First-Order General-Relativistic Viscous Fluid Dynamics

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We present the first generalization of Navier-Stokes theory to relativity that satisfies all of the following properties: (a) the system coupled to Einstein's equations is causal and strongly hyperbolic; (b) equilibrium states are stable; (c) all leading dissipative contributions are present, i.e., shear viscosity, bulk viscosity, and thermal conductivity; (d) nonzero baryon number is included; (e) entropy production is non-negative in the regime of validity of the theory; (f) all of the above hold in the nonlinear regime without any simplifying symmetry assumptions. These properties are accomplished using a generalization of Eckart's theory containing only the hydrodynamic variables, so that no new extended degrees of freedom are needed as in Müller-Israel-Stewart theories. Property (b), in particular, follows from a more general result that we also establish, namely, sufficient conditions that when added to stability in the fluid's rest frame imply stability in any reference frame obtained via a Lorentz transformation All of our results are mathematically rigorously established. The framework presented here provides the starting point for systematic investigations of general-relativistic viscous phenomena in neutron star mergers.

Subject Areas: Fluid Dynamics, Gravitation

I. INTRODUCTION

Relativistic fluid dynamics has been successfully used as an effective description of long wavelength, long time phenomena in a multitude of different physical systems, ranging from cosmology [1] to astrophysics [2] and also high-energy nuclear physics [3]. In the latter, relativistic viscous fluid dynamics has played an essential role in the description of the dynamical evolution of the quark-gluon plasma formed in ultrarelativistic heavy-ion collisions [4] and also in the quantitative extraction of its transport properties (see, for instance, Ref. [5]). More recently, with the observation of binary neutron star mergers [6–8], the modeling of the different dynamical stages experienced by the hot and dense matter formed in these collisions requires extending of our current understanding of viscous fluids toward the strong gravity regime where general relativistic effects are important (see, e.g., Refs. [9–14]).

The ubiquitousness of fluid dynamics stems from the existence of general conservation laws (such as energy, momentum, and baryon number) and their consequences to systems where there is a large separation of scales, such that the macroscopic behavior of conserved quantities can be understood without precise knowledge of all the details that govern the system's underlying microscopic properties [15]. Ideal fluid dynamics is the extreme situation where dissipative effects are neglected and the theory's basic properties in this limit are reasonably well understood, both in a fixed background as well as when coupling to Einstein's equations is taken into account [2,16,17]. We remark that because all sources of dissipation relevant for our discussion stem from bulk viscosity, shear viscosity, and heat conduction, and following standard practice in the field [3], we will use the terms viscous fluid and dissipative fluid interchangeably. In particular, other sources of dissipation, such as anomalous dissipation [18,19], will not be discussed.

When dissipative effects are taken into account, the behavior of fluids is far less understood (unless stated otherwise, fluids, hydrodynamics, etc., henceforth mean relativistic fluids, relativistic hydrodynamics, etc.), despite the importance of viscous dissipation in cutting-edge scientific experiments such as in studies of the quark-gluon plasma or their expected relevance for neutron star mergers, as mentioned above. Historically, a stumbling block has been the difficulty of modeling dissipative phenomena while

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preserving *causality*. Causality is a central postulate in special and general relativity, stating that the speed which information can propagate in any system cannot be larger than the speed of light [20]. This implies that a solution to the equations of motion at a given spacetime point x is completely determined by the spacetime region that is in the past of and causally connected to x [20–22]. Of course, this property must hold in relativity regardless of whether dissipation is present or not [20]. While causality is typically not an issue for most matter models under reasonable assumptions [21], including the case of ideal fluids [23], ensuring causality of fluid theories in the presence of dissipation turned out to be a major challenge [2].

The challenges one encounters when modeling fluids with dissipation, however, are not restricted to enforcing causality. Another hallmark property of dissipative fluids is *stability*. By this we mean that perturbations of a system that is in thermodynamic equilibrium should decay in time. This expresses the basic intuition that if dissipation is present, the system will dissipate energy and, consequently, small deviations from equilibrium will be damped, leading the dynamics to return to equilibrium within some characteristic timescale. Naturally, in order to implement this idea in a given formalism one needs to specify what is meant by equilibrium and perturbations. We will consider homogeneous (nonrotating) equilibrium states and our perturbations will be plane-wave solutions to the equations of motion linearized about such homogeneous states. Although this is not the most general definition of equilibrium [24], it captures the most basic intuition about how deviations from equilibrium should behave in a dissipative theory and, consequently, in practice this has been the definition most often used in the literature [25,26]. Like causality, stability is a property that is difficult to incorporate in theories of relativistic fluids with dissipation.

Aside from causality and stability, a third fundamental property required for a theory of relativistic viscous fluids is that the equations of motion be *locally well posed*. This means that given initial conditions, there must exist one and only one solution to the equations of motion taking the prescribed initial conditions [27] and defined for some time [28]. Physically, this means that the system has a well-defined evolution determined by the initial conditions. Like causality, local well posedness is a property required of any field theory [2,20–22], but we emphasize it here since, also like causality, this is a property that is difficult to achieve in theories of fluids with dissipation.

Needless to say, there is little use for a theory of fluids that is causal, stable, and locally well posed if it is not able to make connections with real physical phenomena. Thus, a theory of relativistic viscous fluids must in addition be suitable for empirical studies. This means, at the least, that the theory must agree with well-established physical facts, but also that one needs to be able to extract quantitative predictions from such a theory.

The interplay between theory and experiment is, of course, at the heart of physics. In the context of relativistic fluid dynamics, such interplay has been heavily guided by complex numerical simulations [3]. Moreover, it is clear that simulations will continue to be at the center of developments in the field, particularly when it comes to the investigations of viscous effects in neutron star mergers. In this regard, while there is no one-size-fits-all approach for implementing numerical simulations of general relativistic systems [2,29], in the numerical general relativity community one concept that has been very important for the construction of numerical algorithms is that of *strong hyperbolicity* [30]. This means that the principal part of the equations of motion can be diagonalized; see Sec. V for details. Although a discussion of the role of strong hyperbolicity in general relativistic numerical simulations is beyond the scope of this work (the reader can consult the above references for details), we stress that strong hyperbolicity is a highly desirable feature for numerical studies of general relativistic systems (see also Ref. [31] for more discussion on potential caveats of numerical simulations).

In summary, a physically meaningful theory of relativistic viscous fluids must be (I) causal, (II) stable, and (III) locally well posed. In addition, it is highly desirable to have a theory that is (IV) strong hyperbolic.

While property II is, by definition, concerned with the equations linearized about equilibrium in Minkowski background, we emphasize that whenever referring to causality, local well posedness, and strong hyperbolicity, i.e., properties I, III, and IV, we are always talking about the equations of motion in the *full nonlinear regime*. It is important to stress this point because a substantial body of theoretical work in relativistic viscous fluids is restricted to analyzing the equations linearized about equilibrium and, thus, the corresponding claims about causality, local well posedness, etc., are restricted to this particular, linearized-about-equilibrium case (see Sec. II B). Furthermore, for applications in general relativity (in particular the study of viscous effects in neutron star mergers), one is interested in the case where properties I–IV hold with dynamical coupling to Eintein's equations (again, with exception of property II).

At this point, we should stress that when we say that a theory is causal, stable, etc., we do not mean it unconditionally, but rather under a specific set of assumptions. Obviously, one is interested in cases where the assumptions are physically reasonable, even if they do not cover all cases of physical interest. For simplicity, however, in the remaining of this Introduction and in Sec. II, we avoid discussion of specific hypotheses. Thus, when we say that a certain theory is causal, stable, etc., we mean "causal under a specific set of assumptions," and unless stated otherwise, it will be implicitly understood that the assumptions in question are of physical interest. An exception to this will be made only later in Sec. II B, when we summarize the

extent to which different theories of viscous fluids satisfy one or more of the properties I–IV, since in this case mentioning the assumptions under which such theories fulfill some of these requirements will be important for comparison among them and also with our results. Even in this case, however, we will refer to those assumptions only at a high level (e.g., we will say that a certain property holds for nonzero shear viscosity but without specifying the precise range of nonzero values that is in fact required for the result to hold). We believe that this will suffice to give the reader a panoramic view of the state of affairs in the field. All the precise assumptions for the results that will be discussed can be found in the references we provide or, in the case of the results of this paper, in the remainder of the text.

The goal of this work is to provide the first example of a theory of relativistic viscous fluids that simultaneously satisfies all the properties I–IV. All our results are mathematically rigorous, hold with or without dynamical coupling to Einstein's equations, are valid in the full nonlinear regime, and do not make any symmetry or simplifying assumptions. We establish these results without the need for additional (extended) variables (see Sec. II B for details).

Section II provides a more or less self-contained exposition of our results and how they fit within studies of relativistic fluids with viscosity. We hope that such an exposition will be helpful to readers interested in the subject here investigated but who are not necessarily specialists in all the topics covered by our methods. In order to keep our account as simple as possible, we carry out the discussion in Sec. II at a high level, writing few formulas and omitting several details, but we provide full references for interested readers. More precisely, in Sec. II A, we discuss some important concepts underlying the investigation of relativistic viscous fluids. None of the ideas discussed in Sec. II A are new, but they play a key role in our constructions. Therefore, it is convenient to revisit such ideas here. In Sec. II B, we review the state of affairs in the field regarding properties I–IV. This review is not intended to be exhaustive; rather, our goal is to provide enough context for our results. Finally, in Sec. II C, we provide a summary and discussion of our results. Specialists might skip Sec. II without compromising understanding (although some specialists might still be interested in some aspects of the discussion in Sec. II C).

Definitions.—The spacetime metric $g_{\mu\nu}$ has a mostly plus signature $(-+++)$. Greek indices run from 0 to 3, latin indices from 1 to 3. The spacetime covariant derivative is denoted as ∇_μ. We use natural units, $c = \hbar = k_B = 1$.

A. Organization of the paper

This paper is organized as follows. In Sec. II we provide an overview of our results and the context surrounding them. In Sec. III, we formulate a generalization of Navier-Stokes (NS) theory using the Bemfica-Disconzi-Noronha-Kovtun (BDNK) formalism [32–35]. In Sec. IV, we provide necessary and sufficient conditions that must be fulfilled by the parameters of the theory for causality to hold. In Sec. V, we prove that the full nonlinear system of equations in general relativity is strongly hyperbolic, the solutions are unique, and the initial-value problem is well posed in general relativity. A new theorem concerning the linear stability properties of relativistic fluids in flat spacetime is given in Sec. VI. We employ this theorem in Sec. VII to obtain conditions that ensure that the new theory presented here is stable. The rigorous mathematical proofs of Theorem I, Proposition I, Theorem II, and Theorem III are found in the Appendixes A, B, C, and D, respectively. Our conclusions and outlook can be found in Sec. VIII.

II. BACKGROUND AND DISCUSSION

A. Definition of out-of-equilibrium variables: Hydrodynamic frames

In the modern perspective, relativistic fluid dynamics is understood as an effective theory for the evolution of conserved densities, such as the energy-momentum tensor $T^{\mu\nu}$. (We could include, in this introductory part, other conserved quantities such as the baryon current J^μ and those associated with higher moments. In fact, conservation of J^μ will be implicitly understood later in the discussion of Secs. II A and II B and thereafter since we will often refer to the presence of a chemical potential. For simplicity, however, we will often refer only to $T^{\mu\nu}$ in this part, since this will suffice for the aspects we want to highlight.) To say that $T^{\mu\nu}$ is conserved means that

$$\nabla_\mu T^{\mu\nu} = 0,$$

which provides equations of motion governing the dynamics of the fluid.

The energy-momentum tensor $T^{\mu\nu}$ is understood as the expectation value of the microscopic quantum operator $\hat{T}^{\mu\nu}$, which is an observable that can be defined for any nonequilibrium state. In equilibrium, the state of the system can be parametrized by the temperature $T_{\rm eq}$, the flow velocity $u^\mu_{\rm eq}$ (observe that this is the four-velocity of the fluid, although we will often refer to it simply as the velocity; the fluid velocity is always assumed to be normalized; see Sec. II B), and the chemical potential $\mu_{\rm eq}$. One of the assumptions that forms the basis of a fluid dynamics description is that for states not very far from equilibrium, the physical observable $T^{\mu\nu} = \langle \hat{T}^{\mu\nu} \rangle$ can still be parametrized in terms of a "temperature" T, a "flow velocity" u^μ, and a "chemical potential" μ that reduce to $T_{\rm eq}$, $u^\mu_{\rm eq}$, and $\mu_{\rm eq}$ in equilibrium. We write quotation marks to emphasize the fact that the quantities T, u^μ, and μ have no first-principles microscopic definitions. Therefore, while it is useful to interpret T, u^μ, and μ as out-of-equilibrium

macroscopic temperature, velocity, and chemical potential, since they are close to T_{eq}, u^{μ}_{eq}, and μ_{eq} and reduce to the latter in equilibrium, we should ultimately understand T, u^{μ}, and μ as auxiliary variables that are used to parametrize the physical observable $T^{\mu\nu}$—the latter enjoying a first-principles, microscopic definition even when the system is out of equilibrium.

It follows that there exists an ambiguity in the definition of the out-of-equilibrium quantities T, u^{μ}, and μ, since there are different ways of parametrizing $T^{\mu\nu}$ subject to the constraint that one recovers the unambiguous parametrization in terms of T_{eq}, u^{μ}_{eq}, and μ_{eq} in equilibrium. In other words, different *out-of-equilibrium* choices of T, u^{μ}, and μ to parametrize $T^{\mu\nu}$ are allowed as long as they agree in equilibrium. This is sometimes expressed by saying that T, u^{μ}, and μ correspond to a "fictitious" temperature, flow velocity, and chemical potential [2,36].

A particular choice of parametrization of $T^{\mu\nu}$ in terms of T, u^{μ}, and μ has been historically called a choice of a *hydrodynamic frame,* or simply frame. (It is a bit unfortunate the word "frame" has also other meanings in relativity theory, e.g., reference frames related by a Lorentz transformation, frames in a tetrad formalism, null frames, or a local rest frame (LRF), etc. However, all these different meanings can be distinguished from the context.) A choice of frame is, therefore, a definition of what one means by temperature, velocity, and chemical potential out of equilibrium. Consequently, a choice of frame is always involved whenever we describe a fluid out of equilibrium in terms of temperature, velocity, and chemical potential. This is still the case even if further, extended variables are introduced (see Sec. II B for the notion of extended variables). The notion of hydrodynamic frame and how it represents a choice of out-of-equilibrium variables is discussed extensively in the literature. An incomplete list is given by Refs. [34,36–51]. References [34,48], in particular, contain a detailed discussion of the topic.

Observe that once $T^{\mu\nu}$ is cast in terms of T, u^{μ}, and μ, the energy-momentum conservation equations $\nabla_{\mu}T^{\mu\nu} = 0$ can be equivalently written as evolution equations for those quantities. We also remark that one can choose other thermodynamic quantities, e.g., the energy density or the pressure, to parametrize $T^{\mu\nu}$, and we will in fact do so later on in the paper. Of course, not all thermodynamic scalars are independent; they are connected by the first law of thermodynamics and a prescription of an equation of state [2]. Obviously, the nonuniqueness in the definition of the variables used to parametrize $T^{\mu\nu}$ out of equilibrium remains if we choose a parametrization in terms of other thermodynamic variables such as the energy density, pressure, etc.

In order to pass from this qualitative argument about the ambiguity of T, u^{μ}, and μ away from equilibrium to a more precise assessment of such ambiguity, one needs to be more specific about how one formalizes the idea that fluid dynamics arises as a long time, long wavelength limit of an underlying microscopic theory, i.e., as a description of the macroscopic dynamics of the system for small deviations from equilibrium. Such a formalization can be accomplished in the framework of the so-called *gradient expansion,* which was used a century ago by Chapman and Enskog in the derivation of fluid dynamics from the (nonrelativistic) Boltzmann equation and that has since then been adapted to the relativistic setting [38]. We remark that the gradient expansion is not the only way to formalize the idea that fluid dynamics is an effective description that emerges from a more fundamental microscopic behavior; see Sec. II B for a discussion of ideas involving the so-called moment expansion and holographic techniques. Nevertheless, the gradient expansion, while not fundamental, is a very convenient and powerful formalism based on effective field theory ideas that allows one to track how different parametrizations of $T^{\mu\nu}$ lead to different fluid descriptions.

The gradient expansion is based on the idea that one can write

$$T^{\mu\nu} = \mathcal{O}(1) + \mathcal{O}(\partial) + \mathcal{O}(\partial^2) + \cdots,$$

where $\mathcal{O}(\partial^n)$ denotes terms with n derivatives of T, u^{μ}, and μ [so, e.g., $\mathcal{O}(\partial^2)$ involves both terms of the form $\partial^2 T$ and $\partial T \partial \mu$, etc.] and $\mathcal{O}(1)$ corresponds to the terms that reduce to $T^{\mu\nu}_{eq}$, the energy-momentum tensor parametrized in terms of T_{eq}, u^{μ}_{eq}, and μ_{eq}. Schematically, this is an expansion in powers of the Knudsen number $\text{Kn} \sim \ell_{micro}\partial$, i.e., the ratio between the relevant microscopic scale ℓ_{micro} and the inverse macroscopic scale L, associated with the derivative of the hydrodynamic fields. In this sense, the gradient expansion corresponds to the well-known Knudsen number expansion used in the description of kinetic systems [38,39]. In particular, since the expansion truncated at $\mathcal{O}(1)$ corresponds to ideal hydrodynamics, viscous contributions require considering at least $\mathcal{O}(\partial)$ terms, which is consistent with the basic intuition that dissipation is a phenomenon associated with deviations from equilibrium.

In order to construct a fluid theory out of the gradient expansion, one truncates it at a certain order. This truncation necessarily defines a scale at which the effective description is supposed to be valid, with higher-order effects encoded by the terms neglected in the expansion which are considered outside the limit of validity of the truncated theory. Aside from the truncation order, one also needs to specify the *constitutive relations,* i.e., the specific form of each term $\mathcal{O}(\partial^n)$ in terms of T, u^{μ}, μ, up to the truncation order (see Secs. II B and III for examples). By specifying the truncation order and the constitutive relations, one is in fact defining what is meant by T, u^{μ}, and μ out of equilibrium; i.e., one is making a choice of hydrodynamic frame.

Different frame choices, therefore, correspond to different effective descriptions of the same truncated theory. At

this point, it seems almost unnecessary to talk about frames, and one might be tempted to simply say that one has distinct theories of fluids. The key word here, however, is *effective*. Indeed, when we consider two distinct constitutive relations truncated at a given order,

$$T^{\mu\nu} = T^{\mu\nu}(T, u^\alpha, \mu) \quad \text{and} \quad \tilde{T}^{\mu\nu} = \tilde{T}^{\mu\nu}(\tilde{T}, \tilde{u}^\alpha, \tilde{\mu}),$$

one obviously has different fluid theories: the equations of motion $\nabla_\mu T^{\mu\nu} = 0$ and $\nabla_\mu \tilde{T}^{\mu\nu} = 0$ are not the same. Consequently (upon writing these conservation laws in terms of T, u^α, μ and $\tilde{T}, \tilde{u}^\alpha, \tilde{\mu}$, respectively), the quantities T, u^α, μ and $\tilde{T}, \tilde{u}^\alpha, \tilde{\mu}$ satisfy different evolution equations and, thus, cannot represent the same definition of temperature, fluid velocity, and chemical potential. However, one needs to keep in mind that the temperature, flow velocity, and chemical potential are not fundamental quantities, whereas the energy-momentum tensor is (it does have a first-principles definition). Thus, $T^{\mu\nu}(T, u^\alpha, \mu)$ and $\tilde{T}^{\mu\nu}(\tilde{T}, \tilde{u}^\alpha, \tilde{\mu})$ differ because they represent *distinct* coarse-grained or low-energy limits of the actual, microscopically uniquely defined, energy-momentum tensor. Therefore, the language of frames signals the key fact that one is always considering one possible effective description among many.

Summarizing, there exists an intrinsic ambiguity in how one parametrizes $T^{\mu\nu}$ in terms of out-of-equilibrium temperature T, velocity u^μ, and chemical potential μ. Such ambiguity simply expresses the fact that these quantities do not have first-principles microscopic definitions away from equilibrium. What is not ambiguous away from equilibrium is the definition of $T^{\mu\nu}$. One resolves this ambiguity by choosing a definition of T, u^μ, μ. Such a choice is known as a frame choice. Different parametrizations of $T^{\mu\nu}$, therefore, correspond to different frame choices. Not all frame choices, however, are equally useful. In our work, we explore suitable definitions of temperature, flow velocity, and chemical potential to construct effective theories describing fluids that lead to sensible theories in terms of satisfying properties I–IV.

At this point, the attentive reader will probably have noticed that much of the above discussion does not depend on relativistic principles. In other words, the fact that there is no first-principles definition of out-of-equilibrium quantities such as temperature, flow velocity, and chemical potential applies to nonrelativistic theories as well. In the nonrelativistic setting, however, there exists a highly successful theory of dissipative (Newtonian) fluids, namely, the Navier-Stokes-Fourier theory. In light of its success, it is fair to say that for all practical purposes, one can take the definitions of out-of-equilibrium quantities in the Navier-Stokes-Fourier theory as the correct ones in a nonrelativistic context. Had an equivalently successful theory of relativistic viscous fluids been available (where success would in particular incorporate properties I–IV),

we could similarly take the definitions of out-of-equilibrium quantities in such a theory as the correct ones for all practical purposes. Nevertheless, as we explain in the next section, there is not, at the moment, a theory of relativistic viscous fluids that can claim such a level of success. Hence, exploring how different frame choices can lead to different fluid descriptions becomes a topic of uttermost interest (see Sec. II C).

B. Brief overview of viscous theories

The first proposal for a relativistic viscous fluid theory was done by Eckart [52] in 1940, with a closely related formulation by Landau and Lifshitz [15] in the 1950s. In these works, the authors postulated a form for the energy-momentum tensor (and also of the baryon current J^μ, but, as in the previous section, here we simplify the discussion by focusing on $T^{\mu\nu}$ only) based on ideas from thermodynamics and following a covariant generalization of the nonrelativistic Navier-Stokes-Fourier theory. For example, in Eckart's theory, one has

$$T^{\mu\nu}_{\text{Eckart}} = \varepsilon u^\mu u^\nu + (P - \zeta \nabla_\lambda u^\lambda)\Delta^{\mu\nu} + q^\mu u^\nu + q^\nu u^\mu - 2\eta\sigma^{\mu\nu},$$

where ε, T, and u^μ are the (out-of-equilibrium) energy density [53], temperature, and velocity of the fluid, with the latter normalized [54] by $u^\mu u_\mu = -1$, $\Delta^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$ is the projection onto the space orthogonal to u^μ, P is the equilibrium pressure (see below) given by an equation of state (the choice of which depends on the nature of the fluid; for example, for a conformal fluid one has $P = \frac{1}{3}\varepsilon$), ζ is the coefficient of bulk viscosity, η is the coefficient of shear viscosity, $q_\mu = -\kappa T(\Delta^\nu_\mu \nabla_\nu \ln T + u^\nu \nabla_\nu u_\mu)$ represents energy diffusion, with κ being the coefficient of heat conduction, and $\sigma^{\mu\nu} = \Delta^{\mu\nu\alpha\beta}\nabla_\alpha u_\beta$ is the shear tensor, with $\Delta^{\mu\nu}_{\alpha\beta} = \frac{1}{2}(\Delta^\mu_\alpha \Delta^\nu_\beta + \Delta^\mu_\beta \Delta^\nu_\alpha - \frac{2}{3}\Delta^{\mu\nu}\Delta_{\alpha\beta})$ (so $\Delta^{\mu\nu}_{\alpha\beta}$ projects a two-tensor on the space of two-tensors traceless and orthogonal to u^μ). In the absence of viscous effects, when $\zeta = \eta = \kappa = 0$, one recovers the energy-momentum tensor of an ideal fluid.

According to the standard physical interpretation of the energy-momentum tensor of a fluid, the fluid's total pressure is given by $\frac{1}{3}\Delta_{\mu\nu}T^{\mu\nu}$. It is convenient to write the total pressure as a sum of an "equilibrium" part, which is assumed to be given by an equation of state whose functional form follows that assigned to the fluid in the limit when viscous effects are absent, and a "nonequilibrium" part that contains explicitly the viscous contributions. In the case of $T^{\mu\nu}_{\text{Eckart}}$, the latter is given by $-\zeta\nabla_\mu u^\mu$. This term clearly illustrates the fact that only terms of first order in Knudsen number were kept in this case because ζ/P gives the relevant microscopic length scale associated with particle-number changing processes, while $\nabla_\mu u^\mu$ accounts for the inverse length scale associated with the gradient of the hydrodynamic fields.

As said, Eckart and Landau and Lifshitz were seeking a covariant version of the nonrelativistic Navier-Stokes equation compatible with thermodynamic principles, most notably, the second law of thermodynamics; i.e., their choice of $T^{\mu\nu}$ ensured that entropy production (for a suitable definition of out-of-equilibrium entropy) is non-negative. From a modern perspective, however, these theories are better understood as effective theories that arise from a gradient expansion truncated at first order and with a specific choice of hydrodynamic frame, i.e., a specific choice of constitutive relation that parametrizes the energy-momentum tensor in terms of out-of-equilibrium variables. In fact, it is possible to show that the Eckart and Landau and Lifshitz theories can be obtained from kinetic theory as an expansion in gradients truncated at first order [38]. Constraints on the coefficients that appear in such truncated series are found by imposing the second law of thermodynamics. In accordance with the notion of hydrodynamic frames, the specific choices that lead to the theories of Eckart and of Landau and Lifshitz are known in the literature as the Eckart and Landau and Lifshitz frames [2]. One can immediately see that other frame choices are possible for an energy-momentum tensor truncated at first order upon noticing that $T^{\mu\nu}_{\text{Eckart}}$ does not contain all possible terms that are linear in derivatives of T, u^μ, and μ—terms that are allowed in a truncation at first order. Theories arising from a gradient expansion truncated at first order are known as *first-order theories.* The Eckart and Landau theories are, thus, examples of first-order theories.

The Eckart and Landau and Lifshitz theories are very intuitive and natural at first sight. They correspond to immediate covariant generalizations of the nonrelativistic Navier-Stokes-Fourier theory (in fact, they recover it in the nonrelativistic limit), satisfy the second law of thermodynamics, preserve many features present in the ideal case (e.g., the energy density is recovered from the energy-momentum tensor by double contraction with the velocity), are relatively simple, and, as already said, can be derived from kinetic theory. Yet, they are remarkably at odds with fundamental physical principles in that they are known to violate causality and are unstable [25,55]. Consequently, the Eckart and Landau and Lifshitz theories cannot be taken as viable theories of relativistic viscous fluids. In fact, a large class of first-order theories, of which the theories of Eckart and of Landau and Lifshitz are particular cases, are known to be acausal and unstable [25]. One naturally wonders what are the root causes of the failures of these theories, especially when at first sight they look very intuitive. We return to this point in Sec. II C.

A different approach for the construction of relativistic viscous fluid theories was taken by Israel and Stewart in a series of works [36,37,56–58], adapting ideas developed by Müller in the nonrelativistic setting [59]. The resulting theory is referred to Israel-Stewart or Müller-Israel-Stewart (MIS) theory, or sometimes simply Israel-Stewart theory. In the MIS theory, the energy momentum takes the form

$$T^{\mu\nu}_{\text{MIS}} = \varepsilon u^\mu u^\nu + (P + \Pi)\Delta^{\mu\nu} + \mathcal{Q}^\mu u^\nu + \mathcal{Q}^\nu u^\mu + \pi^{\mu\nu}.$$

The quantities Π, $\pi^{\mu\nu}$, and $\mathcal{Q}^\mu$ represent the bulk viscosity, shear viscosity, and energy diffusion of the fluid, and are referred to as viscous fluxes. We see that $T^{\mu\nu}_{\text{Eckart}}$ corresponds to the choices where the bulk scalar $\Pi = -\zeta\nabla_\mu u^\mu$, the shear-stress tensor is given by $\pi^{\mu\nu} = -2\eta\sigma^{\mu\nu}$, and the energy diffusion reads $\mathcal{Q}^\mu = q^\mu \equiv -\kappa T(\Delta^\nu_\mu \nabla_\nu \ln T + u^\nu \nabla_\nu u_\mu)$. In the MIS theory, however, the viscous fluxes are taken to be new variables on the same par as the "ordinary" variables T, u^μ, etc., (see below). Because $\Pi, \pi^{\mu\nu}, \mathcal{Q}^\mu$ add to the number of variables, hence extending the state space, they are known as extended (thermodynamic) variables and theories that investigate extended variables are referred to as extended (thermodynamic) theories [60,61]. An important point to make (already alluded to earlier) is that one cannot dispense with a choice of hydrodynamic frame even in extended theories, since one still needs to make a definition of out-of-equilibrium temperature, flow velocity, and chemical potential.

At this point, it is convenient to make the following definition. The variables T, u^μ, μ and those derived from them via the first law of thermodynamics and a choice of equation of state are known as *hydrodynamic variables or fields.* In other words, the hydrodynamic variables are the "ordinary" fields already present in the case of an ideal fluid (although the physical interpretation of these variables is not precisely the same as in the ideal fluid case; as discussed, the meaning of, e.g., temperature is different in or out of equilibrium). In this language, we can say that the Eckart and Landau and Lifshitz theories involve only the hydrodynamic variables, whereas the MIS theory involves both hydrodynamic and extended fields. In addition, the gradient expansion is always an expansion in the hydrodynamic variables [62].

Because the MIS formalism introduces new variables in addition to the hydrodynamic fields, it also requires new equations of motion besides the standard conservation laws such as $\nabla_\mu T^{\mu\nu}_{\text{MIS}} = 0$. The desired equations are postulated to be relaxation-type equations whose precise form is chosen so that entropy production is non-negative—where the entropy current is also extended from its usual form used in ideal fluids to include the extended variables $\Pi, \pi^{\mu\nu}, \mathcal{Q}^\mu$. For example, Π satisfies

$$\tau_\Pi u^\mu \nabla_\mu \Pi + \Pi = -\zeta\nabla_\mu u^\mu - \frac{1}{2}\zeta T\Pi\nabla_\mu\left(\frac{\tau_\Pi}{\zeta T}u^\mu\right),$$

where τ_Π is a relaxation time. See, e.g., Ref. [2] for the full set of equations satisfied by $\Pi, \pi^{\mu\nu}, \mathcal{Q}^\mu$, the form of the entropy current including these fields, and the derivation of

the equations of motion from the second law of thermodynamics [63].

The MIS theory enjoys the following good properties: the equations of motion are stable, thus satisfying property II, and their linearization about equilibrium states is causal, thus satisfying property I [24,64]. Also, it can, in certain limits, be derived from kinetic theory [2,57,58].

