}\Delta\right|}{\sqrt{\left|\Omega_{11}\Delta\right|\left|\Omega_{22}\Delta\right|}}\right)\\
=\ &\frac{1}{\pi^2\left(1-S(u)^2\right)^{3/2}}H(u)\arcsin\left(\frac{H(u)}{G(u)}\right)+o(1),
\end{aligned}$$

by (6.2) - (6.5).

Step 3: T_3

From (5.4),

$$\begin{aligned}
&\frac{T_3}{\kappa^2}\\
=\ &\frac{1}{\pi^2\kappa^2}\left(\frac{K_{n+1}(x,y)^2}{\Delta K_{n+1}(x,x)K_{n+1}(y,y)}\right)\sqrt{\Psi(x)\Psi(y)}\\
=\ &\frac{1}{\pi^2}\left[\frac{K_{n+1}(x,y)e^{-\tau u}}{K_{n+1}(x,x)}\right]^2\left[\frac{K_{n+1}(y,y)}{K_{n+1}(x,x)}e^{-2\tau u}\right]^{-1}\left[\frac{K_{n+1}(x,x)^2}{\Delta e^{-2\tau u}}\right]\left[\sqrt{\frac{\Psi(x)\Psi(y)}{K_{n+1}(x,x)^4}\frac{e^{-4\tau u}}{\kappa^4}}\right]\\
=\ &\frac{1}{\pi^2}\left(\frac{S(u)^2}{1-S(u)^2}\right)\frac{\pi^2}{3}+o(1),
\end{aligned}$$

by (1.12), (6.2), (6.8), and (6.9). Substituting the asymptotics for $T_j, j=1,2,3$ into (6.7) gives

$$\begin{aligned}
&\frac{1}{\kappa^2}\left\{\rho_2(x,y)-\rho_1(x)\rho_1(y)\right\}\\
=\ &\frac{1}{\pi^2\left(1-S(u)^2\right)}\left\{\sqrt{F(u)}-\frac{\pi^2}{3}\left(1-S(u)^2\right)+\frac{H(u)}{\sqrt{1-S(u)^2}}\arcsin\left(\frac{H(u)}{G(u)}\right)\right\}+o(1)\\
=\ &\Xi(u)+o(1),
\end{aligned}$$

recall (1.6). ■

We next deal with u near 0, which turns out to be challenging. First, we prove

Lemma 6.2

(a) $\Delta\left(x,x+\frac{u}{\kappa}\right)$ *has a double zero at* $u=0$*, and there is* $\rho>0$ *such that for all* $x\in[a,b]$ *and* n *large enough,* $\Delta\left(x,x+\frac{u}{\kappa}\right)$ *has no other zeros in* $|u|\le\rho$*. Moreover, uniformly for* u *in compact subsets of* $\mathbb{C}$*, and* $|x|\le 1-\varepsilon$*,*

$$\lim_{n\to\infty}\frac{\Delta\left(x,x+\frac{u}{\kappa}\right)}{K_{n+1}(x,x)^2u^2}e^{-2\tau u}=\frac{1-S(u)^2}{u^2}. \tag{6.10}$$

The right-hand side is interpreted as its limiting value at $u=0$*.*

(b) $\left[\left(\Omega_{11}\Omega_{22}-\Omega_{12}^2\right)\Delta\right]\left(x,x+\frac{u}{\kappa}\right)$ *has a zero of even order at least* 4 *at* $u=0$*.*

Moreover, uniformly for u in compact subsets of $\mathbb{C}$, and $|x| \leq 1-\varepsilon$,

$$\lim_{n\to\infty} \frac{\left(\Omega_{11}\Omega_{22}-\Omega_{12}^2\right)}{\Delta}\frac{1}{\kappa^4} = \frac{F(u)}{\left(1-S(u)^2\right)^2}.$$

The right-hand side is interpreted as its limiting value at $u=0$.

Proof

(a) First,

$$\begin{aligned} &\Delta\left(x, x+\frac{u}{\kappa}\right) \\ = \;& K_{n+1}(x,x)\, K_{n+1}\left(x+\frac{u}{\kappa}, x+\frac{u}{\kappa}\right) - K_{n+1}\left(x, x+\frac{u}{\kappa}\right)^2 \end{aligned}$$

is a polynomial in u, and by Cauchy-Schwarz is non-negative for real u, with a zero at $u=0$. This then must be a zero of even multiplicity. But since

$$\lim_{n\to\infty} \frac{\Delta\left(x, x+\frac{u}{\kappa}\right)}{K_{n+1}(x,x)^2} e^{-2\tau u} = 1 - S(u)^2,$$

uniformly in compact sets by Lemma 6.1(b), and the right-hand side has an isolated double zero at 0, it follows from Hurwitz' Theorem and the considerations above, that necessarily for large enough n, $\Delta\left(x, x+\frac{u}{\kappa}\right)$ has a double zero at 0, and no other zeros in some neighborhood of 0 that is independent of n. Since the convergence is uniform in x, the neighborhood may also be taken independent of x. But then $\left\{\frac{\Delta\left(x,x+\frac{u}{\kappa}\right)}{K_{n+1}(x,x)^2u^2}e^{-2\tau u}\right\}_{n\geq 1}$ is a sequence of entire functions in u that converges uniformly in compact subsets of $\mathbb{C}\setminus\{0\}$ and hence also in compact subsets of $\mathbb{C}$.

