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Asymptotics for orthogonal polynomials and separation of their zeros[☆]

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Abstract

Let $\{p_n\}$ denote the orthonormal polynomials associated with a measure μ with compact support on the real line. In a recent paper, we showed there is a close relationship between the spacing of zeros of successive orthogonal polynomials p_n, p_{n-1}, and uniform bounds on the orthogonal polynomials in subintervals of the support. In this paper, we show there is also a relationship between asymptotics for the spacing of zeros of p_{n-1}, p_n, and pointwise asymptotics for the orthogonal polynomials.

Keywords: Orthogonal polynomials; Pointwise asymptotics

1. Results

Let μ be a finite positive Borel measure with compact support, which we denote by $\mathrm{supp}[\mu]$. Then we may define orthonormal polynomials

$$p_n(x) = \gamma_n x^n + \cdots, \gamma_n > 0,$$

$n = 0, 1, 2, \ldots$ satisfying the orthonormality conditions

$$\int p_n p_m d\mu = \delta_{mn}.$$

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The zeros $\{x_{jn}\}$ of p_n are real and simple. We list them in decreasing order:

$$x_{1n} > x_{2n} > \cdots > x_{n-1,n} > x_{nn}.$$

We shall make substantial use of the fact that the zeros of p_n and p_{n-1} interlace. Thus for $1 \leq j \leq n-1$,

$$x_{j,n-1} \in \left(x_{j+1,n}, x_{jn}\right).$$

The three term recurrence relation has the form

$$(x - b_n)\, p_n(x) = a_{n+1} p_{n+1}(x) + a_n p_{n-1}(x),$$

where for $n \geq 1$,

$$a_n = \frac{\gamma_{n-1}}{\gamma_n} = \int x p_{n-1}(x)\, p_n(x)\, d\mu(x)\, ;\;\; b_n = \int x p_n^2(x)\, d\mu(x)\, .$$

In a recent paper [7], we analyzed the relationship between the spacing of zeros of successive orthogonal polynomials p_{n-1}, p_n, namely $x_{jn} - x_{j,n-1}$ and uniform bounds on orthogonal polynomials in subintervals of the support. Spacing of zeros for the same orthogonal polynomial, namely $x_{j-1,n} - x_{jn}$, has been intensively studied for decades [6,11,15,16]. Bounds on orthogonal polynomials is also a classic topic in orthogonal polynomials [1,2,4,5,9].

The results from [7] require more terminology: we let $dist\,(a, \mathbb{Z})$ denote the distance from a real number a to the integers. We say that μ is regular (in the sense of Stahl, Totik, and Ullmann) if

$$\lim_{n\to\infty} \gamma_n^{1/n} = \frac{1}{cap\,(\operatorname{supp}[\mu])},$$

where cap denotes logarithmic capacity. If the support consists of finitely many intervals, and $\mu' > 0$ a.e. in each subinterval, then μ is regular, though much less is required [13].

Recall that the equilibrium measure for the compact set supp$[\mu]$ is the probability measure that minimizes the energy integral

$$\iint \log \frac{1}{|x-y|} d\nu(x)\, d\nu(y)$$

amongst all probability measures ν supported on supp$[\mu]$. If I is an interval contained in supp$[\mu]$, then the equilibrium measure is absolutely continuous in I, and moreover its density, which we denote throughout by ω, is positive and continuous in the interior I^o of I [10, p.216, Thm. IV.2.5]. Given sequences $\{x_n\}, \{y_n\}$ of non-0 real numbers, we write

$$x_n \sim y_n$$

if there exists $C > 1$ such that for $n \geq 1$,

$$C^{-1} \leq x_n / y_n < C.$$

Similar notation is used for functions and sequences of functions.

In [7, Theorem 1.1], we proved:

Theorem A. *Let μ be a regular measure on $\mathbb{R}$ with compact support. Let I be a closed subinterval of the support and assume that in some open interval containing I, μ is absolutely continuous, while μ' is positive and continuous. Let ω be the density of the equilibrium measure for the support of μ. Let $A > 0$. The following are equivalent:*

(a) There exists $C > 0$ such that for $n \geq 1$ and $x_{jn} \in I$,

$$dist\left(n\omega\left(x_{jn}\right)\left(x_{jn} - x_{j,n-1}\right), \mathbb{Z}\right) \geq C. \tag{1.1}$$

(b) There exists $C > 0$ such that for $n \geq 1$ and $x \in I$,

$$\|p_{n-1}\|_{L_\infty[x-\frac{A}{n},x+\frac{A}{n}]} \|p_n\|_{L_\infty[x-\frac{A}{n},x+\frac{A}{n}]} \leq C. \tag{1.2}$$

Moreover, under either (a) or (b), we have

$$\sup_{n\geq 1}\sup_{x\in I}\left||x-b_n|^{1/2}\, p_n\left(x\right)\right| < \infty. \tag{1.3}$$

Under additional assumptions on the spacing of the zeros of p_{n-2} and p_n, the factor $|x - b_n|^{1/2}$ in (1.3) was removed.

