

Pole Skipping in Holographic Theories with Bosonic Fields

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We study pole skipping in holographic conformal field theories dual to diffeomorphism invariant theories containing an arbitrary number of bosonic fields in the large N limit. Defining a weight to organize the bulk equations of motion, a set of general pole skipping conditions are derived. In particular, the frequencies simply follow from general covariance and weight matching. In the presence of higher-spin fields, we find that the imaginary frequency for the highest-weight pole skipping point equals the higher-spin Lyapunov exponent which lies outside of the chaos bound. Without higher-spin fields, we show that the energy density Green's function has its highest-weight pole skipping happening at a location related to the out-of-time-order correlator for arbitrary higher-derivative gravity, with a Lyapunov exponent saturating the chaos bound and a butterfly velocity matching that extracted from a shockwave calculation. We also suggest an explanation for this matching at the metric level by obtaining the on-shell shockwave solution from a regularized limit of the metric perturbation at the skipped pole.

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Introduction.—The out-of-time-order correlator (OTOC), an important quantity containing characteristics of chaos, can be calculated holographically in a shockwave spacetime [1–5]. For a localized perturbation to a chaotic system at temperature T , the OTOC between a perturbation W at $x = t = 0$ and a probe operator V at a later time t behaves as

$$\langle V(x, t) W V(x, t) W \rangle \sim 1 - e^{\lambda_L(t-t_* - |x|/v_B)}, \quad (1)$$

where t_* is called the scrambling time. This defines the Lyapunov exponent λ_L and the butterfly velocity v_B . For classical bulk gravitational theories, λ_L saturates the chaos bound $\lambda_L \leq 2\pi T$ [6], so they are said to be maximally chaotic. The butterfly velocity, however, depends on the theory [2,7–11].

More recently, it was discovered that the quantities λ_L and v_B may already show up in features of the energy density retarded Green's function through a phenomenon called *pole skipping* [12–14]. It was first found numerically for pure Einstein gravity [12] and later studied analytically for Einstein gravity with matter [14]. See also Refs. [15–31] for holographic and Refs. [32–37] for boundary studies.

The retarded Green's function is the relation between a source and its response. Holographically, the Green's function of an operator dual to a bulk dynamic field X (suppressing indices) is given by [38,39]

$$G_R(\omega, k) = \left(\lim_{r \rightarrow \infty} \frac{\Pi(r; \omega, k)|_{X_R}}{X_R(r; \omega, k)} \right) \Big|_{X_0=0}, \quad (2)$$

where X_R is the bulk solution satisfying Dirichlet boundary condition $X_R \rightarrow X_0$ at infinity and ingoing wave boundary condition at the horizon and Π is its conjugate variable in a radial foliation. In terms of an asymptotic expansion, it is proportional to the ratio between the coefficient of the normalizable falloff and that of the non-normalizable falloff. A quasinormal mode, by definition, does not have a non-normalizable divergence, so the poles of the Green's function are identified with the quasinormal spectrum.

Generically, X_R is uniquely determined from X_0 , and G_R is, therefore, well defined. However, a would-be pole can sometimes get multiplied by a zero, resulting in an ill-defined limit. This happens at a special frequency and momentum:

$$\omega = i\lambda_L, \quad k = \frac{i\lambda_L}{v_B}, \quad (3)$$

where λ_L and v_B are, respectively, the Lyapunov exponent and the butterfly velocity extracted from a holographic OTOC calculation (1) in Einstein gravity minimally coupled to a large class of matter fields [12,14].

To explain this universality, Ref. [14] discovered a feature of Einstein's equation at the horizon. Expanding metric perturbations around a stationary planar black hole in terms of Fourier modes, a particular component of Einstein's equation evaluated at the horizon was found to be trivial at Eq. (3) so that there exists one fewer constraint. This implies an extra degree of freedom of the

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ingoing modes and, consequently, an ambiguity in the bulk solution X_R and, in turn, the Green's function G_R .

Later, it was discovered that pole skipping happens more generally at other locations and for other types of Green's functions [17,18,30,40–49]. See also Refs. [50,51] for higher-derivative corrections and Ref. [52] for a zero-temperature example. However, unlike the one at Eq. (3), the other skipped poles are unrelated to chaos.