We next discuss three other theories of great interest that employ extended variables: Denicol-Niemi-Molnar-Rischke (DNMR), resumed Baier-Romatschke-Son-Starinets-Stephanov (rBRSSS), and anisotropic hydrodynamics (AHYDRO) theories. The DNMR theory is an effective theory derived from kinetic theory via an expansion in moments [65]. The moment expansion goes back to Grad in his work on nonrelativistic fluids [66,67]. Applying this formalism to the relativistic Boltzmann equations, together with a new power-counting scheme involving Knudsen and inverse Reynolds number expansions, DNMR arrived at a set of equations for the hydrodynamic fields and a set of extended variables Π, $\pi^{\mu\nu}$, and $\mathcal{Q}^\mu$ that represent the bulk viscosity, shear viscosity, and energy diffusion, similarly to the MIS equations. Also similar to the MIS equation is the fact that the equations satisfied by the viscous fluxes in the DNMR theory are relaxation-type equations. Despite their similarities, it is important to stress that the MIS and DNMR equations are not the same.

The DNMR theory enjoys many good properties. It is stable and its linearization about equilibrium states [68] is causal [26]. When only bulk viscosity is present, the DNMR theory is causal, locally well posed, and strongly hyperbolic; these properties hold with and without dynamical coupling to Einstein's equations [71]. When all viscous fluxes are present, but chemical potential is absent, the DNMR equations have recently been shown to be causal (again, with or without coupling to Einstein's equations) [72] (see Refs. [26,73,74] for related results under symmetry assumptions). Hence, property II holds in general for the DNMR equations; properties I, III, and IV hold if shear viscosity and heat conduction are absent (with or without dynamical coupling to Einstein's equations); and property I holds with all viscous fluxes present but in the absence of chemical potential [75] (with or without dynamical coupling to Einstein's equations). Most importantly, the DNMR theory has been very successful in phenomenological studies of the quark-gluon plasma, particularly in numerical simulations of its dynamical behavior; see, e.g., Refs. [5,76].

We now move to discuss the resumed Baier-Romatschke-Son-Starinets-Stephanov theory [77]. In order to do so, we need to start with the (plain, not resumed) BRSSS theory [77]. This is an effective theory obtained from the gradient expansion truncated at second order. As such, it involves only the hydrodynamic fields, and the equations of motion were chosen in Ref. [77] to be defined in the Landau frame. This effective theory-based approach

was originally developed for conformal fluids in Ref. [77], and the same equations of motion for a conformal system were concurrently derived in Ref. [44] through the fluid-gravity correspondence, a powerful technique introduced in that work which was motivated by the holographic duality of string theory [78]. In order to address the issues with causality and stability, Baier *et al.* [77] proposed a MIS-like theory with transport coefficients that ensure its agreement with the gradient expansion at second order. In the context of Ref. [77], this approach provides a resummation of higher-order terms and the latter explains the differences found, for instance, between rBRSSS and DNMR. However, at the linearized level, this resummed BRSSS theory shares the same properties of DNMR. Furthermore, the techniques used in Ref. [72] can be adapted to establish causality for this theory in the nonlinear regime. The local well-posedness and hyperbolicity aspects of rBRSSS have not yet been established.

Because the MIS, DNMR, and rBRSSS theories share many properties, in particular, the use of extended variables that satisfy similar relaxation-type equations, and their linearizations about equilibrium agree, they are sometimes collectively referred to as Israel-Stewart or Müller-Israel-Stewart theories, Israel-Stewart-like or Müller-Israel-Stewart-like theories, or yet generalized Israel-Stewart or Müller-Israel-Stewart theories. They are sometimes also collectively referred to as second-order theories. While there is no harm in grouping these theories together in this fashion, especially if one is concerned only with their general qualitative behavior, it is important to note that when it comes to specific features, including properties I–IV, the exact form of the equations matters and, therefore, the differences among these theories become important.

The fourth extended theory we would like to briefly discuss is the anisotropic hydrodynamics theory [79–83]. The latter is, in principle, more general than most approaches as it investigates the problem of small deviations around a given anisotropic nonequilibrium state. Formally, this approach involves a resummation in both Knudsen and inverse Reynolds numbers, which may be interpreted as a generalization of DNMR's power-counting ideas [84]. The equations of motion, which are in practice derived using kinetic theory, can be approximated to give rise to a MIS-like theory. As such, causality and stability in the linearized regime follow from previous results. Nothing is known about causality in the nonlinear regime of this theory. The local well-posedness and hyperbolicity aspects of AHYDRO have not yet been established.

The above summary highlights how the use of extended variables has led to many successes in the study of relativistic viscous fluids. These accomplishments seem even more impressive when they are contrasted with the fact already mentioned that first-order theories (which do not employ extended variables) had been largely ruled out for decades due to instabilities and lack of causality [25,55].

Such successes nonetheless, it is important to keep in mind several actual or potential limitations of the extended theories discussed above, as we now discuss.

First of all, observe that none of the theories MIS, DNMR, rBRSSS, or AHYDRO is known to satisfy all the properties I–IV. To the extent that they satisfy some of these properties, this happens under *restrictive* assumptions. Indeed, in the case of the quark-gluon plasma it is abundantly clear that one needs to consider situations when all viscous fluxes are present and the chemical potential is nonzero [85] (and it is likely that this is also true in neutron star mergers [12,14]), in which case none of these theories is known to be causal and locally well posed. Moreover, while numerical simulations of the dynamics of the quark-gluon plasma based on the DNMR equations have been carried out for a long time [86–88], only recently, with the aforementioned causality results [71,72], one can determine regions in the parameter and state spaces for which causality holds or fails. When such constraints are taken into consideration, it is found that state-of-the-art numerical simulations of the quark-gluon plasma violate causality [89,90], especially at early times [90]. Although further research is required to find out the implications of such causality violations to our current understanding of those properties of the quark-gluon plasma that have been extracted from numerical simulations, such results should serve as a definite cautionary tale about running numerical simulations of relativistic viscous fluids whose causality properties are poorly understood. Furthermore, if causality violations can be a real issue in numerical simulations of the quark-gluon plasma, which are carried out in flat spacetime, the situation is even more precarious in simulations of general relativistic viscous fluids, such as in neutron star mergers. While some simulations have been implemented in this setting [11], they rely on a formulation for which the key properties I, III, and IV are not known to hold.

Another potential limitation of the extended theories discussed above is that they do not seem appropriate for describing shock waves [91–93]. This is a potentially important limitation given the preponderance of shock waves in fluid dynamics, which is aggravated by the recent discovery that solutions to MIS-like equations can become singular in finite time [94]. Additionally, MIS-like and AHYDRO theories are only expected to describe the transient regime of dilute gases as their derivation is most naturally understood within kinetic theory [36,65]. Therefore, their use in other types of systems, such as in strongly coupled relativistic fluids, is *a priori* not justified. In fact, it is known that MIS-like equations do not generally describe the complex transient regime of holographic strongly coupled gauge theories [95–97] (see Ref. [98] for the case of higher-derivative corrections). In this aspect, we anticipate that the causal and stable first-order theory developed here does not describe this transient regime

either, despite satisfying properties I–IV. However, this is not an issue *per se* given that the description of such a far-from-equilibrium state is certainly beyond the regime of applicability of first-order hydrodynamics.

Finally, MIS-like theories lack the degree of universality expected to hold in hydrodynamics as the equations of motion themselves change depending on the derivation. For instance, the equations of motion in Ref. [77] have different terms than in Ref. [65], which is explained by the different power-counting scheme employed in those works. This situation should be contrasted with theories derived from the gradient expansion: although, of course, a plethora of different effective theories can be derived in the gradient expansion formalism, these different theories can always be viewed as particular cases, obtained via different frame choices, of the most general expansion truncated at a certain order. In fact, an approach of this type is employed in this paper; see Sec. II C.

Summarizing, despite its undeniable success in advancing our understanding of relativistic viscous fluids in general, and of the quark-gluon plasma in particular, MIS-like and AHYDRO theories still face many challenges, especially when it comes to settings where general relativity is involved. Thus, it is extremely important to also consider alternative theories of relativistic viscous fluids. This is especially the case when pursuing the study of viscous effects in neutron star mergers [12,14,99,100] and, as already mentioned, it is far from clear that the MIS-like and AHYDRO approach are the correct approaches for this setting.

In view of the above, it is not surprising that researchers have explored other theories of relativistic viscous fluids than those discussed so far. A natural place to start such an investigation is the gradient expansion, and the simplest possibility that includes viscous effects is that of first-order theories, i.e., effective fluid descriptions arising as a truncation of the gradient expansion at first order. On the other hand, since, as said, large classes of first-order theories are acausal and unstable, one might naturally wonder whether such an approach would be doomed to fail. In order to answer this, it is important to understand the assumptions involved. While it is true that the acausality and instability results [25,64] cover large classes of first-order theories, these results apply only to theories that satisfy

$$u_\mu u_\nu T^{\mu\nu} = \varepsilon, \tag{1}$$

i.e., only to frame choices that preserve the relation (1). In other words, the latter means that an observer moving with the fluid always sees the energy density as if it were in equilibrium, even for states where entropy is produced. Therefore, the construction of stable and causal first-order theories remains a distinct possibility as long as one avoids constitutive relations that imply (1). First-order theories for

which (1) holds are often collectively referred to as the (relativistic) Navier-Stokes theory [65], although there is no universal agreement on the terminology [35].

The physical meaning of (1), as well as of not satisfying it, is discussed in Sec. II C. We also remark that the assumptions in Refs. [25,64] imply other special relations than (1). But here, for simplicity, we focus only on (1), since our goal is not to have a detailed discussion of the assumptions involved in those works but rather to illustrate how their conclusions apply only for a particular class of theories that employ very specific frame choices and, therefore, say nothing about first-order theories that employ other hydrodynamic frames. In other words, here the reader can take (1) as a placeholder for the class of frames that are assumed in the instability and acausality results [25,64]. Such a class of frames is far from exhaustive. Consequently, the results in Refs. [25,64] simply do not apply if different constitutive relations are used.

This motivated researchers to construct stable and causal first-order theories of viscous fluids. Important attempts in this direction go back to the first decade of this century [21,42,47,101]. The first more formal indication that causal and stable first-order theories could be constructed if the frame choice (1) is avoided is given in Refs. [102–104]. These works were also the first ones to carry out a systematic study of viscous shocks in relativistic theories, a topic that in fact seems to be one of the main goals in these references.

The first construction of a stable and causal first-order theory of viscous fluids was carried out by Bemfica *et al.* [32] for the case of conformal fluids (see also Ref. [105] for some of the mathematical details of Ref. [32]). These results hold with or without dynamical coupling to Einstein's equations. Although Ref. [32] was restricted to conformal fluids, it provided an unequivocal proof that first-order stable and causal theories are possible, provided that one avoids the frame choice (1). Soon thereafter, causal and stable first-order theories were obtained by Kovtun [34] and by Bemfica *et al.* [33] for the case of nonconformal fluids without a chemical potential [106]—although stability was obtained only with the help of a numerical investigation, so it might be more precise to say that stability was only strongly suggested and not established. The resulting first-order theory became known in the literature as the BDNK theory [35]. Its local well posedness and strong hyperbolicity was established in Refs. [107,108]. The stability and causality of the BDNK theory in the presence of a chemical potential was obtained in Ref. [35] (again, stability in this case was inferred only numerically). We also mention the closely related results [109,110]. Of course, all these results are obtained using frame choices different than (1). Perhaps not surprisingly, after these results, the community took a renewed interest in first-order theories. See, e.g., Refs. [51,98,111–119], and references therein. We remark

that choices of frames other than (1) have been studied before BDNK in Refs. [42,101,102,120], but, as said, the first construction of a stable and causal first-order theory was done in Ref. [32] in the case of a conformal fluid. We return to the BDNK theory in Sec. II C. In what follows, we continue with our brief review of viscous theories.

Another first-order theory of interest is the Lichnerowicz theory [121], introduced in the 1950s but not investigated in detail until recently (see references that follow). The Lichnerowicz theory has been shown to be causal in the (very special) case of irrotational fluids [122] by the second author of this paper (see also Ref. [123]). While irrotationality is too strong of a constraint to be useful for most physical applications, Ref. [122] is of interest because it initiated the techniques that have since then been employed to study the causality of the BDNK theory, including the techniques employed in this work. We should also mention that the Lichnerowicz theory has found some interesting applications in the study of dissipative cosmological models [124–127].

Another formalism of importance in the study of viscous theories is that of divergence-type (DT) theories [128]. In this approach, all the conserved quantities describing the dynamics of the fluid are obtained from a single generating function χ which is a function of a dynamical set of variables $\zeta_A = (\zeta, \zeta_\mu, \zeta_{\mu\nu})$ (with $\zeta_{\mu\nu}$ trace-free and symmetric) representing the degrees of freedom of the fluid. For example, in the DT approach the energy-momentum tensor is obtained as

$$T_{\mathrm{DT}}^{\mu\nu} = \frac{\partial \chi}{\partial \zeta_\mu \partial \zeta_\nu}.$$

DT theories provide a far-reaching subject with many important contributions to the physics of fluids, kinetic theory, and out-of-equilibrium phenomena. Here, we limit ourselves to discuss DT theories with respect to properties I–IV. See Refs. [2,61,92,128–131] for further discussion of DT theories and Refs. [132–134] for applications of DT theories to the quark-gluon plasma.

All information of DT theories is contained in the generating function χ. Unfortunately, there is no prescription on how to construct χ, not to speak of how to construct a generating function that leads to a theory satisfying properties I–IV. In fact, we think it would be more accurate to consider the DT approach as a general formalism instead of a precisely defined theory or set of theories. That is because radically different theories, such as Eckart's and certain types of extended theories, can be cast in divergence-type by the choice of a suitable generating function [128].

Properties I–IV have been investigated in the context of DT theories in Ref. [128]. The authors constructed a DT theory that satisfies properties I–IV for states in equilibrium, i.e., when $\zeta_A = \zeta_A|_{\mathrm{eq}}$. Next, they argued that, by continuity, these properties will also hold for ζ_A sufficiently

close to $\zeta_A|_{\text{eq}}$. However, no estimate is obtained for how close to $\zeta_A|_{\text{eq}}$ the state ζ_A needs to be. Thus, given *any* nonequilibrium state ζ_A, this continuity result does not provide any information on whether this specific system satisfies the desired properties I–IV. In particular, without a quantitative estimate on how small $\zeta_A - \zeta_A|_{\text{eq}}$ needs to be, one does not know whether the states ζ_A for which properties I–IV hold include states of physical interest. It could in principle happen that this continuity argument only guarantees the desired properties in a neighborhood of $\zeta_A|_{\text{eq}}$ that is orders of magnitude smaller than the size of any deviation from equilibrium that one typically considers in viscous fluid dynamics.

Another way of saying this is that the results in Ref. [128] are purely qualitative, not providing a quantitative assessment of their applicability to physical systems. This should be contrasted with the precise quantitative results we establish here (see Secs. IV–VI) and in the predecessor works [32,33,72], which are obtained by employing substantially more refined techniques than a general continuity argument. In Refs. [92,130–132,134], further results have been obtained, but they are all of the same qualitative nature as above, relying on precisely the same continuity argument. Thus, we believe that a fair assessment of DT theories is that they can in principle accommodate properties I–IV, but precise conditions ensuring that such properties hold—in particular, conditions that allow application to concrete physical problems—are yet unknown.

We finally briefly mention recent formulations of viscous fluids [135,136] inspired by Carter's formalism and the variational principle [112]. Such formulations address some of the properties I–IV but do not establish them in completeness.

Although the review here provided is not exhaustive, we believe that it suffices to get across the following main point, namely, despite intense work on the subject and many different proposals made in the last 80 years, one still does not have a theory of relativistic viscous fluids that incorporates all relevant viscous fluxes and chemical potential while satisfying all the properties I–IV. Constructing such a theory is the goal of the present paper.

C. Summary and discussion of our results

In this paper, we consider the BDNK theory with chemical potential and all relevant viscous fluxes, namely, bulk viscosity, shear viscosity, and heat conduction, and show that it satisfies all the properties I–IV, i.e., causality, stability, local well posedness, and strong hyperbolicity. Our results hold in the full nonlinear regime for the fluid equations in a fixed background or dynamically coupled to Einstein's equations. We work in $3 + 1$ dimensions and do not make any symmetry or simplifying assumptions. As explained in the previous section, this is the first time that a theory of relativistic viscous fluids with all these properties

is constructed. In addition, all our results are mathematically rigorous and we provide a set of precise inequalities among scalar quantities (e.g., shear and bulk viscosity) that determine the regions in parameter and state space for which properties I–IV hold. Such inequalities are useful for numerical simulations as they allow us to check, at each time step, whether conditions for causality and stability are fulfilled.

The key conceptual ingredient that allows us to establish our results is the realization that the causality and stability properties of a theory are intrinsically tied to its hydrodynamic frame. This happens because different choices affect the properties of the corresponding partial differential equations (PDEs) that describe the evolution of the fluid. In particular, we avoid the frame choice (1), which in first-order theories leads to acausality and instability. The frame choice (1) has a natural intuitive appeal; namely, it states that the energy density measured by an observer moving with the fluid (i.e., in the fluid's local rest frame), $u_\mu u_\nu T^{\mu\nu}$, can be parametrized by a single scalar that can be identified with the energy density of the fluid in equilibrium [note that (1) holds for an ideal fluid]. It is not surprising, therefore, that Eckart and Landau and Lifshitz adopted frames satisfying (1). On the other hand, such a simplicity in the definition of the hydrodynamic fields out of equilibrium, while desirable, is by no means a fundamental property. The key idea underlying the BDNK theory is that one should let the fundamental principle of causality (and also of stability and local well posedness) dictate which frame choices (i.e., parametrizations of $T^{\mu\nu}$) are allowed, rather than choose a frame based on nonfundamental principles and only then investigate properties such as causality. In passing, we note that the MIS-like theories discussed in this section also adopt (1), although, as just said, other frame choices can be made. Different frames have been recently investigated in the context of extended theories in Refs. [137,138].

The idea of exploring different frame choices to construct a first-order theory that satisfies properties I–IV is not entirely new to this work. It was, in fact, the key idea employed in the earlier versions of the BDNK theory that have been showed to satisfy those properties in some particular cases (see Sec. II B). We next explain what the new aspects of this work are, but in order to do so, we need to first review some other key ideas employed in the earlier constructions of the BDNK theory.

Since we do not want to make premature frame choices, our first step is to consider the most general frame; i.e., we write down the most general expression for $T^{\mu\nu}$ (and also J^μ in the case of the present work since we here consider nonzero chemical potential) compatible with the gradient expansion truncated at first order; see Eqs. (5) and (6) for the precise expression. By considering the most general constitutive relations compatible with the symmetries of the problem as our starting point, we are in fact applying the

basic tenets behind the construction of effective theories [48,139–141] to formulate hydrodynamics as a classical effective theory that describes the near equilibrium, long time, long wavelength behavior of many-body systems in terms of the same variables $\{T, \mu, u^\nu\}$ already present in equilibrium. For completeness, we remind the reader that an effective theory is constructed to capture the most general dynamics among low-energy degrees of freedom that is consistent with the assumed symmetries. When this procedure is done using an action principle, the action must include all possible fields consistent with the underlying symmetries up to a given operator dimension and the coefficients of this expansion can then be computed from the underlying microscopic theory. These coefficients are ultimately constrained by general physical principles such as unitarity, *CPT* (charge, parity, and time reversal) invariance, and vacuum stability. Analogously, in an effective theory formulation of relativistic viscous hydrodynamics, the equations of motion must take into account all the possible terms in the constitutive relations up to a given order in derivatives that describe deviations from equilibrium. The coefficients that appear in this expansion can then be computed from the underlying microscopic theory (using, for instance, linear response theory [48]), being ultimately constrained by general physical principles such as causality in the case of relativistic fluids [20] and also by the fact that the equilibrium state must be stable; i.e., small disturbances from equilibrium in an interacting (unitary) many-body system should decrease with time [142].

Observe that by considering the most general energy-momentum tensor at first order, we are allowing viscous corrections to the equilibrium energy density; i.e., one has

$$u_\mu u_\nu T^{\mu\nu} = \varepsilon + \partial(T, \mu).$$

[See Eq. (7) for the precise expression.] Even though this is in sharp contrast with (1), in hindsight it seems the natural thing to do. After all, it is standard to do precisely the same with the pressure, i.e., to split $\frac{1}{3}\Delta_{\mu\nu}T^{\mu\nu}$ into an "equilibrium" part and a "viscous part" (see Sec. II B) [143]. There is no reason not to follow a similar recipe for the energy density seen by a comoving observer.

We next investigate how causality constrains the constitutive relations. The idea that one should let causality determine which frames are allowed in a theory, while conceptually powerful, does not tell us how to in practice find the appropriate frames. Causality of a theory can be determined by computing its characteristics [144]. Roughly, the characteristics are hypersurfaces in spacetime that correspond to the propagation modes of a theory. For example, in the case of Einstein's equations, the characteristics are simply the light cones $g_{\mu\nu}v^\mu v^\nu = 0$. While in principle we can always compute the characteristics of a system of PDEs, in practice a brute-force calculation of the characteristics seems unattainable for a nonlinear system of

PDEs as complex as the BDNK system. In order to be able to compute the characteristics, we take a cue from the system's underlying geometric properties. Inspired by structures found in the case of ideal fluids by Disconzi and Speck in Ref. [145], which need to be recovered in the ideal limit, we look for acoustical-metric-like structures. In addition, knowing what the characteristics of the system should be in some particular limit (e.g., in the conformal case that had already been treated) is also helpful to guide the calculations. In the case treated here, in particular, we already know what needs to be recovered in the limit of zero chemical potential. Finally, physical intuition also tells us what kinds of modes of propagation should be present in the system. In a nutshell, by relying on geometrical and physical intuition and an understanding of the causal properties of the theory in some particular limits, we can have a good educated guess for what the characteristics should look like. This allows us to look for a specific factorization of the characteristic determinant that points in that direction. This is the reason why, in our calculations, we group certain terms in certain ways, leading to expressions that can be managed in the end. Naturally, a brute-force approach would not be able to anticipate how one should group and factor terms in a way that would allow an explicit determination of the characteristics.

The next step is to carry out a diagonalization of the principal part of the equations of motion in order to establish strong hyperbolicity. We are able to do so because we have a precise understanding of the system's characteristics. Even so, in order to carry out the diagonalization, we need to write the system as a system of first-order PDEs (notice that $\nabla_\mu T^{\mu\nu} = 0$ is a system of second-order PDEs because $T^{\mu\nu}$ involves up to first derivatives of the hydrodynamic fields). In doing so, there is the risk of introducing spurious characteristics. For example, in the standard linear wave equation the characteristics are the light cones. However, when one writes it as a first-order system in the standard way, the resulting system has a spurious characteristic (it corresponds, in the language of eigenvalues that can be applied to first-order systems, to a zero eigenvalue). While the presence of spurious characteristics *per se* is not an obstacle to diagonalization, the more of them there are, the more likely there will be obstacles to the diagonalization. Thus, we seek to choose as variables for our first-order system quantities that have direct physical or geometrical meaning, so that the roots of the resulting characteristic polynomial resemble as closely as possible the ones of the original system. Of course, this does not guarantee diagonalizability. We still need to carry out some work mostly technical in nature to assure that the system is diagonalizable. But mutilating the equations upon rewriting them as first order by introducing new, fake features is likely to only make the technical work harder or even insurmountable.

With diagonalization at hand, we can proceed to establish local well posedness. The basic idea is that once the

system is diagonalized, one can rely on techniques of diagonal systems of PDEs. There is a catch, though. The diagonalization of the system is at the level of the so-called principal symbol (i.e., it is a purely algebraic procedure that does not deal directly with differential operators). In order to apply it to the actual system of PDEs, one needs to introduce pseudo-differential operators, and the quasilinear nature of the equations causes further complications as we need to deal with pseudo-differential operators with limited smoothness. While there are results available in the literature for such situations (see, e.g., Ref. [146]), we have not found a result that could be directly applied to our case. Thus, Bemfica and co-workers developed the necessary tools in Refs. [107,108] with applications to the BDNK equations with zero chemical potential in mind. From these techniques and the diagonalization, local well posedness follows.

Finally, let us address stability. For this, one needs to find the roots of the polynomial determining the Fourier modes of the perturbations. More precisely, only the sign of the roots is relevant. Since the corresponding polynomial is of high order, there is little hope of determining its roots exactly, and even the analysis of the sign of the roots is very challenging. Moreover, differently than what happens to the causality analysis, geometrical intuition is not of much help here because the Fourier modes are not covariant quantities. Because of these difficulties, in previous works the stability of the BDNK equations was not determined rigorously, being obtained numerically or only in the homogeneous Lorentz boosted frame [33,35]. Because of a new result demonstrated in this paper, this limitation is eliminated, as we discuss below.

We are now ready to discuss specific novelties of the present work. While we continue to employ the ideas described above and in fact improve on them, especially with respect to some of the technical aspects that are more challenging for the complete system here considered, we want to highlight what are the truly new aspects introduced in this work. First, we are able to completely and rigorously determine the stability of the system. For this, we rely on a new stability theorem, which roughly says that stability in the fluid's local rest frame (which can in general be determined because in this case the polynomial for the modes simplifies considerably) implies stability in any Lorentz boosted frame provided that the system is causal and strong hyperbolic; see Sec. VI for the precise assumptions and statement of the theorem. The theorem thus establishes a close relationship between causality and stability. While connections between causality and stability have been discussed before, see Refs. [24,26] and references therein, these results focused on specific theories, thus making unclear whether they were due to the specific form of the equations of motion or if they were examples of a yet undiscovered connection between causality and stability as general physical principles. Our theorem, in contrast, is a general theorem that can be applied to many different systems, showing that the relationship between causality and stability runs deeper and is not a feature of specific systems. In fact, we obtain stability of the BDNK system by showing that it satisfies the assumptions of the general theorem.

Interestingly, recently, a related theorem was proven in Ref. [147], albeit using entirely different methods. The results in Ref. [147] also provide further physical intuition on the relationship between causality and stability, showing that lack of causality allows that dissipation in one Lorentz frame be viewed as "antidissipation" (i.e., dissipation running "backward in time") in another Lorentz frame. We also note the related work, Ref. [69]. Combined, our paper and the works of Refs. [69,147] provide a comprehensive picture of the relationship between causality and stability, an idea that was hinted at several times before in the literature (see above references) but that had eluded the community until now.

We now discuss strong hyperbolicity. While strong hyperbolicity has been obtained for the BDNK theory before in the absence of a chemical potential [33,107,108], the introduction of a chemical potential causes new severe difficulties and the approach used in the case without chemical potential does not seem to work. Indeed, in Refs. [33,107,108], the choice of variables to write the system as first order was based primarily on their physical interpretation. For example, the viscous correction to the equilibrium energy density was one of the variables chosen. As just said, a similar approach does not work here. While it is often a good idea to consider variables with a physical meaning, the first-order reduction we seek to establish itself does not need to carry much physical meaning, so an approach employing easily identifiable physical variables might not bear any fruit. The first-order system does carry, however, some intrinsic *geometric* properties, such as natural decompositions in the directions parallel and perpendicular to u^μ or the fact that the characteristics of the original system are preserved by the reduction to first order. Thus, a choice of geometric variables seems more appropriate. That is what we have done, considering new variables that involve several tensorial decompositions of the original variables. This has the extra advantage that several tensorial and geometric properties of the fields can be used to carry out the difficult calculations needed to diagonalize the system. Yet another advantage is that while the previous physical choice of variables was specific to the form of the BDNK equations, the geometric approach is much more general and, thus, can be adapted to other theories in that similar tensorial decompositions hold for several fluids equations. Therefore, a second novel aspect of this work is a new framework to investigate strong hyperbolicity in relativistic fluids. We remark that once the diagonalization is carried out, we can rely on the techniques developed in Refs. [107,108] to establish local well

posedness. Thus, while local well posedness is probably the most technical and mathematical aspect of our results, we were able to rely more on previous techniques than any other of the results we obtain here.

In addition, it should by no means be overlooked that, although the proof of causality provided here follows similar ideas as in our earlier work [33], the fact that we are now considering the full set of equations makes the analysis much more difficult. Thus, a third novelty of our work is a substantial improvement of the techniques previously employed to analyze causality. From our causality analysis, it follows that the characteristics of the BDNK theory are the flow lines, sound waves, the so-called second sound, corresponding to the propagation of temperature perturbations [24], and shear waves (plus heat diffusion). In addition, when coupling to Einstein's equations is considered, we find another set of characteristics corresponding to gravitational waves.

Finally, as already stressed many times, the main end product of this paper is itself a major novelty, namely, the first construction of a viscous theory containing all relevant fields and satisfying properties I–IV. We accomplish so by building and expanding on several previous ideas and also by introducing a series of novel ideas, as described above.

Having discussed the new aspects of our work, we move on to discuss how they combine with other aspects of the BDNK theory to provide a promising theoretical tool for the study of general relativistic viscous phenomena. We begin by pointing out that the BDNK theory has been shown to be derivable from kinetic theory and holographic arguments [32,33,148]. While derivation from kinetic theory by itself is not guarantee that a theory is physically meaningful since the coarse-grain procedure might introduce nonphysical features—indeed, recall that the Eckart and Landau-Lifshitz theories are derivable from kinetic theory—it is reassuring to establish this connection with a microscopic theory. As shown in Ref. [148], the derivation of BDNK theory from holography can be done in the context of the fluid-gravity correspondence [44] by carefully taking into account the presence of zero modes of the corresponding differential operators in the holographic bulk.

Next, we should point out that, contrary to MIS-like theories, the BDNK theory is capable of handling shocks. By this, we mean that Rankine-Hugoniot-type conditions can in principle be obtained for the BDNK theory simply due to the fact that the BDNK equations are written as the conservation laws $\nabla_\mu T^{\mu\nu} = 0$ and $\nabla_\mu J^\mu = 0$. Aside from this simple observation, viscous shocks have been recently studied for the BDNK theory in the case of a conformal fluid using numerical methods in Ref. [149], while mathematically rigorously properties were established in Ref. [150].

At this point, we need to explain the role of shocks in the BDNK theory. Since the BDNK theory is an effective

theory truncated at first order in the gradient expansion, it is expected to be valid when gradients are not very large, which is precisely the opposite of shocks. In order to explain what we mean by a description of shocks in the BDNK formalism, let us consider for a moment an ideal fluid. In this case, one also is assuming that gradients are small. Alternatively, one may also see this as the limit where microscospic length scales are much smaller than the length scales associated with the gradients. However, shocks are known to develop in solutions of ideal hydrodynamics, and the study of shocks is indeed an important topic within the community. To what extent such shocks are accurate depictions of the state of the physical system is a legitimate question. Nevertheless, once we have decided to study shocks in the context of ideal hydrodynamics, the formalism allows us to do so in that the equations of motion of ideal fluids can accommodate weak solutions (also known as distributional solutions) using the Rankine-Hugoniot conditions [23]. The same situation happens with BDNK: the formalism in principle allows for the study of shocks. Whether or not such solutions are physical, or accurate in the sense that the results would change significantly if the formalism was extended to second order, is an important question that is beyond the scope of our paper. However, the point we are making is that we can, in principle, study shock solutions in the BDNK theory.