In this paper, we investigate the relationship between pointwise asymptotics of orthogonal polynomials, and the spacing $x_{jn} - x_{j,n-1}$. As a pointer to what might be possible, let us recall the form of the classical pointwise asymptotic for orthogonal polynomials inside supp$[\mu]$. Let us suppose our support is $[-1, 1]$, that μ satisfies Szegő's condition, and in some subinterval $I \subset (-1, 1)$, μ is absolutely continuous, μ' is continuous, while the local modulus of continuity of μ' satisfies a suitable Dini condition. Badkov [3] generalized many earlier results, proving that as $n \to \infty$, uniformly in closed subintervals of I^o,

$$p_n\left(x\right)\mu'\left(x\right)^{1/2}\left(1-x^2\right)^{1/4} = \sqrt{\frac{2}{\pi}}\cos\left(n\theta + h\left(\theta\right)\right) + o\left(1\right), \tag{1.4}$$

where $x = \cos\theta$, and h is a continuous function. See [3] for a precise statement of the hypotheses. It is straightforward to prove:

Proposition 1.1. *Assume the asymptotic (1.4) holds uniformly for x in $I \subset (-1, 1)$. Fix $k \geq 0, \ell \in \mathbb{Z}$. Let J be a closed subinterval of I^o. Then uniformly for x_{jn} in J,*

$$n\left(x_{jn} - x_{j-\ell,n-k}\right) = \sqrt{1-x_{jn}^2}\left[k\arccos\left(x_{jn}\right) - \ell\pi\right] + o\left(1\right). \tag{1.5}$$

We shall prove this in Section 2. One can compare this to the much studied asymptotic for spacing of successive zeros of p_n when the support is $[-1, 1]$, namely

$$n\left(x_{jn} - x_{j+1,n}\right) = \pi\sqrt{1-x_{jn}^2} + o\left(1\right),$$

or for more general supports with equilibrium density ω, [6,12,15]

$$n\left(x_{jn} - x_{j+1,n}\right) = \frac{1}{\omega\left(x_{jn}\right)} + o\left(1\right).$$

We prove the following partial converse. In its formulation, we need the zeros of p_n', which are denoted by y_{jn}, ordered so that

$$y_{jn} \in \left(x_{j+1,n}, x_{jn}\right),\ 1 \leq j \leq n-1.$$

Theorem 1.2. *Let μ be a regular measure on $\mathbb{R}$ with compact support. Let I be a closed subinterval of the support in which μ is absolutely continuous, while μ' is positive and continuous. Let ω be the density of the equilibrium measure for the support of μ. Let $\mathcal{S}$ be an infinite sequence of positive integers. Assume that uniformly for $n \in \mathcal{S}$, $m = n, n+1$, and $x_{jn} \in I$,*

$$m\omega\left(x_{jm}\right)\left(x_{jm} - x_{j,m-1}\right) = g\left(x_{jm}\right) + o\left(1\right), \tag{1.6}$$

where $g : I \to (0,1)$ *is continuous. Let*

$$f_n(x) = \omega(x_{jn})(x - y_{jn}) - \frac{j}{n}, x \in [x_{j+1,n}, x_{jn}) \cap I. \tag{1.7}$$

Let J *be a compact subinterval of* I^o. *Then uniformly for* x *in* J, *as* $n \to \infty, n \in \mathcal{S}$,

$$|x - b_n|^{1/2} p_n(x) \mu'(x)^{1/2} = \sqrt{\frac{2}{\pi}} |\cot \pi g(x)|^{1/2} [\cos n\pi f_n(x) + o(1)]. \tag{1.8}$$

Corollary 1.3. *If* J *is a compact subinterval of* I, *as* $n \to \infty, n \in \mathcal{S}$,

$$\sup_{x \in J} \left| p_n(x) \mu'(x)^{1/2} |x - b_n|^{1/2} \right| = \sqrt{\frac{2}{\pi}} \sup_{x \in J} |\cot \pi g(x)|^{1/2} + o(1). \tag{1.9}$$