(b) Recall (3.11). Here $\det(\Sigma)$ is also a polynomial in u when $y = x+\frac{u}{\kappa}$. As in the proof of Lemma 2.2 in the Appendix in [18], Σ is a positive definite matrix when $x \neq y$, so is nonegative definite for all x, y. Then $\det(\Sigma) \geq 0$ for real x, y while $\det(\Sigma) = 0$ when $u = 0$. Thus as a polynomial in u, $\det(\Sigma)$ can only have an even multiplicity zero at $u=0$. We need to show that it has a zero of multiplicity at least 4 when $u=0$. By a classical inequality for determinants of positive definite matrices and their leading submatrices [3, p. 63, Thm. 7], when y is real,

$$0 \leq \det(\Sigma) \leq \Delta(x,y) \det\begin{bmatrix} K_{n+1}^{(1,1)}(x,x) & K_{n+1}^{(1,1)}(x,y) \\ K_{n+1}^{(1,1)}(x,y) & K_{n+1}^{(1,1)}(y,y) \end{bmatrix}.$$

We already know that Δ has a double zero at $u=0$ for $y = x + \frac{u}{n\omega(x)}$. But the second determinant also vanishes when $y = x$, that is $u = 0$. It follows that necessarily as a polynomial in u, $\det(\Sigma)$ has a zero of multiplicity at least 4 at $u=0$. Then

$$\frac{\Omega_{11}\Omega_{22}-\Omega_{12}^2}{\Delta} = \frac{\det(\Sigma)}{\Delta^2}$$

has a removable singularity at 0, since the zero of multiplicity 4 in the denominator is cancelled by the zero of multiplicity ≥ 4 in the numerator. Then from (6.1), (6.2), uniformly for $x \in [-1+\varepsilon, 1-\varepsilon]$ and u in some neighborhood of 0,

$$\begin{aligned}&\frac{\Omega_{11}\Omega_{22}-\Omega_{12}^2}{\Delta}\frac{1}{\kappa^4}\\
&= \frac{\left(\Omega_{11}\Omega_{22}-\Omega_{12}^2\right)\Delta}{K_{n+1}(x,x)^4}\left(\frac{e^{-\tau u}}{\kappa}\right)^4\left[\frac{K_{n+1}(x,x)^2}{\Delta e^{-2\tau u}}\right]^2\\
&= \frac{F(u)}{\left(1-S(u)^2\right)^2}+o(1).\end{aligned}$$

Moreover, since $S(u) = 1$ only at $u = 0$, this limit actually holds uniformly for u in compact subsets of $\mathbb{C}$. ■

Next, we deal with the most difficult term Ω_{12} :

Lemma 6.3
There exist $C, n_0, \rho > 0$ such that uniformly for $n \geq n_0$, $|u| \leq \rho$, and $|x| \leq 1-\varepsilon$,

$$\frac{|\Omega_{12}|}{\sqrt{\Delta}\kappa^2} \leq C.$$

Moreover, uniformly for $|u| \leq \rho$,

$$\lim_{n\to\infty}\frac{\Omega_{12}}{\sqrt{\Delta}}\frac{1}{\kappa^2} = \frac{H(u)}{(1-u^2)^{3/2}}.$$

Proof
We note that this proof is simpler than the corresponding one in [18]. First, from the previous lemma, there exists $\rho > 0$ and n_0 such that for $n \geq n_0$ and $|u| \leq \rho$, $\Delta(x,y) = \Delta\left(x, x+\frac{u}{\kappa}\right)$ has a double zero at 0 and no other zeros in the disk $|u| \leq \rho$. Then we may choose a branch of $\sqrt{\Delta\left(x, x+\frac{u}{\kappa}\right)}$ in u that is single valued and analytic in $|u| \leq \rho$, with a simple zero at $u = 0$. Then inasmuch as $\Omega_{12}\Delta$ is a polynomial in u, by (3.10),

$$\frac{\Omega_{12}}{\sqrt{\Delta}}\frac{1}{\kappa^2} = \frac{\Omega_{12}\Delta}{\left(\sqrt{\Delta}\right)^3}\frac{1}{\kappa^2}$$

is for $n \geq n_0$ analytic in the deleted disc $0 < |u| \leq \rho$ with at worst a pole of order at most 3 at 0. We now show that $\Omega_{12}\Delta$ has a zero of order at least 3 at $u = 0$, so that in fact $\frac{\Omega_{12}}{\sqrt{\Delta}}\frac{1}{\kappa^2}$ has a removable singularity at 0, and thus after redefinition at 0, is analytic in the disc $|u| \leq \rho$. First recall that

$$\Delta\Omega_{12} = \det\begin{bmatrix} K_{n+1}(x,x) & K_{n+1}(x,y) & K_{n+1}^{(0,1)}(x,x) \\ K_{n+1}(y,x) & K_{n+1}(y,y) & K_{n+1}^{(0,1)}(y,x) \\ K_{n+1}^{(1,0)}(y,x) & K_{n+1}^{(0,1)}(y,y) & K_{n+1}^{(1,1)}(y,x)\end{bmatrix}.$$