In other words, while the derivation of BDNK theory rests on the assumption of small gradients, one might try to apply it to situations where in principle gradients are not small (like shocks), just like it was done before in the context of ideal fluids. Although this seems inconsistent, it is precisely what it is done when one employs the equations of ideal fluids to the study of shocks. Moreover, it is also the case that MIS-like theories are often applied to situations where gradients are not so small; see, e.g., Refs. [84,90,151–155]. It is an intriguing, almost philosophical, question why one can sometimes still obtain meaningful results in such cases, even though shocks are formally beyond the regime of validity of any known approach to viscous fluids—an important question, however, that is beyond our scope here.

We now discuss another aspect of importance in viscous theories, which is entropy production. Naturally, one needs the second law of thermodynamics to be satisfied; i.e., entropy production for physically realizable states of the system must be non-negative. Before addressing this point in the BDNK theory, however, some important points need to be highlighted. Strictly speaking, there is no universally understood expression for the entropy of a given system out of equilibrium, aside from the one given by the Boltzmann equation. Thus, while it is useful to define an out-of-equilibrium entropy (which must, of course, reduce to the definition of equilibrium entropy in the absence of dissipation), we need to keep in mind that such a definition is

not fundamental or even unique. Moreover, the requirement that entropy production be non-negative on-shell unconditionally, i.e., to all orders in gradients, is certainly too stringent. In fact, since a fluid description is an effective description, it has a certain limit of applicability. Therefore, one should require that entropy production be non-negative only within the regime of validity of the theory (which is constructed within a certain approximation scheme). This point was stressed in Ref. [102] and discussed in detail in Ref. [34]. In fact, enforcing non-negative entropy production even in the presence of any size of gradients was part of the Eckart and Landau and Lifshitz theories, but the resulting theory is unstable and acausal, as seen, showing that this requirement by itself is not guaranteed to lead to sensible theories in the context of the gradient expansion. Non-negative entropy production to all gradients is also a guiding principle in the construction of the MIS theory, but so far properties I, III, and IV remain open for it. On the other hand, the DNMR equations, that are stable, causal (in the absence of chemical potential), and are extensively used in numerical simulations of the quark-gluon plasma, do not have entropy production non-negative to all orders in Knudsen and inverse Reynolds numbers, but they should have non-negative entropy production within the limit of validity of the theory [65]. The same is true for the BDNK theory, as pointed out in Ref. [34] and shown in Sec. III A. A thorough discussion of the role of entropy in viscous theories can be found in Ref. [51].

We finally comment on the ability of the BDNK theory to describe realistic physical systems. In order to go beyond theoretical aspects and make connection with experiments, one needs to carry out realistic numerical simulations of the BDNK equations. Not surprisingly, given how recent the theory is, such investigations are at an initial stage, but the results so far have been encouraging. In Ref. [149], the authors carry out numerical simulations of the BDNK theory in $1 + 1$ dimensions in the case of a conformal fluid and compare the results with simulations of MIS (rBRSSS) equations in the same setting. They found that for small values of the coefficient of shear viscosity, BDNK and MIS provide essentially the same evolution, but their dynamics differ for larger viscosity values. Given that small viscosity is one of the main regimes of interest of both theories (higher-order corrections might become relevant in both theories if viscosity is not small), this shows that at least in this test case the BDNK theory reproduces the well-studied and considerably successful behavior of MIS theory. In addition, the BDNK theory also reproduces well-known behavior considering Bjorken [156] and Gubser [157–159] flows, including the presence of a hydrodynamic attractor [32]. Further numerical studies of BDNK theory can be found in Refs. [160,161].

We also stress the obvious point that being a causal, stable, and locally well-posed theory are themselves fundamental properties that need to be satisfied as a prerequisite for describing actual physical phenomena. Thus, while on the one hand a theory possessing these properties is only of formal interest if it is not connected to experiments, on the other hand, a theory that has some phenomenological success but violates, say, causality, cannot be taken as an accurate description of real relativistic physical phenomena. In this regard, we once more remark that, in view of the results presented in this paper, the BDNK theory is currently the only theory that satisfies the fundamental requirements I–III and the additional property IV when all viscous contributions and chemical potential are incorporated, including in the case when dynamical coupling to Einstein's equations is considered.

III. GENERALIZED NAVIER-STOKES THEORY

We consider a general-relativistic fluid described by an energy-momentum tensor $T^{\mu\nu}$ and a timelike conserved current J^{μ} associated with a global $U(1)$ charge that we take to represent baryon number. In our approach, the equations of relativistic fluid dynamics are given by the conservation laws,

$$\nabla_{\mu}J^{\mu} = 0 \quad \text{and} \quad \nabla_{\mu}T^{\mu\nu} = 0, \tag{2}$$

which are dynamically coupled to Einstein's field equations:

$$R_{\mu\nu} - \frac{R}{2}g_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{3}$$

For the sake of completeness, we begin by recalling the case of a fluid in local equilibrium [2]. In this limit, one uses the following expressions in the conservation laws:

$$T^{\mu\nu} = \varepsilon u^{\mu}u^{\nu} + P\Delta^{\mu\nu} \quad \text{and} \quad J^{\mu} = nu^{\mu}, \tag{4}$$

where ε is the equilibrium energy density, n is the equilibrium baryon density, $P = P(\varepsilon, n)$ is the thermodynamical pressure defined by the equation of state, u^{μ} is a normalized timelike vector (i.e., $u_{\mu}u^{\mu} = -1$) called the flow velocity, and $\Delta_{\mu\nu} = g_{\mu\nu} + u_{\mu}u_{\nu}$ is a projector onto the space orthogonal to u^{μ}. The thermodynamical quantities in equilibrium are connected via the first law of thermodynamics $\varepsilon + P = Ts + \mu n$, where T is the temperature, s is the equilibrium entropy density, and μ is the chemical potential associated with the conserved baryon charge. We note that $u^{\mu}\nabla_{\mu}\varepsilon = 0$ and $u^{\mu}\nabla_{\mu}n = 0$ in global equilibrium. These are much stronger constraints on the dynamical variables than in the case of local equilibrium where, e.g., only the combination $u^{\mu}\nabla_{\mu}\varepsilon + (\varepsilon + P)\nabla_{\mu}u^{\mu}$ vanishes. In local equilibrium, both $u_{\mu}T^{\mu\nu}$ and J^{ν} are proportional to u^{ν} and, thus, the flow velocity may be defined using either quantity [2].

The system of equations (2) and (3) for an ideal fluid [defined by Eq. (4)] is causal in the full nonlinear regime.

Furthermore, given suitably defined initial data for the dynamical variables, solutions for the nonlinear problem exist and are unique. The latter properties establish that the equations of motion of ideal relativistic fluid dynamics are locally well posed in general relativity [16,17].

Let us now consider the effects of dissipation. Without any loss of generality, one may decompose the current and the energy-momentum tensor in terms of an arbitrary future-directed unit timelike vector u^μ as follows [48]:

$$J^\mu = \mathcal{N}u^\mu + \mathcal{J}^\mu, \tag{5}$$

$$T^{\mu\nu} = \mathcal{E}u^\mu u^\nu + \mathcal{P}\Delta^{\mu\nu} + u^\mu \mathcal{Q}^\nu + u^\nu \mathcal{Q}^\mu + \mathcal{T}^{\mu\nu}, \tag{6}$$

where $\mathcal{N} = -u_\mu J^\mu$, $\mathcal{E} = u_\mu u_\nu T^{\mu\nu}$, and $\mathcal{P} = \Delta_{\mu\nu}T^{\mu\nu}/3$ are Lorentz scalars while the vectors $\mathcal{J}^\nu = \Delta^\nu_\mu J^\mu$, $\mathcal{Q}^\nu = -u_\mu T^{\mu\lambda}\Delta^\nu_\lambda$, and the traceless symmetric tensor $\mathcal{T}^{\mu\nu} = \Delta^{\mu\nu}_{\alpha\beta}T^{\alpha\beta}$, with $\Delta^{\mu\nu}_{\alpha\beta} = \frac{1}{2}(\Delta^\mu_\alpha \Delta^\nu_\beta + \Delta^\mu_\beta \Delta^\nu_\alpha - \frac{2}{3}\Delta^{\mu\nu}\Delta_{\alpha\beta})$, are all transverse to u_ν. Observe that this decomposition is purely algebraic and simply expresses the fact that a vector and a symmetric two-tensor can be decomposed relatively to a future-directed unit timelike vector. The physical content of the theory is prescribed by relating the several components in this decomposition to physical observables, which will then evolve [162] according to Eqs. (5) and (6).

The general decomposition in Eqs. (5) and (6) expresses $\{J^\mu, T^{\mu\nu}\}$ in terms of 17 variables $\{\mathcal{E}, \mathcal{N}, \mathcal{P}, u^\mu, \mathcal{J}^\mu, \mathcal{Q}^\mu, \mathcal{T}^{\mu\nu}\}$, and the conservation laws in Eq. (2) give five equations of motion for these variables. Therefore, additional assumptions must be made to properly define the evolution of the fluid. As mentioned before, the NS theory, including the standard approach in Refs. [15,52], assumes that $\mathcal{E} = \varepsilon$ and $\mathcal{N} = n$. The same assumption is usually made in the MIS theory [36], though different prescriptions can be easily defined in the context of kinetic theory [46,65,163]. A further constraint is usually imposed on the transverse vectors, i.e., either $\mathcal{J}^\mu = 0$ or $\mathcal{Q}^\mu = 0$ throughout the evolution. For instance, the former gives $J^\mu = nu^\mu$ and $T^{\mu\nu} = \varepsilon u^\mu u^\nu + (P + \Pi)\Delta^{\mu\nu} + u^\mu \mathcal{Q}^\nu + u^\nu \mathcal{Q}^\mu + \mathcal{T}^{\mu\nu}$, where Π is the bulk viscous pressure (in equilibrium, $\Pi = 0$, $\mathcal{Q}^\nu = 0$, and $\mathcal{T}^{\mu\nu} = 0$). In this case, in an extended variable approach such as MIS [36], Π, $\mathcal{Q}^\nu$, and $\mathcal{T}^{\mu\nu}$ obey additional equations of motion that must be specified and solved together with the conservation laws, whereas in the NS approach these quantities are expressed in terms of u^μ, ε, and its derivatives.

In this paper, we investigate the problem of viscous fluids in general relativity using the BDNK formulation of relativistic fluid dynamics. See Secs. II B and II C for a detailed discussion of the origins of the BDNK theory and the conceptual framework that it entails. As explained in those sections, the starting point in the formulation of the BDNK theory is the most general expression for the energy-momentum tensor and the baryon current at first order.

In practice, the most general expressions for the *constitutive relations* that define the quantities in Eqs. (5) and (6), truncated to first order in derivatives, are (following the notation in Ref. [34])

$$\mathcal{E} = \varepsilon + \varepsilon_1 \frac{u^\alpha \nabla_\alpha T}{T} + \varepsilon_2 \nabla_\alpha u^\alpha + \varepsilon_3 u^\alpha \nabla_\alpha(\mu/T), \tag{7a}$$

$$\mathcal{P} = P + \pi_1 \frac{u^\alpha \nabla_\alpha T}{T} + \pi_2 \nabla_\alpha u^\alpha + \pi_3 u^\alpha \nabla_\alpha(\mu/T), \tag{7b}$$

$$\mathcal{N} = n + \nu_1 \frac{u^\alpha \nabla_\alpha T}{T} + \nu_2 \nabla_\alpha u^\alpha + \nu_3 u^\alpha \nabla_\alpha(\mu/T), \tag{7c}$$

$$\mathcal{Q}^\mu = \theta_1 \frac{\Delta^{\mu\nu}\nabla_\nu T}{T} + \theta_2 u^\alpha \nabla_\alpha u^\mu + \theta_3 \Delta^{\mu\nu}\nabla_\nu(\mu/T), \tag{7d}$$

$$\mathcal{J}^\mu = \gamma_1 \frac{\Delta^{\mu\nu}\nabla_\nu T}{T} + \gamma_2 u^\alpha \nabla_\alpha u^\mu + \gamma_3 \Delta^{\mu\nu}\nabla_\nu(\mu/T), \tag{7e}$$

$$\mathcal{T}^{\mu\nu} = -2\eta\sigma^{\mu\nu}, \tag{7f}$$

where $\sigma^{\mu\nu} = \Delta^{\mu\nu\alpha\beta}\nabla_\alpha u_\beta$ is the shear tensor. The transport parameters $\{\varepsilon_i, \pi_i, \theta_i, \nu_i, \gamma_i\}$ and the shear viscosity η are functions of T and μ. Thermodynamic consistency of the equilibrium state (i.e., that ε, P, and n have the standard interpretations of equilibrium quantities connected via well-known thermodynamic relations) imposes that $\gamma_1 = \gamma_2$ and $\theta_1 = \theta_2$ [34]. The final equations of motion for $\{T, \mu, u^\alpha\}$, which are of second order in derivatives, are found by substituting the expressions above in the conservation laws. In the language of Sec. II A, expressions (7) for Eqs. (5) and (6) correspond to the most general choice of a hydrodynamic frame for a first-order theory. As stressed in Ref. [34], it is of course impossible to not choose a hydrodynamic frame since the latter actually defines the meaning of the variables $\{T, \mu, u^\mu\}$ out of equilibrium (see Sec. II A for details).

In fact, in the regime of validity of the first-order theory, one may shift $\{T, \mu, u^\mu\}$ by adding terms that are of first order in derivatives, shifting also the transport parameters $\{\varepsilon_i, \pi_i, \theta_i, \nu_i, \gamma_i\}$, without formally changing the physical content of $T^{\mu\nu}$ and J^μ [34]. However, there are combinations of the transport parameters that remain invariant under these field redefinitions. In fact, the shear viscosity η and the combination of coefficients that give the bulk viscosity ζ and charge conductivity σ are invariant under first-order field redefinitions, as explained in Ref. [34]. Additional constraints among the transport parameters appear when the underlying theory displays conformal invariance, as discussed in detail in Ref. [32] at $\mu = 0$, and at finite chemical potential in Refs. [34,35] (see also Ref. [110]).

Hoult and Kovtun [35] investigated Eq. (7) at nonzero chemical potential using a class of hydrodynamic frames where $\varepsilon_3 = \pi_3 = \theta_3 = 0$. This corresponds to the case where there are nonequilibrium corrections to both the

conserved current and the heat flux. This choice is useful when considering relativistic fluids where the net baryon density is not very large, as in high-energy heavy-ion collisions. Conditions for causality were derived and limiting cases were studied that strongly indicated that this choice of hydrodynamic frame is stable against small disturbances around equilibrium. Further studies are needed to better understand the nonlinear features of its solutions (well posedness) and also the stability properties of this class of hydrodynamic frames at nonzero baryon density in a wider class of equilibrium states.

In this paper, we consider another class of hydrodynamic frames that we believe can be more naturally implemented in simulations of the baryon-rich matter formed in neutron star mergers or in low-energy heavy-ion collisions. Our choice for the hydrodynamic frame is closer to Eckart's as we define the flow velocity using the baryon current; i.e., $J^\mu = nu^\mu$ holds throughout the evolution ($\gamma_i = \nu_i = 0$). Clearly, this limits the domain of applicability of the theory to problems where there are many more baryons than antibaryons so the net baryon charge is large.

In this case, it is more convenient to use ε and n as dynamical variables instead of T and μ/T because the most general expressions for the Lorentz scalar contributions to the constitutive relations involve only linear combinations of $u^\mu \nabla_\mu \varepsilon$ and $\nabla_\mu u^\mu$, given that current conservation implies that the replacement $u^\lambda \nabla_\lambda n = -n \nabla_\lambda u^\lambda$ is valid. For simplicity, we choose to parametrize the out-of-equilibrium corrections to the scalars as follows [we note that $\theta_1 = \theta_2$ and $\gamma_1 = \gamma_2$, and in practice, 8 out of the 14 parameters in Eq. (7) can be set using first-order field redefinitions [34], so one is then left with η, ζ, σ, and three other parameters]:

$$\mathcal{E} = \varepsilon + \tau_\varepsilon[u^\lambda \nabla_\lambda \varepsilon + (\varepsilon + P)\nabla_\lambda u^\lambda], \tag{8a}$$

$$\mathcal{P} = P - \zeta \nabla_\lambda u^\lambda + \tau_P[u^\lambda \nabla_\lambda \varepsilon + (\varepsilon + P)\nabla_\lambda u^\lambda], \tag{8b}$$

where τ_ε and τ_P have dimensions of a relaxation time and ζ is the bulk viscosity transport coefficient. When evaluated on the solutions of the equations of motion, one can see that these quantities assume their standard form as in Eckart's theory up to second order in derivatives because $\mathcal{E} \sim \varepsilon + \mathcal{O}(\partial^2)$ and $\mathcal{P} = P - \zeta \nabla_\mu u^\mu + \mathcal{O}(\partial^2)$ on shell (we follow traditional terminology where a given quantity is said to be on shell when it is evaluated using the solutions to the equations of motion).

In fact, we remind the reader that in Eckart's theory [52] the energy-momentum tensor is given by $T_{\mu\nu} = \varepsilon u_\mu u_\nu + (P - \zeta \nabla_\lambda u^\lambda)\Delta_{\mu\nu} - 2\eta\sigma_{\mu\nu} + u_\mu \mathcal{Q}_\nu + u_\nu \mathcal{Q}_\mu$, with heat flux $\mathcal{Q}_\mu = -\kappa T(u^\lambda \nabla_\lambda u_\mu + \Delta_\mu^\lambda \nabla_\lambda T/T)$, where $\kappa = (\varepsilon + P)^2 \sigma/(n^2 T)$ is the thermal conductivity coefficient. However, as remarked in Ref. [34], in the domain of validity of the first-order theory one may rewrite the Eckart expression for the heat flux as $\mathcal{Q}_\nu = \sigma T[(\varepsilon + P)/n]\Delta_\nu^\lambda \nabla_\lambda(\mu/T)$ plus second-order terms. This is done by noticing that

$(\varepsilon + P)u^\lambda \nabla_\lambda u^\mu + \Delta^{\mu\lambda} \nabla_\lambda P = 0 + \mathcal{O}(\partial^2)$ on shell, which implies that one may write, using the standard thermodynamic relation $[(dP)/(\varepsilon + P)] = [(dT)/T] + [(nT)/(\varepsilon + P)] \times d(\mu/T)$,

$$u^\lambda \nabla_\lambda u^\alpha + \frac{\Delta^{\alpha\lambda}\nabla_\lambda T}{T} = -\frac{nT}{\varepsilon + P}\Delta^{\alpha\lambda}\nabla_\lambda(\mu/T) + \mathcal{O}(\partial^2). \tag{9}$$

Therefore, one can always choose the coefficients such that the heat flux $\mathcal{Q}^\mu$ has the same physical content of Eckart's theory plus terms that are of second order on shell. We use this to write this quantity as

$$\mathcal{Q}_\nu = \sigma T \frac{(\varepsilon + P)}{n}\Delta_\nu^\lambda \nabla_\lambda(\mu/T)$$
$$+ \tau_Q[(\varepsilon + P)u^\lambda \nabla_\lambda u_\nu + \Delta_\nu^\lambda \nabla_\lambda P], \tag{10}$$

where τ_Q has dimensions of a relaxation time.

In this work, we make the following choice for the constitutive relations that give the energy-momentum tensor and the baryon current:

$$J^\mu = nu^\mu, \tag{11a}$$

$$T^{\mu\nu} = (\varepsilon + \mathcal{A})u^\mu u^\nu + (P + \Pi)\Delta^{\mu\nu} - 2\eta\sigma^{\mu\nu}$$
$$+ u^\mu \mathcal{Q}^\nu + u^\nu \mathcal{Q}^\mu, \tag{11b}$$

$$\mathcal{A} = \tau_\varepsilon[u^\lambda \nabla_\lambda \varepsilon + (\varepsilon + P)\nabla_\lambda u^\lambda], \tag{11c}$$

$$\Pi = -\zeta \nabla_\lambda u^\lambda + \tau_P[u^\lambda \nabla_\lambda \varepsilon + (\varepsilon + P)\nabla_\lambda u^\lambda], \tag{11d}$$

$$\mathcal{Q}^\nu = \tau_Q(\varepsilon + P)u^\lambda \nabla_\lambda u^\nu + \beta_\varepsilon \Delta^{\nu\lambda}\nabla_\lambda \varepsilon + \beta_n \Delta^{\nu\lambda}\nabla_\lambda n, \tag{11e}$$

where

$$\beta_\varepsilon = \tau_Q \left(\frac{\partial P}{\partial \varepsilon}\right)_n + \frac{\sigma T(\varepsilon + P)}{n}\left(\frac{\partial(\mu/T)}{\partial \varepsilon}\right)_n, \tag{12a}$$

$$\beta_n = \tau_Q \left(\frac{\partial P}{\partial n}\right)_\varepsilon + \frac{\sigma T(\varepsilon + P)}{n}\left(\frac{\partial(\mu/T)}{\partial n}\right)_\varepsilon, \tag{12b}$$

and τ_ε, τ_P, and τ_Q quantify the magnitude of second-order corrections to the out-of-equilibrium contributions to the energy-momentum tensor given by the energy density correction $\mathcal{A}$, the bulk viscous pressure Π, and the heat flux $\mathcal{Q}^\mu$. In other words, Eqs. (11) and (12) correspond to the frame we consider in this work; thus they provide a definition of what we mean by the nonequilibrium hydrodynamic fields.

The reason for considering the constitutive relations (11) and (12) is that they lead to a theory satisfying properties I–IV, as we show below. We refer the reader to Sec. II C for a discussion of the ideas and techniques that led to the particular choice of Eqs. (11) and (12).

The equations of motion for the fluid variables are obtained from the conservation laws and they can be written explicitly as

$$u^\lambda \nabla_\lambda n + n\nabla_\lambda u^\lambda = 0, \tag{13a}$$

$$
\begin{aligned}
u^\lambda \nabla_\lambda \varepsilon + (\varepsilon + P)\nabla_\lambda u^\lambda = &-u^\lambda \nabla_\lambda \mathcal{A} - (\mathcal{A} + \Pi)\nabla_\lambda u^\lambda \\
&- \nabla_\mu \mathcal{Q}^\mu - \mathcal{Q}^\mu u^\lambda \nabla_\lambda u_\mu + 2\eta\sigma_{\mu\nu}\sigma^{\mu\nu},
\end{aligned}
\tag{13b}
$$

$$
\begin{aligned}
(\varepsilon + P)&u^\nu \nabla_\nu u^\beta + \Delta^{\beta\lambda}\nabla_\lambda P \\
&= -(\mathcal{A} + \Pi)u^\nu \nabla_\nu u^\beta - \Delta^{\beta\lambda}\nabla_\lambda \Pi + \Delta_\lambda^\beta \nabla_\mu(2\eta\sigma^{\mu\lambda}) \\
&\quad - u^\lambda \nabla_\lambda \mathcal{Q}^\beta - \frac{4}{3}\nabla_\lambda u^\lambda \mathcal{Q}^\beta - \mathcal{Q}_\mu \sigma^{\mu\beta} - \mathcal{Q}_\mu \omega^{\mu\beta},
\end{aligned}
\tag{13c}
$$

where $\omega_{\mu\nu} = \frac{1}{2}(\Delta_\mu^\lambda \nabla_\lambda u_\nu - \Delta_\nu^\lambda \nabla_\lambda u_\mu)$ is the kinematic vorticity tensor [2]. The equations above show that, on shell, $\mathcal{A} \sim 0 + \mathcal{O}(\partial^2)$, $\Pi \sim -\zeta\nabla_\mu u^\mu + \mathcal{O}(\partial^2)$, and $\mathcal{Q}_\nu = \sigma T[(\varepsilon + P)/n]\Delta_\nu^\lambda \nabla_\lambda(\mu/T) + \mathcal{O}(\partial^2)$. Equations (11)–(13) define a causal and stable generalization of Eckart's theory that is fully compatible with general relativity, as we prove in the next sections. We remark that when one neglects the effects of a conserved current altogether, the theory reduces to the case studied in Refs. [33,34]. For additional discussion about the case without a chemical potential, including far-from-equilibrium behavior and also the presence of analytical solutions, see Refs. [111,116,117].

A. Entropy production

It is instructive to investigate how the second law of thermodynamics is obeyed in this general first-order approach. This was discussed in detail by Kovtun in Ref. [34] and, more recently, by other authors in Ref. [51].

The standard covariant definition of the entropy current based on the first law of thermodynamics $T\mathbb{S}^\mu = Pu^\mu - u_\nu T^{\nu\mu} - \mu J^\mu$ [36], together with Eq. (11), can be used to show that the entropy density measured by a comoving observer is given by

$$-u_\mu \mathbb{S}^\mu = s + \frac{\mathcal{A}}{T}. \tag{14}$$

Note that in our system one finds that $\mathcal{A} = 0 + \mathcal{O}(\partial^2)$ on shell. Furthermore, using Eqs. (11) and (13) one finds that the divergence of the entropy current is given by

$$
\begin{aligned}
\nabla_\mu \mathbb{S}^\mu = &\frac{2\eta\sigma_{\mu\nu}\sigma^{\mu\nu}}{T} - \frac{\Pi}{T}\nabla_\mu u^\mu + \frac{n}{\varepsilon + P}\mathcal{Q}^\nu \Delta_\nu^\lambda \nabla_\lambda(\mu/T) \\
&- \frac{\mathcal{Q}^\nu}{T}\left[u^\lambda \nabla_\lambda u_\nu + \frac{\Delta_\nu^\lambda \nabla_\lambda P}{\varepsilon + P}\right] - \frac{\mathcal{A}}{T}\frac{u^\lambda \nabla_\lambda T}{T}.
\end{aligned}
\tag{15}
$$

It is crucial to note [34] that in a first-order approach $\nabla_\mu \mathbb{S}^\mu$ can only be correctly determined up to second order in

derivatives [recall that in this argument terms such as $\nabla_\mu \nabla_\nu \phi$ and $(\nabla_\mu \phi)(\nabla_\nu \phi)$, for any field ϕ, count as second-order terms; see Sec. II A]. This means that not all the terms in Eq. (15) actually contribute to this expression at second order. For instance, when evaluating Eq. (15) *on shell* one must keep in mind that the last two terms in Eq. (15) are already at least of third order and must, thus, be dropped. A similar argument can be used to show that the term $\Pi\nabla_\mu u^\mu = -\zeta(\nabla_\mu u^\mu)^2 + \mathcal{O}(\partial^3)$. Therefore, one can see that

$$
\begin{aligned}
\nabla_\mu \mathbb{S}^\mu = &\frac{2\eta\sigma_{\mu\nu}\sigma^{\mu\nu}}{T} + \frac{\zeta(\nabla_\mu u^\mu)^2}{T} \\
&+ \sigma T[\Delta_\nu^\lambda \nabla_\lambda(\mu/T)][\Delta^{\nu\alpha}\nabla_\alpha(\mu/T)] + \mathcal{O}(\partial^3),
\end{aligned}
\tag{16}
$$

which is non-negative when $\eta, \zeta, \sigma \geq 0$. Hence, there are no violations of the second law of thermodynamics in the domain of validity of the first-order theory—higher-order derivative terms $\mathcal{O}(\partial^3)$ in the entropy production can only be understood by considering terms of higher order in derivatives in the constitutive relations in $T^{\mu\nu}$ and J^μ, which is beyond the scope of the first-order approach.

IV. CAUSALITY

In order to determine the conditions under which causality holds in this theory, we need to understand the system's characteristics. Our system is a mixed first-second-order system of PDEs. While the principal part and characteristics of systems of this form can be investigated using Leray's theory [21,105,164], here it is simpler to transform our equations into a system where all equations are of second order. We thus apply $u^\mu \nabla_\mu$ on Eq. (13a). In this case, the conservation laws (2) coupled to Einstein's equations (3) written in harmonic gauge, $g^{\mu\nu}\Gamma_{\mu\nu}^\alpha = 0$, read

$$u^\beta u^\alpha \partial_{\alpha\beta}^2 n + n\delta_\nu^\alpha u^\beta \partial_{\alpha\beta}^2 u^\nu + \tilde{\mathcal{B}}_1(n, u, g)\partial^2 g = \mathcal{B}_1(\partial n, \partial u, \partial g), \tag{17a}$$

$$
\begin{aligned}
(\tau_\varepsilon u^\alpha u^\beta &+ \beta_\varepsilon \Delta^{\alpha\beta})\partial_{\alpha\beta}^2 \varepsilon + \beta_n \Delta^{\alpha\beta}\partial_{\alpha\beta}^2 n \\
&+ \rho(\tau_\varepsilon + \tau_Q)u^{(\alpha}\delta_\nu^{\beta)}\partial_{\alpha\beta}^2 u^\nu + \tilde{\mathcal{B}}_2(\varepsilon, n, u, g)\partial^2 g \\
&= \mathcal{B}_2(\partial\varepsilon, \partial n, \partial u, \partial g),
\end{aligned}
\tag{17b}
$$

$$
\begin{aligned}
(\beta_\varepsilon + \tau_P)u^{(\alpha}\Delta^{\beta)\mu}\partial_{\alpha\beta}^2 \varepsilon &+ \beta_n u^{(\alpha}\Delta^{\beta)\mu}\partial_{\alpha\beta}^2 n + \mathcal{C}_\nu^{\mu\alpha\beta}\partial_{\alpha\beta}^2 u^\nu \\
&+ \tilde{\mathcal{B}}_3^\mu(\varepsilon, n, u, g)\partial^2 g = \mathcal{B}_3^\mu(\partial\varepsilon, \partial n, \partial u, \partial g),
\end{aligned}
\tag{17c}
$$

$$g^{\alpha\beta}\partial_{\alpha\beta}^2 g^{\mu\nu} = \mathcal{B}_4^{\mu\nu}(\partial\varepsilon, \partial n, \partial u, \partial g), \tag{17d}$$

where $\partial_{\alpha\beta}^2 = \partial_\alpha \partial_\beta$ (using standard partial derivatives), $\rho = (\varepsilon + P)$, and $A_{(\alpha}B_{\beta)} = (A_\alpha B_\beta + A_\beta B_\alpha)/2$. The remaining notation is as follows. We use $\partial^\ell \phi$ to indicate that a term depends on at most ℓ derivatives of ϕ. A term of the

form $\mathcal{B}(\partial^{\ell_1}\phi_1, \ldots, \partial^{\ell_k}\phi_k)\partial^{\ell}\phi_i$, $i \in \{1, \ldots, k\}$, indicates an expression that is linear in $\partial^{\ell}\phi_i$ with coefficients depending on at most ℓ_1 derivatives of $\phi_1, \ldots, \ell_k$ derivatives of ϕ_k. For example, the term $(u^\mu \partial_\mu \varepsilon + \partial_\mu u^\mu) g^{\alpha\beta} \partial^2_{\alpha\beta} g_{\gamma\delta}$ would be written as $\mathcal{B}(\partial\varepsilon, \partial u, g)\partial^2 g$ (a term of this form is not present in our system; we write it here only for illustration). The terms $\tilde{\mathcal{B}}$ above are top order in derivatives of g and thus belong to the principal part, although, as we will see, their explicit form is not needed for our argument, whereas the $\mathcal{B}$ terms are lower order and do not contribute to the principal part. We have also defined

$$C^{\mu\alpha\beta}_\nu = \left(\tau_P \rho - \zeta - \frac{\eta}{3}\right)\Delta^{\mu(\alpha}\delta^{\beta)}_\nu + (\rho\tau_Q u^\alpha u^\beta - \eta\Delta^{\alpha\beta})\delta^\mu_\nu. \quad (18)$$

We notice that by taking $u^\mu \nabla_\mu$ of Eq. (13a) we are not introducing new characteristics in the system. This can be viewed from the characteristic determinant computed below which contains an overall factor of $u^\mu \xi_\mu$ to a power greater than one. Theorem I below establishes necessary and sufficient conditions for causality to hold in our system of equations. We show that the assumptions of Theorem I are not empty in Sec. VII A. Throughout this paper, we use the following definition for the speed of sound c_s:

$$c^2_s = \left(\frac{\partial P}{\partial \varepsilon}\right)_{\bar{s}} = \left(\frac{\partial P}{\partial \varepsilon}\right)_n + \frac{n}{\rho}\left(\frac{\partial P}{\partial n}\right)_\varepsilon, \quad (19)$$

where $\bar{s}$ is the equilibrium entropy per particle. Also, we define

$$\kappa_s = \frac{\rho^2 T}{n}\left[\frac{\partial(\mu/T)}{\partial \varepsilon}\right]_{\bar{s}} = \frac{\rho^2 T}{n}\left[\frac{\partial(\mu/T)}{\partial \varepsilon}\right]_n + T\rho\left[\frac{\partial(\mu/T)}{\partial n}\right]_\varepsilon. \quad (20)$$

Theorem I.—Let $(\varepsilon, n, u^\mu, g_{\alpha\beta})$ be a solution to Eqs. (3) and (13), with $u^\mu u_\mu = -1$, defined in a globally hyperbolic spacetime $(M, g_{\alpha\beta})$. Assume that Assumption 1

$$\text{(A1)} \quad \rho = \varepsilon + P, \tau_\varepsilon, \tau_Q, \tau_P > 0 \quad \text{and} \quad \eta, \zeta, \sigma \geq 0.$$

Then, causality holds for $(\varepsilon, n, u^\mu, g_{\alpha\beta})$ if, and only if, the following conditions are satisfied:

$$\rho\tau_Q > \eta, \quad (21a)$$

$$\left[\tau_\varepsilon\left(\rho c^2_s \tau_Q + \zeta + \frac{4\eta}{3} + \sigma\kappa_s\right) + \rho\tau_P\tau_Q\right]^2$$
$$\geq 4\rho\tau_\varepsilon\tau_Q\left[\tau_P(\rho c^2_s \tau_Q + \sigma\kappa_s) - \beta_\varepsilon\left(\zeta + \frac{4\eta}{3}\right)\right] \geq 0, \quad (21b)$$

$$2\rho\tau_\varepsilon\tau_Q > \tau_\varepsilon\left(\rho c^2_s \tau_Q + \zeta + \frac{4\eta}{3} + \sigma\kappa_s\right) + \rho\tau_P\tau_Q \geq 0, \quad (21c)$$

$$\rho\tau_\varepsilon\tau_Q + \sigma\kappa_s\tau_P > \tau_\varepsilon\left(\rho c^2_s \tau_Q + \zeta + \frac{4\eta}{3} + \sigma\kappa_s\right)$$
$$+ \rho\tau_P\tau_Q(1 - c^2_s) + \beta_\varepsilon\left(\zeta + \frac{4\eta}{3}\right). \quad (21d)$$

The same result holds true for Eqs. (13) if the metric is not dynamical.