Remarks.
(a) If $g(x) = \frac{1}{\pi} \arccos x$, as is the case in Proposition 1.1, while $b_n = 0$, Theorem 1.2 simplifies to

$$p_n(x) \mu'(x)^{1/2} \left(1 - x^2\right)^{1/4} = \sqrt{\frac{2}{\pi}} [\cos n\pi f_n(x) + o(1)], \tag{1.10}$$

uniformly in compact subsets of $I \setminus \{0\}$.
(b) Note that nf_n is "asymptotically continuous" in some sense. Indeed, as shown by Lemma 3.1(c),

$$\lim_{n \to \infty} \lim_{x \to x_{jn}-} nf_n(x) = \frac{1}{2} - j = \lim_{n \to \infty} nf_n(x_{jn}).$$

(c) Lemma 3.4 below shows that we can recast the asymptotic as

$$\begin{aligned} &|x - b_n|^{1/2} p_n(x) \mu'(x)^{1/2} \\ &= \sqrt{\frac{2}{\pi}} \left| \frac{\left(\frac{x - b_{n-1}}{2a_n}\right)\left(\frac{x - b_n}{2a_n}\right)}{1 - \left(\frac{x - b_{n-1}}{2a_n}\right)\left(\frac{x - b_n}{2a_n}\right)} \right|^{1/4} [\cos n\pi f_n(x) + o(1)] \end{aligned} \tag{1.11}$$

and hence that

$$\begin{aligned} &\sqrt[4]{1 - \left(\frac{x - b_{n-1}}{2a_n}\right)\left(\frac{x - b_n}{2a_n}\right)} p_n(x) \mu'(x)^{1/2} \\ &= \sqrt{\frac{1}{a_n \pi}} \left| \frac{x - b_{n-1}}{x - b_n} \right|^{1/4} [\cos n\pi f_n(x) + o(1)]. \end{aligned}$$

except close to zeros of $\cos \pi g(x)$.
(d) One may replace y_{jn} in the definition (1.7) of f_n by $\frac{1}{2}\left(x_{jn} + x_{j+1,n}\right)$, since these differ by $o\left(\frac{1}{n}\right)$. See (3.3).

We prove Proposition 1.1 in the next section and Theorem 1.2 and Corollary 1.3 in Section 3. We close this section with some notation. In the sequel $C, C_1, C_2, \ldots$ denote constants independent of n, x, θ. The same symbol does not necessarily denote the same constant in different occurrences. The nth reproducing kernel for μ is

$$K_n(x, y) = \sum_{k=0}^{n-1} p_k(x) p_k(y).$$

2. Proof of Proposition 1.1

We turn to

Proof of Proposition 1.1. Write for $x_{jn} \in J$,

$$x_{jn} = \cos\left(\theta_{jn}\right), \ \theta_{jn} \in (0, \pi).$$

Note that $\left\{\theta_{jn}\right\}$ lie in a closed subinterval of $(0, \pi)$ as J is a closed subinterval of $(-1, 1)$. Then from (1.4),

$$\cos\left(n\theta_{jn} + h\left(\theta_{jn}\right)\right) = o(1).$$

This gives for some integer $j_1 = j_1(j, n)$ that may depend on both j and n,

$$n\theta_{jn} + h\left(\theta_{jn}\right) = -\frac{\pi}{2} + j_1\pi + o(1)$$

so

$$\theta_{jn} = \frac{1}{n}\left(-\frac{\pi}{2} + j_1\pi - h\left(\theta_{jn}\right)\right) + o\left(\frac{1}{n}\right). \tag{2.1}$$

We claim that for fixed $k, \ell \geq 0$,

$$\theta_{j-\ell,n-k} = \frac{1}{n-k}\left(-\frac{\pi}{2} + (j_1 - \ell)\pi - h\left(\theta_{j-\ell,n-k}\right)\right) + o\left(\frac{1}{n}\right). \tag{2.2}$$

To see this first note that for $k = 0$ and $\ell \geq 0$, this follows from the asymptotics (1.4) and our ordering of the zeros. Next let us prove this for $k = 1$ and $\ell \geq 0$. Assume the notation (2.2) for $k = 0$, and for a given $\ell \geq 0$, write from (2.1),

$$\theta_{j-\ell,n-1} = \frac{1}{n-1}\left(-\frac{\pi}{2} + m\pi - h\left(\theta_{j-\ell,n-1}\right)\right) + o\left(\frac{1}{n}\right). \tag{2.3}$$

