An Explicit Reference Governor for Linear Sampled-Data Systems With Disturbance Inputs and Uncertain Time Delays

Nan Li, Sijia Geng, Ilya Kolmanovsky, and Anouck Girard

***Abstract*— In this paper, we propose a novel reference governor (RG) scheme for pre-stabilized linear sampled-data systems to satisfy pointwise-in-time constraints in the presence of bounded disturbances and uncertain input and/or measurement delays. Based on an explicit bound on the system response to step changes in the reference signal derived using the logarithmic norm, this RG scheme yields a closed-form solution for updating the reference signal at sample time instants that guarantees both sample-time and inter-sample constraint satisfaction. Due to its closed-form expression, the proposed RG scheme requires minimum computational effort and is thereby suitable for systems with limited computing capability.**

I. INTRODUCTION

Many control requirements can be expressed as pointwise-in-time constraints. Control approaches that can handle such constraints include Model Predictive Control (MPC) [1], [2] and invariant set-based approaches [3]–[5]. Another framework for handling constraints is the use of Reference Governors (RGs) to augment (rather than replace) nominal controllers [6]. Specifically, the RG enforces constraints by monitoring, and modifying when necessary, the reference signal to the nominal controller. From practical perspective, the RG is an add-on scheme that preserves desirable small-signal characteristics of the nominal closed-loop system, which typically does not account for the constraints, while protecting the system against constraint violations for larger signals. In most of the above approaches, an optimization problem, typically a linear or quadratic program, needs to be solved online. In applications such as to small spacecraft [7] and small-scale robotic systems [8], on-board computing capabilities are limited and electrical power consumed during intensive computations becomes a concern. For these and other certification-related reasons, closed-form/explicit solutions without the need for an online optimization solver are highly desirable. Such solutions are available in the case of the scalar RG for discrete-time linear systems [6], [9].

For continuous-time systems, the Explicit Reference Governor (ERG) [10], [11] uses a continuous-time dynamic feedback law to adjust the reference signal for guarding the system from constraint violations. In this paper, we propose a novel ERG scheme in a sampled-data setting for linear systems with disturbance inputs and time delays, which exploits the *Logarithmic Norm* and is thus referred to as *ERG-LN*. The logarithmic norm has been exploited in [12] to bound the errors between the responses of a nonlinear system and its linearized model. This bound was then used to define an RG scheme based on online prediction and optimization. In this paper, the ERG-LN scheme is proposed for a different class of systems, in which the logarithmic norm is exploited in a different way.

Compared to the traditional ERG, the proposed ERG-LN has the following differences: 1) The traditional ERG is inherently a continuous-time reference adjustment law. Typically, the issues related to discrete-time implementation are not explicitly addressed assuming high sampling rates. While constraint satisfaction for all times as well as reference convergence can be established through an extra algorithm based on one-step prediction [13], this introduces extra computing tasks. In contrast, the proposed ERG-LN is applicable to a sampled-data setting [14]. Specifically, while applied to a continuous-time system, ERG-LN is inherently a digital device – it measures the state and updates the reference signal only at periodic sample time instants. Such a sampled-data setting is also different from a discrete-time setting. A discrete-time model typically does not capture the inter-sample behavior of the continuous-time system, while the ERG-LN guarantees both sample-time and inter-sample constraint satisfaction. 2) The ERG-LN can account for bounded but uncertain constant delays in the reference input and/or state measurement, which, to the best of our knowledge, has not been addressed in the ERG literature. The formulation of ERG for continuous-time linear systems that are subject to known input delay has been studied in [15]. For second-order linear systems with known reference input delay, a prediction-based RG scheme that has low computational footprint was proposed in [16]. An uncertain reference input delay was treated within the prediction-based RG framework in [17], where updates to the reference signal are determined through online optimization. However, uncertain state measurement delay was not addressed in [15]–[17], which can also be handled by our ERG-LN scheme.

We note that the ability to account for uncertain reference input and/or measurement delays is important. On the one hand, such delays commonly exist in real systems due to, e.g., time needed for measurement, computation and communication (see Fig. 1 and also [17]). On the other hand, although these delays are often small, ignoring them in the design may cause the constraints to not be strictly enforced during the operation (see the example provided in [17]). In this context, the contributions of this paper are as follows:

1) We propose a novel optimization-free, closed-form/explicit RG solution, referred to as ERG-LN, for pre-stabilized linear sampled-data systems with bounded disturbances and uncertain delays affecting reference input and/or state measurement channels (see Fig. 1) to satisfy

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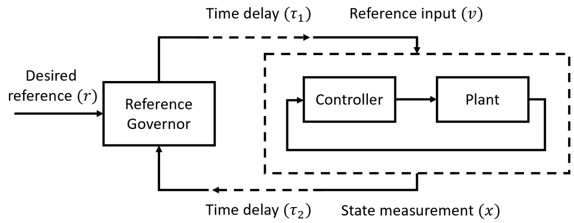

Fig. 1: Application of reference governor to pre-stabilized system subject to input and measurement delays.

pointwise-in-time constraints. To the best of our knowledge, this is the first explicit RG solution in the literature that is designed in a sampled-data setting and can simultaneously handle disturbances and uncertain delays.

2) This ERG-LN scheme employs the logarithmic norm to derive explicit bounds on the system response and a closed-form solution to the reference signal that guarantees constraint satisfaction. Such an application shows that the logarithmic norm is a useful tool for constrained control, especially in a sampled-data setting.
3) We establish important theoretical properties of the proposed ERG-LN, including guaranteed both sample-time and inter-sample constraint enforcement and finite-time convergence of the reference signal to constant, strictly steady-state constraint admissible desired reference.
4) We illustrate the proposed ERG-LN through simulation examples and compare it with other relevant schemes.

This paper is organized as follows: In Section II, we introduce the system model considered in this paper and the logarithmic norm, which will be exploited to define our ERG-LN scheme. In Section III, we present our ERG-LN scheme, together with a discussion of its theoretical properties. We illustrate the proposed ERG-LN scheme and compare it with other relevant schemes through simulation examples in Section IV. The paper is concluded in Section V.

The notations used in this paper are standard. For a set $X \subset \mathbb{R}^n$, $\text{int}(X)$ denotes its interior, $\text{cl}(X)$ its closure, and $X^C = \mathbb{R}^n \setminus X$ its complement. For a map $\mathscr{F} : [a,b] \to \mathbb{R}^n$, $\text{Im}(\mathscr{F})$ denotes its image, i.e., $\text{Im}(\mathscr{F}) = \{\mathscr{F}(x) : x \in [a,b]\}$. To facilitate exposition in a sampled-data setting, we use (t) with round brackets to represent a continuous time $t \in [0,\infty)$, and use $[t]$ with square brackets to represent a sample time instant $t \in \mathbb{N}_0 T$, where $\mathbb{N}_0$ denotes the set of natural numbers including 0, and T denotes the sampling period.