We subtract the first column from the second and $\frac{u}{\kappa}\times$the third column from the second. We use that as $u\to 0$,

$$\begin{aligned} K_{n+1}(x,y)-\left[K_{n+1}(x,x)+\frac{u}{\kappa}K_{n+1}^{(0,1)}(x,x)\right] &= \frac{1}{2}\left(\frac{u}{\kappa}\right)^2 K_{n+1}^{(0,2)}(x,x)+O\left(u^3\right); \\ K_{n+1}(y,y)-\left[K_{n+1}(y,x)+\frac{u}{\kappa}K_{n+1}^{(0,1)}(y,x)\right] &= \frac{1}{2}\left(\frac{u}{\kappa}\right)^2 K_{n+1}^{(0,2)}(y,x)+O\left(u^3\right); \\ K_{n+1}^{(0,1)}(y,y)-\left[K_{n+1}^{(0,1)}(y,x)+\frac{u}{\kappa}K_{n+1}^{(1,1)}(y,x)\right] &= \frac{1}{2}\left(\frac{u}{\kappa}\right)^2 K_{n+1}^{(1,2)}(y,x)+O\left(u^3\right) \end{aligned}$$

Using symmetry of K_n, we then obtain as $u\to 0$,

$$\Delta\Omega_{12}=\frac{1}{2}\left(\frac{u}{\kappa}\right)^2\det\begin{bmatrix} K_{n+1}(x,x) & K_{n+1}^{(0,2)}(x,x) & K_{n+1}^{(0,1)}(x,x) \\ K_{n+1}(y,x) & K_{n+1}^{(0,2)}(y,x) & K_{n+1}^{(0,1)}(y,x) \\ K_{n+1}^{(1,0)}(y,x) & K_{n+1}^{(1,2)}(y,x) & K_{n+1}^{(1,1)}(y,x) \end{bmatrix}+O\left(u^3\right).$$

Next we subtract the first row from the second and see that each of the resulting terms in the second row is $O(u)$. So indeed, $\Delta\Omega_{12}=O\left(u^3\right)$ as $u\to 0$. As Δ has a double zero at $u=0$, $\frac{\Omega_{12}}{\sqrt{\Delta}}\frac{1}{\kappa^2}=\frac{\Delta\Omega_{12}}{\left(\sqrt{\Delta}\right)^3}\frac{1}{\kappa^2}$ is analytic and single valued in $|u|\le\rho$. Next, from Lemma 6.1(e), (perhaps with a smaller ρ)

$$\begin{aligned} \lim_{n\to\infty}\frac{\Omega_{12}}{\sqrt{\Delta}}\frac{1}{\kappa^2} &= \lim_{n\to\infty}\left[\frac{\Delta\Omega_{12}}{K_{n+1}(x,x)^3}\frac{e^{-3\tau u}}{\kappa^2}\right]\left[\frac{\Delta}{K_{n+1}(x,x)^2}e^{-2\tau u}\right]^{-3/2} \\ &= \frac{H(u)}{\left(1-u^2\right)^{3/2}}, \end{aligned}$$

uniformly for u in compact subsets of the deleted disc $0<|u|\le\rho$. Here $H(u)/\left(1-u^2\right)^{3/2}$ is analytic in $|u|\le\rho$, as is $\frac{\Omega_{12}}{\sqrt{\Delta}}\frac{1}{\kappa^2}$, so the maximum modulus principle shows that the convergence is uniform in compact subsets of $|u|\le\rho$. Then the result follows. ■

Now we can deduce the desired bound near the diagonal:

Proof of Lemma 3.4(b)

Recall that ρ_2 was defined by (3.6). Then for $|x|\le 1-\varepsilon$, and $u\in[-\eta,\eta]$,

$$\begin{aligned} &|\rho_2(x,y)|\frac{1}{\kappa^2} \\ \le\ & \frac{1}{\pi^2}\left(\sqrt{\frac{\Omega_{11}\Omega_{22}-\Omega_{12}^2}{\Delta}}+\frac{|\Omega_{12}|}{\sqrt{\Delta}}\arcsin\left(\frac{|\Omega_{12}|}{\sqrt{\Omega_{11}\Omega_{22}}}\right)\right)\frac{1}{\kappa^2}\le C, \end{aligned}$$

by Lemmas 6.2 - 6.3. Next, from (5.1), followed by (6.8),

$$\frac{\rho_1(x)}{\kappa}=\frac{1}{\pi}\sqrt{\frac{\Psi(x)}{K_{n+1}(x,x)^2\kappa^2}}=\frac{1}{\sqrt{3}}+o(1), \tag{6.11}$$

and a similar asymptotic holds for $\rho_1(y)$. It follows that

$$|\rho_2(x,y)-\rho_1(x)\rho_1(y)|\frac{1}{\kappa^2}\le C,$$

in view of (1.11), which gives the result.

Proof of Lemma 3.4(c)
This follows directly from (6.11), (1.11), and (4.1). ■

7. Proof of Theorem 2.1

We note that the measures in Theorem 2.1 belong to the class $\mathcal{Q}$ defined in [15, p. 6]. We turn to verifying the hypotheses (I) - (V) in Section 1. We first recall some results from [15]. There we made substantial use of the Christoffel function $\lambda_n(\mu_n, x) = K_n(x,x)^{-1}$.