Here $m = m(j, n-1, \ell)$ is an integer. We must show that for large enough n, $m = j_1 - \ell$. We use the interlacing

$$x_{j-\ell,n-1} \in \left(x_{j-\ell+1,n}, x_{j-\ell,n}\right).$$

Since h is continuous and $\theta_{j-\ell,n-1} - \theta_{j-\ell,n} = o(1)$, $\theta_{j-\ell,n-1} - \theta_{j-\ell+1,n} = o(1)$, the interlacing gives, after dropping the h terms, and taking account that cos is decreasing,

$$\begin{aligned}\frac{1}{n}\left(-\frac{\pi}{2} + (j_1 - \ell + 1)\pi\right) &> \frac{1}{n-1}\left(-\frac{\pi}{2} + m\pi\right) + o\left(\frac{1}{n}\right) \\ &> \frac{1}{n}\left(-\frac{\pi}{2} + (j_1 - \ell)\pi\right) + o\left(\frac{1}{n}\right).\end{aligned}$$

We multiply this relation by $\frac{n}{\pi}$ and cancel a factor of $-\frac{1}{2}$, giving

$$j_1 - \ell + 1 > \frac{n}{n-1}m + o(1) > j_1 - \ell + o(1)$$

$$\Rightarrow 1 > m - (j_1 - \ell) + \frac{m}{n-1} + o(1) > o(1).$$

Since our zeros are confined to a closed subinterval of $(-1, 1)$, there exists $\varepsilon > 0$ such that $\frac{m}{n-1} \in [\varepsilon, 1 - \varepsilon]$. The left-hand inequality then shows that we cannot have $m - (j_1 - \ell) \geq 1$. But then the right-hand inequality shows we cannot have $m - (j_1 - \ell) \leq -1$. So $m = j_1 - \ell$

as desired. This establishes (2.2) for $k = 1$. By repeating this argument, we can establish it for any fixed k.

Now that we have (2.2), we turn to the proof of (1.5). Fix ℓ, k. We have then

$$\frac{1}{2}\left(\theta_{j-\ell,n-k} + \theta_{jn}\right) = \theta_{jn} + O\left(\frac{1}{n}\right)$$

and hence also, as $\sin\theta_{jn}$ is bounded away from 0,

$$\sin\left(\frac{1}{2}\left(\theta_{j-\ell,n-k} + \theta_{jn}\right)\right) = \left(\sin\theta_{jn}\right)\left(1 + O\left(\frac{1}{n}\right)\right).$$

Also

$$\begin{aligned}
&\frac{1}{2}\left(\theta_{j-\ell,n-k} - \theta_{jn}\right) \\
&= \left(-\frac{\pi}{2} + j_1\pi\right)\frac{1}{2}\left(\frac{1}{n-k} - \frac{1}{n}\right) - \frac{\ell\pi}{2\left(n-k\right)} + \frac{1}{2}\left(\frac{h\left(\theta_{jn}\right)}{n} - \frac{h\left(\theta_{j-\ell,n-k}\right)}{n-k}\right) + o\left(\frac{1}{n}\right) \\
&= \left(-\frac{\pi}{2} + j_1\pi\right)\frac{k}{2}\frac{1}{n\left(n-k\right)} - \frac{\ell\pi}{2n} + o\left(\frac{1}{n}\right) \\
&= \frac{k}{2\left(n-k\right)}\left[\theta_{jn} + \frac{h\left(\theta_{jn}\right)}{n}\right] - \frac{\ell\pi}{2n} + o\left(\frac{1}{n}\right) \\
&= \frac{k\theta_{jn}}{2n} - \frac{\ell\pi}{2n} + o\left(\frac{1}{n}\right).
\end{aligned}$$

Then

$$\begin{aligned}
&x_{jn} - x_{j-\ell,n-k} \\
&= \cos\left(\theta_{jn}\right) - \cos\left(\theta_{j-\ell,n-k}\right) \\
&= -2\sin\left(\frac{1}{2}\left(\theta_{j-\ell,n-k} + \theta_{jn}\right)\right)\sin\left(\frac{1}{2}\left(\theta_{jn} - \theta_{j-\ell,n-k}\right)\right) \\
&= 2\left(\sin\theta_{jn}\right)(1 + O\left(\frac{1}{n}\right))\sin\left(\frac{k\theta_{jn}}{2n} - \frac{\ell\pi}{2n} + o\left(\frac{1}{n}\right)\right) \\
&= \left(\sin\theta_{jn}\right)\left(1 + O\left(\frac{1}{n}\right)\right)\left[\frac{k\theta_{jn}}{n} - \frac{\ell\pi}{n} + o\left(\frac{1}{n}\right)\right],
\end{aligned}$$

so that

$$\begin{aligned}
&n\left(x_{jn} - x_{j-\ell,n-k}\right) \\
&= \left(\sin\theta_{jn}\right)\left[k\theta_{jn} - \ell\pi + o\left(1\right)\right] + o\left(1\right) \\
&= \sqrt{1 - x_{jn}^2}\left[k\arccos\left(x_{jn}\right) - \ell\pi\right] + o\left(1\right). \quad \blacksquare
\end{aligned}$$