Proof.—The proof can be reduced to a computation of the characteristics of Eq. (17) [164]. Technical details are found in Appendix A.

V. STRONG HYPERBOLICITY AND LOCAL WELL POSEDNESS

In this section, we investigate the initial-value problem for Eqs. (3) and (13). The goal is to show that the system is causal and locally well posed under very general conditions. First, we briefly discuss the initial data required to solve the system of equations. Then, we rewrite our system as a first-order system. We show that this first-order system is diagonalizable in the sense of Proposition I. This means, in particular, that the system is *strong hyperbolic* according to the usual definition of the term, as in, e.g., Refs. [2,23]. The importance of having strongly hyperbolic equations is due to its implications for the initial-value problem. As already mentioned, one is generally interested in evolution equations that are locally well posed [165]. For equations with constant coefficients, local well posedness is equivalent to strong hyperbolicity [166]. For nonconstant coefficients and nonlinear systems, such an equivalence does not hold [167–169]. However, there remains a close connection between strong hyperbolicity and local well posedness. For most reasonable systems, once diagonalizability is available, one can use known techniques to derive energy estimates which, in turn, can be used to prove local well posedness; see Sec. II C for more discussion on the techniques involved. This is precisely the case for our system of equations. Even though our equations consist of a system of second-order PDEs, we can use the diagonalized system of first-order equations to derive energy estimates. Once these estimates are available, we use a standard approximation argument as in Refs. [17,170] to obtain local well posedness (see Theorem II).

A. Initial data

Equations (13) are second order in ε, n, and u^μ. Thus, initial data along a noncharacteristic hypersurface consist of the values of ε, n, u^μ and their first-order time derivatives. Clearly, the initial u^μ has to satisfy $u^\mu u_\mu = -1$. Also, it is important to note that Eq. (13a) is first order and, thus, the initial data cannot be arbitrary but must satisfy a compatibility condition ensuring that Eq. (13a) holds at $t = 0$. Therefore, one can use Eq. (13a) to write the time derivative of n in terms of the time derivative of u^μ

(this feature would also appear in Navier-Stokes theory in the Eckart hydrodynamic frame).

A natural choice to determine the initial conditions for the matter sector is to set an initial state that is within the regime of validity of the first-order theory and closely reproduces Eckart's theory. First, one can directly extract n and u^μ from J^μ at the initial spacelike hypersurface. Then, one sets the nonequilibrium correction to the energy density $\mathcal{A}$ in Eq. (11) to zero in the initial state, so then the initial value for ε equals $T^{\mu\nu}u_\mu u_\nu$ and the first-order time derivative of ε is defined in terms of the first-order time derivative of the flow velocity (plus spatial derivatives that are known in the initial state). Clearly, $\mathcal{A}$ will be different than zero later during the actual evolution, and its value can be used to check if the simulations remain within the regime of validity of the first-order approach (i.e., $|\mathcal{A}|/\varepsilon$ must remain less than unity). Finally, the time derivative of the flow velocity can be set by imposing that the second-order on-shell term $(\varepsilon + P)u^\lambda\nabla_\lambda u^\nu + \Delta^{\nu\lambda}\nabla_\lambda P$ vanishes. Hence, one can obtain the time derivative of the flow velocity and all the other required initial data in the regime of validity of the first-order approach, emulating Eckart's theory as much as possible.

We recall that the initial data for the gravitational sector has to further satisfy the well-known Einstein constraint equations. We briefly make some comments on this in Sec. VIII.

B. Diagonalization and eigenvectors

In this section, we write Eqs. (3) and (13) as a first-order system, as discussed above. For this, we begin defining the variables $V = u^\alpha\partial_\alpha\varepsilon$, $\mathcal{V}^\mu = \Delta^{\mu\alpha}\partial_\alpha\varepsilon$, $W = u^\alpha\partial_\alpha n$, $\mathcal{W}^\mu = \Delta^{\mu\alpha}\partial_\alpha n$, $S^\mu = u^\alpha\nabla_\alpha u^\mu$, $\mathcal{S}^\nu_\lambda = \Delta^\alpha_\lambda\nabla_\alpha u^\nu$, $F_{\mu\nu} = u^\alpha\partial_\alpha g_{\mu\nu}$, and $\mathcal{F}^\lambda_{\mu\nu} = \Delta^{\lambda\alpha}\partial_\alpha g_{\mu\nu}$. Then, the equations of motion can be cast as

$$\tau_\varepsilon u^\alpha\partial_\alpha V + \tau_Q\rho\partial_\nu S^\nu + \tau_\varepsilon\rho u^\alpha\partial_\alpha S^\nu_\nu + \beta_\varepsilon\partial_\nu\mathcal{V}^\nu + \beta_n\partial_\nu\mathcal{W}^\nu = r_1,$$
(22a)

$$\tau_P\Delta^{\mu\alpha}\partial_\alpha V + \tau_Q\rho u^\alpha\partial_\alpha S^\mu + \beta_\varepsilon u^\alpha\partial_\alpha\mathcal{V}^\mu + \beta_n u^\alpha\partial_\alpha\mathcal{W}^\mu$$
$$+ \eta\Pi^{\mu\lambda\alpha}_\nu\partial_\alpha S^\nu_\lambda = r^\mu_2,$$
(22b)

$$u^\alpha\partial_\alpha\mathcal{V}^\mu - \Delta^{\mu\alpha}\partial_\alpha V = r^\mu_3,$$
(22c)

$$u^\alpha\partial_\alpha\mathcal{W}^\mu + n\Delta^{\mu\alpha}\partial_\alpha\mathcal{S}^\nu_\nu = r^\mu_4,$$
(22d)

$$u^\alpha\partial_\alpha\mathcal{S}^\nu_\lambda - \Delta^\alpha_\lambda\partial_\alpha S^\nu - \mathcal{X}^{\nu A\alpha}_\lambda\partial_\alpha F_A - \mathcal{Y}^{\nu A\alpha}_{\lambda\delta}\partial_\alpha\mathcal{F}^\delta_A = r^\nu_{5\lambda},$$
(22e)

$$u^\alpha\partial_\alpha F_A - \Delta^\alpha_\delta\mathcal{F}^\delta_A = r_{6A},$$
(22f)

$$u^\alpha\partial_\alpha\mathcal{F}^\delta_A - \Delta^{\delta\alpha}\partial_\alpha F_A = r^\delta_{7A},$$
(22g)

$$u^\alpha\partial_\alpha\varepsilon = r_8,$$
(22h)

$$u^\alpha\partial_\alpha n = r_9,$$
(22i)

$$u^\alpha\partial_\alpha u^\mu = r^\mu_{10},$$
(22j)

$$u^\alpha\partial_\alpha g_A = r_{11A},$$
(22k)

where the r's are functions of the fields $\varepsilon, u^\nu, \ldots, \mathcal{F}^\lambda_{\mu\nu}$ but not its derivatives and $A = \sigma\beta$ for $\sigma \geq \beta$; i.e., A takes the 10 independent values 00,01,02,03,11,12,13,22,23,33 with repeated index A summing from 00 to 33,

$$\Pi^{\mu\lambda\alpha}_\nu = -\eta(\Delta^{\mu\lambda}\delta^\alpha_\nu + \Delta^{\alpha\lambda}\delta^\mu_\nu) + \left(\rho\tau_P - \zeta + \frac{2\eta}{3}\right)\Delta^{\mu\alpha}\delta^\lambda_\nu, \quad (23a)$$

$$\mathcal{X}^{\nu A(=\sigma\beta)\alpha}_\lambda = \frac{1}{2}[g^{\nu(\sigma}\Delta^{\beta)}_\lambda u^\alpha - u^{(\sigma}\Delta^{\beta)}_\lambda g^{\nu\alpha} - u^{(\sigma}\Delta^{\beta)\nu}\Delta^\alpha_\lambda](2 - \delta_A),$$
(23b)

$$\mathcal{Y}^{\nu A(=\sigma\beta)\alpha}_{\lambda\delta} = \frac{1}{2}u^{(\sigma}u^{\beta)}\Delta^\alpha_\lambda\delta^\nu_\delta(2 - \delta_A). \quad (23c)$$

By δ_A we mean the Kronecker delta in the sense that when $A = \sigma\beta$, then $\delta_A = \delta_{\sigma\delta}$, which equals one when $\sigma = \beta$ and zero otherwise. Also, the terms r may be functions of the 95 variables. Equations (22) were obtained as follows. Equations (22a) and (22b) come from the conservation law $\nabla_\nu T^{\mu\nu} = 0$ when projected into the directions parallel and perpendicular to u^ν, respectively. Equations (22c), (22d), (22e), and (22g) correspond, respectively, to the identities $\nabla_\alpha\nabla_\beta\varepsilon - \nabla_\beta\nabla_\alpha\varepsilon = 0$, $\nabla_\alpha\nabla_\beta n - \nabla_\beta\nabla_\alpha n = 0$, $\nabla_\alpha\nabla_\beta u^\nu - \nabla_\beta\nabla_\alpha u^\nu = R^\nu_{\alpha\beta\sigma}u^\sigma = (\partial_\alpha\Gamma^\nu_{\beta\sigma} - \partial_\beta\Gamma^\nu_{\alpha\sigma})u^\sigma +$ terms of order zero in derivatives, and $\partial_\alpha\partial_\beta g_{\mu\nu} - \partial_\beta\partial_\alpha g_{\mu\nu} = 0$, all contracted with $u^\alpha\Delta^\beta_\lambda$. Equations (22f) is the Einstein equation in the harmonic gauge, i.e., $g^{\alpha\beta}\partial_\alpha\partial_\beta g_{\mu\nu} =$ terms of lower order in derivatives, while Eqs. (22h)–(22k) are the definitions of V, W (also using the identity $u^\alpha\nabla_\alpha n + n\nabla_\alpha u^\alpha = W + n\mathcal{S}^\alpha_\alpha = 0$ to eliminate W thoroughly), S^μ, and F_A, respectively. We may now define the 95×1 column vectors Ψ and $\mathfrak{B}$ as

$$\Psi = \begin{bmatrix} \psi_m \\ \psi_g \\ \psi_d \end{bmatrix}, \quad (24)$$

and $\mathfrak{B} = (r_1, \ldots, r_{11A})^T$, where $\psi_m = (V, S^\nu, \mathcal{V}^\nu, \mathcal{W}^\nu, \mathcal{S}^\nu_0, \mathcal{S}^\nu_1, \mathcal{S}^\nu_2, \mathcal{S}^\nu_3)^T \in \mathbb{R}^{29}$, $\psi_g = (F_A, \mathcal{F}^0_A, \mathcal{F}^1_A, \mathcal{F}^2_A, \mathcal{F}^3_A)^T \in \mathbb{R}^{50}$, and $\psi_d = (\varepsilon, n, u^\nu, g_A)^T \in \mathbb{R}^{16}$, to write the quasilinear first-order system (22) in matrix form as

$$\mathfrak{A}^\alpha\partial_\alpha\Psi = \mathfrak{B}, \quad (25)$$

where, here, $\mathfrak{A}^\alpha = \mathbb{A}^\alpha \oplus u^\alpha I_{16}$ ($\oplus$ being the direct sum). The matrix $\mathbb{A}^\alpha$ is split in the following way:

$$\mathbb{A}^\alpha = \begin{bmatrix} \mathbb{A}_m^\alpha & -L^\alpha \\ 0_{50\times29} & \mathbb{A}_g^\alpha \end{bmatrix}, \tag{26}$$

where

$$\mathbb{A}_m^\alpha = \begin{bmatrix} \tau_\varepsilon u^\alpha & \rho\tau_Q\delta_\nu^\alpha & \beta_\varepsilon\delta_\nu^\alpha & \beta_n\delta_\nu^\alpha & \rho\tau_\varepsilon u^\alpha\delta_\nu^0 & \rho\tau_\varepsilon u^\alpha\delta_\nu^1 & \rho\tau_\varepsilon u^\alpha\delta_\nu^2 & \rho\tau_\varepsilon u^\alpha\delta_\nu^3 \\ \tau_P\Delta^{\mu\alpha} & \rho\tau_Q u^\alpha\delta_\nu^\mu & \beta_\varepsilon u^\alpha\delta_\nu^\mu & \beta_n u^\alpha\delta_\nu^\mu & \Pi_\nu^{\mu0\alpha} & \Pi_\nu^{\mu1\alpha} & \Pi_\nu^{\mu2\alpha} & \Pi_\nu^{\mu3\alpha} \\ -\Delta^{\mu\alpha} & 0_{4\times4} & u^\alpha\delta_\nu^\mu & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} \\ 0_{4\times1} & 0_{4\times4} & 0_{4\times4} & u^\alpha\delta_\nu^\mu & n\Delta^{\mu\alpha}\delta_\nu^0 & n\Delta^{\mu\alpha}\delta_\nu^1 & n\Delta^{\mu\alpha}\delta_\nu^2 & n\Delta^{\mu\alpha}\delta_\nu^3 \\ 0_{4\times1} & -\Delta_0^\alpha\delta_\nu^\mu & 0_{4\times4} & 0_{4\times4} & u^\alpha\delta_\nu^\mu & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} \\ 0_{4\times1} & -\Delta_1^\alpha\delta_\nu^\mu & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & u^\alpha\delta_\nu^\mu & 0_{4\times4} & 0_{4\times4} \\ 0_{4\times1} & -\Delta_2^\alpha\delta_\nu^\mu & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & u^\alpha\delta_\nu^\mu & 0_{4\times4} \\ 0_{4\times1} & -\Delta_3^\alpha\delta_\nu^\mu & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & u^\alpha\delta_\nu^\mu \end{bmatrix}, \tag{27}$$

while

$$\mathbb{A}_g^\alpha = \begin{bmatrix} u^\alpha I_{10} & -\Delta_0^\alpha I_{10} & -\Delta_1^\alpha I_{10} & -\Delta_2^\alpha I_{10} & -\Delta_3^\alpha I_{10} \\ -\Delta^{0\alpha}I_{10} & u^\alpha I_{10} & 0_{10\times10} & 0_{10\times10} & 0_{10\times10} \\ -\Delta^{1\alpha}I_{10} & 0_{10\times10} & u^\alpha I_{10} & 0_{10\times10} & 0_{10\times10} \\ -\Delta^{2\alpha}I_{10} & 0_{10\times10} & 0_{10\times10} & u^\alpha I_{10} & 0_{10\times10} \\ -\Delta^{3\alpha}I_{10} & 0_{10\times10} & 0_{10\times10} & 0_{10\times10} & u^\alpha I_{10} \end{bmatrix} \tag{28}$$

and

$$L^\alpha = \begin{bmatrix} 0_{1\times10} & 0_{1\times10} & 0_{1\times10} & 0_{1\times10} & 0_{1\times10} \\ 0_{4\times10} & 0_{4\times10} & 0_{4\times10} & 0_{4\times10} & 0_{4\times10} \\ 0_{4\times10} & 0_{4\times10} & 0_{4\times10} & 0_{4\times10} & 0_{4\times10} \\ 0_{4\times10} & 0_{4\times10} & 0_{4\times10} & 0_{4\times10} & 0_{4\times10} \\ \mathcal{X}_0^{\mu A\alpha} & \mathcal{Y}_{00}^{\mu A\alpha} & \mathcal{Y}_{01}^{\mu A\alpha} & \mathcal{Y}_{02}^{\mu A\alpha} & \mathcal{Y}_{03}^{\mu A\alpha} \\ \mathcal{X}_1^{\mu A\alpha} & \mathcal{Y}_{10}^{\mu A\alpha} & \mathcal{Y}_{11}^{\mu A\alpha} & \mathcal{Y}_{12}^{\mu A\alpha} & \mathcal{Y}_{13}^{\mu A\alpha} \\ \mathcal{X}_2^{\mu A\alpha} & \mathcal{Y}_{20}^{\mu A\alpha} & \mathcal{Y}_{21}^{\mu A\alpha} & \mathcal{Y}_{22}^{\mu A\alpha} & \mathcal{Y}_{23}^{\mu A\alpha} \\ \mathcal{X}_3^{\mu A\alpha} & \mathcal{Y}_{30}^{\mu A\alpha} & \mathcal{Y}_{31}^{\mu A\alpha} & \mathcal{Y}_{32}^{\mu A\alpha} & \mathcal{Y}_{33}^{\mu A\alpha} \end{bmatrix}. \tag{29}$$

We are now ready to establish that, when written as a first-order system as above, the equations of motion are strongly hyperbolic. In Sec. VII A, we show that the assumptions of Proposition I are not empty.

Proposition I.—Consider the system (22). Assume that (A1) with $\eta > 0$ holds and that Eq. (21) in Theorem I holds in strict form, i.e., with $>$ instead of $\geq$. Let ξ be a timelike covector. Then, (i) $\det(\mathfrak{A}^\alpha\xi_\alpha) \neq 0$, and (ii) for any spacelike vector ζ, the eigenvalue problem $(\zeta_\alpha + \Lambda\xi_\alpha)\mathfrak{A}^\alpha R = 0$ has only real eigenvalues Λ and a complete set of right eigenvectors R.

Proof.—The proof of this proposition is very lengthy and we refer the interested reader to check all the details and the proof presented in Appendix B.

C. Local well posedness

In this section, we establish the local existence and uniqueness of solutions to the nonlinear equations of motion in Eqs. (3) and (13).

We begin by noticing that Eq. (13) used the normalization $u^\mu u_\mu = -1$ to project the divergence of $T_{\mu\nu}$ and J^μ onto the directions parallel and orthogonal to u^μ. In order to show that the condition $u^\mu u_\mu = -1$ is propagated by the flow, it is more convenient to work directly with Eqs. (2) and (3). In order to complete the system, we differentiate $u^\mu u_\mu = -1$ twice in the u^μ direction:

$$u^\beta\nabla_\beta[u^\alpha\nabla_\alpha(u^\alpha u_\alpha)] = 0. \tag{30}$$

We also differentiate $\nabla_\mu J^\mu = 0$ once, as in Sec. IV:

$$u^\mu\nabla_\mu(\nabla_\nu J^\nu) = 0. \tag{31}$$

Observe that Eqs. (30) and (31) imply that $u^\mu u_\mu = -1$ and $\nabla_\mu J^\mu = 0$ hold at later times if these hold at the initial time.

The main result of this section can be found below.

Theorem II.—Let $(\Sigma, \mathring{g}_{\alpha\beta}, \hat{\kappa}_{\alpha\beta}, \mathring{\varepsilon}, \hat{\varepsilon}, \mathring{n}, \hat{n}, \mathring{u}^\alpha, \hat{u}^\alpha)$ be an initial-data set for the system composed of Einstein's equations (2) and $\nabla_\mu J^\mu = 0$, where $T_{\alpha\beta}$ and J^μ are given in Eq. (11). Assume that $\mathring{u}^\mu\mathring{u}_\mu = -1$, $\mathring{n} > 0$ [171], and that $\nabla_\mu J^\mu = 0$ holds for the initial data. Assume (A1) with $\eta > 0$ and suppose that Eqs. (21) of Theorem I hold in strict form and that the transport coefficients are analytic functions of their arguments. Finally, assume $\mathring{g}_{\alpha\beta}, \mathring{\varepsilon}, \mathring{n}, \mathring{u}^\alpha \in H^N(\Sigma)$ and that $\hat{\kappa}_{\alpha\beta}, \hat{\varepsilon}, \hat{n}, \hat{u}^\alpha \in H^{N-1}(\Sigma)$, $N \geq 5$, where H^N is the Sobolev space. Then, there exists a globally hyperbolic

development of the initial data. This globally hyperbolic development is unique if taken to be the maximum globally hyperbolic development of the initial data.

Proof.—The proof is found in Appendix C.

VI. NEW THEOREM ABOUT LINEAR STABILITY

Any ordinary fluid must be stable against small deviations from the thermodynamic equilibrium state [15]. (We only consider systems such that the equilibrium state is unique and has a finite correlation length. Therefore, in principle, our discussion does not apply to systems where the correlation length in equilibrium can become arbitrarily large, such as at a critical point.) We recall that in equilibrium $\beta_\mu = u_\mu/T$ must be a Killing vector, i.e., $\nabla_\mu \beta_\nu + \nabla_\nu \beta_\mu = 0$, and also $\nabla_\alpha(\mu/T) = 0$ [36,172,173]. In Minkowski spacetime, nonrotating equilibrium corresponds to a class of states with constant T and μ and background flow velocity $u^\mu = \gamma(1, \mathbf{v})$ defined by a constant subluminal three-velocity $\mathbf{v}$, where $\gamma = 1/\sqrt{1 - \mathbf{v}^2}$. (In this paper, we neglect the constant thermal vorticity term; see Ref. [172] for a nice discussion of its physical content and consequences.) In the local rest frame $\mathbf{v} = 0$ and the background flow is simply $u^\mu = (1, 0, 0, 0)$. In a stable theory, small disturbances from the general equilibrium state $T \to T + \delta T(t, \mathbf{x})$, $\mu \to \mu + \delta\mu(t, \mathbf{x})$, and $u^\mu \to u^\mu + \delta u^\mu(t, \mathbf{x})$ (with $u_\mu \delta u^\mu = 0$) lead to small variations in the energy-momentum tensor and current, $\delta T^{\mu\nu}(t, \mathbf{x})$ and $\delta J^\mu(t, \mathbf{x})$, which decay with time.

The standard theories from Eckart and Landau-Lifshitz are unstable, as shown by Hiscock and Lindblom many years ago [25]. This instability appears because such theories possess exponentially growing, hence *unstable*, nonhydrodynamic modes, which spoil linear stability around equilibrium even at vanishing wave number. (The frequency of a hydrodynamic mode, such as a sound wave, vanishes in a spatially uniform state. On the other hand, a nonhydrodynamic mode correspond to a collective excitation that possesses nonzero frequency even at zero wave number.) For Landau-Lifshitz theory at zero chemical potential, this instability is only observed when considering a general equilibrium state with nonzero $\mathbf{v}$ [25,26,73], while in the case of Eckart the instability already appears even when $\mathbf{v} = 0$. The lack of causality in these approaches implies that it is not sufficient to investigate only the static $\mathbf{v} = 0$ case in order to determine the stability properties of a general equilibrium state where $\mathbf{v} \neq 0$, even though such states are in principle connected via a simple Lorentz transformation.

The necessity to investigate the stability properties of general equilibrium states where $\mathbf{v} \neq 0$ makes linear stability analyses of viscous hydrodynamic theories very complicated. Already in the local rest frame, finding whether the linear modes of the system are stable requires determining the sign of the imaginary part of the roots of a high-order polynomial, which becomes a daunting task when $\mathbf{v} \neq 0$ (see Refs. [35,70] for recent examples of how complicated a $\mathbf{v} \neq 0$ analysis can become in BDNK and MIS theory, respectively).

We prove below a new theorem that gives sufficient conditions for causal fluid dynamic equations to be linearly stable against disturbances of a general nonrotating equilibrium state with arbitrary background velocity. In this case, proving stability for the local rest frame implies stability in any other frame (note that the word frame here is used in the standard context of special relativity, i.e., to refer to an inertial observer, and has nothing to do with the concept of a hydrodynamic frame discussed in previous sections, which concerned the definition of hydrodynamic variables out of equilibrium) connected to the local rest frame via a Lorentz transformation. This general feature is expected to hold in any interacting relativistic system; i.e., no issues should appear if one simply observes a given system in another inertial frame. We then use this theorem in Sec. VII to find conditions under which the hydrodynamic theory presented here is stable. We remark that our results can be used to establish stability at nonzero $\mathbf{v} \neq 0$ in other theories as well, e.g., MIS, as long as the conditions discussed below are fulfilled.

A. Transforming a second-order system of linear differential equations into a first-order one

We begin by showing how one may convert a system of linear second-order PDEs into a first order one, as this is needed for the theory discussed in this paper. Let the system of linearized second-order PDEs be given by

$$\sum_b \mathfrak{M}(\partial)^a_b \delta\psi^b(X) = \mathfrak{N}(\partial\delta\psi)^a, \qquad (32)$$

where a and b run from 1 to n, $\mathfrak{M}(\partial)^b_a$ are differential linear operators of order 2, $\mathfrak{N}(\partial\Psi)$ are linear terms containing derivatives of the perturbed fields $\delta\Psi$ up to order 1, and $\delta\psi^1(X), \ldots, \delta\psi^n(X)$ are the perturbed fields (for instance, $\delta\varepsilon$, δn, etc.). We suppose that Eq. (32) arises from the conservation laws $-u_\alpha \partial_\beta \delta T^{\alpha\beta} = 0$, $\Delta^\mu_\alpha \partial_\beta \delta T^{\alpha\beta} = 0$, and $\partial_\alpha \delta J^\alpha = -u_\beta u^\alpha \partial_\alpha \delta J^\beta + \Delta^{\alpha\beta} \partial_\alpha \delta J_\beta = 0$, where the first two come from $\partial_\alpha \delta T^{\alpha\beta} = 0$, while the last equation appears only when J^μ is included. In this manner, the derivatives in the equations of motion in Eq. (32) shall always appear as combinations of $u^\alpha \partial_\alpha$ and $\Delta^{\alpha\beta} \partial_\beta$ only. Thus, if the system in Eq. (32) has one or more second-order equations, it can be rewritten as a first-order system in the $N \equiv 5n$ new variables $\delta\bar{\psi}^a(X) = u^\alpha \partial_\alpha \psi^a(X)$ and $\delta\tilde{\psi}^a_\mu(X) = \Delta^\nu_\mu \partial_\nu \psi^a(X)$. These definitions automatically lead (32) to n first-order linear equations. One then needs to supplement those with the $4n$ dynamical equations that are missing. By means of the identity $\partial_\alpha \partial_\beta \psi^a(X) - \partial_\beta \partial_\alpha \psi^a(X) = 0$, one may find the extra $4n$ dynamical equations

$u^\alpha \partial_\alpha \delta \tilde{\psi}^a_\mu(X) - \Delta^\alpha_\mu \partial_\alpha \delta \bar{\psi}^a(X) = 0$, giving the needed $5n$ first-order dynamical equations, as required. In matrix form it becomes

$$\mathbb{A}^\alpha \partial_\alpha \delta \Psi(X) + \mathbb{B} \delta \Psi(X) = 0, \tag{33}$$

where $\mathbb{A}^\alpha$ and $\mathbb{B}$ are $N \times N$ constant real matrices and $\delta \Psi(X)$ is a $N \times 1$ column vector with entries $\delta \bar{\psi}^1, \delta \tilde{\psi}^1_\nu, \ldots, \delta \bar{\psi}^n, \delta \tilde{\psi}^n_\nu$. This ends the procedure. However, if one of the equations in Eq. (32) is already of first order but contains variables that have second-order derivative in other equations, then one can eliminate this equation by using it as a constraint to eliminate one of the variables. For example, consider the case of the ideal current $J^\mu = n u^\mu$. In this case, the conservation equation $\partial_\alpha J^\alpha = 0$ becomes $u^\alpha \partial_\alpha \delta n(X) + n \partial_\alpha \delta u^\alpha(X) = 0$. If $T^{\mu\nu}$ has shear or bulk contributions, for example, then the other equations must have second-order derivatives of δu^μ. Thus, one must write $\partial_\alpha \delta J^\alpha = 0$ as $\delta \bar{\psi} + n \delta \tilde{\psi}^\mu_\mu = 0$, where $\delta \tilde{\psi}^\mu_\nu = \Delta^\alpha_\nu \partial_\alpha u^\mu$ and $\delta \bar{\psi} = u^\alpha \partial_\alpha n$. This is a zeroth-order equation in the new variables and, therefore, is just a constraint. One may use this constraint in order to eliminate the variable $\delta \bar{\psi}$ in the other dynamical equations. Then, in this case one ends up with $5n - 1$ dynamical equations for the $5n - 1$ fields.