II. Problem statement and preliminaries

In this paper, we consider a pre-stabilized continuous-time linear system in the following form,

$$\dot{x}(t) = Ax(t) + Bv(t-\tau_1) + Dw(t), \tag{1a}$$

$$\hat{x}(t) = x(t-\tau_2), \tag{1b}$$

where $x(t) \in \mathbb{R}^{n_x}$ denotes the system state, $v(t) \in \mathbb{R}^{n_v}$ denotes a delayed reference input determining the set-point of the system, $\hat{x}(t) \in \mathbb{R}^{n_x}$ denotes a delayed measurement of the system state, and $w(t) \in \mathbb{R}^{n_w}$ denotes an unmeasured disturbance input, which can also be used to represent modeling errors. By pre-stabilization, we mean that (1a) represents a closed-loop system consisting of a possibly unstable plant and a stabilizing controller such that the matrix A in (1a) is Hurwitz (i.e., all eigenvalues of A have strictly negative real parts). Note that the nominal stabilizing controller can be designed without considering any constraints or delays that affect the RG channels (see Fig. 1). In this case, a variety of approaches such as pole placement and linear quadratic regulator (LQR) can be used for its design. Then, the proposed ERG-LN will be used as an add-on scheme to augment the nominal system (1a) with constraint handling capability (see Fig. 1). More details on this RG approach to handling constraints can be found in [6].

When A is Hurwitz, for any constant reference input $v(t) \equiv v \in \mathbb{R}^{n_v}$, there is a corresponding steady state $x_e(v) = -A^{-1}Bv$ such that $x(t) \to x_e(v)$ as $t \to \infty$ when disturbance-free (i.e., $Dw(t) \equiv 0$). We now make the following assumptions on the delays τ_1, τ_2, and on the disturbance input $w(t)$:

Assumption 1: The values of the constant time delays $\tau_1, \tau_2 \geq 0$ are uncertain but satisfy $\tau_1 + \tau_2 \leq T$ with T being a known constant.

We note that the delays τ_1, τ_2 may be attributed to various sources, such as sensing, data processing, and communication between the inner-loop (the nominal closed-loop system) and the outer-loop (the augmented ERG-LN). For instance, the nominal system (1) may be monitored and commanded remotely in a networked systems setting (see Fig. 1), in which case significant communication delays may occur. For a real system, exact values of such delays may not be easy to measure, but establishing a bound T for them is realistic.

Assumption 2: The disturbance input $w(t)$ is Lebesgue measurable in $t \in [0,\infty)$ and takes values in a bounded set $W \subset \mathbb{R}^{n_w}$ for almost all t.

Note that we assume $w(t)$ to take values in the bounded set W for almost all t (rather than for all t) because the values of $w(t)$ on a null set of time instants t do not change the solution of the differential equation (1a), i.e., we allow $w(t)$ to take any values on a null set of t [18].

We consider the following pointwise-in-time constraints,

$$x(t) \in X, \quad \forall t \in [0,\infty), \tag{2}$$

where $X \subset \mathbb{R}^{n_x}$ is a closed set. The following assumption on the constraint set X is further made:

Assumption 3: For any two reference input values $v, v' \in \mathbb{R}^{n_v}$ of interest, there is a continuous function $\mathscr{P} : [0,1] \to \mathbb{R}^{n_v}$ connecting $\mathscr{P}(0) = v$ and $\mathscr{P}(1) = v'$ (called a "path") that satisfies $-A^{-1}B\,\text{Im}(\mathscr{P}) \subset \text{int}(X)$.

For instance, if $\text{int}(X)$ is convex and the images of v, v' under the linear transformation $-A^{-1}B$ both belong to $\text{int}(X)$, then $\mathscr{P}$ can be the linear path $\mathscr{P}(\rho) = (1-\rho)v + \rho v'$. Note that the proposed ERG-LN can also be applied for non-convex constraint sets X, as long as a path $\mathscr{P}$ satisfying Assumption 3 is available. An ERG for continuous-time systems that can handle non-convex constraints has been proposed in [19], but the development in [19] is not in a sampled-data setting and does not address delays.

We aim to develop a RG scheme applicable in a sampled-data setting that governs the reference input $v(t)$ based on the delayed state measurement $\hat{x}(t) = x(t-\tau_2)$ and, under Assumptions 1–3, is able to enforce the constraint (2). For this, we exploit the *logarithmic norm*.

Definition 1 [20], [21]: A functional $\mu(\cdot,\|\cdot\|):\mathbb{R}^{n\times n}\to\mathbb{R}$ defined by

$$\mu(F,\|\cdot\|)=\lim_{h\to 0^+}\frac{\|I_n+hF\|-1}{h},\quad F\in\mathbb{R}^{n\times n}, \tag{3}$$

is called a *logarithmic norm*, where $\|\cdot\|$ denotes both an arbitrary vector norm on $\mathbb{R}^n$ and its induced matrix norm on $\mathbb{R}^{n\times n}$ (defined by $\|F\|=\max_{\|x\|=1}\|Fx\|$).

Note that a logarithmic norm is not a *norm* on a vector space, but a real-valued functional on linear operators $\mathbb{R}^n\to\mathbb{R}^n$ identified by square matrices in $\mathbb{R}^{n\times n}$. Importantly, a logarithmic norm can take negative values. Furthermore, a logarithmic norm can be induced from an arbitrary vector norm using the formula (3), allowing a large flexibility in the design. For common vector norms, such as ℓ^p-norms with $p=1,2,\infty$, their corresponding logarithmic norms admit explicit expressions [22]. In particular, for a quadratic norm defined by $\|\cdot\|_P=\sqrt{(\cdot)^\top P(\cdot)}$ with a positive-definite matrix P, its corresponding logarithmic norm is given by

$$\mu(F,\|\cdot\|_P)=\lambda_{\max}\left(\frac{(P^{1/2}FP^{-1/2})+(P^{1/2}FP^{-1/2})^\top}{2}\right), \tag{4}$$

where $\lambda_{\max}(\cdot)$ represents the largest eigenvalue of a real symmetric matrix. The logarithmic norm has several useful properties, presented as follows. For the sake of completeness, we include their proofs in the Appendices.

Proposition 1: If $V(\theta)=\theta^\top P\theta$ is a Lyapunov function proving the asymptotic stability of $\theta=0$ for the system $\dot\theta(t)=F\theta(t)$, then $\mu(F,\|\cdot\|_P)<0$.

Proof: See Appendix A. ■

Since the matrix A of our pre-stabilized system (1a) is Hurwitz, for any positive-definite matrix Q, we can obtain a positive-definite matrix P by solving the Lyapunov equation

$$A^\top P+PA+Q=0, \tag{5}$$

such that $V(x)=x^\top Px$ is a Lyapunov function proving the asymptotic stability of $x=0$ for the system $\dot x(t)=Ax(t)$. Then, Proposition 1 implies that $\mu(A,\|\cdot\|_P)<0$.

Proposition 2: Consider $\dot\theta(t)=F\theta(t)+\gamma(t)$, where $F\in\mathbb{R}^{n\times n}$ and $\gamma\in\mathscr{L}^\infty\big([0,\infty)\to\mathbb{R}^n\big)$. We have (i) $D_t^+\|\theta(t)\|\le\mu(F,\|\cdot\|)\|\theta(t)\|+\|\gamma(t)\|$, where $D_t^+\|\theta(t)\|:=\limsup_{h\to0^+}\frac{\|\theta(t+h)\|-\|\theta(t)\|}{h}$. If $\mu(F,\|\cdot\|)<0$ and $\|\gamma(t)\|\le\gamma_{\max}$ for almost all $t\in[0,\infty)$, then we also have (ii) $\lim_{t\to\infty}\operatorname{dist}\big(\theta(t),\Theta\big)=0$ for all $\theta(0)\in\mathbb{R}^n$ and (iii) $\theta(0)\in\Theta\implies\theta(t)\in\Theta$ for all $t\in[0,\infty)$, where $\Theta:=\Big\{\theta\in\mathbb{R}^n\,|\,\|\theta\|\le-\frac{\gamma_{\max}}{\mu(F,\|\cdot\|)}\Big\}$ and $\operatorname{dist}(\theta,\Theta):=\inf_{\theta_0\in\Theta}\|\theta-\theta_0\|$.