3. The converse

We begin with some established asymptotics and bounds for orthogonal polynomials:

Lemma 3.1. *Assume that μ is a regular measure with compact support. Let I be a closed subinterval of the support in which μ is absolutely continuous, and μ' is positive and continuous. Let J be a compact subset of the interior I^o of I. Let ω denote the equilibrium density for the support of μ.*

(a) Then

$$\lim_{n\to\infty} \frac{p_n\left(y_{jn}+\frac{z}{n\omega\left(y_{jn}\right)}\right)}{p_n\left(y_{jn}\right)} = \cos \pi z \tag{3.1}$$

uniformly for $y_{jn} \in J$ *and* z *in compact subsets of* $\mathbb{C}$*. Here* ω *is the equilibrium density for the support of* μ*.*
(b) Uniformly for $x \in J$*,*

$$\lim_{n\to\infty} \frac{1}{n} K_n(x,x)\,\mu'(x) = \omega(x). \tag{3.2}$$

(c) Uniformly for $y_{jn} \in J$*,*

$$n\omega\left(x_{jn}\right)\left(x_{jn}-y_{jn}\right) = \frac{1}{2}+o(1)\,;\, n\omega\left(x_{jn}\right)\left(y_{jn}-x_{j+1,n}\right) = \frac{1}{2}+o(1). \tag{3.3}$$

(d) Uniformly for $y_{jn} \in J$*,*

$$n\omega\left(x_{jn}\right)\left(x_{jn}-x_{j+1,n}\right) = 1+o(1)\,;\, n\omega\left(x_{jn}\right)\left(y_{jn}-y_{j+1,n}\right) = 1+o(1). \tag{3.4}$$

Proof. (a) See [8, Theorem 1.1].
(b) See [14].
(c), (d) See [7, Lemma 3.1]. ■

Lemma 3.2. *Assume that* μ *is a regular measure with compact support. Let* I *be a closed subinterval of the support in which* μ *is absolutely continuous, and* μ' *is positive and continuous. Let* J *be a compact subset of the interior* I^o *of* I*. Let* ω *denote the equilibrium density for the support of* μ*. Assume in addition the hypothesis (1.6).*
(a) Uniformly for $y_{jn} \in J$*,*

$$n\omega\left(x_{jn}\right)\left(x_{jn}-y_{j,n-1}\right) = g\left(x_{jn}\right)+\frac{1}{2}+o(1)\,; \tag{3.5}$$

$$n\omega\left(x_{jn}\right)\left(y_{jn}-y_{j,n-1}\right) = g\left(x_{jn}\right)+o(1). \tag{3.6}$$

(b) Fix $A > 0$*. Uniformly for* $n \geq 1$*,*

$$\sup_{x\in J} \|p_n\|_{L_\infty\left[x-\frac{A}{n},x+\frac{A}{n}\right]} \|p_{n-1}\|_{L_\infty\left[x-\frac{A}{n},x+\frac{A}{n}\right]} \leq C. \tag{3.7}$$

(c)

$$\|p_n\|_{L_\infty(J)} = o\left(n^{1/2}\right). \tag{3.8}$$

Proof. (a) Using continuity of ω, our hypothesis (1.6), and (3.3),

$$\begin{aligned} & n\omega\left(x_{jn}\right)\left(x_{jn}-y_{j,n-1}\right) \\ &= n\omega\left(x_{jn}\right)\left(x_{jn}-x_{j,n-1}\right)+n\omega\left(x_{j,n-1}\right)\left(x_{j,n-1}-y_{j,n-1}\right)+o(1) \\ &= g\left(x_{jn}\right)+\frac{1}{2}+o(1). \end{aligned}$$

Next, from this last asymptotic and (3.3),

$$n\omega\left(x_{jn}\right)\left(y_{jn}-y_{j,n-1}\right) = n\omega\left(x_{jn}\right)\left(y_{jn}-x_{jn}+x_{jn}-y_{j,n-1}\right) = g\left(x_{jn}\right)+o(1).$$

(b) See [7, Theorem 1.1]. Note that our hypothesis (1.6) implies the spacing hypotheses required for Theorem 1.1 there, since $g(J)$ is a compact subset of (0, 1).
(c) This follows from the asymptotic (3.2) for the Christoffel function: as $n \to \infty$, uniformly for $x \in J$,

$$\begin{aligned}\frac{1}{n} p_n^2(x)\,\mu'(x) &= \left(1+\frac{1}{n}\right)\frac{1}{n+1} K_{n+1}(x,x)\,\mu'(x) - \frac{1}{n} K_n(x,x)\,\mu'(x)\\ &\to \omega(x) - \omega(x) = 0. \quad \blacksquare\end{aligned}$$

Here is the main ingredient for our Theorem 1.2: we assume the hypotheses there.