Lemma 7.1
Assume that $\{Q_n\}$ are as in Theorem 2.1. Let $L \geq 0$.
(a) For $m = n, n+1$,

$$\sup_{x \in I_n} |p_{n,m}(x)| e^{-nQ_n(x)} \left[|1 - |x|| + n^{-2/3} \right]^{1/4} \sim 1. \tag{7.1}$$

(b) For $|x| \leq 1$,

$$K_{n+1}(\mu_n, x, x) \mu_n'(x) \sim n \max\left\{1 - |x|, n^{-2/3}\right\}^{1/2}. \tag{7.2}$$

(c) There exists $c > 0$ such that for $|x| \leq 1 - n^{-c}$,

$$\frac{1}{n} K_n(\mu_n, x, x) = \sigma_{Q_n}(x) + o(1). \tag{7.3}$$

(d) Uniformly for $n \geq 1$ and for $x \in (-1,1)$,

$$\sigma_{Q_n(x)} \sim \sqrt{1 - x^2}. \tag{7.4}$$

(e) Uniformly for $n \geq 1$ and for $x, y \in (-1,1)$,

$$\left| \sigma_{Q_n(x)} - \sigma_{Q_n(x)} \right| \leq C |x - y|^{\alpha}. \tag{7.5}$$

(f) There exists $\tau > 0$ such that for $|x| \leq 1 - n^{-\tau}$ and for u, v in compact subsets of the real line,

$$\frac{\tilde{K}_n\left(x + \frac{u}{\kappa}, x + \frac{v}{\kappa}\right)}{\tilde{K}_n(x,x)} = S(v - u) + O\left(n^{-\tau}\right). \tag{7.6}$$

(g) For polynomials P of degree $\leq n + L$,

$$\left\| P' e^{-nQ_n} \right\|_{L_\infty(I_n)} \leq Cn \left\| P e^{-nQ_n} \right\|_{L_\infty(I_n)}. \tag{7.7}$$

(h)

$$\frac{\gamma_{n,n}}{\gamma_{n,n+1}} = \frac{1}{2} + o(1). \tag{7.8}$$

(i) For polynomials P of degree $\leq n + L$

$$\left\| P e^{-nQ_n} \right\|_{L_\infty(I_n)} \leq C \left\| P e^{-nQ_n} \right\|_{L_\infty[-1,1]}. \tag{7.9}$$

(j)

$$\sup_n \|Q_n'\|_{L_\infty[-1,1]} < \infty. \tag{7.10}$$

Proof
(a) See Theorem 2.1(a) in [15, p. 9].
(b) See Theorem 2.1(b) in [15, p. 9]. Note that there $\lambda_n(x) = 1/K_n(x,x)$.

(c) See Theorem 2.2(c) in [15, p. 11].
(d) See Theorem 3.1(a) in [15, p. 15] and recall that there $a_{n,1} = 1$ while $a_{-n,1} = -1$.
(e) See Theorem 3.1(b) in [15, p. 15].
(f) See Theorem 15.1 in [15, p. 155].
(g) See Theorem 8.1(b) in [15, p. 63].
(h) See Theorem 13.4 in [15, p. 124].
(i) Apply Theorem 4.2(a) in [15, p. 30] with $T = 1$.
(j) It is shown in Lemma 3.2(a) in [15, p. 16] that $|Q_n'(\pm 1)| \sim 1$. Since Q_n' is increasing, we obtain (7.10). ■

We proceed to verify the hypotheses (I) - (V) in Section 1.

Lemma 7.2 - Verification of (I)
Let $0 < \varepsilon < 1$. Then for $|x| \leq 1 - \varepsilon$, and $m = n, n+1$,

$$\left|p_{n,m}'(x)\right| e^{-nQ_n(x)} \leq Cn. \tag{7.11}$$

Proof
Note that (7.1) implies the bound (1.9) for $j = 0$. From the restricted range inequality Lemma 7.1(i),

$$\begin{aligned} & \sup_{x \in I_n} \left|p_{n,m}(x)\left(1-x^2\right)\right| e^{-nQ_n(x)} \\ \leq \quad & C_1 \sup_{x \in [-1,1]} \left|p_{n,m}(x)\left(1-x^2\right)\right| e^{-nQ_n(x)} \leq C_2, \end{aligned}$$

by (7.1). Then by the Bernstein inequality Lemma 7.1(g),

$$\sup_{x \in I_n} \left| \frac{d}{dx}\left[p_{n,m}(x)\left(1-x^2\right)\right]\right| e^{-nQ_n(x)} \leq Cn.$$

Then for $|x| \leq 1 - \varepsilon$,

$$\begin{aligned} & \left|p_{n,m}'(x)\left(1-x^2\right)\right| e^{-nQ_n(x)} \\ \leq \quad & \left|p_{n,m}(x)\, 2x\right| e^{-nQ_n(x)} + Cn \leq C_1 n \end{aligned}$$

and then as $1 - x^2 \geq \varepsilon$, we obtain (7.11) and hence (1.9). ■

Next we turn to establishing the universality limit for complex u, v. We use Theorem 1.2 from [14] with $h = 1$ there. We continue to use the notation for κ, τ.

Lemma 7.3
For $n \geq 1$, let μ_n be a positive Borel measure on the real line, with at least the first $2n+1$ power moments finite. Let I be a compact interval in which each μ_n is absolutely continuous. Assume moreover that in I,

$$d\mu_n(x) = e^{-2nQ_n(x)}dx = W_n^{2n}(x)\,dx, \tag{7.12}$$

is continuous on I. Let σ_{Q_n} denote the equilibrium measure for the restriction of W_n to I. Let J be a compact subinterval of I^o. Assume that
(a) $\{\sigma_{Q_n}\}_{n=1}^{\infty}$ are positive and uniformly bounded in some open interval containing J;
(b) $\{Q_n'\}_{n=1}^{\infty}$ are equicontinuous and uniformly bounded in some open interval containing J; or