Proof: See Appendix B. ■

Proposition 2 represents a way of using the logarithmic norm to estimate a bound on the response of a continuous-time linear system subject to a bounded input. We choose to exploit the logarithmic norm for this purpose because it has been shown in [20], [21] that the logarithmic norm offers a less conservative estimate than some other tools such as Lipschitz constants. This result will be relied upon in our RG design to achieve constraint enforcement.

To summarize, the problem treated in this paper is to develop a RG scheme for system (1), which is subject to a bounded disturbance input $w(t)$ and uncertain delays τ_1,τ_2, to satisfy constraint (2) while achieving desirable reference and state convergence properties. Furthermore, we require this RG scheme to be applicable in a sampled-data setting and is computationally light. For the latter requirement, we pursue an optimization-free, closed-form/explicit solution.

III. Explicit reference governor using logarithmic norm for constraint handling

The basic idea of our design is to update the reference input $v(t)$ periodically until $v(t)$ converges to its desired value, r, while guaranteeing that the system response $x(t)$ to the periodically updated $v(t)$ satisfies the constraint (2). The desired reference value r is typically an input provided by either a human operator or a higher-level planning algorithm. In principle, r may also be a periodically updated signal, in which case the constraint enforcement result of our designed ERG-LN scheme still holds true. Note that in many applications the update period of r is typically much longer than the update period of $v(t)$ [6]. In view of this fact, we assume r to be a constant throughout this section to simplify the exposition. Under the assumption of constant r, we also establish the finite-time convergence result of $v(t)$.

Let $T\ge\tau_1+\tau_2$ and consider updating $v(t)$ at the sample time instants $\mathbb{N}_0T$.[1] Let $v[t_0]$ denote the reference input value determined at the sample time instant $t_0=kT$. Note that due to the input delay τ_1, the reference $v[t_0]$ acts on the system (1a) starting from $t=t_0+\tau_1$. Then, define $y(t)=x(t)-x_e(v[t_0])=x(t)+A^{-1}Bv[t_0]$ and write (1a) in the y coordinates as

$$\begin{aligned}\dot y(t)&=\dot x(t)+\frac{\mathrm{d}}{\mathrm{d}t}\left(A^{-1}Bv[t_0]\right)=\dot x(t)=Ax(t)+Bv(t-\tau_1)+Dw(t)\\&=A\left(y(t)-A^{-1}Bv[t_0]\right)+Bv(t-\tau_1)+Dw(t)\\&=Ay(t)+Bv(t-\tau_1)-Bv[t_0]+Dw(t).\end{aligned} \tag{6}$$

Note that in the above expressions $v[t_0]$ denotes a value for the reference input and is treated as a constant. Now consider updating the reference input from $v[t_0]$ to some value $v[t_1]$ at the sample time instant $t_1=(k+1)T$. The solutions to (6) satisfy the following result:

Proposition 3: Let $y(t)$ be a solution to (6) over $[t_1-\tau_2,\infty)$ with a given initial condition $y(t_1-\tau_2)$, where $v(t-\tau_1)=v[t_0]$ for $t\in[t_1-\tau_2,t_1+\tau_1)$, $v(t-\tau_1)=v[t_1]$ for $t\in[t_1+\tau_1,\infty)$, and $w(t)$ satisfies Assumption 2. Then, $y(t)$ must satisfy

$$\|y(t)\|_P\le\max\left(-\frac{\|B(v[t_1]-v[t_0])\|_P+\Omega}{\mu(A,\|\cdot\|_P)},\|y(t_1-\tau_2)\|_P\right), \tag{7}$$

for all $t\in[t_1-\tau_2,\infty)$, where P satisfies (5) and $\Omega=\sup_{w\in W}\|Dw\|_P$.

Proof: For $t\in[t_1-\tau_2,t_1+\tau_1)$, (6) reduces to

$$\dot y(t)=Ay(t)+Dw(t). \tag{8}$$

Firstly, from Proposition 2(i), if $\|y(t)\|_P\ge-\frac{\Omega}{\mu(A,\|\cdot\|_P)}\ge-\frac{\|Dw(t)\|_P}{\mu(A,\|\cdot\|_P)}$, then $D_t^+\|y(t)\|_P\le0$. Secondly, from Proposition 2(iii), if $\|y(t_1-\tau_2)\|_P\le-\frac{\Omega}{\mu(A,\|\cdot\|_P)}$, then $\|y(t)\|_P\le$

[1] For systems in which a processor performing RG computations is capable of a shorter sampling period T, one can update the reference $v(t)$ every N sample instants where $NT\ge\tau_1+\tau_2$. Without loss of generality one can then treat (and rename) NT as T in the analysis. We require the update period of $v(t)$ to satisfy $T\ge\tau_1+\tau_2$ in order to guarantee constraint satisfaction for all times without knowing exact values of the delays τ_1 and τ_2, as will be shown in what follows.

$-\frac{\Omega}{\mu(A,\|\cdot\|_P)}$ for all $t \in [t_1-\tau_2, t_1+\tau_1]$. Combining these two cases, we must have

$$\|y(t)\|_P \le \max\left(-\frac{\Omega}{\mu(A,\|\cdot\|_P)}, \|y(t_1-\tau_2)\|_P\right), \tag{9}$$

for all $t \in [t_1-\tau_2, t_1+\tau_1]$.

Similarly, for $t \in [t_1+\tau_1, \infty)$, (6) reduces to

$$\dot{y}(t) = Ay(t) + B(v[t_1]-v[t_0]) + Dw(t), \tag{10}$$

and we must have

$$\|y(t)\|_P \le \max\left(-\frac{\|B(v[t_1]-v[t_0])\|_P+\Omega}{\mu(A,\|\cdot\|_P)}, \|y(t_1+\tau_1)\|_P\right), \tag{11}$$

for all $t \in [t_1+\tau_1, \infty)$.

Note $-\frac{\Omega}{\mu(A,\|\cdot\|_P)} \le -\frac{\|B(v[t_1]-v[t_0])\|_P+\Omega}{\mu(A,\|\cdot\|_P)}$ and (9) implies

$$\|y(t_1+\tau_1)\|_P \le \max\left(-\frac{\Omega}{\mu(A,\|\cdot\|_P)}, \|y(t_1-\tau_2)\|_P\right). \tag{12}$$

Then, combining (9), (11) and (12), we obtain (7). ■

Proposition 3 provides an explicit bound on the system response to a step change in the reference input $v(t)$. Then, the following assumption is made and will be relied on to enforce the constraint (2).

Assumption 4: The P-weighted Euclidean distance of $x_e[t_0] = x_e(v[t_0]) = -A^{-1}Bv[t_0]$ to the constraint boundary,

$$\mathrm{dist}(x_e[t_0], X^C) = \inf_{x \in X^C} \|x_e[t_0]-x\|_P, \tag{13}$$

can be measured, and the measurement will be available at the sample time instant t_1.