Finally, we remark that other approaches to viscous relativistic fluids, such as MIS, are already written in the format (33) in the linearized regime so the procedure to reduce the order of the equations of motion described above is not needed and one can move directly to the part below.

B. New linear stability theorem

To study linear stability, let us expand the perturbed fields in the Fourier modes $K^\mu = (i\Gamma, k^i)$ by substituting $\delta \Psi(X) \rightarrow \exp(iK_\mu X^\mu) \delta \Psi(K) = \exp(\Gamma t + ik_i x^i) \delta \Psi(K)$ in Eq. (33). The result is

$$iK_\mu \mathbb{A}^\mu \delta \Psi(K) + \mathbb{B} \delta \Psi(K) = 0. \tag{34}$$

Since K^μ appears, as aforementioned, as combinations of $-u^\alpha K_\alpha = \gamma(i\Gamma - k_i v^i)$ and $\Delta^{\mu\nu} K_\mu K_\nu = (u^\mu K_\mu)^2 + \Gamma^2 + k^2$, where $k^2 = k_i k^i$, then the direction of k^i is not relevant once one keeps v^i arbitrary. Thus, we may write $K^\mu = -n^\mu n_\nu K^\nu + \zeta^\mu \zeta_\nu K^\nu$, where n_μ is timelike and ζ_μ is spacelike, with $n^\mu n_\mu = -1$, $n_\mu \zeta^\mu = 0$, and $\zeta_\mu \zeta^\mu = 1$ [for example, it is common to choose $K^\mu = (K^0, k, 0, 0)$ so that n_μ and ζ_ν are $(-1, 0, 0, 0)$ and $(0,1,0,0)$, respectively]. In this case we define $\Omega = n_\alpha K^\alpha$ and $\kappa = \zeta_\alpha K^\alpha$ such that $K^\mu = -\Omega n^\mu + \kappa \zeta^\mu$ [70]. Then, Eq. (34) can be written as

$$i\Omega(-n_\alpha \mathbb{A}^\alpha) \delta \Psi(K) = -i\kappa \zeta_\alpha \mathbb{A}^\alpha \delta \Psi(K) - \mathbb{B} \delta \Psi(K). \tag{35}$$

The general form of the covectors n and ζ is $n_\alpha = \gamma_n(-1, c^i)$ for any c^i such that $0 \le c^i c_i < 1$ and where

$\gamma_n = 1/\sqrt{1 - c^i c_i} \ge 1$, and $\zeta_\alpha = \gamma_\zeta(-\hat{d}^j c_j, \hat{d}^i) \ge 1$, where $\hat{d}^i \hat{d}_i = 1$ for an arbitrary unitary $\hat{d}^i$ and $\gamma_\zeta = 1/\sqrt{1 - (\hat{d}^i c_i)^2} \ge 1$. From the Cauchy-Schwarz inequality $(\hat{d}^i c_i)^2 \le |c^i|^2$ (here, $|c^i| = \sqrt{c^i c_i}$), then one obtains that

$$\gamma_n \ge \gamma_\zeta. \tag{36}$$

Stability demands that the perturbed modes $\Gamma = \Gamma(k^i)$ are such that $\Gamma_R \le 0$. Now, consider the eigenvalue problem,

$$(\Lambda n_\alpha + \zeta_\alpha) \mathbb{A}^\alpha \mathfrak{r} = 0, \tag{37}$$

where here Λ is the eigenvalue associated with the right eigenvector $\mathfrak{r}$.

Proposition II.—If Eq. (33) is causal, then the eigenvalues Λ are real and lie in the range $[-1, 1]$. Furthermore, $\det(n_\alpha \mathbb{A}^\alpha) \ne 0$.

Proof.—Causality demands that the roots of $Q(\xi) = \det(\xi_\alpha \mathbb{A}^\alpha) = 0$ are such that (i) $\xi_0 = \xi_0(\xi_i) \in \mathbb{R}$ and that (ii) the curves ξ_0 lie outside or over the light cone. In other words, $\xi^\alpha \xi_\alpha \ge 0$. If one writes $\xi_\alpha = \Lambda n_\alpha + \zeta_\alpha$, where n and ζ are real, then condition (i) means that Λ is real. On the other hand, since n and ζ are orthonormal, then condition (ii) means that $\xi_\alpha \xi^\alpha = -\Lambda^2 + 1 \ge 0$, which demands that $\Lambda^2 \le 1$, i.e., $\Lambda \in [-1, 1]$. Now, since $Q(\xi) = 0$ if and only if ξ is spacelike or lightlike, this means that $\det(n_\alpha \mathbb{A}^\alpha) \ne 0$. ∎

Theorem III.—Let Eq. (37) have a set of N linearly independent (LI) real eigenvectors $\{\mathfrak{r}_1, \ldots, \mathfrak{r}_N\}$. If Eq. (33) is causal and stable in the local rest frame $\mathcal{O}$, then it is also stable in any other Lorentz frame $\mathcal{O}'$ connected to $\mathcal{O}$ by a Lorentz transformation.

Proof.—The details of the proof are found in Appendix D. However, we summarize some steps here. Note that causality enables us to invert the matrix $(-n_\alpha \mathbb{A}^\alpha)$. Then, it is possible to rewrite Eq. (35) as

$$\begin{aligned} i\Omega \delta \Psi(K)^\dagger (R^T)^{-1} R^{-1} \delta \Psi(K) \\ = -i\kappa \delta \Psi(K)^\dagger (R^T)^{-1} R^{-1} (-n_\alpha \mathbb{A}^\alpha)^{-1} (\zeta_\alpha \mathbb{A}^\alpha) \delta \Psi(K) \\ - \delta \Psi(K)^\dagger (R^T)^{-1} R^{-1} (-n_\alpha \mathbb{A}^\alpha)^{-1} \mathbb{B} \delta \Psi(K), \end{aligned} \tag{38}$$

where the dagger stands for the matrix transpose and complex conjugate operations altogether, T stands for matrix transpose operation, while R is the square matrix that diagonalizes $(-n_\alpha \mathbb{A}^\alpha)^{-1} (\zeta_\alpha \mathbb{A}^\alpha)$, since Eq. (37) has a complete set of real eigenvectors in $\mathbb{R}^n$ with only real eigenvalues. Then, we can expand $\delta \Psi(K)$ in terms of these eigenvectors. In the proof, it is shown that $\delta \Psi(K)^\dagger (R^T)^{-1} R^{-1} \delta \Psi(K)$ and $\delta \Psi(K)^\dagger (R^T)^{-1} R^{-1} (-n_\alpha \mathbb{A}^\alpha)^{-1} (\zeta_\alpha \mathbb{A}^\alpha) \delta \Psi(K)$ are real for any Lorentz frame. After some work, we demonstrate that, under the theorem's statements, stability reduces to the condition that the term $\delta \Psi(K)^\dagger (R^T)^{-1} R^{-1} (-n_\alpha \mathbb{A}^\alpha)^{-1} \mathbb{B} \delta \Psi(K)$ must be greater than or equal to zero. Since this is proven to be a scalar

under Lorentz boosts, it can be computed in any frame. Thus, this implies that if the theory is stable in the LRF and obeys the other conditions of the theorem, it is stable in any other Lorentz frame.

We note that this result implies that the original system of linearized second-order PDEs in Eq. (32) is stable under the stated assumptions.

1. Applying the stability theorem to a toy model

To illustrate the application of the stability theorem, consider the simple model described by the fields ϕ and ψ^μ that obey the first-order dynamical linear equations:

$$u^\alpha \partial_\alpha \phi - \alpha \Delta_\nu^\alpha \partial_\alpha \psi^\nu + \lambda \phi = 0, \qquad (39a)$$

$$u^\alpha \partial_\alpha \psi^\mu - \beta \Delta^{\mu\alpha} \partial_\alpha \phi = 0. \qquad (39b)$$

We consider the case where u^μ is constant [$u^\mu = \gamma(1, v^i)$, with $\gamma = 1/\sqrt{1-v^2}$ and $v^2 = v^i v_i < 1$] as done in the stability theorem of the last section. If we write Eq. (39) in matrix form as

$$\mathbb{A}^\alpha \partial_\alpha \Psi(X) + \mathbb{B}\Psi(X) = 0, \qquad (40)$$

where $\Psi(X) = (\phi, \psi^\nu)$ is a 5×1 column vector,

$$\mathbb{B} = \begin{bmatrix} \lambda & 0_{1\times 4} \\ 0_{4\times 1} & 0_{4\times 4} \end{bmatrix}$$

and

$$\mathbb{A}^\alpha = \begin{bmatrix} u^\alpha & -\alpha\Delta_\nu^\alpha \\ -\beta\Delta^{\mu\alpha} & u^\alpha \delta_\nu^\mu \end{bmatrix} \qquad (41)$$

are 5×5 matrices, the propagation modes $\omega = \omega(k^i)$ are obtained by means of the Fourier transform $\Psi(X) \to e^{iK_\mu X^\mu} \tilde{\Psi}(K)$, where $K^\mu = (\omega, k^i)$, and are the roots of $\det[iK_\alpha \mathbb{A}^\alpha + \mathbb{B}] = 0$. Let us write $\omega = i\Gamma$. Then, stability requires that $\mathrm{Re}(\Gamma) \leq 0$. In the local rest frame, these equations are $\Gamma = 0$ and $\Gamma^2 + \lambda\Gamma + \alpha\beta k^2 = 0$, where $k^2 = k_i k^i$. Then, stability in the LRF implies the conditions

$$\alpha\beta \geq 0, \qquad (42a)$$

$$\lambda \geq 0. \qquad (42b)$$

As for the boosted frame obtained by the Lorentz transform $\Gamma \to \gamma(\Gamma + iv^i k_i)$ and $k^2 \to \Gamma^2 + k^2 - \gamma^2(\Gamma + iv^i k_i)^2$, the first root is $\Gamma = -iv^i k_i$, which is stable, while the remaining two roots demand (after a long but straightforward computation)

$$\lambda \geq 0, \qquad (43a)$$

$$0 \leq \alpha\beta \leq 1. \qquad (43b)$$

To verify stability via the stability theorem proven in this paper, we must verify conditions where Eq. (39) is causal and if the matrix $\Phi_\alpha \mathbb{A}^\alpha$ (with $\Phi_\alpha = \Lambda n_\alpha + \zeta_\alpha$, n and ζ are the unitary timelike and spacelike covectors defined in the text) has a complete set of eigenvectors in $\mathbb{R}^5$. Proposition I guarantees that if Eq. (39) is causal, then $\Lambda \in \mathbb{R}$. In order to study causality, we compute the characteristics ξ_α of the system, which reduces to the roots of $\det(\mathbb{A}^\alpha \xi_\alpha) = (u^\alpha \xi_\alpha)^3 [(u^\beta \xi_\beta)^2 - \alpha\beta\Delta^{\mu\nu}\xi_\mu \xi_\nu] = 0$. Causal roots must be real and obey $\xi_\mu \xi^\mu \geq 0$, which gives the conditions $0 \leq \alpha\beta \leq 1$. These conditions, together with stability in the LRF, coincide with the conditions obtained by means of the above direct calculation. However, if we did not know, *a priori*, the conditions for stability in any frame (which is the case when considering higher-order polynomials for the modes), we would still have to obtain the eigenvectors of

$$\Phi_\alpha \mathbb{A}^\alpha = \begin{bmatrix} u^\alpha \Phi_\alpha & -\alpha\Delta_\nu^\alpha \Phi_\alpha \\ -\beta\Delta^{\mu\alpha}\Phi_\alpha & u^\alpha \Phi_\alpha \delta_\nu^\mu \end{bmatrix}. \qquad (44)$$

We can do it firstly by obtaining the eigenvalues Λ, which may be easily obtained by changing $\xi_\alpha \to \Phi_\alpha$ in the computation of the characteristics. With that result one obtains the eigenvalue $\Lambda^{(1)}$ that is the root of $u^\alpha \Phi_\alpha^{(1)} = 0$ with multiplicity 3 and the eigenvalue $\Lambda_\pm^{(2)}$, which give the two roots of $(u^\beta \Phi_{\pm\beta}^{(2)})^2 - \alpha\beta\Delta^{\mu\nu}\Phi_{\pm\mu}^{(2)}\Phi_{\pm\nu}^{(2)} = 0$. The corresponding eigenvectors are as follow.

(i) For $u^\alpha \Phi_\alpha^{(1)} = 0$, the system $\Phi_\alpha^{(1)}\mathbb{A}^\alpha r_a^{(1)} = 0$ has as eigenvectors the three linearly independent vectors given by

$$r_a^{(1)} = \begin{bmatrix} 0 \\ w_a^\nu \end{bmatrix}, \qquad (45)$$

where $\{w_a^\nu\}_{a=1}^3$ is a set of three linearly independent vectors orthogonal to the vector $\Delta^{\mu\alpha}\Phi_\alpha^{(1)}$.

(ii) For $(u^\beta \Phi_{\pm\beta}^{(2)})^2 - \alpha\beta\Delta^{\mu\nu}\Phi_{\pm\mu}^{(2)}\Phi_{\pm\nu}^{(2)} = 0$, we assume $\alpha\beta \neq 0$ and obtain the two eigenvectors,

$$r_\pm^{(2)} = \begin{bmatrix} u^\alpha \Phi_{\pm\alpha}^{(2)} \\ \beta\Delta^{\nu\alpha}\Phi_{\pm\alpha}^{(2)} \end{bmatrix}. \qquad (46)$$

[Note that in the special case $\alpha\beta = 0$, the root $u^\alpha\Psi_\alpha = 0$ is the only root with multiplicity 5. We end up with two distinct situations: first, if $\alpha \neq 0$ or $\beta \neq 0$ with $\alpha\beta = 0$, then one obtains four LI eigenvectors as can be seen from Eqs. (40) and (41). On the other hand, if $\alpha = \beta = 0$, then the system is already diagonal and the theorem applies directly.] Thus, Eq. (46) completes the remaining two linearly independent eigenvectors since $\Lambda_\pm$ are distinct

eigenvalues, giving the five LI eigenvectors. Then, the stability theorem states that the system is stable if $\lambda \geq 0$ and $0 < \alpha\beta \leq 1$ or if $\lambda \geq 0$ and $\alpha = \beta = 0$. Note that there is a slight difference from the condition obtained from the direct calculation. To wit, it does not include the case $\alpha\beta = 0$ with α or β different from zero. The conclusion is that stability in any frame does not necessarily imply strong hyperbolicity. However, strong hyperbolicity plus causality plus stability in the LRF implies stability in any boosted frame. In other words, stability may occur outside the conditions imposed by the theorem.

2. Applying the stability theorem to the MIS system

As another example of the usefulness of Theorem III, let us briefly comment how it can be used to recover the stability conditions of the MIS equations [24] in the presence of bulk viscosity. More precisely, we take the MIS-like equations studied in Ref. [71] where only bulk viscous effects have been considered. In that case, it was proven that there exist conditions such that the system of PDEs is nonlinearly causal and symmetric hyperbolic; hence the principal part of the equations is diagonalizable. The linear version of such equations forms a system that is also symmetric hyperbolic and the conditions for stability needed for the application of Theorem III can be shown to agree with those found in Ref. [24] for the case where only bulk viscosity is present.

VII. CONDITIONS FOR LINEAR STABILITY

We now apply the theorem proved in the last section to determine conditions that ensure the stability of the hydrodynamic theory proposed in this paper. Let us first define

$$D \equiv \rho c_s^2(\tau_\varepsilon + \tau_Q) + \zeta + \frac{4\eta}{3} + \sigma\kappa_\varepsilon \qquad (47)$$

and

$$E \equiv \sigma[p'_\varepsilon\kappa_s - c_s^2\kappa_\varepsilon]$$
$$= \sigma T\rho\left[\left(\frac{\partial P}{\partial\varepsilon}\right)_n\left(\frac{\partial(\mu/T)}{\partial n}\right)_\varepsilon - \left(\frac{\partial P}{\partial n}\right)_\varepsilon\left(\frac{\partial(\mu/T)}{\partial\varepsilon}\right)_n\right], \qquad (48)$$

where $\kappa_s = (T\rho^2/n)[\partial(\mu/T)/\partial\varepsilon]_{\bar{s}} = \kappa_\varepsilon + \kappa_n$, $\kappa_\varepsilon = (T\rho^2/n) \times [\partial(\mu/T)/\partial\varepsilon]_n$, $\kappa_n = (T\rho)[\partial(\mu/T)/\partial n]_\varepsilon$, and $p'_\varepsilon = (\partial P/\partial\varepsilon)_n$. Standard thermodynamic identities imply that $p'_\varepsilon\kappa_s - c_s^2\kappa_\varepsilon > 0$, then $E \geq 0$ from (A1). By assuming the Cowling approximation [174] with $g_{\mu\nu} = \eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$ and $\delta g_{\mu\nu} = 0$, we find that the system described by Eq. (13) is linearly stable if it is causal within the strict form of the inequalities in Eq. (21) together with the additional restriction $\eta > 0$ in (A1) and

$$(\tau_\varepsilon + \tau_Q)|B| \geq \tau_\varepsilon\tau_Q D \geq \rho c_s^2\tau_\varepsilon\tau_Q(\tau_\varepsilon + \tau_Q), \qquad (49a)$$

$$(\tau_\varepsilon + \tau_Q)|B|D + \rho\tau_\varepsilon\tau_Q(\tau_\varepsilon + \tau_Q)E$$
$$> \tau_\varepsilon\tau_Q D^2 + \rho(\tau_\varepsilon + \tau_Q)^2 C, \qquad (49b)$$

$$c_s^2 D - E \geq \rho c_s^4(\tau_\varepsilon + \tau_Q), \qquad (49c)$$

$$(\tau_\varepsilon + \tau_Q)[|B|(c_s^2 D - 2E) + 2c_s^2\rho\tau_\varepsilon\tau_Q E + CD]$$
$$> 2c_s^2\rho(\tau_\varepsilon + \tau_Q)^2 C + \tau_\varepsilon\tau_Q D(c_s^2 D - E), \qquad (49d)$$

$$|B|D[C(\tau_\varepsilon + \tau_Q) + E\tau_\varepsilon\tau_Q] + 2\rho\tau_\varepsilon\tau_Q(\tau_\varepsilon + \tau_Q)CE$$
$$> \rho C^2(\tau_\varepsilon + \tau_Q)^2 + \tau_\varepsilon\tau_Q(CD^2 + \rho\tau_\varepsilon\tau_Q E^2)$$
$$+ B^2 E(\tau_\varepsilon + \tau_Q), \qquad (49e)$$

where B and C are given by

$$B \equiv -\tau_\varepsilon\left(\rho c_s^2\tau_Q + \zeta + \frac{4\eta}{3} + \sigma\kappa_s\right) - \rho\tau_P\tau_Q, \qquad (50a)$$

$$C \equiv \tau_P(\rho c_s^2\tau_Q + \sigma\kappa_s) - \beta_\varepsilon\left(\zeta + \frac{4\eta}{3}\right), \qquad (50b)$$

as in Eq. (A5), with $|B| = -B > 0$ from Eq. (21c) in the strict form.

To prove the statement above, as before we may expand the perturbations $\delta\Psi = (\delta\varepsilon, \delta u^\mu, \delta n)$ in Fourier modes by means of the substitution $\delta\Psi(X) \to \exp[T(\Gamma t + k_i x^i)]\delta\Psi(K)$, where $K^\mu = (i\Gamma, k^i)$ is dimensionless due to the introduction of background temperature T in the exponent. We begin by proving stability in the local rest frame, where the modes are the roots of the shear and sound polynomials,

$$\text{shear channel:} \quad \bar{\tau}_Q\Gamma^2 + \bar{\eta}k^2 + \Gamma = 0, \qquad (51a)$$

$$\text{sound channel:} \quad a_0\Gamma^5 + a_1\Gamma^4 + a_2\Gamma^3 + a_3\Gamma^2 + a_4\Gamma + a_5 = 0, \qquad (51b)$$

where $k^2 = k^i k_i$ and

$$a_0 = \bar{\tau}_\varepsilon\bar{\tau}_Q, \qquad (52a)$$

$$a_1 = \bar{\tau}_\varepsilon + \bar{\tau}_Q, \qquad (52b)$$

$$a_2 = 1 + k^2|\bar{B}|, \qquad (52c)$$

$$a_3 = k^2\bar{D}, \qquad (52d)$$

$$a_4 = c_s^2 k^2 + k^4\bar{C}, \qquad (52e)$$

$$a_5 = k^4\bar{E}. \qquad (52f)$$

We defined the dimensionless quantities $\bar{\tau}_Q = T\tau_Q$, $\bar{\tau}_\varepsilon = T\tau_\varepsilon$, $\bar{\eta} = T\eta/\rho$, $\bar{B} = (T^2/\rho)B$, $\bar{C} = (T^2/\rho)C$, $\bar{D} = (T/\rho)D$, and $\bar{E} = (T/\rho)E$. From the second inequality in Eq. (21c) in its strict form one obtains that $\bar{B} < 0$ (see the definition of a_2). The analysis of stability in the LRF goes as follows.

Shear stability conditions.—The second-order polynomial (51a) has two roots with $\Gamma_R \leq 0$ only if $\tau_Q > 0$ and $\eta \geq 0$, which is in accordance with assumption (A1). One can see that τ_Q clearly acts as a relaxation time (the same role is played by the shear relaxation time coefficient τ_π present in MIS theory) for the shear channel, which ensures causality. In fact, the condition $\tau_Q > 0$ is clear since the leading contribution to the nonhydrodynamic frequency in this channel goes as $1/\tau_Q$ at zero wave number.

Sound stability conditions.— As for the sound channel in the rest frame, by means of the Routh-Hurwitz criterion [175], the necessary and sufficient conditions for $\Gamma_R < 0$ are (i) a_0, $a_1 > 0$, (ii) $a_1a_2 - a_0a_3 > 0$, (iii) $a_3(a_1a_2 - a_0a_3) - a_1(a_1a_4 - a_0a_5) > 0$, (iv) $(a_1a_4 - a_0a_5)[a_3(a_1a_2 - a_0a_3) - a_1(a_1a_4 - a_0a_5)] - a_5(a_1a_2 - a_0a_3)^2 > 0$, and (v) $a_5 > 0$. Condition (i) is already satisfied from (A1). Condition (ii) corresponds to the first inequality in Eq. (49a), while (iii) is the second inequality in Eqs. (49a) and (49b). Condition (iv) corresponds to Eqs. (49c)–(49e). Given that $E \geq 0$, thus, when $E = 0$ and (i)–(iv) are observed, then $\Gamma_R \leq 0$, which is in accordance with stability. Also, if $k = 0$, then $\Gamma_R \leq 0$ (three zero roots and two negative roots) because $a_0, a_1, a_2 > 0$ from (A1). Hence, the system is linearly stable in the local rest frame.

We remark that our system displays three types of hydrodynamic modes and three nonhydrodynamic modes. In the small k expansion that typically defines the linearized hydrodynamic regime, our shear channel gives a diffusive hydrodynamic mode with (real) frequency $\omega(k) = -ik^2\eta/(\varepsilon + P) + \cdots$, while in the sound channel one finds proper sound waves with $\omega(k) = \pm c_s k - ik^2\Gamma_s/2 + \cdots$ and also a heat diffusion mode with $\omega(k) = -iDk^2 + \cdots$, where $D \sim \sigma$, and $\Gamma_s = \Gamma_s(\eta, \zeta, \sigma)$ just as in Eckart theory (see Ref. [35] for their detailed expressions). Therefore, our theory has the same physical content of Eckart's theory in the hydrodynamic regime. On the other hand, the shear channel has a nonhydrodynamic mode with frequency given by $\omega(k) = -i/\tau_Q + \cdots$, while the sound channel has two nonhydrodynamic modes with frequency $\omega(k) = -i/\tau_\varepsilon + \cdots$ and $\omega(k) = -i/\tau_Q + \cdots$ in the low k limit. These nonhydrodynamic modes parametrize the UV behavior of the system in a way that ensures causality and

stability, making sure that the theory is well defined (though, of course, not accurate) even outside the typical domain of validity of hydrodynamics.

The complete proof of linear stability demands an analysis of the linearized system around an equilibrium state at nonzero velocity. In this regard, we shall use the results presented in Sec. VI B. We first write the system in Eq. (13) as a first-order linear system of PDEs. Then, since we already have proven causality and also linear stability in the LRF, it remains to be shown that the first-order counterpart of Eq. (13) is diagonalizable in the sense of Eq. (D2). This is done below.

First-order system.—Following Sec. VI A, we may define $\delta V = u^\alpha \partial_\alpha \delta\varepsilon$, $\delta\mathcal{V}^\mu = \Delta^{\mu\alpha}\partial_\alpha\delta\varepsilon$, $\delta W = u^\alpha\partial_\alpha\delta n$, $\delta\mathcal{W}^\mu = \Delta^{\mu\alpha}\partial_\alpha\delta W$, $\delta S^\mu = u^\alpha\partial_\alpha\delta u^\mu$, $\delta\mathcal{S}^\nu_\lambda = \Delta^\alpha_\lambda\partial_\alpha\delta u^\nu$. Since the current is ideal, i.e., $J^\mu = nu^\nu$, then the linearized conservation equation $\partial_\mu\delta J^\mu = \delta W + n\delta\mathcal{S}^\nu_\nu = 0$ enables us to eliminate δW from the new system of equations. Hence, the first-order equations become

$$\tau_\varepsilon u^\alpha\partial_\alpha\delta V + \rho\tau_Q\partial_\alpha\delta S^\alpha + \beta_\varepsilon\partial_\alpha\delta\mathcal{V}^\alpha + \beta_n\partial_\alpha\delta\mathcal{W}^\alpha$$
$$+ \rho\tau_\varepsilon u^\alpha\partial_\alpha\delta\mathcal{S}^\nu_\nu + \delta V + \rho\delta\mathcal{S}^\nu_\nu = 0, \qquad (53a)$$

$$\tau_P\Delta^{\mu\alpha}\partial_\alpha\delta V + \rho\tau_Q u^\alpha\partial_\alpha\delta S^\mu + \beta_\varepsilon u^\alpha\partial_\alpha\delta\mathcal{V}^\mu + \beta_n u^\alpha\partial_\alpha\delta\mathcal{W}^\mu$$
$$+ \Pi^{\mu\lambda\alpha}_\nu\partial_\nu\delta\mathcal{S}^\nu_\lambda + p'_\varepsilon\delta\mathcal{V}^\mu + p'_n\delta\mathcal{W}^\mu + \rho\delta S^\mu = 0, \qquad (53b)$$

$$u^\alpha\partial_\alpha\delta\mathcal{V}^\mu - \Delta^{\mu\alpha}\partial_\alpha\delta V = 0, \qquad (53c)$$

$$u^\alpha\partial_\alpha\delta\mathcal{W}^\mu + n\Delta^{\mu\alpha}\partial_\alpha\delta\mathcal{S}^\nu_\nu = 0, \qquad (53d)$$

$$u^\alpha\partial_\alpha\delta\mathcal{S}^\mu_\lambda - \Delta^\alpha_\lambda\partial_\alpha\delta S^\mu = 0, \qquad (53e)$$

where $p'_n = (\partial P/\partial n)_\varepsilon$ and

$$\Pi^{\mu\lambda\alpha}_\nu = -\eta(\Delta^{\mu\lambda}\delta^\alpha_\nu + \Delta^{\lambda\alpha}\delta^\mu_\nu) + \left(\rho\tau_P - \zeta + \frac{2\eta}{3}\right)\Delta^{\mu\alpha}\delta^\lambda_\nu. \quad (54)$$

The supplemental equations (53c)–(53e) come from the identities $\partial_\alpha\partial_\beta\delta\varepsilon - \partial_\beta\partial_\alpha\delta\varepsilon = 0$, $\partial_\alpha\partial_\beta\delta n - \partial_\beta\partial_\alpha\delta n = 0$, and $\partial_\alpha\partial_\beta\delta u^\mu - \partial_\beta\partial_\alpha\delta u^\mu = 0$, respectively, when contracted with $u^\alpha\Delta^{\beta\lambda}$. In particular, in Eq. (53d) we have substituted $\delta W = -n\delta\mathcal{S}^\nu_\nu$ that comes from the conservation equation of J^μ. Then, we may write Eq. (53) in matrix form, $\mathbb{A}^\alpha\partial_\alpha\delta\Psi(X) + \mathbb{B}\Psi(X) = 0$, were $\delta\Psi(X)$ is the 29×1 column matrix with entries $\delta V, \delta S^\nu, \delta\mathcal{V}^\nu, \delta\mathcal{W}^\nu, \delta\mathcal{S}^\nu_0$, $\delta\mathcal{S}^\nu_1, \delta\mathcal{S}^\nu_2, \delta\mathcal{S}^\nu_3$,

$$
\mathbb{A}^\alpha = \begin{bmatrix}
\tau_\varepsilon u^\alpha & \rho\tau_Q\delta_\nu^\alpha & \beta_\varepsilon\delta_\nu^\alpha & \beta_n\delta_\nu^\alpha & \rho\tau_\varepsilon u^\alpha\delta_\nu^0 & \rho\tau_\varepsilon u^\alpha\delta_\nu^1 & \rho\tau_\varepsilon u^\alpha\delta_\nu^2 & \rho\tau_\varepsilon u^\alpha\delta_\nu^3 \\
\tau_P\Delta^{\mu\alpha} & \rho\tau_Q u^\alpha\delta_\nu^\mu & \beta_\varepsilon u^\alpha\delta_\nu^\mu & \beta_n u^\alpha\delta_\nu^\mu & \Pi_\nu^{\mu0\alpha} & \Pi_\nu^{\mu1\alpha} & \Pi_\nu^{\mu2\alpha} & \Pi_\nu^{\mu3\alpha} \\
-\Delta^{\mu\alpha} & 0_{4\times4} & u^\alpha\delta_\nu^\mu & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} \\
0_{4\times4} & 0_{4\times4} & 0_{4\times4} & u^\alpha\delta_\nu^\mu & n\Delta^{\mu\alpha}\delta_\nu^0 & n\Delta^{\mu\alpha}\delta_\nu^1 & n\Delta^{\mu\alpha}\delta_\nu^2 & n\Delta^{\mu\alpha}\delta_\nu^3 \\
0_{4\times1} & -\Delta_0^\alpha\delta_\nu^\mu & 0_{4\times4} & 0_{4\times4} & u^\alpha\delta_\nu^\mu & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} \\
0_{4\times1} & -\Delta_1^\alpha\delta_\nu^\mu & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & u^\alpha\delta_\nu^\mu & 0_{4\times4} & 0_{4\times4} \\
0_{4\times1} & -\Delta_2^\alpha\delta_\nu^\mu & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & u^\alpha\delta_\nu^\mu & 0_{4\times4} \\
0_{4\times1} & -\Delta_3^\alpha\delta_\nu^\mu & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & u^\alpha\delta_\nu^\mu
\end{bmatrix},
\tag{55}
$$

and

$$
\mathbb{B} = \begin{bmatrix}
1 & 0_{1\times4} & 0_{1\times4} & 0_{1\times4} & \rho\delta_\nu^0 & \rho\delta_\nu^1 & \rho\delta_\nu^1 & \rho\delta_\nu^3 \\
0_{4\times1} & \rho\delta_\nu^\mu & p_\varepsilon'\delta_\nu^\mu & p_n'\delta_\nu^\mu & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} \\
0_{4\times1} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} \\
0_{4\times1} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} \\
0_{4\times1} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} \\
0_{4\times1} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} \\
0_{4\times1} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} \\
0_{4\times1} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4}
\end{bmatrix}.
\tag{56}
$$

We must now obtain the eigenvectors of Eq. (37). However, note that $\mathbb{A}^\alpha$ above is exactly the same as the matrix $\mathbb{A}_m^\alpha$ in Eq. (27) with the difference that now the coefficients of $\mathbb{A}^\alpha$ are constants. We have already proven in Sec. V that the matrix $\mathbb{A}_m^\alpha$ in Eq. (37) has real eigenvalues and a complete set of eigenvectors in $\mathbb{R}^{29}$. The same solution is true for $\mathbb{A}^\alpha$ in Eq. (37) if we change $\xi_\alpha \to n_\alpha$ (and also $\mathbb{A}_m^\alpha \to \mathbb{A}^\alpha$) in the results for the matter sector in Sec. V. Thus, the 29×29 matrix $(-n_\alpha\mathbb{A}^\alpha)\zeta_\beta\mathbb{A}^\beta$ is diagonalizable, completing the requirements from Theorem III. This shows that the theory is linearly stable in any other reference frame $\mathcal{O}'$ connected via a Lorentz transformation. Therefore, one then obtains that our set of linearized second-order PDEs is stable in any equilibrium state.