(b′) more generally, for some open interval J_2 containing J, and for each fixed $a > 0$,

$$\sup_{t\in J_2, |h|\le a} \left| Q_n'(t) - Q_n'\left(t + \frac{h}{n}\right)\right| \to 0 \text{ as } n \to \infty. \tag{7.13}$$

(c) For some $C_1, C_2 > 0$, and for $n \ge 1$ and $x \in I$,

$$C_1 \le K_n(\mu, x, x) W_n^{2n}(x)/n \le C_2. \tag{7.14}$$

(d) Uniformly for $x \in J$ and a in compact subsets of the real line,

$$\lim_{n\to\infty} \frac{K_n\left(\mu, x + \frac{a}{n}, x + \frac{a}{n}\right)}{K_n(\mu, x, x)} \frac{W_n^{2n}(x)}{W_n^{2n}\left(x + \frac{a}{n}\right)} = 1. \tag{7.15}$$

Then uniformly for $x \in J$, and u, v in compact subsets of the complex plane, we have

$$\lim_{n\to\infty} \frac{K_n\left(x + \frac{u}{\kappa}, x + \frac{v}{\kappa}\right)}{K_n(x, x)} e^{-\tau(u+v)} = S(v - u).$$

Proof

The result is stated as (1.13) in [14, p. 749]. The weaker condition (b′) is noted in the remarks on page 749 in [14]. See (1.12) there. ■

Lemma 7.4 - Verification of (IV)

Assume that $\{Q_n\}$ are as in Theorem 2.1. Let $0 < \varepsilon < 1$. Then uniformly for $|x| \le 1 - \varepsilon$ and u, v in compact subsets of $\mathbb{C}$,

$$\lim_{n\to\infty} \frac{K_n\left(x + \frac{u}{\kappa}, x + \frac{v}{\kappa}\right)}{K_n(x, x)} e^{-\tau(u+v)} = S(v - u).$$

Proof

From Lemma 7.1(d),we have the requirements of Lemma 7.3(a). From Lemma 7.1(j), and the assumed smoothness (2.2) of $\{Q_n'\}$, we have the requirements of Lemma 7.3(b,b'). From Lemma 7.1(b), we have the requirements of Lemma 7.3(c). From Lemma 7.1(c), (e), we have the requirements of Lemma 7.3(d). Then we have the conclusion of Lemma 7.3. ■

Lemma 7.5 - Verification of (II), (III), (V)

The estimates (1.10), (1.11), (1.13), (1.14) are valid.

Proof

Firstly, (1.10) follows directly from Lemma 7.1(h). Next, (1.11) follows from Lemma 7.1(c), (d). Next, (1.13) follows from Lemma 7.1(j). Finally, (1.14) follows easily from the Lipschitz condition (2.2). ■

Proof of Theorem 2.1

We have verified all the hypotheses of Theorem 1.1 in Lemmas 7.2, 7.4, 7.5. ■

8. Proof of Theorem 2.3 and Corollary 2.4

Recall the notation (2.6) - (2.13). We also need the function φ_n :

$$\varphi_n(x) = \frac{|x - a_{-2n}|\,|x - a_{2n}|}{n\sqrt{\left[|x - a_{-n}| + |a_{-n}|\,\eta_{-n}\right]\left[|x - a_{-n}| + |a_{-n}|\,\eta_{-n}\right]}}, x \in [a_{-n}, a_n] \tag{8.1}$$

while $\varphi_n(x) = \varphi_n(a_n)$, $x > a_n$, and $\varphi_n(x) = \varphi_n(a_{-n})$, $x < a_{-n}$. We let $p_n(W^2, x)$ denote the nth orthonormal polynomial for W^2, so that

$$\int p_n(W^2, x)\, p_m(W^2, x)\, W^2(x)\, dx = \delta_{mn}.$$

Moreover, for non-negative integers r, s, we let

$$K_n^{(r,s)}(W^2, x, t) = \sum_{j=0}^{n-1} p_j^{(r)}(W^2, x)\, p_j^{(s)}(W^2, t)$$

and

$$\tilde{K}_n^{(r,s)}(W^2, x, t) = W(x)\, W(t)\, K_n^{(r,s)}(W^2, x, t).$$

Lemma 8.1
Let $0 < \varepsilon < 1$. Assume that $W = \exp(-Q) \in \mathcal{F}(C^2)$.
(i)

$$\sup_{x \in \mathbb{R}} |p_n(x)|\, e^{-Q(x)} \left[|x - a_n|\, |x - a_{-n}|\right]^{1/4} \sim 1 \tag{8.2}$$

(ii) Uniformly for $x \in J_n(\varepsilon)$,

$$K_n(W^2, x, x)\, W^2(x) \sim \frac{n}{\delta_n}. \tag{8.3}$$

(iii) Uniformly for $x \in J_n(\varepsilon)$,

$$K_n(W^2, x, x)\, W^2(x) = \sigma_n(x) + o(1). \tag{8.4}$$

(iv) Uniformly for $n \geq 1$ and for $x \in (-1 + \varepsilon, 1 - \varepsilon)$,

$$\sigma_n^*(x) \sim 1, \tag{8.5}$$

and uniformly for $x \in J_n(\varepsilon)$,

$$\sigma_n(x) \sim \frac{n}{\delta_n}. \tag{8.6}$$

(v) Uniformly for $n \geq 1$ and for $x, y \in (-1 + \varepsilon, 1 - \varepsilon)$,

$$|\sigma_n^*(x) - \sigma_n^*(y)| \leq C\, |x - y|^{1/4}. \tag{8.7}$$

(vi) For polynomials P of degree $\leq n$,

$$\left\|(PW)'\, \varphi_n\right\|_{L_\infty(\mathbb{R})} \leq C\, \|PW\|_{L_\infty(\mathbb{R})}. \tag{8.8}$$