Remark 1: For many well-structured constraint sets X, $\mathrm{dist}(x', X^C) = \inf_{x\in X^C}\|x'-x\|_P$ admits an analytical expression. For instance, for a polyhedral set $X = \{x \in \mathbb{R}^{n_x} : M_i^\top x \le m_i,\ i = 1,\cdots,n_m\}$, $\mathrm{dist}(x', X^C)$ can be computed as

$$\mathrm{dist}(x', \{M_i^\top x = m_i\}) = \frac{m_i - M_i^\top x'}{\|M_i\|_P^*} = \frac{m_i - M_i^\top x'}{\sqrt{M_i^\top P^{-1} M_i}}, \tag{14a}$$

$$\mathrm{dist}(x', X^C) = \min_{i=1,\cdots,n_m} \mathrm{dist}(x', \{M_i^\top x = m_i\}), \tag{14b}$$

where $\|\cdot\|_P^*$ is the dual norm of $\|\cdot\|_P$ defined as $\|\cdot\|_P^* = \sqrt{(\cdot)^\top P^{-1}(\cdot)}$. For a non-structured or a priori unknown constraint set X, $\mathrm{dist}(x_e[t_0], X^C)$ may be estimated online through sampling points on the constraint boundary ∂X, i.e.,

$$\mathrm{dist}(x_e[t_0], X^C) \approx \min_{i=1,\cdots,n_s} \|x_e[t_0]-x_i\|_P, \quad \{x_i\}_{i=1}^{n_s} \subset \partial X. \tag{15}$$

Such estimation may require non-negligible processing time. However, Assumption 4 says that we need the estimate of $\mathrm{dist}(x_e[t_0], X^C)$ to be available at the next sample time instant $t_1 = t_0 + T$, i.e., a time period of length $T \ge \tau_1 + \tau_2$ is available for such processing.

On the basis of Proposition 3 and Assumption 4, the safety of any value for $v[t_1]$ in terms of constraint enforcement can be evaluated using the following result:

Proposition 4: Let $x(t)$ be a solution to (1a) over $[t_1-\tau_2, \infty)$ with a given initial condition $x(t_1-\tau_2)$, where $v(t-\tau_1) = v[t_0]$ for $t \in [t_1-\tau_2, t_1+\tau_1)$, $v(t-\tau_1) = v[t_1]$ for $t \in [t_1+\tau_1, \infty)$, and $w(t)$ satisfies Assumption 2. If the following conditions hold,

$$\|B(v[t_1]-v[t_0])\|_P \le -\mu(A,\|\cdot\|_P)\,\mathrm{dist}(x_e[t_0], X^C) - \Omega, \tag{16a}$$

$$\|\hat{x}(t_1) - x_e[t_0]\|_P \le \mathrm{dist}(x_e[t_0], X^C), \tag{16b}$$

where $\hat{x}(t_1) = x(t_1-\tau_2)$ is the measurement of the initial condition at the delayed time instant t_1 according to (1b), then the constraint $x(t) \in X$ is guaranteed to be satisfied for all $t \in [t_1-\tau_2, \infty)$.

Proof: Firstly, the combination of (16a) and (16b) yields

$$\mathrm{dist}(x_e[t_0], X^C) \ge \eta := \max\left(-\frac{\|B(v[t_1]-v[t_0])\|_P+\Omega}{\mu(A,\|\cdot\|_P)}, \|\hat{x}(t_1)-x_e[t_0]\|_P\right). \tag{17}$$

If $\eta > 0$, then (17) implies that the open ball centered at $x_e[t_0]$ with radius η, $\mathscr{B}(x_e[t_0], \eta) = \{x : \|x - x_e[t_0]\|_P < \eta\}$, is contained entirely in X. More specifically, suppose $\exists x_0 \in \mathscr{B}(x_e[t_0], \eta)$ such that $x_0 \in X^C$. In this case, we would have $\mathrm{dist}(x_e[t_0], X^C) = \inf_{x\in X^C}\|x_e[t_0]-x\|_P \le \|x_e[t_0]-x_0\|_P < \eta$, which contradicts (17). Because $\mathscr{B}(x_e[t_0], \eta) \subset X$ and X is closed, the closure of $\mathscr{B}(x_e[t_0], \eta)$, $\mathrm{cl}(\mathscr{B}(x_e[t_0], \eta)) = \{x : \|x - x_e[t_0]\|_P \le \eta\}$, is also contained entirely in X. Recall we have shown in Proposition 3 that for $t \in [t_1-\tau_2, \infty)$, $\|x(t) - x_e[t_0]\|_P = \|y(t)\|_P \le \eta$, i.e., $x(t) \in \mathrm{cl}(\mathscr{B}(x_e[t_0], \eta))$. Thus, it holds that $x(t) \in X$. According to the definition of η in (17), the only possible case for $\eta = 0$ is with $v[t_1] = v[t_0]$, $\Omega = 0$ (disturbance-free), and $\hat{x}(t_1) = x(t_1-\tau_2) = x_e[t_0]$. In this case, $x(t)$ starts at the steady state $x_e[t_0] = x_e(v[t_0])$ and stays there forever. Then, according to Assumption 3, $x_e(v[t_0]) \in X$, which implies $x(t) \in X$. ■

Now, based on the result of Proposition 4 and following the idea of optimization-based RG [6], we present the following design for determining $v[t_1]$: If $\|\hat{x}(t_1) - x_e[t_0]\|_P \le \mathrm{dist}(x_e[t_0], X^C)$, then

$$v[t_1] = \operatorname*{argmin}_v \|v - r\|_S^2 \quad \text{s.t.} \tag{18}$$

$$(v - v[t_0])^\top (B^\top PB)(v - v[t_0]) \le \Big(\mu(A,\|\cdot\|_P)\,\mathrm{dist}(x_e[t_0], X^C) + \Omega\Big)^2,$$

where S is a positive-definite matrix penalizing the difference between v and the desired reference value r; $v[t_1] = v[t_0]$ otherwise.

The online optimization problem (18) is a convex quadratically constrained quadratic program. To reduce the computational complexity of the design, we derive a closed-form/explicit alternative in what follows.

We restrict the periodically updated values of v to the path in Assumption 3, $\mathscr{P} : [0,1] \to \mathbb{R}^{n_v}$ connecting $v[0]$ and r. Then, we define $\mathscr{T}[t_0] : [\rho[t_0], 1] \to \mathbb{R}$ as

$$\mathscr{T}[t_0](\rho) = \|B(\mathscr{P}(\rho) - v[t_0])\|_P = \|B(\mathscr{P}(\rho) - \mathscr{P}(\rho[t_0]))\|_P, \tag{19}$$

where $\rho[t_0]$ is the coordinate of $v[t_0]$ on the path, and make the following assumption:

Assumption 5: $\mathscr{T}[t_0]$ is a monotone increasing function.

Assumption 5 holds for many cases. For instance, if $\mathscr{P}$ is a linear path, $\mathscr{P}(\rho) = (1-\rho)v[0] + \rho r$, then

$$\begin{aligned}\mathscr{T}[t_0](\rho) &= \|B((1-\rho)v[0] + \rho r - (1-\rho[t_0])v[0] - \rho[t_0]r)\|_P \\ &= (\rho - \rho[t_0])\|B(r - v[0])\|_P\end{aligned} \tag{20}$$

is monotone increasing in ρ.