We put these in the same framework by considering general diffeomorphism invariant bulk theories with matter fields that are not necessarily minimally coupled. For simplicity, we consider only bosonic fields and leave a discussion of fermionic fields to the last section. By defining a weight, we can separate the equations of motion into different groups and evaluate them in a given order. This allows us to find the frequencies of the skipped poles and the corresponding momenta, in general. This is done in Sec. “General pole skipping conditions”. Furthermore, we observe a relation between higher-weight pole skipping frequencies and higher-spin Lyapunov exponents and use it to justify the removal of a bounded tower of higher-spin fields from consideration in the remaining sections.

In Sec. “Matching of butterfly velocities”, it is shown that, for general higher-derivative gravitational theories, the butterfly velocity can be obtained from the highest-weight equation of motion, and it agrees with the butterfly velocity obtained via a shockwave calculation. This generalizes the matching for Gauss-Bonnet gravity and Einstein gravity with a string theory correction at $O(\alpha^3)$ [15]. We also try to explain this matching between pole skipping and chaos in the same section. By regularizing the metric perturbation at the chaotic skipped pole with a Gaussian distribution in the frequency Fourier space, we obtain a metric that is regular at the horizon. Extending it to a Kruskal-Szekeres coordinate patch and taking the regulator away, we show that this metric perturbation localizes to the past horizon in a distributional sense, like the shockwave metric. We end with a summary and a discussion of potential future directions in Sec. “Discussion”.

General pole skipping conditions.—The metric for a general stationary planar black hole can be written in ingoing Eddington-Finkelstein coordinates as

$$ds^2 = -f(r)dv^2 + 2dvdr + h(r)dx^i dx^i, \quad (4)$$

where $f(r_0) = 0$ at the horizon $r = r_0$ and $i = 1, \dots, d$. The nonvanishing Christoffel components are given by

$$\begin{aligned} \Gamma_{vv}^v &= \frac{1}{2}f', & \Gamma_{ij}^v &= -\frac{1}{2}h'\delta_{ij}, & \Gamma_{vr}^r &= -\frac{1}{2}f', \\ \Gamma_{rj}^i &= \frac{h'}{2h}\delta_j^i, & \Gamma_{vv}^r &= \frac{1}{2}ff', & \Gamma_{ij}^r &= -\frac{1}{2}fh'\delta_{ij}. \end{aligned} \quad (5)$$

For simplicity, we assume that background matter fields are stationary, isotropic, and homogeneous in x^i and regular at both past and future horizons, like the metric.

Now, if we define a *pseudo-weight* for any tensor component as the number of lower v indices minus that of lower r indices, where an upper v is considered a lower r and vice versa, then any background tensor component (ones constructed from the stationary background metric and matter fields) with positive weight needs to vanish at the horizon. We prove this next.

In Kruskal-Szekeres coordinates, defined via

$$U = -e^{-f'(r_0)(v-2r_*)/2}, \quad V = e^{f'(r_0)v/2}, \quad (6)$$

where $dr_*/dr = 1/f(r)$, one can similarly define a *boost weight* as the number of lower V indices minus that of lower U indices [53]. Then, the boost symmetry ($V \mapsto aV$, $U \mapsto U/a$) requires that a background quantity with boost weight $n > 0$ must scale like U^n times a function of the product UV , and regularity at the bifurcate horizon requires this function to be nonsingular as $UV \rightarrow 0$. Therefore, at the future horizon ($U = 0$), this vanishes. Relating this to quantities in ingoing Eddington-Finkelstein coordinates, using

$$dv = \frac{2}{f'(r_0)} \frac{dV}{V}, \quad dr = \frac{f(r)}{f'(r_0)} \left(\frac{dV}{V} + \frac{dU}{U} \right) \quad (7)$$

for each lower index V or U of a tensor T , we have (suppressing other indices)

$$T_V = \frac{\partial v}{\partial V} T_v + \frac{\partial r}{\partial V} T_r = \frac{2}{f'(r_0)V} \left(T_v + \frac{1}{2}f(r)T_r \right) \quad (8)$$

and

$$T_U = \frac{\partial r}{\partial U} T_r = \frac{1}{U f'(r_0)} T_r. \quad (9)$$

We see that each V index maps to a v index and each U index maps to an r index (all lower indices here but upper ones work similarly) up to terms that are of higher order in f . Given that background quantities with positive boost weight and f vanish at the horizon, we arrive at the conclusion that the same is true if we replace boost weight with pseudo-weight. From now on, we no longer need to mention boost weight and will refer to pseudo-weight simply as *weight* [54].