Moreover, given $\varepsilon \in (0, 1)$, for $x \in J_n(\varepsilon)$,

$$|P'(x)|\, W(x) \leq C \frac{n}{\delta_n}\, \|PW\|_{L_\infty(\mathbb{R})}. \tag{8.9}$$

(vii)

$$\frac{\gamma_n}{\gamma_{n+1}} = \frac{\delta_n}{2}\, (1 + o(1)). \tag{8.10}$$

(viii) For polynomials P of degree $\leq n$,

$$\|PW\|_{L_\infty(\mathbb{R})} = \|PW\|_{L_\infty[a_{-n}, a_n]}. \tag{8.11}$$

Proof
(i) See Theorem 1.17 in [13, p. 22].

(ii) This follows from Corollary 1.14(c) in [13, p. 20], where estimates were provided for $\lambda_n\left(W^2,x\right)=1/K_n\left(W^2,x,x\right)$. Note that the class of weights above is contained in the class $\mathcal{F}\left(lip\frac{1}{2}\right)$ mentioned there (cf. [13, p. 12]). More precisely, it was shown that for $x\in[a_{-n},a_n]$,

$$K_n\left(W^2,x,x\right)W^2(x)\sim\varphi_n(x)^{-1},$$

where $\varphi_n(x)$ is defined by (8.1). Here if $x\in J_n(\varepsilon)=[a_{-n}+\varepsilon\delta_n,a_n-\varepsilon\delta_n]$, we see that $|x-a_{\pm n}|\geq C\delta_n$, so

$$\varphi_n(x)\sim\frac{\delta_n}{n}. \tag{8.12}$$

(iii) See Theorem 1.25 in [13, p. 26]. Note that if $0<\alpha<1$, then for large enough n, we have $J_n(\varepsilon)\subset[a_{-\alpha n},a_{\alpha n}]$.
(iv) See Theorems 1.10 and 1.11 in [13, pp. 17-18].
(v) See Theorem 6.3 in [13, pp. 147-8] and the discussion on page 149.
(vi) The first assertion is a special case of Theorem 10.1 in [13, p. 293]. For the second we see that

$$|P'W|(x)\varphi_n(x)\leq|PW|(x)Q'(x)\varphi_n(x)+\|PW\|_{L_\infty(\mathbb{R})}.$$

From Lemma 3.8(a) in [13, p. 77]

$$Q'(x)\leq C\frac{n}{\delta_n}. \tag{8.13}$$

Then the second estimate follows from this and (8.12).
(vii) See Theorem 1.23 in [13, p. 26] and note that there $A_n=\frac{\gamma_{n-1}}{\gamma_n}$, while $\frac{\delta_n}{\delta_{n+1}}=1+o(1)$.
(viii) See Theorem 4.1 in [13, p. 95].
■

To apply Theorem 1.1, we introduce a sequence of measures $\{\mu_n\}$ as follows: for $n\geq1$, let

$$Q_n(x)=\frac{1}{n}Q\left(L_n^{[-1]}(x)\right);$$
$$W_n(x)=e^{-Q_n(x)};$$
$$d\mu_n(x)=e^{-2nQ_n(x)}dx.$$

Note that

$$W_n^{2n}=W^2\circ L_n^{[-1]}; \tag{8.14}$$

and

$$Q_n'=\frac{\delta_n}{n}Q'\circ L_n^{[-1]}. \tag{8.15}$$

We denote the orthonormal polynomials for μ_n by $\{p_{n,j}\}_{j=0}^\infty$ as in Section 1. We also use the notation for the reproducing kernels and other quantities there. A substitution shows that

$$p_{n,j}(x)=\delta_n^{1/2}p_j\left(W^2,L_n^{[-1]}(x)\right) \tag{8.16}$$

and

$$K_n(\mu_n,x,y)=\delta_nK_n\left(W^2,L_n^{[-1]}(x),L_n^{[-1]}(y)\right). \tag{8.17}$$

As in Section 1, we use the abbreviation $K_n(x,y) = K_n(\mu_n, x, y)$. Now we turn to the derivatives.

Lemma 8.2
Let $0 < \varepsilon < 1$.
(a) For $x \in J_n(\varepsilon)$ *and* $\ell = 0, 1$,

$$\left| p_n^{(\ell)}(x) \right| W(x) \leq \frac{C}{\delta_n^{1/2}} \left(\frac{n}{\delta_n} \right)^{\ell}. \tag{8.18}$$

(b) For $|t| \leq 1 - \varepsilon$, $\ell = 0, 1$, *and* $k = n, n+1$,

$$\left| p_{n,k}^{(\ell)}(t) \right| W_n^n(t) \leq C n^{\ell}. \tag{8.19}$$

Proof
(a) The case $\ell = 0$ follows from (8.2). Now

$$(x - a_{-n})(a_n - x) = \delta_n^2 \left(1 - L_n(x)^2 \right),$$

so we can reformulate part of our bound (8.2) on p_n as

$$\delta_n^{1/2} |p_n(x)| W(x) \left| 1 - L_n^2(x) \right|^{1/4} \leq C, \; x \in \mathbb{R},$$

and then also ,

$$\delta_n^{1/2} |p_n(x)| W(x) \left| 1 - L_n^2(x) \right| \leq C, \; x \in [a_{-n-2}, a_{n+2}]. \tag{8.20}$$