With $\mathscr{T}[t_0]$ and Assumption 5, we present the following design, referred to as the *Explicit Reference Governor using Logarithmic Norm (ERG-LN)*, for generating the continuous-time signal of v:

$$v(t) = v[t_1], \tag{21}$$

for each time interval $[t_1, t_1+T)$, $t_1 \in \mathbb{N}_0 T$, where $v[t_1]$ is determined as

$$v[t_1] = \mathscr{P}(\rho[t_1]), \tag{22a}$$

$$\rho[t_1] = \chi[t_1]\rho^*[t_1] + (1-\chi[t_1])\rho[t_0], \tag{22b}$$

$$\chi[t_1] = \Big[\|\hat{x}(t_1) - x_e[t_0]\|_P \le \text{dist}(x_e[t_0], X^C) \Big], \tag{22c}$$

$$\rho^*[t_1] = (\mathscr{T}[t_0])^{-1}\Big(\min\Big(-\mu(A, \|\cdot\|_P)\,\text{dist}(x_e[t_0], X^C) - \Omega, \\ \|B(r - v[t_0])\|_P \Big)\Big), \tag{22d}$$

where $\chi[t_1]$ is an indicator function, taking its value as 1 if the condition on the right-hand side of (22c) holds true and taking 0 otherwise. The second term in (22d) ensures that (22) outputs $v[t_1] = r$ whenever r is a feasible solution.

Note first that the above design updates the value of v at the sample time instants $t_1 \in \mathbb{N}_0 T$ based on the delayed state measurement $\hat{x}(t_1) = x(t_1 - \tau_2)$ rather than the instantaneous state value $x(t_1)$, which is not available due to the measurement delay τ_2 (see (1b)). Note also that Assumption 5 guarantees the function $\mathscr{T}[t_0]$ to be invertible, and for many path types $\mathscr{P}$, (22d) admits an analytical expression. For instance, for a linear path $\mathscr{P}(\rho) = (1-\rho)v[0] + \rho r$ with the expression of $\mathscr{T}[t_0]$ given as (20), (22d) reduces to

$$\rho^*[t_1] = \min\left(-\frac{\mu(A, \|\cdot\|_P)\,\text{dist}(x_e[t_0], X^C) + \Omega}{\|B(r - v[0])\|_P} + \rho[t_0], 1 \right). \tag{23}$$

We now discuss theoretical properties of the reference input signal $v(t)$ generated by our ERG-LN (21)-(22). The following results demonstrate its safety, in terms of constraint enforcement, and liveness, in terms of convergence to r, respectively.

Proposition 5: Suppose that for a constant reference input, $v(t) \equiv v[0]$ over $[-\tau_1, \infty)$, and for $w(t)$ satisfying Assumption 2, the state trajectory $x(t)$ of (1a) satisfies (2). Then, for $v(t)$ generated by (21)-(22) and $w(t)$ satisfying Assumption 2, $x(t)$ is guaranteed to satisfy (2).

Proof: Let $t' \in [0, \infty)$ be arbitrary and let us consider the following two possible cases. Firstly, suppose there exists an earlier sample time instant $t_1 \in \mathbb{N}_0 T$, $t_1 \le t' - \tau_1$, such that the value of $v(t - \tau_1)$ gets adjusted at $t = t_1 + \tau_1$ (i.e., $v((t_1+\tau_1) - \tau_1) = v(t_1) \neq \lim_{t \to t_1^-} v(t)$), and after that $v(t - \tau_1)$ remains constant over $t \in [t_1 + \tau_1, t']$. According to (21), $v(t_1) \neq \lim_{t \to t_1^-} v(t)$ implies $v[t_1] \neq v[t_0]$, where $t_0 = t_1 - T$ denotes the sample time instant previous to t_1. In this case, according to the reference adjustment law (22), the following two conditions must hold true at t_1: 1) $\chi[t_1]$ must be equal to 1, which implies the right-hand side of (22c) holds true, i.e., $\|\hat{x}(t_1) - x_e[t_0]\|_P \le \text{dist}(x_e[t_0], X^C)$.[2] 2) Under Assumption 5, (22d) ensures $\|B(\mathscr{P}(\rho^*[t_1]) - v[t_0])\|_P \le -\mu(A, \|\cdot\|_P)\,\text{dist}(x_e[t_0], X^C) - \Omega$. Then, according to Proposition 4, $v[t_1] = \mathscr{P}(\rho^*[t_1])$ generated by (22) guarantees $x(t) \in X$ to be satisfied for all t starting from $t = t_1 - \tau_2$ to the first time instant after $t = t_1 + \tau_1$ at which $v(t - \tau_1)$ gets adjusted again. Since $v(t - \tau_1)$ remains constant over $t \in [t_1 + \tau_1, t']$ (see the definition of t_1 above), this implies $x(t') \in X$.

Secondly, suppose the $t_1 \in \mathbb{N}_0 T$ defined in the first case does not exist. This immediately implies $v(t - \tau_1) = v[0]$ for all $t \in [0, t']$ (i.e., no adjustment has occurred). In this case, the assumption in the proposition statement guarantees $x(t) \in X$ for all $t \in [0, t']$. Therefore, we have shown that in either case, $x(t') \in X$. Since $t' \in [0, \infty)$ is arbitrary, this proves the satisfaction of (2). ■

In Proposition 5, we assume that the system trajectory corresponding to the initial reference value $v[0]$ satisfies the constraint (2). This is a reasonable assumption [6]. For instance, the system may start operating from a constraint-admissible steady state (or, the time instant when the system trajectory converges to a sufficiently small neighborhood of a constraint-admissible steady state is considered as the initial time $t = 0$). However, this does not mean one can simply maintain the reference input $v(t)$ at its initial value $v[0]$ and not update it, because $v[0]$ and its corresponding steady state $x_e(v[0])$ may not be equal to the desired reference r and the desired steady state $x_e(r)$. Indeed, the latter two represent the current set-point/operational task for the system. Next, we show the finite-time convergence property of $v(t)$ to r and the exponential convergence property of $x(t)$ to a bounded neighborhood of $x_e(r)$ using our ERG-LN.

Proposition 6: Suppose that $\mathscr{P}$ is uniformly continuous and there exists $\delta > 0$ such that for every $\rho \in [0, 1]$, $\text{dist}\big(x_e(\mathscr{P}(\rho)), X^C\big) \ge -\frac{\Omega}{\mu(A, \|\cdot\|_P)} + \delta$. Then, (i) there exists $t_f \in \mathbb{N}_0 T$ such that $v(t) = r$ for all $t \ge t_f$, and (ii) $\text{dist}(x(t), \Omega(r))$ converges exponentially to 0 as $t \to \infty$, where $\Omega(r) = \big\{x \in \mathbb{R}^{n_x} : \|x - x_e(r)\|_P \le -\frac{\Omega}{\mu(A, \|\cdot\|_P)}\big\}$.