Lemma 3.3. *Uniformly for $y_{jn} \in J$,*

$$\left|y_{jn} - b_n\right|^{1/2} p_n\left(y_{jn}\right) \mu'\left(y_{jn}\right)^{1/2} (-1)^j = \sqrt{\frac{2}{\pi}} \left|\cot \pi g\left(y_{jn}\right)\right|^{1/2} + o(1). \tag{3.9}$$

Proof. We multiply the recurrence relation by $p_n\left(y_{jn}\right)$:

$$\left(y_{jn} - b_n\right) p_n^2\left(y_{jn}\right) = a_{n+1}\left(p_{n+1} p_n\right)\left(y_{jn}\right) + a_n\left(p_n p_{n-1}\right)\left(y_{jn}\right). \tag{3.10}$$

We use the local limit (3.1) and the Christoffel–Darboux formula to simplify the right-hand side. First, from the confluent form of the Christoffel–Darboux formula,

$$K_n\left(y_{jn}, y_{jn}\right) = -a_n p'_{n-1}\left(y_{jn}\right) p_n\left(y_{jn}\right). \tag{3.11}$$

Since the local limit (3.1) holds uniformly in compact subsets of the plane, we can differentiate it:

$$\lim_{n\to\infty} \frac{p'_{n-1}\left(y_{j,n-1} + \frac{z}{(n-1)\omega\left(y_{j,n-1}\right)}\right)}{p_{n-1}\left(y_{j,n-1}\right)(n-1)\,\omega\left(y_{j,n-1}\right)} = -\pi \sin \pi z.$$

Using this and (3.6),

$$\begin{aligned} p'_{n-1}\left(y_{jn}\right) &= p'_{n-1}\left(y_{j,n-1} + \left(y_{jn} - y_{j,n-1}\right)\right)\\ &= p'_{n-1}\left(y_{j,n-1} + \frac{g\left(x_{jn}\right) + o(1)}{n\omega\left(y_{j,n-1}\right)}\right)\\ &= -p_{n-1}\left(y_{j,n-1}\right)(n-1)\,\omega\left(y_{j,n-1}\right)\pi\left(\sin \pi g\left(x_{jn}\right) + o(1)\right)\\ &= -\pi p_{n-1}\left(y_{j,n-1}\right) n\omega\left(y_{j,n-1}\right)\left(\sin \pi g\left(x_{jn}\right)\right)(1 + o(1)), \end{aligned} \tag{3.12}$$

recall that $\sin \pi g\left(x_{jn}\right)$ is bounded away from 0. We substitute this in (3.11), multiply by $\frac{1}{n}\mu'\left(y_{jn}\right)$, and use the asymptotic (3.2):

$$\omega\left(y_{jn}\right) + o(1) = a_n \pi p_{n-1}\left(y_{j,n-1}\right) p_n\left(y_{jn}\right) \mu'\left(y_{jn}\right) \omega\left(y_{j,n-1}\right)\left(\sin \pi g\left(x_{jn}\right)\right)(1 + o(1))$$

and hence

$$a_n \pi p_{n-1}\left(y_{j,n-1}\right) p_n\left(y_{jn}\right) \mu'\left(y_{jn}\right)\left(\sin \pi g\left(x_{jn}\right)\right) = (1 + o(1)). \tag{3.13}$$

To replace $p_{n-1}\left(y_{j,n-1}\right)$ by $p_{n-1}\left(y_{jn}\right)$, we again use (3.1) (as in (3.12)):

$$p_{n-1}\left(y_{jn}\right) = p_{n-1}\left(y_{j,n-1}\right)\left[\cos \pi g\left(x_{jn}\right) + o(1)\right]. \tag{3.14}$$

Thus after multiplying (3.13) by $\cos \pi g\left(x_{jn}\right)+o(1)$, it can be recast as

$$\begin{aligned}
& a_n \pi p_n\left(y_{jn}\right) p_{n-1}\left(y_{jn}\right) \mu'\left(y_{jn}\right) \sin \pi g\left(x_{jn}\right) \\
= & \cos \pi g\left(x_{jn}\right)+o(1)+o\left(p_n\left(y_{jn}\right) p_{n-1}\left(y_{j,n-1}\right)\right) \\
= & \cos \pi g\left(x_{jn}\right)+o(1),
\end{aligned}$$

by Lemma 3.2(b). Since $\sin \pi g\left(x_{jn}\right)$ is bounded away from 0,

$$a_n \pi p_n\left(y_{jn}\right) p_{n-1}\left(y_{jn}\right) \mu'\left(y_{jn}\right)=\cot \pi g\left(x_{jn}\right)+o(1). \tag{3.15}$$