Here $p_n(x) \left(1 - L_n^2(x) \right)$ is a polynomial of degree $n+2$. Then our restricted range inequality Lemma 8.1(viii) give that

$$\sup_{x \in \mathbb{R}} \delta_n^{1/2} |p_n(x)| W(x) \left| 1 - L_n^2(x) \right| \leq C.$$

Next, we apply (8.9) to the polynomial $p_n(x) \left(1 - L_n^2(x) \right)$, of degree $n+2$: for $x \in J_{n+2}(\varepsilon) \supseteq J_n(\varepsilon)$,

$$\left| \frac{d}{dx} \left\{ \delta_n^{1/2} p_n(x) \left(1 - L_n^2(x) \right) \right\} W(x) \right| \leq C \frac{n}{\delta_n}.$$

Then for $x \in J_n(\varepsilon)$,

$$\delta_n^{1/2} \left| p_n'(x) \left(1 - L_n^2(x) \right) W(x) \right| \leq \delta_n^{-1/2} |p_n(x) 2 L_n(x)| W(x) + C \frac{n}{\delta_n} \leq C \frac{n}{\delta_n},$$

by (8.2). Since $1 - L_n^2(x) \geq C$ in $J_n(\varepsilon)$, we obtain (8.18) for $\ell = 1$.
(b) This follows from the identity (8.16). ■

Next, the universality limits:

Lemma 8.3
Let $0 < \varepsilon < 1$.
(a) *Let* $W = \exp(-Q) \in \mathcal{F}(C^2)$. *Then uniformly for* u, v *in compact subsets of the complex plane, and* $x \in J_n(\varepsilon)$, *we have as* $n \to \infty$,

$$\lim_{n \to \infty} \frac{K_n\left(W^2, x + \frac{u}{\tilde{K}_n(W^2,x,x)}, x + \frac{v}{\tilde{K}_n(W^2,x,x)} \right)}{K_n(W^2, x, x)} e^{-\frac{Q'(x)}{\tilde{K}_n(W^2,x,x)}(u+v)} = S(v-u).$$

(b) For μ_n defined above, we have uniformly for u, v in compact subsets of the complex plane, and $|\xi| \leq 1-\varepsilon$, we have as $n \to \infty$,

$$\lim_{n\to\infty} \frac{K_n\left(\xi+\frac{u}{\widetilde{K}_n(\xi,\xi)}, \xi+\frac{v}{\widetilde{K}_n(\xi,\xi)}\right)}{K_n(\xi,\xi)} e^{-\frac{n}{\widetilde{K}_n(\xi,\xi)}Q_n'(\xi)(u+v)} = S(v-u).$$

Proof

(a), (b) This was established in Theorem 7.4 of [14, p. 771] for a bigger class of weights. It was stated in Theorem 7.4 for real u, v but as noted in Lemma 7.3 above, it was stated in (1.13) in [14] that we have uniformly for u, v in compact subsets of $\mathbb{C}$, and $K_n = K_n(\mu_n)$, and $\xi \in [-1+\varepsilon, 1-\varepsilon]$

$$\lim_{n\to\infty} \frac{K_n\left(\xi+\frac{u}{\widetilde{K}_n(\xi,\xi)}, \xi+\frac{v}{\widetilde{K}_n(\xi,\xi)}\right)}{K_n(\xi,\xi)} e^{-\frac{n}{\widetilde{K}_n(\xi,\xi)}Q_n'(\xi)(u+v)} = S(v-u).$$

Thus we have the conclusion of (b). Here from (8.15), (8.17), if $x = L_n^{[-1]}(\xi) \in J_n(\varepsilon)$,

$$\frac{n}{\widetilde{K}_n(\xi,\xi)} Q_n'(\xi) = \frac{Q'(x)}{\tilde{K}_n(W^2,x,x)}$$

so we also obtain the conclusion of (a), using

$$\xi + \frac{u}{\widetilde{K}_n(\xi,\xi)} = L_n\left(x + \frac{u}{\widetilde{K}_n(W^2,x,x)}\right).$$

■

Finally, we verify the remaining hypotheses (II), (III), (V).

Lemma 8.4

(a) The estimate (1.10) holds true for μ_n.
(b) The estimate (1.11) holds for $|x| \leq 1-\varepsilon$.
(c) The estimates (1.13) and (1.14) hold for $|x| \leq 1-\varepsilon$.

Proof

(a) From (8.16), we have

$$\gamma_{n,j} = \delta_n^{j+1/2}\gamma_j$$

so from Lemma 8.1(vii),

$$\frac{\gamma_{n,n}}{\gamma_{n,n+1}} = \frac{1}{2} + o(1).$$

(b) This follows from Lemma 8.1(ii) and (8.17).
(c) Firstly it is shown in Lemma 7.6(a) in [14, Lemma 7.6, p. 773] that $\{Q_n'\}$ are uniformly bounded in compact subsets of $(-1,1)$. In Lemma 7.6(b) there, it is shown that for fixed $a > 0$,

$$\sup_{|t|\leq 1-\varepsilon, |h|\leq a,} \left| Q_n'(t) - Q_n'\left(t+\frac{h}{n}\right)\right| \to 0 \text{ as } n \to \infty.$$