Proof: Let $t_0 \in \mathbb{N}_0 T$ be arbitrary such that $v[t_0] \neq r$. Since $\mu(A, \|\cdot\|_P) < 0$, from Proposition 2(ii), if $v(t) = v[t_0]$ for a sufficiently long time period, then there must be $t' \ge t_0 + \tau_1 + \tau_2$ such that $\|x(t' - \tau_2) - x_e[t_0]\|_P \le -\frac{\Omega}{\mu(A, \|\cdot\|_P)} + \delta \le \text{dist}(x_e[t_0], X^C)$. This ensures the existence of a sample time instant $t_1 \in \mathbb{N}_0 T$, $t_0 \le t_1 \le t'$, such that the condition (22c) is satisfied at t_1.

Given that $\mu(A, \|\cdot\|_P) < 0$, it follows that

$$-\mu(A, \|\cdot\|_P)\,\text{dist}(x_e[t_0], X^C) - \Omega \\ \ge -\mu(A, \|\cdot\|_P)\left(-\frac{\Omega}{\mu(A, \|\cdot\|_P)} + \delta \right) - \Omega = -\mu(A, \|\cdot\|_P)\delta > 0. \tag{24}$$

Hence, there exists $\rho > \rho[t_0]$ such that

$$\|B(\mathscr{P}(\rho) - \mathscr{P}(\rho[t_0]))\|_P = \|B(\mathscr{P}(\rho) - v[t_0])\|_P \\ \le -\mu(A, \|\cdot\|_P)\,\text{dist}(x_e[t_0], X^C) - \Omega. \tag{25}$$

This ensures that the solution to (22d) at the sample time instant t_1 satisfies $\rho^*[t_1] > \rho[t_0]$. Note that $\rho[t_0] \neq 1$ since we have assumed $v[t_0] \neq r$. Furthermore, by the uniform continuity of $\mathscr{P}$, there exists $\varepsilon = \varepsilon(\delta) > 0$ such that for any ρ with $|\rho - \rho[t_0]| \le \varepsilon$, we have

$$\|B(\mathscr{P}(\rho) - \mathscr{P}(\rho[t_0]))\|_P \le -\mu(A, \|\cdot\|_P)\delta \\ \le -\mu(A, \|\cdot\|_P)\,\text{dist}(x_e[t_0], X^C) - \Omega. \tag{26}$$

This ensures that the solution to (22d) at t_1 must satisfy $\rho^*[t_1] \ge \min(\rho[t_0] + \varepsilon, 1)$. Note that ε here is a constant determined by δ, independent of t_0 or t_1.

Therefore, by combining the above results we have shown that for any $t_0 \in \mathbb{N}_0 T$ with $v[t_0] \neq r$, there exists $t_1 \in \mathbb{N}_0 T$ such that t_1 is the first sample time instant after t_0 where v is updated from $v[t_0] = \mathscr{P}(\rho[t_0])$ to $v[t_1] = \mathscr{P}(\rho[t_1])$ and, in particular, $\rho[t_1]$ satisfies $\rho[t_1] \ge \min(\rho[t_0] + \varepsilon, 1)$. This

[2] If the right-hand side of (22c) does not hold true and consequently $\chi[t_1] = 0$, then according to (22a)-(22b), $v(t_1) = \mathscr{P}(\rho[t_1]) = \mathscr{P}(\rho[t_0]) = v[t_0] = \lim_{t \to t_1^-} v(t)$, which violates the definition of t_1.

ensures that 1) starting from $v[0] = \mathscr{P}(0)$, the piecewise constant signal v must converge to $r = \mathscr{P}(1)$ after a finite number of jumps, and 2) the time periods between consecutive jumps are of finite length. This proves (i).

Then, (ii) follows from (i) and Proposition 2(ii). In particular, the convergence is exponential, which follows from the expression (35). ■

IV. Examples

In this section, we use examples to illustrate our proposed ERG-LN and compare it to relevant schemes, including the prediction-based RG and the traditional ERG.

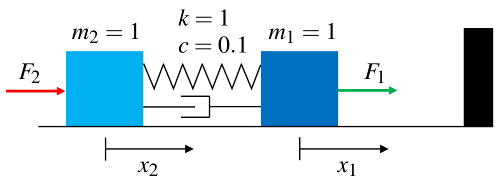

Fig. 2: Two-mass system.

A. Example 1

Consider the two-mass system shown in Fig. 2. The dynamics are represented by the following differential equation:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k}{m_1} & \frac{k}{m_1} & -\frac{c}{m_1} & \frac{c}{m_1} \\ \frac{k}{m_2} & -\frac{k}{m_2} & \frac{c}{m_2} & -\frac{c}{m_2} \end{bmatrix}}_{=:A_o} \begin{bmatrix} x_1 \\ x_2 \\ \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{m_1} & 0 \\ 0 & \frac{1}{m_2} \end{bmatrix}}_{=:[B_o,\, D]} \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}, \tag{27}$$

where the force F_1 is a controlled input and the force F_2 is an uncontrolled disturbance input, which takes values in the range $W = [-w_{\max}, w_{\max}]$. We assume the following feedback control law for F_1 has been designed to stabilize the system,

$$F_1 = K\,[x_1 - r,\, x_2 - r,\, \dot{x}_1,\, \dot{x}_2]^\top, \tag{28}$$

in which r represents a desired position deviation from the nominal for the two masses m_1 and m_2 to track, and K is the LQR gain $K = -R^{-1}B_o^\top P$ where P is the positive-definite solution to the continuous-time algebraic Riccati equation,

$$A_o^\top P + PA_o - PB_oR^{-1}B_o^\top P + Q = 0, \tag{29}$$

with $Q = \mathrm{diag}(1, 1, 0.25, 0.25)$ and $R = 1$. We assume that the following collision-avoidance constraints,

$$x_1(t) \leq 2, \qquad x_1(t) - x_2(t) \geq -0.5, \tag{30}$$

must be satisfied. For enforcing such constraints, we apply the ERG-LN, which replaces the desired position r in (28) with a modified reference signal v.

In this example, we assume the existence of input and measurement delays of $\tau_1 = \tau_2 = 0.1$. To account for these delays, we consider a sampling period of $T = \tau_1 + \tau_2 = 0.2$. It is well-known that the function $V(x) = x^\top Px$ with P obtained from (29) is a Lyapunov function for the LQR closed-loop system. According to Proposition 1, we use this P to define the logarithmic norm $\mu(\cdot, \|\cdot\|_P)$ used in ERG-LN. Moreover, since the constraint set X defined by (30) is convex, we consider a linear path $\mathscr{P}$ connecting the initial reference $v(0) = 0$ and r. Note that in this case Assumption 5 holds true and (22d) admits the analytical solution (23).

We consider the initial condition $x(0) = [0, 0, 0, 0]^\top$ and the desired reference $r = 1.98$. The steady state corresponding to r is $x_e(r) = [1.98, 1.98, 0, 0]^\top$, which is located near the boundary of the constraint $x_1(t) \leq 2$. We consider two cases for the disturbance input F_2, $w_{\max} = 0$ (disturbance-free) and $w_{\max} = 1 \times 10^{-2}$. The simulation results of using ERG-LN to govern the reference signal v versus directly setting $v \equiv r$ are shown in Fig. 3. It can be seen that without modification to the reference signal r, i.e., $v \equiv r$, the system response violates both constraints of (30). In contrast, when using ERG-LN to govern v, the constraints are strictly satisfied, which verifies our constraint enforcement result in Proposition 5. For $w_{\max} = 0$, $v(t)$ converges to r at the time instant $t^* = 22.8$ (indicated by the dotted vertical line in Fig. 3(a)). This observation verifies our result in Proposition 6 on finite-time convergence of v to r. For $w_{\max} = 1 \times 10^{-2}$, $v(t)$ converges to a point whose corresponding steady state is kept away from the constraint boundary by a safety margin. Such a safety margin is kept to guarantee that even under the worst-case disturbance trajectory, constraint satisfaction is ensured.