To describe ingoing quasinormal modes at the horizon, for any dynamic field X , we expand its perturbation around the stationary background in the Fourier space as

$$\delta X(r, v, x) = \delta X(r) e^{-i\omega v + i k x}. \quad (10)$$

For Einstein gravity, writing Einstein's equation as $E_{\mu\nu} = T_{\mu\nu}$, a particular component under perturbation, δE_v^r , is proportional to

$$\left(k^2 - i \frac{d}{2} \omega h' \right) \delta g_{vv} + (\omega - i 2\pi T) [\omega \delta g_{ii} + 2k_i \delta g_{vi}]. \quad (11)$$

On the horizon, for matter perturbations that are regular enough, the stress tensor component $\delta T_v^r = 0$ [14], and prefactors in δE_v^r can be tuned to zero by choosing Eq. (3). As a consequence, Einstein's equation provides one fewer constraint, which serves as an explanation for the universal behavior of the energy density Green's function with low-spin matter fields coupled to Einstein gravity [14].

Now consider an arbitrary diffeomorphism invariant theory defined with a local action $S = S_g + S_M$, where the gravitational part S_g is given by

$$S_g = \int d^{d+2}x \sqrt{-g} \mathcal{L}(g, R, \nabla, \Phi) \quad (12)$$

and S_M is part of the action with only minimally coupled matter fields, artificially separated from the rest for later convenience. Here, $\mathcal{L}$ can be an arbitrary function of the metric, g , and an arbitrary number of bosonic matter fields collectively denoted as Φ . More specifically, $\mathcal{L}$ can be written as a sum of contractions between an arbitrary number of the metric, curvature tensors, matter fields, and an arbitrary number of covariant derivatives of them.

The metric equation of motion is defined as

$$E_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S_g}{\delta g^{\mu\nu}} = -\frac{2}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}} = T_{\mu\nu}. \quad (13)$$

The remaining equations of motion are given by $\delta S/\delta \Phi = 0$, indices suppressed. Now, to obtain Eq. (2), the idea is to perturb the dynamical fields and apply the equations of motion everywhere. However, it turns out sufficient to consider the near-horizon expansion of all perturbations in order to study pole skipping. For readability, we introduce the following compact notation: We use $\delta \mathcal{E} = 0$ to denote collectively all the perturbed equations of motions and their radial derivatives (∇_r) evaluated on the horizon. These are essentially the coefficients of a near-horizon Taylor expansion. We further define $\delta \mathcal{E}_p$ as the subset of $\delta \mathcal{E}$ with weight p , organized into a vector, and denote its number of components as $|\delta \mathcal{E}_p|$.

Similarly, we collect perturbations of all dynamics fields (including both the metric and matter) and their radial derivatives with weight q into $\delta \mathcal{X}_q$ (all evaluated on the horizon). For example, $\delta \mathcal{X}_2 = (\delta g_{vv}, \nabla_r \delta B_{vvv}, \dots)$ and $\delta \mathcal{X}_0 = (\delta g_{ij}, \nabla_r \delta g_{vi}, \nabla_r \nabla_r \delta g_{vv}, \delta A_i, \nabla_r \delta A_v, \dots)$.

With these definitions, we can now write

$$\delta \mathcal{E}_p = \sum_q \mathcal{M}_{p,q}(\omega, k) \delta \mathcal{X}_q, \quad (14)$$

where each $\mathcal{M}_{p,q}(\omega, k)$ is a matrix of size $|\delta \mathcal{E}_p| \times |\delta \mathcal{X}_q|$. To arrive at this form, first commute all ∇_r 's through ∇_i 's and ∇_v 's to the rightmost location before substituting the Fourier expansion and evaluating the ∇_v 's and ∇_i 's. By definition, the radial derivatives are then absorbed into $\delta \mathcal{X}_q$.