A. Fulfilling the causality, local well posedness, and linear stability conditions

We now give a simple example that illustrates that the set of linear stability conditions (and consequently, causality and local well posedness, since those are part of the linear stability conditions) is not empty. Let us analyze the case where $\tau_Q = \tau_\varepsilon$ and $\tau_P = c_s^2\tau_\varepsilon$, assuming an equation of state $P = P(\varepsilon)$, with $c_s^2 = p_\varepsilon' = 1/2$. Also, assume that $\zeta + 4\eta/3 > 0$ (their specific values are not relevant as far as they are positive and $\eta > 0$ for the sake of the stability and well-posedness theorems). Then, one may easily verify that

the causality conditions (21) hold in their strict form, as required, and that the remaining conditions (49) are also observed when $\rho\tau_\varepsilon = 8(\zeta + 4\eta/3)$, $\kappa_\varepsilon = \kappa_s/2 = 1/4$, and in the three different situations, namely, $\sigma/(\zeta + 4\eta/3) = 0$, $1/4$, and 1.

VIII. CONCLUSIONS AND OUTLOOK

In this work, we presented the first generalization of relativistic Navier-Stokes theory that simultaneously satisfies the following properties: the system, with or without coupling to Einstein's equations, is causal, strongly hyperbolic, and locally well posed (see the content of Theorems I and II); equilibrium states in flat spacetime are stable (consequence of Theorem III); all dissipative contributions (shear viscosity, bulk viscosity, and heat conductivity) are included; and finally the effects from nonzero baryon number are also taken into account. All of the above hold without any simplifying symmetry assumptions and are mathematically rigorously established. In addition, entropy production is non-negative in the regime of validity of this effective theory.

This is accomplished in a natural way using a generalized Navier-Stokes theory containing only the original hydrodynamic variables, which is different than other approaches where the space of variables is extended (such as in Müller-Israel-Stewart theory). However, it is important to remark that the meaning of the hydrodynamic variables in our work is different than in standard approaches, such as Refs. [15,52]. In fact, in the context of the formalism put forward by Bemfica *et al.* [32,33] and Kovtun [34], our formulation uses a definition for the hydrodynamic variables (i.e., our choice of hydrodynamic frame) that is not standard as there are nonzero out-of-equilibrium corrections to the energy density and there is energy and heat diffusion even at zero baryon density. Despite these necessary differences (imposed by causality and stability), the theory still provides the simplest causal and strongly hyperbolic generalization of Eckart's original theory [52], sharing the same physical properties in the hydrodynamic regime (for instance, both theories have the same spectrum of hydrodynamic modes). However,

differently than Eckart's approach, our formulation is fully compatible with the postulates of general relativity, and its physical content in dynamical settings can be readily investigated using numerical relativity simulations. In fact, we hope that the framework presented here will provide the starting point for future systematic studies of viscous phenomena in the presence of strong gravitational fields, such as in neutron star mergers.

Motivated by the task of establishing stability of general equilibrium states in flat spacetime, in this work we also proved a new general result (see Theorem III) concerning the stability of relativistic fluids. In fact, we found conditions that causal relativistic fluids should satisfy such that stability around the static equilibrium state directly implies stability in any other equilibrium state at nonzero background velocity. Theorem III is very general and its regime of applicability goes beyond BDNK theories and it could also be relevant when investigating the stability properties of other sets of linear equations of motion as well. In this regard, see the discussion in Sec. II B, and see also Secs VI B 1 and VI B 2 for further examples of the applicability of Theorem III.

Our generalized Navier-Stokes theory can be used to understand how matter in general relativity starts to deviate from equilibrium. An immediate application is in the modeling of viscous effects in neutron star mergers. Our approach can be useful in simulations that aim at determining the fate of the hypermassive remnant formed after the merger of neutron stars, hopefully leading to a better quantitative understanding of their evolution and eventual gravitational collapse toward a black hole. Differently than any other approach in the literature, the new features displayed by our formulation and its strongly hyperbolic character make it a suitable candidate to be used in such simulations. This will be especially relevant also when considering how viscous effects may modify the gravitational wave signals emitted soon after the merger [12,14]. In this regard, we remark that previous simulations performed in Ref. [11] employed a formulation of relativistic viscous hydrodynamics where the key properties studied here (causality, strong hyperbolicity, and local well posedness) are not known to hold in the nonlinear regime.

Our work is applicable in the case of baryon-rich matter, such as that formed in neutron star mergers or in low-energy heavy-ion collisions. The latter include the experimental efforts in the beam energy scan program at RHIC [176], the STAR fixed-target program [176], the HADES experiment at GSI [177], the future FAIR facility at GSI [178], and also NICA [179]. For a discussion of viscous effects in low-energy heavy-ion collisions at nonzero density, see Refs. [85,113,180]. High-energy heavy-ion collisions, such as those studied at the LHC, involve a different regime than the one considered here where the net baryon number can be very small and, thus, that case is better understood using a different formulation such as the

one proposed in Ref. [35], also in the context of the BDNK formalism.

In our approach, we only take into account first-order derivative corrections to the dynamics. Therefore, the domain of validity of our theory is currently limited by the size of such deviations. Hence, further work is needed to extend our analysis, incorporating higher-order derivative corrections, to get a better understanding of what happens as the system gets farther and farther from equilibrium. In this context, it would be interesting to extend our equations to include second-order corrections and consider also, more generally, the large order behavior of the gradient expansion in an arbitrary hydrodynamic frame. The latter will be different than most approaches to the gradient expansion since in BDNK the constitutive relations contain time derivatives even in the local rest frame of the fluid. This essential difference has important consequences in a kinetic theory formulation; see the original references [32,33]. The large order behavior of the relativistic gradient series has been recently the focus of several works [84,181–194], and it would be interesting to extend such analyses to include the type of theories investigated here.

There are a number of ways in which our work could be extended or improved. First, it would be useful to obtain a better qualitative understanding of why some hydrodynamic frames (such as the Landau-Lifshitz frame or the Eckart frame) are not compatible with causality and stability in the BDNK approach, given that the situation is different in other formulations. In fact, the Landau frame seems to display no significant issues in the case of MIS-like theories in the nonlinear regime at least at zero chemical potential, as demonstrated in Ref. [72]. Perhaps a more in-depth investigation of how BDNK emerges in kinetic theory, going beyond the original work done in Refs. [32,33], can be useful in this regard (see also the recent work [148]). Also, it would be interesting to use the BDNK approach to investigate causality and stability in more exotic cases, such as in relativistic superfluids. Furthermore, the inclusion of electromagnetic field effects in the dynamics of relativistic viscous fluids can also be of particular relevance, especially in the context of neutron star mergers [195] and high-energy heavy-ion collisions [196]. This problem has been recently investigated using other formulations of viscous fluid dynamics, see for instance Refs. [197–202], and also most recently in the BDNK approach in Ref. [203]. Consistent modeling of relativistic viscous fluid dynamics coupled to electromagnetic fields can also be relevant to determine the importance of dissipative processes in the dynamics and radiative properties of slowly accreting black holes, as discussed in Ref. [197].

Further work needs to be done to understand the global in-time features of solutions of relativistic viscous fluid dynamics. For instance, one may investigate the presence

of shocks, which is a topic widely investigated in the context of ideal fluids [21,145,204–206] and was done in Ref. [94] for the MIS theory (see Sec. II B for further discussion on shocks). The importance of hydrodynamic shocks has been recognized both in an astrophysical setting [197] as well as in the study of jets in the quark-gluon plasma [207–219]. We also remark that one task that we have not done here was the construction of initial data for the full Einstein plus fluid system by solving the Einstein constraint equations. We believe that standard arguments to handle the constraints [21] will be applicable in our case. This will be investigated in detail in a future work.

We believe our work will also be relevant to give insight into the physics of turbulent fluids embedded in general relativity. The fact that the equations of motion of the viscous fluid must be hyperbolic in relativity stands in sharp contrast to the parabolic nature of the nonrelativistic Navier-Stokes equations, usually employed in studies of turbulence. Recent works in Refs. [18,220] tackled the problem of turbulence in the relativistic regime and our formulation may be very useful in this regard, as it provides a simple strongly hyperbolic generalization of Eckart's theory that is fully compatible with general relativity.

In summary, in this paper we propose a new solution to the question initiated by Eckart in 1940 concerning the motion of viscous fluids in relativity. Our approach is rooted in well-known physical principles and solid mathematics, displays a number of desired properties, and extends the state of the art of the field in a number of ways. Potential applications of the formalism presented here spread across a numbers of areas, including astrophysics, nuclear physics, cosmology, and mathematical physics. This work establishes for the first time a common unifying framework, from heavy-ion collisions to neutron stars, that can be used to discover the novel properties displayed by ultradense baryonic matter as it evolves in spacetime.

ACKNOWLEDGMENTS

APPENDIX A: PROOF OF THEOREM I

We only consider the 10 independent components of the metric and, thus, this system of equations can be written in terms of a 16×1 column vector $\Psi = (\varepsilon, n, u^\nu, g_{\mu\nu})$, and its equation of motion in Eq. (17) can be expressed in matrix form as $\mathfrak{M}(\partial)\Psi = \mathfrak{N}$, where $\mathfrak{N}$ contains the $\mathcal{B}$ terms that do not enter in the principal part. The matrix $\mathfrak{M}(\partial)$ is given by

$$\mathfrak{M}(\partial) = \begin{bmatrix} \mathbb{M}(\partial) & \mathfrak{b}(\partial) \\ 0_{6\times10} & I_{10}g^{\alpha\beta}\partial^2_{\alpha\beta} \end{bmatrix}, \tag{A1}$$

where the 6×10 matrix $\mathfrak{b}(\partial)$ contains the $\tilde{\mathcal{B}}$ terms and

$$\mathbb{M}(\partial) = \begin{bmatrix} 0 & u^\alpha u^\beta & n\delta_\nu^{(\alpha}u^{\beta)} \\ (\tau_\varepsilon u^\alpha u^\beta + \beta_\varepsilon \Delta^{\alpha\beta}) & \beta_n \Delta^{\alpha\beta} & \rho(\tau_\varepsilon + \tau_Q)u^{(\alpha}\delta_\nu^{\beta)} \\ (\beta_\varepsilon + \tau_P)u^{(\alpha}\Delta^{\beta)\mu} & \beta_n u^{(\alpha}\Delta^{\beta)\mu} & C_\nu^{\mu\alpha\beta} \end{bmatrix}$$
$$\times \partial^2_{\alpha\beta}. \tag{A2}$$

The system's characteristics are obtained by replacing $\partial_\alpha \to \xi_\alpha$ and determining the roots of $\det[\mathfrak{M}(\xi)] = 0$. The system is causal when the solutions for $\xi_\alpha = (\xi_0(\xi_i), \xi_i)$ are such that condition 1 (Cond-1) ξ_α is real and condition 2 (Cond-2) $\xi_\mu\xi^\mu \geq 0$ [21]. It is easy to see that $\det[\mathfrak{M}(\xi)] = (\xi_\alpha\xi^\alpha)^{10}\det[\mathbb{M}(\xi)]$. The roots associated with the vanishing of the overall factor $(\xi_\alpha\xi^\alpha)^{10} = 0$ coming from the gravitational sector are clearly causal. The remaining roots come from $\det[\mathbb{M}(\xi)] = 0$, which we will investigate next.

We first define $b \equiv u^\alpha\xi_\alpha$ and $v^\alpha \equiv \Delta^{\alpha\beta}\xi_\beta$, which gives $\xi_\alpha = -bu_\alpha + v_\alpha$ and $\xi_\alpha\xi^\alpha = -b^2 + v \cdot v$, where $v \cdot v = \Delta^{\alpha\beta}\xi_\alpha\xi_\beta$. We proceed by also defining the tensor

$$\mathfrak{D}_\nu^\mu = C_\nu^{\mu\alpha\beta}\xi_\alpha\xi_\beta$$
$$= \left(\tau_P\rho - \zeta - \frac{\eta}{3}\right)v^\mu\xi_\nu + [\rho\tau_Q b^2 - \eta(v \cdot v)]\delta_\nu^\mu, \tag{A3}$$

which gives

$$\det[\mathbb{A}(\xi)] = \det \begin{bmatrix} 0 & b^2 & nb\xi_\nu \\ \tau_\varepsilon b^2 + \beta_\varepsilon(v \cdot v) & \beta_n(v \cdot v) & \rho(\tau_\varepsilon + \tau_Q)bv_\nu \\ (\beta_\varepsilon + \tau_P)bv^\mu & \beta_n bv^\mu & \mathfrak{D}_\nu^\mu \end{bmatrix}$$
$$= -b^2[\rho\tau_Q b^2 - \eta(v \cdot v)]^3$$
$$\times [Ab^4 + Bb^2(v \cdot v) + C(v \cdot v)^2] \tag{A4a}$$
$$= -\rho^4\tau_Q^4\tau_\varepsilon(u^\alpha\xi_\alpha)^2 \prod_{a=1,\pm}[(u^\alpha\xi_\alpha)^2 - c_a\Delta^{\alpha\beta}\xi_\alpha\xi_\beta]^{n_a}, \tag{A4b}$$

where, to shorten notation in Eq. (A4a) we defined

$$A \equiv \rho\tau_\varepsilon\tau_Q, \tag{A5a}$$

$$B \equiv -\tau_\varepsilon\left(\rho c_s^2\tau_Q + \zeta + \frac{4\eta}{3} + \sigma\kappa_s\right) - \rho\tau_P\tau_Q, \tag{A5b}$$

$$C \equiv \tau_P(\rho c_s^2\tau_Q + \sigma\kappa_s) - \beta_\varepsilon\left(\zeta + \frac{4\eta}{3}\right), \tag{A5c}$$

and used the fact that $\beta_\varepsilon + n\beta_n/\rho = \tau_Q c_s^2 + \sigma\kappa_s/\rho$. In Eq. (A4a) it becomes evident that assumption (A1) guarantees that $v^\mu \neq 0$, eliminating one of the possible acausal roots. From Eqs. (A4a)–(A4b) we defined $n_1 = 3$, $n_\pm = 1$, $c_1 = [\eta/(\rho\tau_s)]$, and $c_\pm = [(-B \pm \sqrt{B^2 - 4AC})/2A]$. Note that since $\xi^\alpha\xi_\alpha = -b^2 + (v \cdot v)$, the roots in Eq. (A4b) can be cast as $b^2 = c_a(v \cdot v)$. Then, (Cond-1) demands that $c_a \in \mathbb{R}$ together with $c_a \geq 0$ and (Cond-2) that $c_a < 1$ for causality [221], which comes from the fact that the root $b^2 = c_a v \cdot v$ must obey $\xi_\mu\xi^\mu = -b^2 + v \cdot v = (1 - c_a)v \cdot v > 0$. Thus, causality is ensured if $0 \leq c_a < 1$ in the matter sector. Clearly, the root $b = u^\alpha\xi_\alpha = 0$ is causal. Also, the six roots related to c_1 are causal when Eq. (21a) is observed. As for the roots $c_\pm$, they are real if $B^2 - 4AC \geq 0$, i.e., if the first inequality in Eq. (21b) holds. On the other hand, $c_\pm \geq 0$ is obtained whenever $c_- \geq 0$, which is guaranteed if $-B \geq 0$ [second inequality in condition (21c)] together with $C \geq 0$ [second inequality of Eq. (21a)], while $c_\pm < 1$ is ensured if $c_+ < 1$, which demands that $2A + B > 0$ [first inequality in condition (21c)] and $A + B + C > 0$ [condition (21d)]. ∎

We observe that, although we employed the harmonic gauge to calculate the system's characteristics, the causality established in Theorem I does not depend on any gauge choices. This follows from well-known properties of Einstein's equations [22] and the geometric invariance of the characteristics [144]. See the end of Sec. V C for further comments in this direction.

The analysis above and the conditions we obtained for causality are valid in the full nonlinear regime of the theory. However, we remark in passing that the principal part concerning only the fluid equations would have exactly the same structure if one were to linearize the fluid dynamic equations about equilibrium with nonzero flow in Minkowski spacetime. This is a generic feature of the BDNK approach (at least, when truncated at first order), i.e., the analysis of the system's characteristics, and thus of its causality properties, is formally the same in the nonlinear regime and in the linearization about a generic equilibrium state. This is not, however, a general feature of hydrodynamic models as it does not hold in MIS-like theories. In fact, as discussed at length in Refs. [71,72], in MIS the thermodynamic fluxes explicitly enter in the calculation of the characteristics, but they are not present in the linear analysis.

APPENDIX B: PROOF OF PROPOSITION I

To prove (i) we may compute the determinant $\det(\xi_\alpha\mathfrak{A}^\alpha) = \det(\xi_\alpha\mathbb{A}_m^\alpha)\det(\xi_\alpha\mathbb{A}_g^\alpha)(u^\alpha\xi_\alpha)^{16}$. Note that $u^\alpha\xi_\alpha \neq 0$ if ξ is timelike. We must then look into the matter and gravity sector in what follows. We again define $b = u^\alpha\xi_\alpha$ and $v^\mu = \Delta^{\mu\alpha}\xi_\alpha$, $v \cdot v = \Delta^{\mu\nu}\xi_\mu\xi_\nu$, and introduce

$$\Xi_\nu^\mu = v_\lambda\Pi_\nu^{\mu\lambda\alpha}\xi_\alpha$$

$$= -\eta(v \cdot v)\delta_\nu^\mu - \eta v^\mu\xi_\nu + \left(\rho\tau_P - \zeta + \frac{2\eta}{3}\right)v^\mu v_\nu \quad \text{(B1)}$$

to obtain

$$\det(\xi_\alpha\mathbb{A}_m^\alpha) = \det\begin{bmatrix} \tau_\varepsilon b & \rho\tau_Q\xi_\nu & \beta_\varepsilon\xi_\nu & \beta_n\xi_\nu & \rho\tau_\varepsilon b\delta_\nu^0 & \rho\tau_\varepsilon b\delta_\nu^1 & \rho\tau_\varepsilon b\delta_\nu^2 & \rho\tau_\varepsilon b\delta_\nu^3 \\ \tau_P v^\mu & \rho\tau_Q b\delta_\nu^\mu & \beta_\varepsilon b\delta_\nu^\mu & \beta_n b\delta_\nu^\mu & \Pi_\nu^{\mu 0\alpha}\xi_\alpha & \Pi_\nu^{\mu 1\alpha}\xi_\alpha & \Pi_\nu^{\mu 2\alpha}\xi_\alpha & \Pi_\nu^{\mu 3\alpha}\xi_\alpha \\ -v^\mu & 0_{4\times 4} & b\delta_\nu^\mu & 0_{4\times 4} & 0_{4\times 4} & 0_{4\times 4} & 0_{4\times 4} & 0_{4\times 4} \\ 0_{4\times 1} & 0_{4\times 4} & 0_{4\times 4} & b\delta_\nu^\mu & nv^\mu\delta_\nu^0 & nv^\mu\delta_\nu^1 & nv^\mu\delta_\mu^2 & nv^\mu\delta_\nu^3 \\ 0_{4\times 1} & -v_0\delta_\nu^\mu & 0_{4\times 4} & 0_{4\times 4} & b\delta_\nu^\mu & 0_{4\times 4} & 0_{4\times 4} & 0_{4\times 4} \\ 0_{4\times 1} & -v_1\delta_\nu^\mu & 0_{4\times 4} & 0_{4\times 4} & 0_{4\times 4} & b\delta_\nu^\mu & 0_{4\times 4} & 0_{4\times 4} \\ 0_{4\times 1} & -v_2\delta_\nu^\mu & 0_{4\times 4} & 0_{4\times 4} & 0_{4\times 4} & 0_{4\times 4} & b\delta_\nu^\mu & 0_{4\times 4} \\ 0_{4\times 1} & -v_3\delta_\nu^\mu & 0_{4\times 4} & 0_{4\times 4} & 0_{4\times 4} & 0_{4\times 4} & 0_{4\times 4} & b\delta_\nu^\mu \end{bmatrix}$$

$$= b^{19}\det\begin{bmatrix} \tau_\varepsilon b^2 + \beta_\varepsilon(v \cdot v) & b^2(\rho\tau_Q\xi_\nu + \rho\tau_\varepsilon v_\nu) - n\beta_n(v \cdot v)v_\nu \\ (\tau_P + \beta_\varepsilon)v^\mu & \rho\tau_Q b^2\delta_\nu^\mu + \Xi_\nu^\mu - n\beta_n v^\mu v_\nu \end{bmatrix}$$

$$= b^{19}[\rho\tau_Q b^2 - \eta(v \cdot v)]^3[Ab^4 + Bb^2(v \cdot v) + C(v \cdot v)^2]$$

$$= \rho^4\tau_Q^4\tau_\varepsilon b^{19}\prod_{a=1,\pm}[b^2 - c_a(v \cdot v)]^{n_a}, \quad \text{(B2)}$$

where, as we have obtained in Eqs. (A4) and (A5), and in the text below it,

$$A \equiv \rho\tau_\varepsilon\tau_Q, \quad \text{(B3a)}$$

$$B \equiv -\tau_\varepsilon\left(\rho c_s^2\tau_Q + \zeta + \frac{4\eta}{3} + \sigma\kappa_s\right) - \rho\tau_P\tau_Q, \quad \text{(B3b)}$$

$$C \equiv \tau_P(\rho c_s^2\tau_Q + \sigma\kappa_s) - \beta_\varepsilon\left(\zeta + \frac{4\eta}{3}\right), \quad \text{(B3c)}$$

$n_1 = 3$, $n_\pm = 1$, $c_1 = [\eta/(\rho\tau_s)]$, and $c_\pm = [(-B \pm \sqrt{B^2 - 4AC})/2A]$. It is worth mentioning that the assumptions of Proposition I guarantee that $0 < c_1, c_\pm < 1$. Under assumptions (A1), $\eta > 0$, and conditions (21) in the strict form, then one obtains that $\det(\xi_\alpha A_m^\alpha) = 0$ only if $0 \leq c_a < 1$

(with the equality holding only in the case $a = 0$); i.e., the equation $b_a^2 - c_a(v_a \cdot v_a) = 0$ gives $\xi_{a,\alpha}$ such that $\xi_{a,\alpha}\xi_a^\alpha = -b_a^2 + v_a \cdot v_a = (1 - c_a)v_a \cdot v_a > 0$. Thus, if ξ is timelike, then (i) is guaranteed for the matter sector as well. As for the gravity sector, one obtains that

$$\det(\xi_\alpha \mathbb{A}_g^\alpha) = \det \begin{bmatrix} bI_{10} & -v_0 I_{10} & -v_1 I_{10} & -v_2 I_{10} & -v_3 I_{10} \\ -v^0 I_{10} & bI_{10} & 0_{10\times10} & 0_{10\times10} & 0_{10\times10} \\ -v^1 I_{10} & 0_{10\times10} & bI_{10} & 0_{10\times10} & 0_{10\times10} \\ -v^2 I_{10} & 0_{10\times10} & 0_{10\times10} & bI_{10} & 0_{10\times10} \\ -v^3 I_{10} & 0_{10\times10} & 0_{10\times10} & 0_{10\times10} & bI_{10} \end{bmatrix}$$

$$= \frac{1}{b^{10}} \det \begin{bmatrix} (b^2 - v^\nu v_\nu)I_{10} & 0_{10\times10} & 0_{10\times10} & 0_{10\times10} & 0_{10\times10} \\ -v^0 I_{10} & bI_{10} & 0_{10\times10} & 0_{10\times10} & 0_{10\times10} \\ -v^1 I_{10} & 0_{10\times10} & bI_{10} & 0_{10\times10} & 0_{10\times10} \\ -v^2 I_{10} & 0_{10\times10} & 0_{10\times10} & bI_{10} & 0_{10\times10} \\ -v^3 I_{10} & 0_{10\times10} & 0_{10\times10} & 0_{10\times10} & bI_{10} \end{bmatrix}$$

$$= (u^\alpha \xi_\alpha)^{30}(\xi_\alpha \xi^\alpha)^{10}. \tag{B4}$$

Again, note that if ξ is timelike, then $\det(\xi_\alpha \mathbb{A}_g^\alpha) \neq 0$. This completes the proof of (i).

As for (ii), let us define $\phi_\alpha = \zeta_\alpha + \Lambda\xi_\alpha$ and make the changes $\xi \to \phi$ in the determinant calculations above. Then, the eigenvalues Λ are obtained from the roots of $\det(\phi_\alpha \mathfrak{A}^\alpha) = \det(\phi_\alpha \mathbb{A}_m^\alpha) \det(\phi_\alpha \mathbb{A}_g^\alpha)(u^\alpha\phi_\alpha)^{16} = 0$. Note that the general form of the equations implies that the roots $\phi_\alpha = -u_\alpha u^\beta \phi_\beta + \Delta_\alpha^\beta \phi_\beta$ obey

$$(u^\alpha \phi_\alpha)^2 - \beta\Delta^{\alpha\beta}\phi_\alpha\phi_\beta = 0, \tag{B5}$$

where, from causality, in any of the above cases we have that $0 \leq \beta \leq 1$. Then, for each β, the eigenvalues Λ are

$$\Lambda = \frac{\beta(\Delta^{\alpha\beta}\xi_\alpha\zeta_\beta) - (u^\alpha\xi_\alpha)(u^\alpha\zeta_\alpha) \pm \sqrt{\mathcal{Z}}}{(u^\alpha\xi_\alpha)^2 - \beta\Delta^{\alpha\beta}\xi_\alpha\xi_\beta}, \tag{B6}$$

where, since $\xi_\alpha\xi^\alpha < 0$, then $(u^\alpha\xi_\alpha)^2 - \beta\Delta^{\alpha\beta}\xi_\alpha\xi_\beta > 0$ because $0 \leq \beta \leq 1$ and

$$\begin{aligned}\mathcal{Z} &= \beta\{\Delta^{\alpha\beta}\zeta_\alpha\zeta_\beta(u^\mu\xi_\mu)^2 + \Delta^{\alpha\beta}\xi_\alpha\xi_\beta(u^\mu\zeta_\mu)^2 - 2(u^\alpha\xi_\alpha)(u^\beta\zeta_\beta)\Delta^{\mu\nu}\xi_\mu\zeta_\nu - \beta[(\Delta^{\alpha\beta}\zeta_\alpha\zeta_\beta)(\Delta^{\mu\nu}\xi_\mu\xi_\nu) - (\Delta^{\alpha\beta}\xi_\alpha\zeta_\beta)^2]\} \\ &> \beta[\Delta^{\alpha\beta}\zeta_\alpha\zeta_\beta(u^\mu\xi_\mu)^2 + \Delta^{\alpha\beta}\xi_\alpha\xi_\beta(u^\mu\zeta_\mu)^2 - 2(u^\alpha\xi_\alpha)(u^\beta\zeta_\beta)\Delta^{\mu\nu}\xi_\mu\zeta_\nu - (\Delta^{\alpha\beta}\zeta_\alpha\zeta_\beta)(\Delta^{\mu\nu}\xi_\mu\xi_\nu) + (\Delta^{\alpha\beta}\xi_\alpha\zeta_\beta)^2] \\ &= \beta\{(-\xi^\alpha\xi_\alpha)(\zeta^\beta\zeta_\beta) + [(u^\alpha\xi_\alpha)(u^\beta\zeta_\beta) - \Delta^{\alpha\beta}\xi_\alpha\zeta_\beta]^2\} > 0. \end{aligned} \tag{B7}$$

In the operations above we used the fact that $0 \leq \beta \leq 1$, $(\Delta^{\alpha\beta}\xi_\alpha\zeta_\beta)^2 \leq (\Delta^{\alpha\beta}\xi_\alpha\xi_\beta)(\Delta^{\mu\nu}\zeta_\mu\zeta_\nu)$ from the Cauchy-Schwarz inequality and that ξ is timelike and ζ spacelike. Thus, causality guarantees reality of the eigenvalues.

Now we turn to the problem of completeness of the set of eigenvectors. We begin by counting the linearly independent eigenvectors of $\phi_{a,\alpha}^{(m)}\mathbb{A}_m^\alpha$, where $\phi_{a,\alpha}^{(m)} = \zeta_\alpha + \Lambda_a^{(m)}\xi_\alpha$ and $\Lambda_a^{(m)}$ are the eigenvalues of the matter sector and are obtained by means of Eq. (B6) in the cases $\beta = c_0 = 0$ when $a = 0$ and $\beta = c_a$ when $a = 1, \pm$. Let us define an arbitrary vector,

$$r^{(m)} = \begin{bmatrix} F \\ G^\nu \\ H^\mu \\ I^\mu \\ J_0^\nu \\ J_1^\nu \\ J_2^\nu \\ J_3^\nu \end{bmatrix}. \tag{B8}$$

Then, for each of the eigenvalues $\Lambda_a^{(m)}$, $a = 0, 1, \pm$, we must verify how many of the 29 variables in the vector (B8) are free parameters under the equation $\phi_{a,\alpha}^{(m)} \mathbb{A}_m^\alpha r_a^{(m)} = 0$. In fact, this is the dimension of the null space of the matrix $\phi_{a,\alpha}^{(m)} \mathbb{A}_m^\alpha$ and corresponds to the number of linearly independent eigenvectors of $\Lambda_a^{(m)}$. The eigenvectors are the following.