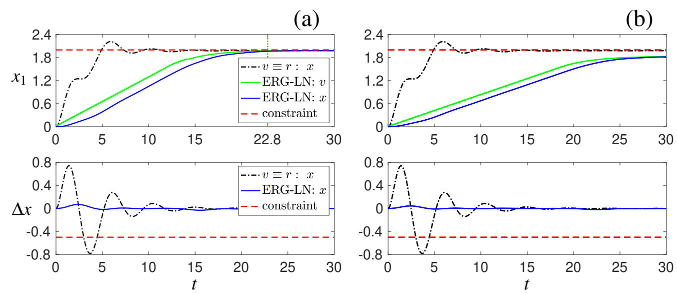

Fig. 3: ERG-LN response versus unconstrained response for (a) $w_{\max} = 0$ and (b) $w_{\max} = 1 \times 10^{-2}$.

As discussed in Section I, the newly proposed ERG-LN and traditional ERG are both closed-form RG schemes, but they have several distinguishing features. To illustrate their differences, we also implement the traditional ERG to the above example. We consider both a continuous-time implementation of ERG (ERG-c) and a digital/discrete-time implementation of ERG (ERG-d) with a sampling period $T = 0.2$. In our ERG implementations, the *dynamic safety margin* used by ERG to enforce constraints [13] is defined based on the same Lyapunov function $V(x) = x^\top Px$ used by our ERG-LN to define the logarithmic norm $\mu(\cdot, \|\cdot\|_P)$. The parameters κ and η [13] have been tuned to achieve a balanced performance between response speed and steady-state oscillation (see Fig. 4). We note that our ERG-d implementation here does not include the extra algorithm based on one-step prediction introduced in [13] for guaranteeing constraint satisfaction for all times as well as reference convergence, which would involve extra computing tasks. This is done to reveal potential issues when a continuous-time scheme is implemented digitally without corresponding modifications, and thereby illustrates the advantage of our ERG-LN, which is formulated for use in a digital micro-controller from the beginning but guarantees both sample-time and inter-sample constraint satisfaction. As a benchmark, we also implement a prediction-based RG [6]. A prediction-based RG operates based on predictions of the system response to reference values and online optimization to determine the most aggressive constraint admissible reference adjustment. As a result, it typically achieves faster responses than optimization-free schemes such as the ERG-LN and the ERG, at the cost of higher online computational footprint.

For the case of $w_{\max} = 0$, the reference v and state x responses using ERG-LN, ERG-c, ERG-d, and prediction-based RG (referred to simply as RG in the figure) are plotted in Fig. 4. It can be seen that the prediction-based RG leads to the fastest response, and the response speeds of the other three schemes are comparable. The prediction-based RG, the ERG-LN with sampling period $T = 0.2$, and the ERG-c all successfully drive the reference signal $v(t)$ to converge to the desired reference $r = 1.98$ and drive the state $x_1(t)$ to converge to its desired steady-state value of 1.98 without violating the constraint $x_1(t) \le 2$ during transience. However, it can be observed in Fig. 4(b) that when the ERG-d with sampling period $T = 0.2$ is used, constraint violation occurs around $t = 20$ and the terminal-phase trajectory of $x_1(t)$ has some oscillation around its desired steady-state value of 1.98. Such behaviors can be understood with the help of Fig. 4(a): When being close to $r = 1.98$, the reference signal $v(t)$ generated by ERG-d exhibits chattering behavior and fails to converge to $r = 1.98$, which are related to overshoots in the forward Euler approximation $v(t+T) = v(t) + T\dot{v}(t)$ of the continuous-time equation governing $v(t)$ used in ERG-c [13]. Such chattering behavior causes the error between $x_1(t)$ and its desired steady-state value of 1.98 and also contributes to the constraint violation.

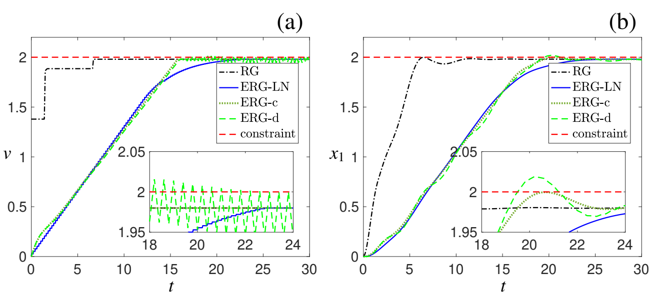

Fig. 4: (a) Reference responses and (b) state responses of prediction-based RG, ERG-LN, ERG-c, and ERG-d.

Note also that unlike our ERG-LN, the traditional ERG-c, ERG-d, and prediction-based RG do not account for delays, meaning that they may not have strict constraint enforcement guarantee when delays are present. In particular, although for $\tau_1 = \tau_2 = 0.1$ no constraint violations are observed when ERG-c and prediction-based RG are used, when we enlarge the delay values to $\tau_1 = \tau_2 = 0.5$, we have observed constraint violation occurrence with the prediction-based RG. Although RG schemes that account for input delays have been proposed in [15]–[17], those schemes rely either on knowledge of the exact delay values [15], [16] or on robust optimization to ensure constraint to be satisfied for all possible delay values [17]. In contrast, the proposed ERG-LN scheme handles uncertain input delays in a straightforward way by using a sufficiently large update period T. Furthermore, our ERG-LN can also handle uncertain state measurement delays, which are not addressed by [15]–[17]. In this regard, the proposed ERG-LN scheme may be particularly attractive to practitioners who want to achieve guaranteed constraint enforcement without involving complicated algorithms.

B. Example 2

The second example we consider is motivated by the motion planning and control problems for small-scale robots [8]. These robots typically have very limited on-board computing capability due to their small sizes. Therefore, closed-form/explicit solutions that involve minimum computational effort are appealing to them. We consider a 2D omni-directional robot with the following integrator-type dynamics

$$\dot{s}(t) = v(t), \qquad \dot{v}(t) = a(t), \qquad \dot{a}(t) = u(t), \tag{31}$$

where $s(t)$, $v(t)$, and $a(t) \in \mathbb{R}^2$ represent its position, velocity, and acceleration on the ground, respectively, and the time derivative of acceleration (jerk), $u(t)$, is the control input.

Suppose the task for the robot is to navigate through the maze shown in Fig. 5(a) from the starting point s_0 to the target point s_t (marked by the green and red points, respectively) without colliding with the walls (the black box and grey shaded areas). We first stabilize the system (31) using the feedback control $u(t) = K[s(t) - s_r, v(t), a(t)]^\top$, in which $s_r \in \mathbb{R}^2$ represents the desired position for the robot to track, and K is the LQR gain corresponding to $Q = \text{diag}(500, 500, 1, 1, 10, 10)$ and $R = \text{diag}(1, 1)$. We assume that a piecewise linear path $\mathscr{P}$, shown by the blue dotted curve in Fig. 5(a), has been planned for the robot. We decompose the task of tracking the endpoint s_t of the entire path into several sub-tasks. Each sub-task corresponds to a linear piece of the path, with the endpoint s_r^j (marked by the orange points) as the immediate desired position to track.