For later convenience, we also commute all ∇_i to the right of ∇_v .

We now prove a useful property that, for $p > q$,

$$\mathcal{M}_{p,q}(\omega, k) \propto [\omega - (p-1)\omega_0] \dots [\omega - q\omega_0], \quad (15)$$

where $\omega_0 = i2\pi T = if'(r_0)/2$.

Begin by noticing that, for a given p and q' ,

$$\delta \mathcal{E}_p \sim F(g, R, \nabla, \Phi) (\nabla_v)^k (\nabla_i)^l (\nabla_r)^m \delta X_{q'+m} \quad (16)$$

before substituting the Fourier expansion, where F is some c -number tensor component constructed out of g, R, ∇ , and Φ such as $R_{virj} A^\mu \nabla_\mu \phi$ evaluated on the horizon of the background configuration and $\delta X_{q'+m}$ is the perturbation to some component of a dynamic field X with weight $q' + m$ —not evaluated on the horizon until acted upon by all the derivative operators in front. Next, notice that the only way to raise weight is with ∇_v , because any background tensor with positive weight vanishes on the horizon. Therefore, to raise the weight of $(\nabla_r)^m \delta X_{q'+m}$ to that of $\delta \mathcal{E}_p$, one needs $k \geq p - q'$. From Eq. (5), it is straightforward to show that, on the horizon,

$$\nabla_v T \propto \left(\partial_v - \frac{n}{2} f'(r_0) \right) T \quad (17)$$

for a general tensor component T with weight n ; therefore, evaluating $(\nabla_v)^k$ and substituting Eq. (10) gives at least a factor of $[\omega - (p-1)\omega_0] \dots [\omega - q'\omega_0]$. Finally, the remaining part $(\nabla_i)^l (\nabla_r)^m \delta X_{q'+m}$ evaluates to a number of terms, each proportional to $\delta \mathcal{X}_q$ for some $q \geq q'$. This follows from Eq. (5), where any Christoffel symbol appearing in $\nabla_i T$ vanishes if multiplying an object with lower weight than T . This concludes our proof of Eq. (15).

We now discuss the general conditions for pole skipping. We take as an assumption that pole skipping happens whenever an equation of motion becomes trivial [55]. Suppose the highest weight of $\delta \mathcal{X}$ is q_0 ; then the highest weight of $\delta \mathcal{E}$ is also q_0 (since the action, being a scalar, has weight zero). Consequently, for any positive integer s , once we set

$$\omega = (q_0 - s)\omega_0, \quad (18)$$

all $\mathcal{M}_{p,q}(\omega, k)$ with $p > q_0 - s \geq q$ are then set to zero (assuming they are not all automatically zero). Now consider the square matrix

$$M_s(k) \equiv \begin{pmatrix} \mathcal{M}_{q_0, q_0} & \cdots & \mathcal{M}_{q_0, q_0-s+1} \\ \cdots & \cdots & \cdots \\ \mathcal{M}_{q_0-s+1, q_0} & \cdots & \mathcal{M}_{q_0-s+1, q_0-s+1} \end{pmatrix}, \quad (19)$$

where Eq. (18) has been substituted. The full set of equations of motion $\delta \mathcal{E}_p$, $\forall p$, does not determine $\delta \mathcal{X}_q$, $\forall q$, when

$$\det M_s(k) = 0. \quad (20)$$

Equations (18) and (20) are, therefore, the generalized pole skipping conditions (for any given $s \geq 1$), assuming the second one has solutions. If the theory has a highest-spin field with bounded spin l , then $q_0 = l$ and the pole skipping frequencies are $(l - s)\omega_0$, consistent with observations made in Refs. [18,32,49] and, in particular, reproducing the positions of pole skipping at Matsubara frequencies first found in Ref. [18]. The second condition is a polynomial equation for k , and the roots are then the pole skipping momenta, which could be more than one. The order of the polynomial increases with the size of the matrix, and, therefore, there will be generically more pole skipping points at larger s (lower ω).