Next, replacing n by $n+1$ in (3.15) and using continuity of μ', g gives

$$a_{n+1} \pi p_{n+1}\left(y_{j,n+1}\right) p_n\left(y_{j,n+1}\right) \mu'\left(y_{jn}\right)=\cot \pi g\left(x_{jn}\right)+o(1). \tag{3.16}$$

Using the local limit (3.1) and the relations (3.3), (3.4) on p_n and then p_{n+1} gives

$$\begin{aligned}
p_{n+1}\left(y_{j,n+1}\right) p_n\left(y_{j,n+1}\right) &= p_{n+1}\left(y_{j,n+1}\right)\left(\cos \pi g\left(x_{jn}\right)+o(1)\right) p_n\left(y_{jn}\right) \\
&= \left(p_{n+1} p_n\left(y_{jn}\right)\right)+o(1),
\end{aligned}$$

by Lemma 3.2(b) again. Thus (3.16) yields

$$a_{n+1} \pi\left(p_{n+1} p_n\left(y_{jn}\right)\right) \mu'\left(y_{jn}\right)=\cot \pi g\left(x_{jn}\right)+o(1). \tag{3.17}$$

We substitute this and (3.15) into the recurrence (3.10):

$$\left(y_{jn}-b_n\right) p_n^2\left(y_{jn}\right) \mu'\left(y_{jn}\right)=\frac{2}{\pi} \cot \pi g\left(x_{jn}\right)+o(1). \tag{3.18}$$

Finally as $y_{jn} \in\left(x_{j+1,n}, x_{jn}\right)$, so

$$(-1)^j p_n\left(y_{jn}\right)>0,\ 1 \leq j \leq n-1,$$

and we obtain the result on taking square roots. ■

Proof of Theorem 1.2. Now from (3.1), (3.9), for $x \in\left[x_{j+1,n}, x_{jn}\right) \cap J$,

$$\begin{aligned}
& \left|y_{jn}-b_n\right|^{1/2} p_n(x) \mu'(x)^{1/2} \\
= & \left[(-1)^j \sqrt{\frac{2}{\pi}}\left|\cot \pi g\left(y_{jn}\right)\right|^{1/2}+o(1)\right]\left[\cos \left(\pi n \omega\left(x_{jn}\right)\left(x-y_{jn}\right)\right)+o(1)\right] \\
= & \sqrt{\frac{2}{\pi}}|\cot \pi g(x)|^{1/2} \cos \left(\pi n\left[\omega(x)\left(x-y_{jn}\right)-\frac{j}{n}\right]\right)+o(1),
\end{aligned}$$

by continuity of g, ω. Next, uniformly for $x \in J$, and with j as above,

$$\begin{aligned}
\left|x-b_n\right|^{1/2} &= \left|y_{jn}-b_n\right|^{1/2}+O\left(\left|x-y_{jn}\right|^{1/2}\right) \\
&= \left|y_{jn}-b_n\right|^{1/2}+O\left(n^{-1/2}\right),
\end{aligned}$$

so

$$\begin{aligned}
& \left|x-b_n\right|^{1/2} p_n(x) \mu'(x)^{1/2} \\
= & \sqrt{\frac{2}{\pi}}|\cot \pi g(x)|^{1/2} \cos \left(\pi n\left[\omega(x)\left(x-y_{jn}\right)-\frac{j}{n}\right]\right)+o(1)+O\left(n^{-1/2}\left|p_n(x)\right|\right) \\
= & \sqrt{\frac{2}{\pi}}|\cot \pi g(x)|^{1/2} \cos \left(\pi n f_n(x)\right)+o(1),
\end{aligned}$$

by Lemma 3.2(c). ■

Proof of Corollary 1.3. This is immediate from Theorem 1.2. ■

Lemma 3.4. *Uniformly for x in compact subsets of J omitting zeros of* $\cos \pi g$,
(a)

$$(x - b_n)(x - b_{n-1}) = 4a_n^2 \cos^2(\pi g(x)) + o(1). \tag{3.19}$$

(b)

$$|\cot \pi g(x)| = \frac{\left| \left(\frac{x-b_{n-1}}{2a_n} \right) \left(\frac{x-b_n}{2a_n} \right) \right|^{1/2}}{\sqrt{1 - \left(\frac{x-b_{n-1}}{2a_n} \right) \left(\frac{x-b_n}{2a_n} \right)}} + o(1) + o(1). \tag{3.20}$$