For enforcing the collision-avoidance constraints, we apply the proposed ERG-LN scheme, which replaces the desired position s_r^j with a modified reference signal v during each sub-task. Since the path corresponding to each sub-task is linear, the explicit solution to (22d) is given by (23). Furthermore, to simplify the computation/estimation of the distance to constraint boundary (13), we consider a box constraint set for each sub-task, represented by the light blue or green shaded rectangles in Fig. 5(a), so that (13) can be computed through the explicit expressions in (14).

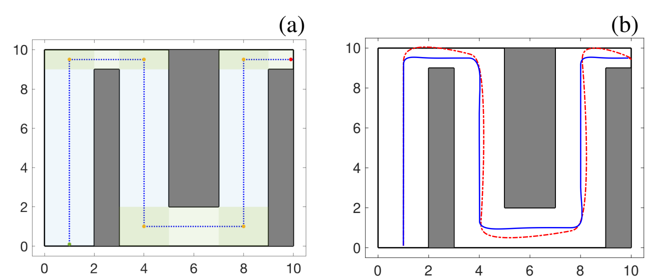

Fig. 5: Omni-directional robot constrained navigation.

The position trajectory of the robot using ERG-LN is shown by the blue curve in Fig. 5(b). For comparison, we also consider a naive navigation strategy, with which the robot directly tracks the endpoint of each sub-path, s_r^j, during each sub-task and switches the endpoint for tracking from s_r^j to the next one s_r^{j+1} when the robot position $s(t)$ satisfies $\|s(t) - s_r^j\| \le 0.5$. The position trajectory corresponding to this naive strategy is shown by the red curve in Fig. 5(b).

It can be seen that the robot using this naive navigation strategy collides with the walls multiple times, while the robot using ERG-LN reaches s_t safely. A cost of guaranteed safety using ERG-LN is slower response speed – it takes the robot only $10\,[\mathrm{s}]$ to get to s_t using the naive strategy and about $30\,[\mathrm{s}]$ using ERG-LN. Approaches to reducing ERG-LN conservativeness will be investigated in our future work.

V. Conclusions

This paper proposed a novel RG scheme, referred to as ERG-LN, for pre-stabilized linear sampled-data systems with uncertain input and/or measurement delays to satisfy pointwise-in-time constraints. We established its theoretical properties, including guaranteed sample-time and inter-sample constraint enforcement as well as guaranteed finite-time convergence of the reference signal to constant, strictly steady-state constraint admissible desired reference in the presence of uncertain delays. Due to its closed-form expression, the proposed ERG-LN scheme operates with minimum computational footprint and is thereby suitable for systems with limited computing capability. The operation and properties of ERG-LN, as well as its potential for practical applications, were illustrated in simulation examples.

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Appendix

A. Proof of Proposition 1

For any t, Taylor expansion of $V((I_n+hF)\theta(t))$ about $\theta(t)$ to the first order yields $V((I_n+hF)\theta(t)) = V(\theta(t)) + h\frac{\partial V}{\partial \theta}(\theta(t))F\theta(t) + O(h^2)$, where $O(h^2)$ denotes higher-order terms. Since $V(\theta)$ is a Lyapunov function proving the asymptotic stability of $\theta = 0$ for the system $\dot{\theta}(t) = F\theta(t)$, we have $\dot{V}(\theta(t)) = \frac{\partial V}{\partial \theta}(\theta(t))F\theta(t) < 0$. Then, for sufficiently small $h > 0$, we have $h\frac{\partial V}{\partial \theta}(\theta(t))F\theta(t) + O(h^2) < 0$, which yields $V\big((I_n+hF)\theta(t)\big) < V(\theta(t))$. Applying (3) yields

$$\mu(F,\|\cdot\|_P) = \lim_{h\to 0^+} \frac{\max_{\|\theta(t)\|_P=1} \|(I_n+hF)\theta(t)\|_P - 1}{h} \tag{32}$$
$$= \lim_{h\to 0^+} \max_{\|\theta(t)\|_P=1} \frac{\sqrt{V((I_n+hF)\theta(t))} - \sqrt{V(\theta(t))}}{h} < 0. \quad \blacksquare$$

B. Proof of Proposition 2

Firstly, we have

$$\begin{aligned} D_t^+\|\theta(t)\| &= \limsup_{h\to 0^+} \frac{\|\theta(t) + h\dot{\theta}(t) + O(h^2)\| - \|\theta(t)\|}{h} \\ &= \limsup_{h\to 0^+} \frac{\|\theta(t) + h(F\theta(t)+\gamma(t)) + O(h^2)\| - \|\theta(t)\|}{h} \\ &\le \limsup_{h\to 0^+} \left(\frac{\|I_n+hF\|-1}{h}\|\theta(t)\| + \|\gamma(t)\| + \frac{\|O(h^2)\|}{h}\right) \\ &= \lim_{h\to 0^+} \frac{\|I_n+hF\|-1}{h}\|\theta(t)\| + \|\gamma(t)\| = \mu(F,\|\cdot\|)\|\theta(t)\| + \|\gamma(t)\|. \end{aligned} \tag{33}$$

This proves (i). Using the Bellman-Gronwall inequality [23], (33) also yields the following bound on the solution $\theta(t)$,

$$\|\theta(t)\| \le e^{\mu(F,\|\cdot\|)t}\|\theta(0)\| + \int_0^t e^{\mu(F,\|\cdot\|)(t-s)}\|\gamma(s)\|\,ds. \tag{34}$$

If $\|\gamma(t)\| \le \gamma_{\max}$ for almost all $t \in [0,\infty)$, then we have

$$\begin{aligned} \|\theta(t)\| &\le e^{\mu(F,\|\cdot\|)t}\|\theta(0)\| + \int_0^t e^{\mu(F,\|\cdot\|)(t-s)}\,ds\,\gamma_{\max} \\ &= e^{\mu(F,\|\cdot\|)t}\|\theta(0)\| + \frac{e^{\mu(F,\|\cdot\|)t}-1}{\mu(F,\|\cdot\|)}\gamma_{\max}. \end{aligned} \tag{35}$$

If $\mu(F,\|\cdot\|) < 0$, then we further have $\lim_{t\to\infty} e^{\mu(F,\|\cdot\|)t}\ \|\theta(0)\| = 0$ and $\frac{e^{\mu(F,\|\cdot\|)t}-1}{\mu(F,\|\cdot\|)}\gamma_{\max} \le -\frac{\gamma_{\max}}{\mu(F,\|\cdot\|)}$. This proves (ii). Finally, if $\|\theta(0)\| \le -\frac{\gamma_{\max}}{\mu(F,\|\cdot\|)}$, then (35) yields

$$\|\theta(t)\| \le -e^{\mu(F,\|\cdot\|)t}\frac{\gamma_{\max}}{\mu(F,\|\cdot\|)} + \frac{e^{\mu(F,\|\cdot\|)t}-1}{\mu(F,\|\cdot\|)}\gamma_{\max} = -\frac{\gamma_{\max}}{\mu(F,\|\cdot\|)}. \tag{36}$$

This proves (iii). $\blacksquare$