The first pole skipping happens at $s = 1$ at frequency $\omega = (q_0 - 1)\omega_0 = i(q_0 - 1)2\pi T$. Suppose there exists an equation of motion with, e.g., three lower v indices. In that case, there will be a skipped pole at $2\omega_0 = i4\pi T$, and the field perturbation (10) will grow like $\exp(4\pi T t)$. On this ground, we expect (finitely many) higher-spin fields to violate the chaos bound. This is supported by an independent calculation of the spin- l Lyapunov exponent $\lambda_L^l = (l - 1)2\pi T$ [56] and is consistent with the findings of Refs. [32,57]. Bounded higher-spin fields also suffer from causality violation [58], which is another reason to exclude them from consideration in the next section. Notice, however, that equations of motion for fields with no dynamics automatically have $\mathcal{M}_{p,q}(\omega, k) = 0$ for $p > q$ due to the nonappearance of ∇_v , so they do not become trivial from nontrivial; therefore, they do not violate the chaos bound, in agreement with Ref. [56], where pure AdS_3 higher-spin gravity was exempt from their argument for bound violation.

If $q_0 = 2$, which is the case for an arbitrary metric theory coupled to matter fields of spin no larger than two, then the bound is satisfied and, in fact, saturated. We will discuss this further in the next section.

For $q_0 < 2$, such as a scalar or vector field without gravitational backreaction, there is no growing mode and, therefore, no relation to chaos, but an infinite number of skipped poles still exist and constrain the structure of Green's functions [17,18].

Matching of butterfly velocities.—For arbitrary higher-derivative gravity coupled to scalar, vector, or form fields, $q_0 = 2$ (from the metric) and the highest-weight skipped pole has $\omega = i2\pi T$. We now show that the corresponding butterfly velocity matches that obtained from the OTOC.

In this case, the only dynamic field with weight 2 is $\delta\mathcal{X}_2 = \delta g_{vv}$, and the corresponding equation of motion is $\delta\mathcal{E}_2 = \delta E_{vv} - \delta T_{vv} = 0$. The perturbation to the stress tensor component δT_{vv} does not necessarily vanish, but $\delta T_v^r (= \delta T_{vv} - T_{rr}\delta g_{vv})$ does vanish for matter fields regular on the horizon [14]. We will make this restriction in order to compare results with OTOC: The metric

shockwave also has vanishing δT_v^r . Therefore, the pole skipping conditions with $s = 1$ are given by

$$\omega = \omega_0, \quad \det M_1 = \frac{\delta E_v^r}{\delta g_{vv}} = 0. \quad (21)$$

This gives a polynomial equation for k with only even powers (by symmetry). In cases where the polynomial is of quartic order or higher, one can take the view that all corrections to Einstein gravity should be treated perturbatively so only the roots continuously connected to Einstein gravity are physical. But, as we will see, the matching is evident without a perturbative treatment.

For the class of theories we consider,

$$\delta E_v^r = \sum_{k,l} H_{k,l}(f, h, \partial_r, \Phi) (\partial_v)^k (\partial_i)^l \delta g_{vv} \quad (22)$$

for some noncovariant c -number coefficients $H_{k,l}$. The nontrivial statement that no ∂_r acts on δg_{vv} and none of the other components such as δg_{vi} can appear follows directly from the weight argument. As an example, consider the Einstein gravity equation of motion (11) studied in Ref. [14]. Since δg_{ij} has weight zero, it has to pick up a factor of ω to get to weight one and then a factor of $(\omega - \omega_0)$ to get to weight two, similarly for δg_{vi} which needs only to raise its weight by one. Another simplification in Einstein gravity is due to the fact of it being two-derivative. It is not possible for Eq. (11) to contain a term like, for example, $\partial_r \delta g_{vi}$: This quantity has weight zero and, therefore, needs two v -derivatives to go to two, but it already has one derivative itself.