(i) $\Lambda_0^{(m)}$: This root has multiplicity 19. The eigenvector that obeys $\phi_{0,\alpha}^{(m)} \mathbb{A}^\alpha r_0^{(m)} = 0$ is

$$
r_0^{(m)} =
\begin{bmatrix}
0 \\
0_{4\times 1} \\
H^\mu \\
I^\mu \\
J_0^\nu \\
J_1^\nu \\
J_2^\nu \\
J_3^\nu
\end{bmatrix},
\qquad \text{(B9)}
$$

$$
\begin{bmatrix}
\tau_\varepsilon b^2 + \beta_\varepsilon(v\cdot v) & b\rho\tau_Q\phi_\nu + b\rho\tau_\varepsilon v_\nu - \frac{n\beta_n(v\cdot v)}{b}v_\nu & 0_{1\times 4} & 0_{1\times 4} & 0_{1\times 4} & 0_{1\times 4} & 0_{1\times 4} & 0_{1\times 4} \\
0_{4\times 1} & \mathcal{K}_\nu v^\mu & 0_{4\times 4} & 0_{4\times 4} & 0_{4\times 4} & 0_{4\times 4} & 0_{4\times 4} & 0_{4\times 4} \\
-v^\mu & 0_{4\times 4} & b\delta_\nu^\mu & 0_{4\times 4} & 0_{4\times 4} & 0_{4\times 4} & 0_{4\times 4} & 0_{4\times 4} \\
0_{4\times 1} & \frac{nv^\mu v_\nu}{b} & 0_{4\times 4} & b\delta_\nu^\mu & 0_{4\times 4} & 0_{4\times 4} & 0_{4\times 4} & 0_{4\times 4} \\
0_{4\times 1} & -v_0\delta_\nu^\mu & 0_{4\times 4} & 0_{4\times 4} & b\delta_\nu^\mu & 0_{4\times 4} & 0_{4\times 4} & 0_{4\times 4} \\
0_{4\times 1} & -v_1\delta_\nu^\mu & 0_{4\times 4} & 0_{4\times 4} & 0_{4\times 4} & b\delta_\nu^\mu & 0_{4\times 4} & 0_{4\times 4} \\
0_{4\times 1} & -v_2\delta_\nu^\mu & 0_{4\times 4} & 0_{4\times 4} & 0_{4\times 4} & 0_{4\times 4} & b\delta_\nu^\mu & 0_{4\times 4} \\
0_{4\times 1} & -v_3\delta_\nu^\mu & 0_{4\times 4} & 0_{4\times 4} & 0_{4\times 4} & 0_{4\times 4} & 0_{4\times 4} & b\delta_\nu^\mu
\end{bmatrix}
r_1^{(m)} = 0, \qquad \text{(B10)}
$$

where

$$
\mathcal{K}_\nu = \left[-\eta\xi_\nu + \left(\rho\tau_P - \zeta + \frac{2\eta}{3} - n\beta_n \right) v_\nu \right]
$$
$$
\times [\tau_\varepsilon b^2 + \beta_\varepsilon(v\cdot v)] - (\tau_P + \beta_\varepsilon)[b^2\rho\tau_Q\xi_\nu
$$
$$
+ b^2\rho\tau_\varepsilon v_\nu - n\beta_n(v\cdot v)v_\nu]. \qquad \text{(B11)}
$$

This enables us to find the eigenvectors,

$$
{}^\pm r_1^{(m)} =
\begin{bmatrix}
F_\pm \\
G_\pm^\nu \\
H_\pm^\nu \\
I_\pm^\nu \\
{}^\pm J_0^\nu \\
{}^\pm J_1^\nu \\
{}^\pm J_2^\nu \\
{}^\pm J_3^\nu
\end{bmatrix},
\qquad \text{(B12)}
$$

where only 19 out of the 24 components $H^\mu, I^\mu, J_\lambda^\nu$ are free variables because of the $1 + 1 + 3$ constraints $\beta_\varepsilon\phi_{0,\nu}^{(m)}H^\nu + \beta_n\phi_{0,\nu}^{(m)}I^\nu = 0$, $J_\lambda^\lambda = 0$, and $\Delta^{\mu\lambda}\phi_{0,\nu}^{(m)}J_\lambda^\nu + \Delta^{\lambda\beta}\phi_{0,\beta}^{(m)}J_\lambda^\mu = 0$ (note that the last four equations are not all independent since the contraction with u_μ is identically zero, resulting in three independent constraints). Thus, the multiplicity of Λ_0 equals the number of LI eigenvectors, i.e., 19.

(ii) $\Lambda_1^{(m)\pm}$: In this case each of the two eigenvalues have multiplicity 3 since $n_1 = 3$ in Eq. (B2) (note that since we assumed here that $\eta > 0$, then $c_1 \neq 0$ and, thus, $c_1 \neq c_0$ and the eigenvalues are different from the case $c_0 = 0$). We may perform some elementary row operations over the linear system $\phi_{1,\alpha}^{(m)} \mathbb{A}^\alpha r_1^{(m)} = 0$ to obtain, by imposing $b^2 - c_1(v\cdot v) = 0$ (remember that $b = u^\alpha\phi_\alpha$ and $v^\alpha = \Delta^{\alpha\beta}\phi_\beta$ after the change $\xi \rightarrow \phi$),

where, from the $29 + 29 = 58$ components of the above eigenvectors (29 for $\Lambda_1^{(m)+}$ and 29 $\Lambda_1^{(m)-}$ cases), they are subjected to the following $26 + 26$ constraints: $1 + 1 = 2$ constraints,

$$
[\tau_\varepsilon b_\pm^2 + \beta_\varepsilon(v_\pm \cdot v_\pm)]F_\pm + b_\pm\rho\tau_Q {}^\pm\phi_{1,\nu}^{(m)}G^\nu
$$
$$
+ b_\pm\rho\tau_\varepsilon v_\nu^\pm G^\nu - \frac{n\beta_n(v_\pm \cdot v_\pm)}{b_\pm}v_\nu^\pm G_\pm^\nu = 0,
$$

$1 + 1 = 2$ constraints $\mathcal{K}_\nu^\pm G_\pm^\nu = 0$, $4 + 4 = 8$ constraints $b_\pm H_\pm^\mu = v_\pm^\mu F_\pm$, $4 + 4 = 8$ constraints $nv_\pm^\mu v_\nu^\pm G^\nu + b_\pm^2 I_\pm^\mu = 0$, and the $16 + 16 = 32$ constraints $b_\pm {}^\pm J_{\pm\lambda}^\mu = v_\lambda^\pm G_\pm^\mu$, where ${}^\pm\phi_{1,\alpha}^{(m)} = {}^\pm\Lambda_1^{(m)}\xi_\alpha + \zeta_\alpha$ and $b^\pm$ and $v_\pm^\alpha$ are defined in terms of ${}^\pm\phi_{1,\nu}^{(m)}$. Hence, there is a total of $3 + 3 = 6$ free parameters. Once again, the degeneracy equals the number of LI eigenvectors.

(iii) $(\Lambda_{\pm})^{\pm}$: Since there is no degeneracy in these four last eigenvalues and they are distinct from the others because $c_{\pm} \neq 0$ in the strict form of the inequalities in Eq. (21) and different among them, then one has four LI eigenvectors.

Thus, the system has $19 + 6 + 4 = 29$ LI eigenvectors. Therefore, there is a complete set in $\mathbb{R}^{29}$, namely, $\{r_b^{(m)}\}_{b=1}^{29}$ such that $\phi_a^{(m)} \mathbb{A}_m^{\alpha} r_b^{(m)} = 0$. Hence, we can use the 29 linearly independent set $\mathcal{S}^{(m)} = \{R_b^{(m)}\}_{b=1}^{29}$ to verify that

$$R_b^{(m)} = \begin{bmatrix} r_b^{(m)} \\ 0_{66 \times 1} \end{bmatrix} \tag{B13}$$

obeys $(\zeta_\alpha + \Lambda_a^{(m)} \xi_\alpha) \mathfrak{A}^\alpha R_b^{(m)} = 0$.

Now, before we discuss the gravity sector $\{F_A, \mathcal{F}_A^\delta\}$, let us look at the sector containing the original fields ε, n, u^ν, and $g_{\mu\nu}$. In this case, let us define

$$R^{(d)} = \begin{bmatrix} 0_{79 \times 1} \\ r^{(d)} \end{bmatrix}, \tag{B14}$$

where $r^{(d)}$ is a 16×1 column vector. Then, $(\zeta_\alpha + \Lambda_a^{(d)} \xi_\alpha) \mathfrak{A}^\alpha R^{(d)} = 0$ reduces to the eigenvalue problem $u^\alpha \phi_\alpha^{(d)} I_{16} r^{(d)} = 0$ whose eigenvalues are $u^\alpha \phi_\alpha^{(d)} = 0$, i.e., $\Lambda^{(d)} = \zeta_\alpha u^\alpha / \xi_\alpha u^\alpha$. Thus, the eigenvectors may be any basis of $\mathbb{R}^{16}$. Let $\{r_a^{(d)}\}_{a=1}^{16}$ be a basis of $\mathbb{R}^{16}$. Then, the set $\mathcal{S}^{(d)} = \{R_a^{(d)}\}_{a=1}^{16}$ is a linearly independent set of 16 eigenvectors of $\phi_\alpha^{(d)} \mathfrak{A}^\alpha$.

To finalize the eigenvector counting we have to analyze the sector containing F_A and $\mathcal{F}_A^\delta$. In this case, let us define

$$R^{(g)} = \begin{bmatrix} w \\ r^{(g)} \\ 0_{16 \times 1} \end{bmatrix}, \tag{B15}$$

where w is some 29×1 columns vector while $r^{(g)}$ is a 50×1 columns vector. The eigenvalues of this sector are in Eq. (B4) and are given by $\Lambda_0^{(g)} = u^\alpha \zeta_\alpha / u^\beta \xi_\beta$, coming from $u^\alpha \phi_{0,\alpha}^{(g)} = 0$ (here $\phi_{a,\alpha}^{(g)} = \zeta_\alpha + \Lambda_a^{(g)} \xi_\alpha$) with multiplicity 30 and corresponding to $\beta = 0$, and the two roots $^{\pm}\Lambda_1^{(g)}$ with multiplicity 10 each coming from $^{\pm}\phi_{1,\alpha}^{(g)\pm}\phi_1^{(g)\alpha} = -[u^{\alpha\pm}\phi_{1,\alpha}^{(g)}]^2 + \Delta^{\alpha\beta\pm}\phi_{1,\alpha}^{(g)\pm}\phi_{1,\beta}^{(g)} = 0$, which corresponds to $\beta = 1$, i.e., gravitational waves moving at the speed of light. Then, the eigenvalue problem $\phi_{a,\alpha}^{(g)} \mathfrak{A}^\alpha R_a^{(g)} = 0$ reduces to the two equations:

$$\phi_{a,\alpha}^{(g)} \mathbb{A}_m^\alpha w_a = L^\alpha r_a^{(g)}, \tag{B16a}$$

$$\phi_{a,\alpha}^{(g)} \mathbb{A}_g^\alpha r_a^{(g)} = 0. \tag{B16b}$$

For the eigenvalues $^{\pm}\Lambda_1^{(g)}$, one obtains that $\det[^{\mp}\phi_{1,\alpha}^{(g)} \mathbb{A}_m^\alpha] \neq 0$ because the root $\beta = 1$ has been eliminated from the matter sector (remember that $c_a < 1$). Thus, there exists a solution of Eq. (B16a) for each $r_a^{(g)}$ in Eq. (B16b). One needs to count the number of linearly independent $r_1^{(g)}$ for $\Lambda_1^{(g)}$, i.e., the number of vectors in the basis of the kernel of $\phi_{1,\alpha}^{(g)} \mathbb{A}_g^\alpha$. In this case, after some elementary row operations [look at the second equality in Eq. (B4) after setting $b^2 = v \cdot v$] one obtains that

$$^{\pm}\phi_{1,\alpha}^{(g)} \mathbb{A}_g^\alpha \sim \begin{bmatrix} 0_{10 \times 10} & 0_{10 \times 10} & 0_{10 \times 10} & 0_{10 \times 10} & 0_{10 \times 10} \\ -\Delta^{0\alpha\pm}\phi_{1,\alpha}^{(g)} I_{10} & (u^{\alpha\pm}\phi_{1,\alpha}^{(g)}) I_{10} & 0_{10 \times 10} & 0_{10 \times 10} & 0_{10 \times 10} \\ -\Delta^{1\alpha\pm}\phi_{1,\alpha}^{(g)} I_{10} & 0_{10 \times 10} & (u^{\alpha\pm}\phi_{1,\alpha}^{(g)}) I_{10} & 0_{10 \times 10} & 0_{10 \times 10} \\ -\Delta^{2\alpha\pm}\phi_{1,\alpha}^{(g)} I_{10} & 0_{10 \times 10} & 0_{10 \times 10} & (u^{\alpha\pm}\phi_{1,\alpha}^{(g)}) I_{10} & 0_{10 \times 10} \\ -\Delta^{3\alpha\pm}\phi_{1,\alpha}^{(g)} I_{10} & 0_{10 \times 10} & 0_{10 \times 10} & 0_{10 \times 10} & (u^{\alpha\pm}\phi_{1,\alpha}^{(g)}) I_{10} \end{bmatrix}, \tag{B17}$$

which has 40 pivots and 10 independent variables (corresponding to the variables associated to the first 10 columns). Thus, there are 10 linearly independent vectors for each eigenvalue $^{\pm}\Lambda_1^{(g)}$; i.e., there is a set $\{^-r_{1,b}^{(g)}, {}^+r_{1,b}^{(g)}\}_{b=1}^{10}$ of 20 linearly independent vectors with corresponding $w_{1,b}^{\pm} = [^{\pm}\phi_{1,\alpha}^{(g)} \mathbb{A}_m^\alpha]^{-1} L^{a\pm} r_{1,b}^{(g)}$ coming from Eq. (B16a) such that $\mathcal{S}_1^{(g)} = \{^+R_{1,b}^{(g)}, {}^-R_{1,b}^{(g)}\}_{b=1}^{10}$, where

$$^{\pm}R_{1,b}^{(g)} = \begin{bmatrix} w_{1,b}^{\pm} \\ ^{\pm}r_{1,b}^{(g)} \\ 0_{16 \times 1} \end{bmatrix}$$

is a linearly independent set of 20 eigenvectors of $\phi_{1,\alpha}^{(g)} \mathfrak{A}^\alpha$.

As for the eigenvalue $\Lambda_0^{(g)}$, note that in this case $\det[\phi_{0,\alpha}^{(g)} \mathbb{A}_m^\alpha] = 0$ because $\beta = c_0 = 0$ is also a root of this equation. Thus, for every solution $r_a^{(g)}$ in Eq. (B16b), Eq. (B16a) can be either undetermined or have infinite solutions. However, for any two different solutions, say, w_a^1 and w_a^2 for one $r_a^{(g)}$, the difference between $R_a^{(g)1} - R_a^{(g)2}$ corresponds to a vector in the space spanned by $\mathcal{S}^{(m)}$, that lies in the kernel of $\phi_{0,\alpha}^{(g)} \mathbb{A}_m^\alpha$. Therefore, since we are counting the number of linearly independent eigenvectors, we must choose one particular solution w_a, if it exists, for each $r_a^{(g)}$. We begin by solving Eq. (B16b). Let $\{l_1^\mu = u^\mu, l_2^\mu, l_3^\mu\}$ be a set of linearly independent vectors that are orthogonal to $\phi_{0,\alpha}^{(g)} = \zeta_\alpha + \Lambda_0^{(g)} \xi_\alpha$, to wit, $l_c^\alpha \phi_{0,\alpha}^{(g)} = 0$ and $\{e_a\}_{a=1}^{10}$ be any basis of $\mathbb{R}^{10}$. Then, one may verify that the 30 linearly independent vectors

$$
r_{0,ac}^{(g)} = \begin{bmatrix} 0_{10\times1} \\ l_c^0 e_a \\ l_c^1 e_a \\ l_c^2 e_a \\ l_c^3 e_a \end{bmatrix}
\tag{B18}
$$

satisfy $\phi_{0,\alpha}^{(g)} \mathbb{A}_g^\alpha r_{0,ac}^{(g)} = 0$. Now we must solve Eq. (B16a), where

$$
\phi_{0,\alpha}^{(g)} L^\alpha r_{0,ac}^{(g)} = \begin{bmatrix} 0_{13\times1} \\ \phi_{0,\alpha}^{(g)} \mathcal{Y}_{0\delta}^{\mu A\alpha} l_c^\delta (e_a)_A \\ \phi_{0,\alpha}^{(g)} \mathcal{Y}_{1\delta}^{\mu A\alpha} l_c^\delta (e_a)_A \\ \phi_{0,\alpha}^{(g)} \mathcal{Y}_{2\delta}^{\mu A\alpha} l_c^\delta (e_a)_A \\ \phi_{0,\alpha}^{(g)} \mathcal{Y}_{3\delta}^{\mu A\alpha} l_c^\delta (e_a)_A \end{bmatrix} = K_a \begin{bmatrix} 0_{13\times1} \\ \phi_{0,0}^{(g)} l_c^\mu \\ \phi_{0,1}^{(g)} l_c^\mu \\ \phi_{0,2}^{(g)} l_c^\mu \\ \phi_{0,3}^{(g)} l_c^\mu \end{bmatrix}, \tag{B19}
$$

where we defined

$$
K_a \equiv \frac{1}{2} \left[\sum_{\substack{\sigma,\beta \\ \sigma \leq \beta}} (2 - \delta_{\sigma\beta}) u^{(\sigma} u^{\beta)} (e_a)_{\sigma\beta} \right].
$$

Let us look for the particular solution

$$
w_{ac} = \begin{bmatrix} 0 \\ -\beta_\varepsilon y_{ac}^\nu \\ \rho\tau_Q y_{ac}^\nu \\ 0_{20\times1} \end{bmatrix}.
\tag{B20}
$$

Note that

$$
\phi_{0,\alpha}^{(g)} \mathbb{A}_m^\alpha w_{ac} = \begin{bmatrix} 0_{13\times1} \\ \beta_\varepsilon \phi_{0,0}^{(g)} y_{ac}^\mu \\ \beta_\varepsilon \phi_{0,1}^{(g)} y_{ac}^\mu \\ \beta_\varepsilon \phi_{0,2}^{(g)} y_{ac}^\mu \\ \beta_\varepsilon \phi_{0,3}^{(g)} y_{ac}^\mu \end{bmatrix},
\tag{B21}
$$

and then, by inserting Eqs. (B19) and (B21) into Eq. (B16a), one finds that

$$
\beta_\varepsilon \phi_{0,\nu}^{(g)} y_{ac}^\mu = K_a \phi_{0,\nu}^{(g)} l_c^\mu.
\tag{B22}
$$

This leads to the solution $y_{ac}^\mu = K_a l_c^\mu / \beta_\varepsilon$ and, thus,

$$
w_{ac} = \begin{bmatrix} 0 \\ -K_a l_c^\nu \\ \frac{\rho\tau_Q}{\beta_\varepsilon} K_a l_c^\nu \\ 0_{20\times1} \end{bmatrix}.
\tag{B23}
$$

As a consequence, the set $\mathcal{S}_0^{(g)} = \{R_{1,1}^{(g)}, R_{1,2}^{(g)}, R_{1,3}^{(g)}, \ldots, R_{10,1}^{(g)}, R_{10,2}^{(g)}, R_{10,3}^{(g)}\}$ with

$$
R_{ac}^{(g)} = \begin{bmatrix} w_{ac} \\ r_{0,ac}^{(g)} \\ 0_{16\times1} \end{bmatrix}
$$

is a linearly independent set of 30 eigenvectors of $\phi_{0,\alpha}^{(g)} \mathfrak{A}^\alpha$. Thus, $\mathfrak{S} = \mathcal{S}^{(m)} \cup \mathcal{S}^{(d)} \cup \mathcal{S}_1^{(g)} \cup \mathcal{S}_0^{(g)}$ contains a complete set of eigenvectors R of $\phi_\alpha \mathfrak{A}^\alpha R = 0$ in $\mathbb{R}^{95}$. This completes the proof. ∎

We remark that the assumption that the inequalities hold in strict form is technical. If equality is allowed, then the multiplicity of the eigenvalues might change. This is because with equality one can have $c_a = 0$ for $a = 1$ or $\pm$ and thus the characteristics defined by $b^2 - c_a(v \cdot v) = 0$ can degenerate into the characteristics $b = 0$. Since the latter is already present in the system, the multiplicity of the characteristics would change. This does not mean that the system would not be diagonalizable. Nor does it imply that local well posedness, established in the next section, would fail [222]. However, a different proof would be needed to show diagonalization in the case $c_a = 0$ in the cases $a = 1$ or $\pm$. We believe that treating this very special case here would be a distraction from the main points of the paper. We also recall that already in the case of an ideal fluid, a different approach to local well posedness has to be employed when the characteristics degenerate [223].

APPENDIX C: PROOF OF THEOREM II

As usual in studies of the initial-value problem for Einstein's equations [22], we embed Σ into $\mathbb{R} \times \Sigma$ and work in harmonic coordinates in the neighborhood of a point. Observe that we already know the system to be causal under our assumptions, thus localization arguments are allowed.

The equations to be studied read

$$u^\alpha u^\beta \partial^2_{\alpha\beta} n + n u^\alpha \delta^\beta_\nu \partial^2_{\alpha\beta} u^\nu + \tilde{\mathcal{B}}_1(n,u,g)\partial^2 g = \mathcal{B}_1(\partial n, \partial u, \partial g),$$
(C1a)

$$u_\nu u^\alpha u^\beta \partial_\alpha \partial_\beta u^\nu + \tilde{\mathcal{B}}_2(n,\varepsilon,u,g)\partial^2 g = \mathcal{B}_2(\partial n, \partial \varepsilon, \partial u, \partial g),$$
(C1b)

$$\beta_n(u^\mu \Delta^{\alpha\beta} + \Delta^{\mu(\alpha}u^{\beta)})\partial_\alpha\partial_\beta n + \mathfrak{G}^{\mu\alpha\beta}\partial_\alpha\partial_\beta \varepsilon + \bar{C}^{\mu\alpha\beta}_\nu \partial_\alpha\partial_\beta u^\nu$$
$$+ \tilde{\mathcal{B}}^\mu_3(n,\varepsilon,u,g)\partial^2 g = \mathcal{B}^\mu_3(\partial n, \partial \varepsilon, \partial u, \partial g),$$
(C1c)

$$g^{\alpha\beta}\partial_\alpha\partial_\beta g_{\mu\nu} = \mathcal{B}_{4\mu\nu}(\partial n, \partial \varepsilon, \partial u, \partial g),$$
(C1d)

where

$$\bar{C}^{\mu\alpha\beta}_\nu = \left(\tau_P \rho - \zeta - \frac{\eta}{3}\right)\Delta^{\mu(\alpha}\delta^{\beta)}_\nu - \eta\Delta^{\alpha\beta}\delta^\mu_\nu$$
$$+ \rho(\tau_\varepsilon + \tau_Q)u^\mu\Delta^{(\alpha}_\nu u^{\beta)} + \tau_Q\rho u^\alpha u^\beta \delta^\mu_\nu,$$
(C2a)

$$\mathfrak{G}^{\mu\alpha\beta} = u^\mu(\beta_\varepsilon \Delta^{\alpha\beta} + \tau_\varepsilon u^\alpha u^\beta) + (\beta_\varepsilon + \tau_P)\Delta^{\mu(\alpha}u^{\beta)},$$
(C2b)

and the notation for the $\tilde{\mathcal{B}}$'s and $\mathcal{B}$'s follow the same construction as in Sec. IV.

We can write Eq. (C1) in matrix form as

$$\mathfrak{M}(\partial)\Psi = \mathfrak{N}(\partial\Psi),$$
(C3)

where $\Psi = (\varepsilon, n, u^\nu, g_{\mu\nu})^T$ is a 16×1 column vector (we count only the 10 independent $g_{\mu\nu}$), $\mathfrak{B}(\partial\Psi)$ is also a 16×1 column vector containing the $\mathfrak{N}$'s, i.e., the lower-order terms in derivatives of each equation, and

$$\mathfrak{M}(\partial) = \begin{bmatrix} \mathbb{M}(\partial) & \mathfrak{b}(\partial) \\ 0_{10\times 6} & g^{\alpha\beta}\partial_\alpha\partial_\beta I_{10} \end{bmatrix}.$$
(C4)

The 6×10 matrix $\mathfrak{b}(\partial)$ contains the terms $\tilde{\mathcal{B}}\partial^2 g$, while

$$\mathbb{M}(\partial) = \begin{bmatrix} 0 & u^\alpha u^\beta & n\delta^{(\alpha}_\nu u^{\beta)} \\ 0 & 0 & u_\nu u^\alpha u^\beta \\ \mathfrak{G}^{\mu\alpha\beta} & \beta_n(u^\mu\Delta^{\alpha\beta} + \Delta^{\mu(\alpha}u^{\beta)}) & \bar{C}^{\mu\alpha\beta}_\nu \end{bmatrix}\partial^2_{\alpha\beta}.$$
(C5)

Let us compute the characteristic determinant of the system and its roots, i.e., $\det[\mathfrak{M}(\xi)] = \det[\mathbb{M}(\xi)](\xi^\alpha\xi_\alpha)^{10} = 0$, where the substitution $\partial \to \xi$ takes place. The pure gravity

sector has the roots $\xi^\alpha\xi_\alpha = 0$. As for the matter sector, by again defining $b = u^\alpha\xi_\alpha$, $v^\mu = \Delta^{\mu\nu}\xi_\nu$, $v \cdot v = v^\mu v_\mu$, and

$$\tilde{C}^\mu_\nu = \bar{C}^{\mu\alpha\beta}_\nu \xi_\alpha\xi_\beta$$
$$= [\tau_Q\rho b^2 - \eta(v \cdot v)]\delta^\mu_\nu + \left(\tau_P\rho - \zeta - \frac{\eta}{3}\right)v^\mu\xi_\nu$$
$$+ \rho(\tau_\varepsilon + \tau_Q)bu^\mu v_\nu,$$
(C6a)

$$\mathfrak{D}^\mu_\nu = \left(\tau_P\rho - \zeta - \frac{\eta}{3} - n\beta_n\right)v^\mu\xi_\nu + [\tau_Q\rho b^2 - \eta(v \cdot v)]\delta^\mu_\nu,$$
(C6b)

$$\tilde{\mathfrak{E}}^\mu = \mathfrak{E}^{\mu\alpha\beta}\xi_\alpha\xi_\beta$$
$$= [\beta_\varepsilon(v \cdot v) + \tau_\varepsilon b^2]u^\mu + (\beta_\varepsilon + \tau_P)bv^\mu,$$
(C6c)

where $\mathfrak{D}^\mu_\nu$ is the same as the one defined in Eq. (A3), we obtain that (by carrying out some elementary row operations)

$$\det[\mathbb{M}(\xi)] = \det\begin{bmatrix} 0 & b^2 & nb\xi_\nu \\ 0 & 0 & b^2 u_\nu \\ \tilde{\mathfrak{E}}^\mu & \beta_n[u^\mu(v \cdot v) + bv^\mu] & \tilde{C}^\mu_\nu \end{bmatrix}$$
$$= \frac{b^3}{\tau_Q\rho b^2 - \eta(v \cdot v)}$$
$$\times \det\begin{bmatrix} 0 & b & n\xi_\nu \\ \tau_\varepsilon b^2 + \beta_\varepsilon(v \cdot v) & \beta_n(v \cdot v) & \rho(\tau_\varepsilon + \tau_Q)bv_\nu \\ (\beta_\varepsilon + \tau + P)bv^\mu & \beta_n bv^\mu & \mathfrak{D}^\mu_\nu \end{bmatrix}.$$
(C7)

The last determinant is the same as the one obtained in Eq. (A4) and the result turns out to be

$$\det[\mathbb{M}(\xi)] = -b^4[\rho\tau_Q b^2 - \eta(v \cdot v)]^2$$
$$\times [Ab^4 + Bb^2(v \cdot v) + C(v \cdot v)^2]$$
$$= -\rho^4\tau_Q^4\tau_\varepsilon(u^\alpha\xi_\alpha)^4 \prod_{a=1,\pm}[(u^\alpha\xi_\alpha)^2 - c_a\Delta^{\alpha\beta}\xi_\alpha\xi_\beta]^{\tilde{n}_a},$$
(C8)

where, as in Eqs. (A5),

$$A \equiv \rho\tau_\varepsilon\tau_Q,$$
(C9a)

$$B \equiv -\tau_\varepsilon\left(\rho c_s^2\tau_Q + \zeta + \frac{4\eta}{3} + \sigma\kappa_s\right) - \rho\tau_P\tau_Q,$$
(C9b)

$$C \equiv \tau_P(\rho c_s^2\tau_Q + \sigma\kappa_s) - \beta_\varepsilon\left(\zeta + \frac{4\eta}{3}\right),$$
(C9c)

while $c_1 = [\eta/(\rho\tau_s)]$ and $c_\pm = [(-B \pm \sqrt{B^2 - 4AC})/2A]$, while $\tilde{n}_1 = 2$ and $\tilde{n}_\pm = 1$. Note that the characteristics are

still the same as in Sec. IV, as expected, although the multiplicity of the roots changed (and there was no reason for the multiplicities to be the same). We conclude that the characteristic determinant of the system is a product of strictly hyperbolic polynomials. We verify at once that the system is a Leray-Ohya system [21,224] for which the results of Ref. [225] (see also Ref. [105]) apply. Thus, if the initial data are quasianalytic (see Ref. [75]), we obtain quasianalytic solutions.

Denote the initial dataset in the theorem by $\mathcal{D}$ and let $\mathcal{D}_\ell$ be a sequence of quasianalytic initial data converging to $\mathcal{D}$ in H^N (see Ref. [27] for the definition of H^N). Let $\boldsymbol{\Psi}_\ell$ be solutions corresponding to $\mathcal{D}_\ell$ (which exist by the foregoing). In order to finish the proof of the theorem, it suffices to show that $\boldsymbol{\Psi}_\ell$ has a limit in H^N. The limit will then be a solution with the desired properties because we can pass to the limit in the equations since $N \geq 5$.

According to the arguments given in Sec. 16.2 of Ref. [146] or in Refs. [107,108], the diagonalization obtained in Sec. V B implies that Ψ defined in Eq. (24) admits a uniform bound in H^{N-1}, and uniform difference bounds in H^{N-2} also holds. We apply these bounds to the vector Ψ_ℓ corresponding to $\boldsymbol{\Psi}_\ell$. We see at once that the uniform H^{N-1} bounds for Ψ_ℓ imply uniform H^N bounds for $\boldsymbol{\Psi}_\ell$, and the difference bounds imply that $\boldsymbol{\Psi}_\ell$ is a Cauchy sequence in H^{N-3}, thus converging in this space. But low-norm convergence combined with high-norm boundedness implies that the limit is in fact in H^N [226]. ∎

We observe that a similar local well-posedness result holds for the fluid equations in a fixed background.

We recall that a standard tensorial argument [22] guarantees that the solution established in Theorem II is intrinsically defined; i.e., given the data, which are defined independently of coordinates or gauge choices, there exists a spacetime where Einstein's equations are satisfied, and this spacetime is defined without any reference to coordinates or gauge choices—even if in the process of proving that this spacetime exists one has to work in a specific gauge and coordinate system. Therefore, even though we used the harmonic gauge in the proof, the existence of the solution is guaranteed for other choices as well. This logic is similar to showing that a map from a finite-dimensional vector space into itself is invertible: one can choose a basis, write the matrix of the linear transformation with respect to that basis, and compute its determinant. The map is invertible if and only if the determinant is nonzero, and this conclusion (the invertibility or not of the linear map) is independent of any basis choice—even if to show that the map is invertible we picked a basis and computed the determinant with respect to that basis.