Proof. From the recurrence relation,

$$\left(y_{jn} - b_{n-1}\right)(p_{n-1}p_n)\left(y_{jn}\right) = a_n p_n^2\left(y_{jn}\right) + a_{n-1}(p_{n-2}p_n)\left(y_{jn}\right). \tag{3.21}$$

We now replace the terms on both sides. First we multiply by $a_n \pi \mu'\left(y_{jn}\right)$ and use (3.15):

$$\begin{aligned} &\left(y_{jn} - b_{n-1}\right)\left[\cot \pi g\left(x_{jn}\right) + o(1)\right] \\ &= a_n^2 \pi p_n^2\left(y_{jn}\right)\mu'\left(y_{jn}\right) + a_{n-1}a_n\pi\,(p_{n-2}p_n)\left(y_{jn}\right)\mu'\left(y_{jn}\right). \end{aligned} \tag{3.22}$$

Next, from (3.13),

$$a_n \pi p_n\left(y_{jn}\right) p_{n-1}\left(y_{j,n-1}\right)\mu'\left(y_{jn}\right)\sin \pi g\left(x_{jn}\right) = 1 + o(1).$$

Replace n by $n - 1$:

$$a_{n-1}\pi p_{n-1}\left(y_{j,n-1}\right) p_{n-2}\left(y_{j,n-2}\right)\mu'\left(y_{j,n-1}\right)\sin \pi g\left(x_{j,n-1}\right) = 1 + o(1).$$

Dividing the two relations, and using continuity of μ', g, and the fact that $\sin \pi g$ is bounded away from 0, gives

$$\frac{a_n p_n\left(y_{jn}\right)}{a_{n-1}p_{n-2}\left(y_{j,n-2}\right)} = 1 + o(1). \tag{3.23}$$

Next, our local limit (3.1) and the spacing (3.6) give

$$p_{n-2}\left(y_{jn}\right) = p_{n-2}\left(y_{j,n-2}\right)\left(\cos\left(2\pi g\left(x_{jn}\right)\right) + o(1)\right),$$

and thus (3.23) gives

$$a_n p_n\left(y_{jn}\right)\left(\cos\left(2\pi g\left(x_{jn}\right)\right) + o(1)\right) = a_{n-1}p_{n-2}\left(y_{jn}\right)(1 + o(1)).$$

Then (3.22) becomes

$$\begin{aligned} &\left(y_{jn} - b_{n-1}\right)\left[\cot \pi g\left(x_{jn}\right) + o(1)\right] \\ &= a_n^2 \pi p_n^2\left(y_{jn}\right)\mu'\left(y_{jn}\right)\{1 + \cos\left(2\pi g\left(x_{jn}\right)\right) + o(1)\}. \end{aligned}$$

Multiplying by $\left(y_{jn} - b_n\right)$ and using Lemma 3.3 gives

$$\begin{aligned} &\left(y_{jn} - b_n\right)\left(y_{jn} - b_{n-1}\right)\left[\cot \pi g\left(x_{jn}\right) + o(1)\right] \\ &= 4a_n^2\left\{\cot \pi g\left(y_{jn}\right) + o(1)\right\}\left\{\cos^2\left(\pi g\left(x_{jn}\right)\right) + o(1)\right\} \end{aligned}$$

so that if $\cos \pi g\left(x_{jn}\right)$ is bounded away from 0,

$$\left(y_{jn} - b_n\right)\left(y_{jn} - b_{n-1}\right) = 4a_n^2 \cos^2\left(\pi g\left(x_{jn}\right)\right) + o(1).$$

Then the result follows from the continuity of g, the density of the $\{y_{jn}\}$ and the boundedness of the $\{b_n\}$.

(b) Away from zeros of $\cos \pi g(x)$, our conclusion in (a) gives

$$\begin{aligned}|\cot \pi g(x)| &= \frac{|\cos \pi g(x)|}{\sqrt{1-\cos^2 \pi g(x)}} \\ &= \frac{\left|\left(\frac{x-b_{n-1}}{2a_n}\right)\left(\frac{x-b_n}{2a_n}\right)\right|^{1/2}}{\sqrt{1-\left(\frac{x-b_{n-1}}{2a_n}\right)\left(\frac{x-b_n}{2a_n}\right)}} + o(1). \quad \blacksquare\end{aligned}$$

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