To compare this with the shockwave calculation, we move to Kruskal-Szekeres coordinates defined in Eq. (6). Then $UV = -e^{f'(r_0)r_*}$, and the metric is given by

$$ds^2 = 2A(UV)dUdV + B(UV)dx^i dx^i, \quad (23)$$

$$A(UV) = \frac{2}{f'(r_0)^2} \frac{f(r)}{UV}, \quad B(UV) = h(r). \quad (24)$$

In general, higher-derivative gravity and for a shockwave along $V = 0$, the only nontrivial component of δE_v^μ perturbed by a local source is δE_V^U [11]. For a general perturbation δg_{vv} , translating to ingoing Eddington-Finkelstein coordinates, this component is given by

$$\delta E_V^U = \frac{U}{V} \left(\frac{2}{f(r)} \delta E_v^r + \delta E_r^r - \delta E_v^v - \frac{f(r)}{2} \delta E_r^r \right). \quad (25)$$

Compared to the first term, others are suppressed with extra factors of $f(r)$, so they vanish when evaluated on the horizon. Similarly, $\delta T_V^U \propto \delta T_v^r$, but recall that this vanishes for regular matter configurations. Therefore,

$$\begin{aligned}
 0 &= \delta E_V^U = \frac{2UV}{f(r)} \frac{1}{V^2} \sum_{k,l} H_{k,l} (\partial_v)^k (\partial_i)^l \delta g_{vv} \\
 &= \frac{2UV}{f(r)} \frac{1}{V^2} \sum_{k,l} H_{k,l} (\partial_i)^l \left(\frac{2}{f'(r_0)} V \partial_V \right)^k \delta g_{vv} \\
 &= \frac{2UV}{f(r)} \sum_{k,l} H_{k,l} (\partial_i)^l \left(\frac{2}{f'(r_0)} (V \partial_V + 2) \right)^k \frac{\delta g_{vv}}{V^2} \\
 &= \frac{2UV}{f(r)} \sum_{k,l} \tilde{H}_{k,l} (\partial_i)^l (V \partial_V)^k \frac{\delta g_{vv}}{V^2} \\
 &= \frac{1}{A} \sum_{k,l} \tilde{H}_{k,l} (\partial_i)^l (V \partial_V)^k \delta g_{VV}, \tag{26}
 \end{aligned}$$

where we used the transformation $\partial_v = [2/f'(r_0)]V\partial_V$ in going to the second line and a trick

$$\frac{1}{V^2} V \partial_V = (V \partial_V + 2) \frac{1}{V^2} \tag{27}$$

in going to the third line. The fourth line follows from a reorganization of the sum with new coefficients $\tilde{H}_{k,l}$, and the last line follows from

$$\delta g_{VV}(V, x) = \frac{4\delta g_{vv}(v, x)}{f'(r_0)^2 V^2}. \tag{28}$$

The special thing about $\omega = \omega_0$ is that

$$\delta g_{vv} \sim e^{-i\omega_0 v} = e^{-[i2\omega_0/f'(r_0)] \log V} = V, \tag{29}$$

and, therefore,

$$\delta g_{VV}(V, x) \sim \frac{1}{V} e^{-ikx}. \tag{30}$$

Compare this with a linearized shockwave perturbation

$$\delta g_{VV} \sim \delta(V) e^{-\mu x}, \tag{31}$$

where $\mu = 2\pi T/v_B$ upon using $\delta E_V^U = 0$ (outside of a localized source in x). Noticing that $\delta(V)$ has the same distributional behavior as $1/V$ under $V\partial_V$ [12], e.g., $V\delta'(V) = -\delta(V)$ and $Vd(1/V)/dV = -1/V$, it follows that $k = i2\pi T/v_B$ upon using Eq. (26) for the perturbation (30), thereby extending Eq. (3) to general higher-derivative gravity and, hence, some of the results of Refs. [14,15].