■

Proof of Theorem 2.3

We have verified the hypotheses (I) - (V) for the measures $\{\mu_n\}$ in Lemmas 8.2, 8.3, 8.4. We can then apply the result of Theorem 1.1 to $\{\mu_n\}$. The transformation formula

$$\begin{aligned} G_n^* (s) &= \sum_{j=0}^{n} a_j p_{n,j} (s) \\ &= \sum_{j=0}^{n} a_j p_j \circ L_n^{[-1]} (s) = G_n \left(L_n^{[-1]} (s) \right) \end{aligned}$$

then gives the result, recalling the asymptotic from Lemma 8.1(iii):

$$\begin{aligned} \frac{1}{n} \tilde{K}_n (s, s) &= \frac{\delta_n}{n} K_n \left(W^2, L_n^{[-1]} (s), L_n^{[-1]} (s) \right) W^2 \left(L_n^{[-1]} (s) \right) \\ &= \frac{\delta_n}{n} \sigma_n \circ L_n^{[-1]} (s) (1 + o(1)) \\ &= \sigma_n^* (s) (1 + o(1)). \end{aligned}$$

■

Proof of Corollary 2.4
It is shown in [17, Lemma 3.2, p. 55] that for $x \in (-1, 1)$,

$$\lim_{n \to \infty} \sigma_n^* (x) = \sigma_\alpha (x).$$

Moreover Lemma 8.1(iv) shows that $\{\sigma_n^*\}$ are uniformly bounded in $[a, b]$. ■

References

[1] J.-M. Azaïs, F. Dalmao and J. R. León, *CLT for the zeros of classical random trigonometric polynomials*, Annales de l'Institut Henri Poincaré, Probabilités et Statistiques, **52** (2016), 804–820.

[2] J.-M. Azaïs and J. R. León, *CLT for crossings of random trigonometric polynomials*, Electronic Journal of Probability, 18 (2013).

[3] E. Beckenbach and R. Bellman, *Inequalities*, Springer, Berlin, 1961.

[4] E. Bogomolny, O. Bohigas, O. and P. Leboeuf, *Quantum chaotic dynamics and random polynomials,* Journal of Statistical Physics, 85(1996), 639–679.

[5] M. Das, *Real zeros of a random sum of orthogonal polynomials*, Proc. Amer. Math. Soc. 27(1971), 147–153.

[6] M. Das and S. S. Bhatt, *Real roots of random harmonic equations*, Indian J. Pure Appl. Math. 13(1982), 411–420.

[7] Y. Do, H. H. Nguyen, O. Nguyen, *Random trigonometric polynomials: universality and non-universality of the variance for the number of real roots*, https://arxiv.org/abs/1912.11901.

[8] Y. Do, O. Nguyen and V. Vu, *Random orthonormal polynomials: Local universality and expected number of real roots,* arXiv:2012.10850.

[9] J. E. A. Dunnage, *The number of real zeros of a random trigonometric polynomial*, Proc. London Math. Soc., 16 (1966), 53–84.

[10] G. Freud, *Orthogonal Polynomials*, Pergamon Press/ Akademiai Kiado, Budapest, 1971.

[11] L. Gass, *Variance of the number of zeros of dependent Gaussian trigonometric polynomials,* Electronic J. Probab., 0(2020), 1-21.

[12] A. Granville and I. Wigman, *The distribution of the zeros of random trigonometric polynomials*, Amer. J. Math. 133(2011), 295–357.

[13] Eli Levin and D.S. Lubinsky, *Orthogonal Polynomials for Exponential Weights,* Springer, New York, 2001.

[14] Eli Levin and D.S. Lubinsky, *Universality Limits in the bulk for varying measures*, Advances in Mathematics, 219(2008), 743-779.

[15] Eli Levin and D.S. Lubinsky, *Bounds and Asymptotics for Orthogonal Polynomials for Varying Weights*, Springer Briefs in Mathematics, Springer, New York, 2018.

[16] D. S. Lubinsky, I. E. Pritsker and X. Xie, *Expected number of real zeros for random linear combinations of orthogonal polynomials*, Proc. Amer. Math. Soc. 144 (2016), 1631–1642.
[17] D. S. Lubinsky, I. E. Pritsker and X. Xie, *Expected number of real zeros for random orthogonal polynomials*, Math. Proc. Camb. Phil. Soc., 164(2018), 47–66.
[18] D.S. Lubinsky and I. Pritsker, *Variance of real zeros of random orthogonal polynomials*, J. Math. Anal. Appl., 498(2021), 124954.
[19] O. Nguyen and V. Vu, *Random polynomials: central limit theorems for the real roots*, https://arxiv.org/abs/1904.04347.
[20] C. Qualls, *On the number of zeros of a stationary Gaussian random trigonometric polynomial,* J. London Math. Soc., 2 (1970), 216–220.
[21] E.B. Saff and V. Totik, *Logarithmic Potentials with External Fields*, Springer, New York, 1997.
[22] J. E. Wilkins, Jr. *An asymptotic expansion for the expected number of real zeros of a random polynomial*, Proc. Amer. Math. Soc. **103** (1988), 1249–1258.
[23] J. E. Wilkins, Jr., *The expected value of the number of real zeros of a random sum of Legendre polynomials*, Proc. Amer. Math. Soc. 125(1997), 1531–1536.
[24] X. Xie, *Statistics of the number of real zeros of random orthogonal polynomials*, Ph.D. Dissertation, Oklahoma State University, Stillwater, 2016.

[1]School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332. lubinsky@math.gatech.edu, [2]Department of Mathematics, Oklahoma State University, Stillwater, Oklahoma 74078. igor@math.okstate.edu