Given the similarity between $1/V$ and $\delta(V)$ and the role this similarity plays in establishing the equivalence of these two calculations of the butterfly velocity, it is natural to wonder whether there is a more direct connection between them. An immediate obstacle is the divergence of the function $1/V$ at the past horizon $V = 0$. We mitigate this problem with an unnormalized [59] regularization of the Fourier space delta function along the real frequency line:

$$\int d\xi \delta(\xi) \rightarrow \int_{-\infty}^{\infty} d\xi e^{-\xi^2/a}, \tag{32}$$

giving rise to a mode

$$\delta g_{vv} = \sqrt{\pi a} e^{-(a/4)(\lambda_L v)^2} e^{\lambda_L v}. \tag{33}$$

To compare with the shockwave metric (31), we convert this to Kruskal-Szekeres coordinates. Using Eq. (28),

$$\delta g_{VV} = \begin{cases} 0, & V < 0, \\ \frac{\sqrt{\pi a}}{\lambda_L^2} \frac{1}{V} e^{-a(\log V)^2/4}, & V \geq 0, \end{cases} \tag{34}$$

where we have used the fact that the perturbation vanishes exactly behind the past horizon. This function is finite and integrates to a constant for finite a , and it vanishes everywhere off the horizon as $a \rightarrow 0$. It, therefore, behaves as a regularized $\delta(V)$. Taking the regulator away, this becomes a shockwave localized at $V = 0$ [60].

Discussion.—We have defined a quantity called weight to organize bulk equations of motion and exploited its convenience to show that pole skipping happens in holographic conformal field theories dual to quite general diffeomorphism invariant bulk theories. As a result, the pole skipping frequencies show up at $(q_0 - s)\omega_0$ for all $s \in \mathbb{Z}^+$, where $\omega_0 = i2\pi T$ and q_0 is defined as the weight of the highest-weight object. In particular, a theory that has a bounded highest spin larger than two, in general, gives rise to $q_0 > 2$, which leads to very fast scrambling that violates the chaos bound. It is, therefore, reasonable to disallow a finite tower of higher-spin fields, in addition to causality reasons [58]. This brings down q_0 to two, and, with this restriction, the metric is the field that can have the highest weight. This is the main reason behind the universality of the special pole skipping point at $\omega = i\lambda_L$ and $k = i\lambda_L/v_B$, where $\lambda_L = 2\pi T$ and v_B is defined via a OTOC calculation.

In other words, for maximally chaotic holographic theories, instead of needing to compute a four-point function, the retarded Green's function already knows about the butterfly velocity, and its dependence on the bulk theory is exactly the same as an OTOC would predict. It would be interesting to test this statement for nonholographic maximally chaotic theories [61]. Furthermore, there are now three ways of computing the butterfly velocity: (i) using an entanglement wedge, (ii) using a shockwave, and (iii) using pole skipping. We proved the equivalence between the second and third *prescriptions* themselves [64].

The restriction of the discussion to bosons is for simplicity, and the generalization to include fermions should be completely analogous. For minimally coupled spinors on a fixed background, pole skipping has been shown to happen at $\omega = (q_0 - s)\omega_0$ for a half integer $q_0 = 1/2$ and positive integers s [41]; with a spin-3/2 Rarita-Schwinger field, q_0 becomes 3/2 [47]. Both of the examples fit the pattern that the leading pole skipping happens at $(q_0 - 1)\omega_0$, and, if one allows both bosonic and

fermionic fields with arbitrary couplings between them, one might expect that both q_0 and s can be half integers. It might be of use to analyze this with the weight argument, perhaps beginning by rephrasing the current discussion in a spin connection language.

We should summarize three assumptions that were used: (i) the existence of a finite q_0 ; (ii) the nontriviality of Eq. (18), i.e., the entries set to zero by this equations are not already all zero; and (iii) Eq. (20) has solutions. We expect that assumption (i) can be lifted with more careful analysis, but assumptions (ii) and (iii) are essential. Given any theory, one needs to check whether these are satisfied. For example, Vasiliev gravity violates assumption (i), as it contains an infinite tower of higher-spin fields; this is consistent with it being dual to a sector of a free theory [65], which does not exhibit chaos.

Another condition implicit in our discussion is the restriction to finite temperatures. Extremal black holes do not have a bifurcate surface, so the property derived from regularity at the bifurcate surface no longer applies. Furthermore, poles in the Green's function get replaced by branch cuts [52,66]. Accordingly, a generalization of our argument to zero temperature will be nontrivial.

We also showed that the shockwave metric could be obtained from a regularized mode of the metric perturbation. This serves as an explanation for the similarities between the two calculations and the equivalence regardless of the theory. One might try different regulators or use different subtraction schemes to find a more regulator-independent relation.

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