We note, however, the following subtlety which is very relevant for numerical simulations. The fact that a unique solution is guaranteed to exist for given initial data, and that this solution is well defined regardless of gauge choices, does not imply that such a solution can always be reconstructed from an arbitrary gauge. In other words, suppose we write the equations in a different gauge. *If* we can numerically integrate them, we will obtain the solution found in Theorem II written on that gauge (modulo numerical accuracy). However, it is possible that the gauge we chose is not adequate to solve the equations numerically, so that our numerical simulation will not produce a solution. This does not mean, of course, that solutions do not exist; it simply means that the guaranteed-to-exist solution given by Theorem II cannot be accessed from that specific gauge. To use again our analogy with determinants, suppose we computed the determinant on a basis b_1 and found it to be nonzero, but now we are interested in computing the determinant numerically using another basis b_2. Depending on the basis b_2 and the numerical algorithm we use, this might not be possible, which, of course, does not mean that the determinant is zero or ill defined.

Thus, the practical matter of solving the equations numerically is not settled by an abstract existence and uniqueness result as Theorem II. Such theorems are naturally important as they provide the foundations on which numerical investigations can be built; i.e., it makes sense to look for solutions numerically because solutions do exist. But these theorems do not, in general, point to how to recover solutions numerically. That is why there is a great deal of work dedicated to writing Einstein's equations in different forms and special gauges, even if basic existence results for Einstein's equations coupled to most matter models are known, as reviewed in Refs. [2,29].

APPENDIX D: PROOF OF THEOREM III

From causality one obtains that $\det(n_\alpha \mathbb{A}^\alpha) \neq 0$ as far as n is timelike. Thus, we can rewrite Eq. (35) as

$$i\Omega\delta\Psi(K) = -i\kappa(-n_\alpha \mathbb{A}^\alpha)^{-1}\zeta_\beta \mathbb{A}^\beta \delta\Psi(K)$$
$$- (-n_\alpha \mathbb{A}^\alpha)^{-1}\mathbb{B}\delta\Psi(K). \tag{D1}$$

Since the eigenvalue problem (37) contains N linearly independent vectors $\mathfrak{r}_a$, one may write Eq. (37) as

$$(-n_\alpha \mathbb{A}^\alpha)^{-1}\zeta_\beta \mathbb{A}^\beta \mathfrak{r}_a = \Lambda_a \mathfrak{r}_a \tag{D2}$$

and define the $N \times N$ invertible matrix $R = [\mathfrak{r}_1 \cdots \mathfrak{r}_N]$ whose columns are the eigenvectors $\mathfrak{r}_1, \ldots, \mathfrak{r}_n$ and the $N \times N$ matrix

$$L \equiv R^{-1} = \begin{bmatrix} \mathfrak{l}_1 \\ \vdots \\ \mathfrak{l}_N \end{bmatrix},$$

where the rows $\mathfrak{l}_a$ are the left eigenvectors of $(-n_\alpha \mathbb{A}^\alpha)\zeta_\beta \mathbb{A}^\beta$ which, consequently, obey $\mathfrak{l}_a \mathfrak{r}_b = \delta_{ab}$ (because $RL = I_N$). Then, we can write

$$\delta\Psi(K) = RL\delta\Psi(K) = \sum_a c_a(K)\mathfrak{r}_a = R\mathfrak{c}, \quad \text{(D3)}$$

where $c_a(K) = \mathfrak{l}_a\delta\Psi(K)$ is a c-number and $\mathfrak{c}$ is the $N \times 1$ matrix

$$\mathfrak{c} = L\delta\Psi(K) = \begin{bmatrix} c_1(K) \\ \vdots \\ c_N(K) \end{bmatrix}.$$

Therefore, Eq. (D1) becomes

$$i\Omega R\mathfrak{c} = -i\kappa R\mathsf{D}\mathfrak{c} - (-n_\alpha\mathbb{A}^\alpha)^{-1}\mathbb{B}R\mathfrak{c}, \quad \text{(D4)}$$

where D is the $N \times N$ real diagonal matrix $\mathsf{D} = \text{diag}(\Lambda_1, \ldots, \Lambda_N)$ and, thus, $(-n_\alpha\mathbb{A}^\alpha)^{-1}\zeta_\beta\mathbb{A}^\beta R = R\mathsf{D}$. By multiplying Eq. (D4) by $\mathfrak{c}^\dagger R^{-1}$ from the left one obtains that

$$i\Omega|\mathfrak{c}|^2 = -i\kappa\mathfrak{c}^\dagger \mathsf{D}\mathfrak{c} - \mathfrak{c}^\dagger R^{-1}(-n_\alpha\mathbb{A}^\alpha)^{-1}\mathbb{B}R\mathfrak{c}. \quad \text{(D5)}$$

Since D is real and diagonal (which gives $\mathfrak{c}^\dagger \mathsf{D}\mathfrak{c} \in \mathbb{R}$), $\Omega = \gamma_n(-i\Gamma + c^i k_i)$, and $\kappa = \gamma_\zeta(-i\hat{\partial}^j c_i\Gamma + \hat{\partial}^j k_j)$, then

$$\Gamma_R\mathfrak{c}^\dagger(\gamma_n I_N + \gamma_\zeta\hat{\partial}^j c_j\mathsf{D})\mathfrak{c} = -\text{Re}[\mathfrak{c}^\dagger R^{-1}(n_\alpha\mathbb{A}^\alpha)^{-1}\mathbb{B}R\mathfrak{c}]. \quad \text{(D6)}$$

On the other hand, note that $\gamma_n I_N + \gamma_\zeta\hat{\partial}^j c_j\mathsf{D}$ is diagonal with elements

$$(\gamma_n I_N + \gamma_\zeta\hat{\partial}^j c_j\mathsf{D})_{aa} = \gamma_n + \gamma_\zeta\hat{\partial}^j c_j\Lambda_a > 0 \quad \text{(D7)}$$

because $|\hat{\partial}^j c_j| \leq |c^i| < 1$, $\Lambda \in [-1, 1]$, and $\gamma_n \geq \gamma_\zeta$ from Eq. (36). Hence, $\gamma_n I_N + \gamma_\zeta\hat{\partial}^j c_j\mathsf{D}$ is a positive Hermitian matrix and $\mathfrak{c}^\dagger(\gamma_n I_N + \gamma_\zeta\hat{\partial}^j c_j\mathsf{D})\mathfrak{c} > 0$. The consequence is that $\Gamma_R \leq 0$ if and only if

$$\text{Re}[\mathfrak{c}^\dagger R^{-1}(n_\alpha\mathbb{A}^\alpha)^{-1}\mathbb{B}R\mathfrak{c}] \geq 0. \quad \text{(D8)}$$

Now, let $\mathcal{O}$ be the LRF and $\mathcal{O}'$ some other boosted frame. The connection between the two frames is given by the Lorentz transform $t' = \gamma(t - v^i x_i)$, $x'^i_\parallel = \gamma(x^i_\parallel - v^i t)$, and $x'^i_\perp = x^i_\perp$, where $\parallel$ and $\perp$ stand for the components parallel and perpendicular to v^i, respectively. This can be compactly written as $X'^\mu = \Lambda^\mu_\nu X^\nu$. Thus, one obtains that $K'^\mu = \Lambda^\mu_\nu K^\nu$ and $\delta\Psi'(K') = M\delta\Psi(K)$ from the structure of Eq. (33) (where M is an $N \times N$ invertible matrix), leading to $\mathbb{A}'^\mu = \Lambda^\mu_\nu M\mathbb{A}^\mu M^{-1}$ and $\mathbb{B}' = M\mathbb{B}M^{-1}$. In particular, $\zeta_\alpha\mathbb{A}^\alpha = M^{-1}\zeta'_\alpha\mathbb{A}'^\alpha M$ and $n_\alpha\mathbb{A}^\alpha = M^{-1}(n'_\alpha\mathbb{A}'^\alpha)M$. From Eq. (37), these relations give $R' = MR$, with the same eigenvalue Λ in both frames. Then, since $\delta\Psi(K) = R\mathfrak{c}$ and $\delta\Psi'(K') = R'\mathfrak{c}' = MR\mathfrak{c}$, one concludes that $\mathfrak{c} = \mathfrak{c}'$, i.e., $c'_a(K') = c_a(K)$. Therefore, one arrives at the following identity:

$$\mathfrak{c}'^\dagger R'^{-1}(-n'_\alpha\mathbb{A}'^\alpha)^{-1}\mathbb{B}'R'\mathfrak{c}' = \mathfrak{c}^\dagger R^{-1}(-n_\alpha\mathbb{A}^\alpha)^{-1}\mathbb{B}R\mathfrak{c}. \quad \text{(D9)}$$

However, if the system is stable in the LRF, then Eq. (D8) holds and, from Eq. (D9), one automatically obtains that $\Gamma'_R \leq 0$, proving that the system is also stable in any other frame $\mathcal{O}'$ obtained via a Lorentz transformation. ∎

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[1] S. Weinberg, *Cosmology* (Oxford University Press, New York, 2008), p. 593.

[2] L. Rezzolla and O. Zanotti, *Relativistic Hydrodynamics* (Oxford University Press, New York, 2013).

[3] P. Romatschke and U. Romatschke, *Relativistic Fluid Dynamics in and out of Equilibrium*, Cambridge Monographs on Mathematical Physics (Cambridge University Press, Cambridge, England, 2019).

[4] U. Heinz and R. Snellings, *Collective Flow and Viscosity in Relativistic Heavy-Ion Collisions*, Annu. Rev. Nucl. Part. Sci. **63**, 123 (2013).

[5] J. E. Bernhard, J. S. Moreland, and S. A. Bass, *Bayesian Estimation of the Specific Shear and Bulk Viscosity of Quark–Gluon Plasma*, Nat. Phys. **15**, 1113 (2019).

[6] B. Abbott *et al.* (LIGO Scientific and Virgo Collaborations), *GW170817: Observation of Gravitational Waves from a Binary Neutron Star Inspiral*, Phys. Rev. Lett. **119**, 161101 (2017).

[7] B. P. Abbott *et al.* (LIGO Scientific Collaboration, Virgo Collaboration, Fermi Gamma-Ray Burst Monitor, and INTEGRAL), *Gravitational Waves and Gamma-Rays from a Binary Neutron Star Merger: GW170817 and GRB 170817A*, Astrophys. J. **848**, L13 (2017).

[8] B. P. Abbott *et al.* (LIGO Scientific Collaboration, Virgo Collaboration, Fermi GBM, INTEGRAL, and others), *Multi-Messenger Observations of a Binary Neutron Star Merger*, Astrophys. J. **848**, L12 (2017).

[9] M. D. Duez, Y. T. Liu, S. L. Shapiro, and B. C. Stephens, *General Relativistic Hydrodynamics with Viscosity: Contraction, Catastrophic Collapse, and Disk Formation in Hypermassive Neutron Stars*, Phys. Rev. D **69**, 104030 (2004).

[10] M. Shibata, K. Kiuchi, and Y.-i. Sekiguchi, *General Relativistic Viscous Hydrodynamics of Differentially Rotating Neutron Stars*, Phys. Rev. D **95**, 083005 (2017).

[11] M. Shibata and K. Kiuchi, *Gravitational Waves from Remnant Massive Neutron Stars of Binary Neutron Star Merger: Viscous Hydrodynamics Effects*, Phys. Rev. D **95**, 123003 (2017).

[12] M. G. Alford, L. Bovard, M. Hanauske, L. Rezzolla, and K. Schwenzer, *Viscous Dissipation and Heat Conduction in Binary Neutron-Star Mergers*, Phys. Rev. Lett. **120**, 041101 (2018).

[13] D. Radice, A. Perego, K. Hotokezaka, S. Bernuzzi, S. A. Fromm, and L. F. Roberts, *Viscous-Dynamical Ejecta from Binary Neutron Star Merger*, Astrophys. J. Lett. **869**, L35 (2018).

[14] E. R. Most, S. P. Harris, C. Plumberg, M. G. Alford, J. Noronha, J. Noronha-Hostler, F. Pretorius, H. Witek, and N. Yunes, *Projecting the Likely Importance of*

Weak-Interaction-Driven Bulk Viscosity in Neutron Star Mergers, Mon. Not. R. Astron. Soc. **509**, 1096 (2021).

[15] L. D. Landau and E. M. Lifshitz, *Fluid Mechanics*, Course of Theoretical Physics, 2nd ed. (Butterworth-Heinemann, London, 1987), Vol. 6, p. 552.

[16] M. M. Disconzi, *Remarks on the Einstein-Euler-Entropy System*, Rev. Math. Phys. **27**, 1550014 (2015).

[17] Y. Fourès-Bruhat, *Théorèmes d'Existence en Mécanique des Fluides Relativistes*, Bull. Soc. Math. Fr. **79**, 155 (1951).

[18] G. L. Eyink and T. D. Drivas, *Cascades and Dissipative Anomalies in Relativistic Fluid Turbulence*, Phys. Rev. X **8**, 011023 (2018).

[19] G. L. Eyink and T. D. Drivas, *Cascades and Dissipative Anomalies in Compressible Fluid Turbulence*, Phys. Rev. X **8**, 011022 (2018).

[20] S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-Time*, Cambridge Monographs on Mathematical Physics (Cambridge University Press, Cambridge, England, 1975), p. 404.

[21] Y. Choquet-Bruhat, *General Relativity and the Einstein Equations* (Oxford University Press, New York, 2009).

[22] R. M. Wald, *General Relativity* (University of Chicago Press, Chicago, 2010).

[23] A. M. Anile, *Relativistic Fluids and Magneto-Fluids: With Applications in Astrophysics and Plasma Physics* Cambridge Monographs on Mathematical Physics, 1st ed. (Cambridge University Press, Cambridge, England, 1990).

[24] W. A. Hiscock and L. Lindblom, *Stability and Causality in Dissipative Relativistic Fluids*, Ann. Phys. (N.Y.) **151**, 466 (1983).

[25] W. A. Hiscock and L. Lindblom, *Generic Instabilities in First-Order Dissipative Fluid Theories*, Phys. Rev. D **31**, 725 (1985).

[26] S. Pu, T. Koide, and D. H. Rischke, *Does Stability of Relativistic Dissipative Fluid Dynamics Imply Causality?*, Phys. Rev. D **81**, 114039 (2010).

[27] For readers interested in technical mathematical aspects of our results, we recall the following point. Local well posedness always refers to local well posedness in a certain function space X. This means that given initial conditions in X, the solution exists in the same function space, in a sense that can be made precise. The relevance of X is that it determines the regularity of the solution, e.g., analytic, infinitely many times differentiable, etc. Here, by local well posedness, we always mean local well posedness in Sobolev spaces, which is the space of functions with square integrable derivatives up to order N (N is a fixed integer depending on the structure of the equations), with norm given by $\|f\|_N^2 = \sum_{k=0}^N \int |\partial^k f|^2 dx$, where ∂^k represent all derivatives of order k and the derivatives are understood in a distributional sense. It is known that Sobolev spaces are the appropriate function spaces for most physical applications, including the investigation of numerical algorithms [2,23].

[28] We say some time instead of all times because solutions can develop singularities and, thus, not be defined for all times. The fact that nonlinear equations of physical significance can indeed develop singularities, e.g., shock waves [2] in fluid dynamics or spacetime singularities in general relativity [20], is well known. The question of how to deal with such singularities, e.g., how to meaningfully extend the solutions past the singularity or replace the singular theory by a more complete one where singularities can be avoided, while of uttermost physical importance, is beyond the scope of this work.

[29] T. W. Baumgarte and S. L. Shapiro, *Numerical Relativity: Solving Einstein's Equations on the Computer* (Cambridge University Press, Cambridge, England, 2010).

[30] O. A. Reula, *Strongly Hyperbolic Systems in General Relativity*, J. Hyperbol. Differ. Equations **01**, 251 (2004).

[31] J.-L. Guermond, F. Marpeau, and B. Popov, *A Fast Algorithm for Solving First-Order PDEs by L^1-Minimization*, Commun. Math. Sci. **6**, 199 (2008).

[32] F. S. Bemfica, M. M. Disconzi, and J. Noronha, *Causality and Existence of Solutions of Relativistic Viscous Fluid Dynamics with Gravity*, Phys. Rev. D **98**, 104064 (2018).

[33] F. S. Bemfica, M. M. Disconzi, and J. Noronha, *Nonlinear Causality of General First-Order Relativistic Viscous Hydrodynamics*, Phys. Rev. D **100**, 104020 (2019).

[34] P. Kovtun, *First-Order Relativistic Hydrodynamics Is Stable*, J. High Energy Phys. 10 (2019) 034.

[35] R. E. Hoult and P. Kovtun, *Stable and Causal Relativistic Navier-Stokes Equations*, J. High Energy Phys. 06 (2020) 067.

[36] W. Israel and J. M. Stewart, *Transient Relativistic Thermodynamics and Kinetic Theory*, Ann. Phys. (N.Y.) **118**, 341 (1979).

[37] W. Israel, *Nonstationary Irreversible Thermodynamics: A Causal Relativistic Theory*, Ann. Phys. (N.Y.) **100**, 310 (1976).

[38] S. R. D. Groot, *Relativistic Kinetic Theory. Principles and Applications* (North-Holland, Amsterdam, 1980), p. 417.

[39] C. Cercignani, *The Boltzmann Equation and Its Applications* (Springer-Verlag, New York, 1988).

[40] D. H. Rischke, *Fluid Dynamics for Relativistic Nuclear Collisions*, Lect. Notes Phys. **516**, 21 (1999).

[41] K. Tsumura, T. Kunihiro, and K. Ohnishi, *Derivation of Covariant Dissipative Fluid Dynamics in the Renormalization-Group Method*, Phys. Lett. B **646**, 134 (2007); *Erratum to: "Derivation of Covariant Dissipative Fluid Dynamics in the Renormalization-Group Method* [Phys. Lett. B **646** 134 (2007)]"; **646**, 134 (2007).

[42] K. Tsumura and T. Kunihiro, *Stable First-Order Particle-Frame Relativistic Hydrodynamics for Dissipative Systems*, Phys. Lett. B **668**, 425 (2008).

[43] K. Tsumura and T. Kunihiro, *Second-Order Relativistic Hydrodynamic Equations for Viscous Systems: How Does the Dissipation Affect the Internal Energy?*, Phys. Lett. B **690**, 255 (2010).

[44] S. Bhattacharyya, V. E. Hubeny, S. Minwalla, and M. Rangamani, *Nonlinear Fluid Dynamics from Gravity*, J. High Energy Phys. 02 (**2008**) 045.

[45] J. Bhattacharya, S. Bhattacharyya, S. Minwalla, and A. Yarom, *A Theory of First Order Dissipative Superfluid Dynamics*, J. High Energy Phys. 05 (2014) 147.

[46] K. Tsumura and T. Kunihiro, *First-Principle Derivation of Stable First-Order Generic-Frame Relativistic Dissipative Hydrodynamic Equations from Kinetic Theory by*

Renormalization-Group Method, Prog. Theor. Phys. **126,** 761 (2011).

[47] P. Van and T. S. Biro, *First Order and Stable Relativistic Dissipative Hydrodynamics*, Phys. Lett. B **709,** 106 (2012).

[48] K. Kovtun, *Lectures on Hydrodynamic Fluctuations in Relativistic Theories*, J. Phys. A **45,** 473001 (2012).

[49] K. Tsumura and T. Kunihiro, *Derivation of Relativistic Hydrodynamic Equations Consistent with Relativistic Boltzmann Equation by Renormalization-Group Method*, Eur. Phys. J. A **48,** 162 (2012).

[50] Y. Kikuchi, K. Tsumura, and T. Kunihiro, *Derivation of Second-Order Relativistic Hydrodynamics for Reactive Multicomponent Systems*, Phys. Rev. C **92,** 064909 (2015).

[51] L. Gavassino, M. Antonelli, and B. Haskell, *When the Entropy Has No Maximum: A New Perspective on the Instability of the First-Order Theories of Dissipation*, Phys. Rev. D **102,** 043018 (2020).

[52] C. Eckart, *The Thermodynamics of Irreversible Processes. III. Relativistic Theory of the Simple Fluid*, Phys. Rev. **58,** 919 (1940).

[53] Note that here the energy density of the system is defined to match that of an equilibrium state. This energy density is then related to the temperature by the (relativistic version of the) first law of thermodynamics.

[54] The fluid velocity is normalized because in relativity observers are characterized by their worldline; different parametrizations of the same worldline can produce different tangent vectors. We fix this ambiguity by normalizing the tangent vectors to unity. The normalization $u^\mu u_\mu = -1$ is assumed in every fluid theory that we discuss.

[55] G. Pichon, *Étude Relativiste de Fluides Visqueux et Chargés*, Ann. Inst. Henri Poincaré Phys. Theor. **2,** 21 (1965).

[56] W. Israel and J. M. Stewart, *Thermodynamics of Nonstationary and Transient Effects in a Relativistic Gas*, Phys. Lett. A **58,** 213 (1976).

[57] J. M. Stewart, *On Transient Relativistic Thermodynamics and Kinetic Theory*, Proc. R. Soc. A **357,** 59 (1977).

[58] W. Israel and J. M. Stewart, *On Transient Relativistic Thermodynamics and Kinetic Theory. II*, Proc. R. Soc. A **365,** 43 (1979).

[59] I. Müller, *Zum Paradox der Wärmeleitungstheorie*, Z. Phys. **198,** 329 (1967).

[60] D. Jou, J. Casas-Vazquez, and G. Lebon, *Extended Irreversible Thermodynamics*, 3rd ed. (Springer, New York, 2001), p. 463.

[61] I. Müller and T. Ruggeri, *Rational Extended Thermodynamics* (Springer, New York, 1998).

[62] More precisely, while in principle it makes sense to consider gradient expansions involving extended variables, all instances of the gradient expansion we consider here are expansions involving only the hydrodynamic fields. Thus, for our purposes we can say that a gradient expansion is always an expansion in the hydrodynamic variables.

[63] We remark that a simplified version of the MIS equations where certain terms are neglected are known as the Maxwell-Cattaneo equations; see Ref. [2].

[64] T. S. Olson, *Stability and Causality in the Israel-Stewart Energy Frame Theory*, Ann. Phys. (N.Y.) **199,** 18 (1990).

[65] G. S. Denicol, H. Niemi, E. Molnar, and D. H. Rischke, *Derivation of Transient Relativistic Fluid Dynamics from the Boltzmann Equation*, Phys. Rev. D **85,** 114047 (2012); **91,** 039902(E) (2015).

[66] H. Grad, Approximation to the Boltzmann equation by moments, Ph.D. thesis, New York University, ProQuest LLC, Ann Arbor, MI, 1948, p. 170.

[67] H. Grad, *On the Kinetic Theory of Rarefied Gases*, Commun. Pure Appl. Math. **2,** 331 (1949).

[68] Although the DNMR and MIS equations are different, they coincide when linearized about equilibrium [69,70].

[69] L. Gavassino, M. Antonelli, and B. Haskell, *Thermodynamic Stability Implies Causality*, Phys. Rev. Lett. **128,** 010606 (2022).

[70] C. V. Brito and G. S. Denicol, *Linear Stability of Israel-Stewart Theory in the Presence of Net-Charge Diffusion*, Phys. Rev. D **102,** 116009 (2020).

[71] F. S. Bemfica, M. M. Disconzi, and J. Noronha, *Causality of the Einstein-Israel-Stewart Theory with Bulk Viscosity*, Phys. Rev. Lett. **122,** 221602 (2019).

[72] F. S. Bemfica, M. M. Disconzi, V. Hoang, J. Noronha, and M. Radosz, *Nonlinear Constraints on Relativistic Fluids Far from Equilibrium*, Phys. Rev. Lett. **126,** 222301 (2021).

[73] G. S. Denicol, T. Kodama, T. Koide, and P. Mota, *Stability and Causality in Relativistic Dissipative Hydrodynamics*, J. Phys. G **35,** 115102 (2008).

[74] S. Floerchinger and E. Grossi, *Causality of Fluid Dynamics for High-Energy Nuclear Collisions*, J. High Energy Phys. 08 (2018) 186.

[75] In addition, a weak form of local well posedness was also obtained under these assumptions. Recalling that local well posedness always refers to a certain function space, existence and uniqueness in the space of quasianalytic functions has been obtained. This is the space of functions satisfying $\partial^k f(x) \sim (k!)^s |x|^k$, $s > 1$; $s = 1$ corresponds to analytic functions. Unfortunately, this space of functions is too restrictive for physical applications. That is why throughout this work by local well posedness we always mean local well posedness in Sobolev spaces; see Ref. [27].

[76] S. Ryu, J. F. Paquet, C. Shen, G. S. Denicol, B. Schenke, S. Jeon, and C. Gale, *Importance of the Bulk Viscosity of QCD in Ultrarelativistic Heavy-Ion Collisions*, Phys. Rev. Lett. **115,** 132301 (2015).

[77] R. Baier, P. Romatschke, D. T. Son, A. O. Starinets, and M. A. Stephanov, *Relativistic Viscous Hydrodynamics, Conformal Invariance, and Holography*, J. High Energy Phys. 04 (2008) 100.

[78] J. M. Maldacena, *The Large N Limit of Superconformal Field Theories and Supergravity*, Adv. Theor. Math. Phys. **2,** 231 (1998).

[79] W. Florkowski and R. Ryblewski, *Highly Anisotropic and Strongly Dissipative Hydrodynamics for Early Stages of Relativistic Heavy-Ion Collisions*, Phys. Rev. C **83,** 034907 (2011).

[80] M. Martinez and M. Strickland, *Dissipative Dynamics of Highly Anisotropic Systems*, Nucl. Phys. **A848,** 183 (2010).

[81] D. Bazow, U. W. Heinz, and M. Strickland, *Second-Order (2 + 1)-Dimensional Anisotropic Hydrodynamics*, Phys. Rev. C **90,** 054910 (2014).

[82] D. Almaalol, M. Alqahtani, and M. Strickland, *Anisotropic Hydrodynamic Modeling of 200 GeV Au-Au Collisions*, Phys. Rev. C **99**, 044902 (2019).

[83] M. Alqahtani, M. Nopoush, and M. Strickland, *Relativistic Anisotropic Hydrodynamics*, Prog. Part. Nucl. Phys. **101**, 204 (2018).

[84] M. Strickland, J. Noronha, and G. S. Denicol, *Anisotropic Nonequilibrium Hydrodynamic Attractor*, Phys. Rev. D **D97**, 036020 (2018).

[85] G. S. Denicol, C. Gale, S. Jeon, A. Monnai, B. Schenke, and C. Shen, *Net Baryon Diffusion in Fluid Dynamic Simulations of Relativistic Heavy-Ion Collisions*, Phys. Rev. C **98**, 034916 (2018).

[86] A. Muronga, *Second Order Dissipative Fluid Dynamics for Ultrarelativistic Nuclear Collisions*, Phys. Rev. Lett. **88**, 062302 (2002); Phys. Rev. Lett. **88**, 062302(E) (2002); **89**, 159901(E) (2002).

[87] A. Muronga, *Causal Theories of Dissipative Relativistic Fluid Dynamics for Nuclear Collisions*, Phys. Rev. C **69**, 034903 (2004).

[88] P. Romatschke and U. Romatschke, *Viscosity Information from Relativistic Nuclear Collisions: How Perfect Is the Fluid Observed at RHIC?*, Phys. Rev. Lett. **99**, 172301 (2007).

[89] C. Chiu and C. Shen, *Exploring Theoretical Uncertainties in the Hydrodynamic Description of Relativistic Heavy-Ion Collisions*, Phys. Rev. C **103**, 064901 (2021).

[90] C. Plumberg, D. Almaalol, T. Dore, J. Noronha, and J. Noronha-Hostler, *Causality Violations in Realistic Simulations of Heavy-Ion Collisions*, arXiv:2103.15889.

[91] T. S. Olson and W. A. Hiscock, *Plane Steady Shock Waves in Israel-Stewart Fluids*, Ann. Phys. (N.Y.) **204**, 331 (1990).

[92] R. Geroch and L. Lindblom, *Causal Theories of Dissipative Relativistic Fluids*, Ann. Phys. (N.Y.) **207**, 394 (1991).

[93] T. S. Olson and W. A. Hiscock, *Plane Steady Shock Waves in Israel-Stewart Fluids*, Ann. Phys. (N.Y.) **204**, 331 (1990).

[94] M. M. Disconzi, V. Hoang, and M. Radosz, *Breakdown of Smooth Solutions to the Müller-Israel-Stewart Equations of Relativistic Viscous Fluids*, arXiv:2008.03841.

[95] G. S. Denicol, J. Noronha, H. Niemi, and D. H. Rischke, *Origin of the Relaxation Time in Dissipative Fluid Dynamics*, Phys. Rev. D **83**, 074019 (2011).

[96] M. P. Heller, R. A. Janik, M. Spalinski, and P. Witaszczyk, *Coupling Hydrodynamics to Nonequilibrium Degrees of Freedom in Strongly Interacting Quark-Gluon Plasma*, Phys. Rev. Lett. **113**, 261601 (2014).

[97] W. Florkowski, M. P. Heller, and M. Spalinski, *New Theories of Relativistic Hydrodynamics in the LHC Era*, Rep. Prog. Phys. **81**, 046001 (2018).

[98] S. Grozdanov, A. Lucas, and N. Poovuttikul, *Holography and Hydrodynamics with Weakly Broken Symmetries*, Phys. Rev. D **99**, 086012 (2019).

[99] M. G. Alford and S. P. Harris, *Damping of Density Oscillations in Neutrino-Transparent Nuclear Matter*, Phys. Rev. C **100**, 035803 (2019).

[100] M. G. Alford and A. Haber, *Hyperon Bulk Viscosity in Neutron Star Mergers*, Phys. Rev. C **103**, 045810 (2021).

[101] P. Van and T. S. Biro, *Relativistic Hydrodynamics—Causality and Stability*, Eur. Phys. J. Special Topics **155**, 201 (2008).

[102] H. Freistühler and B. Temple, *Causal Dissipation and Shock Profiles in the Relativistic Fluid Dynamics of Pure Radiation*, Proc. R. Soc. A **470**, 20140055 (2014).

[103] H. Freistühler and B. Temple, *Causal Dissipation for the Relativistic Dynamics of Ideal Gases*, Proc. R. Soc. A **473**, 20160729 (2017).

[104] H. Freistühler and B. Temple, *Causal Dissipation in the Relativistic Dynamics of Barotropic Fluids*, J. Math. Phys. (N.Y.) **59**, 063101 (2018).

[105] M. M. Disconzi, *On the Existence of Solutions and Causality for Relativistic Viscous Conformal Fluids*, Commun. Pure Appl. Anal. **18**, 1567 (2019).

[106] The results of Ref. [33] also allow for dynamical coupling to Einstein's equations.

[107] F. S. Bemfica, M. M. Disconzi, and P. J. Graber, *Local Well-Posedness in Sobolev Spaces for First-Order Barotropic Causal Relativistic Viscous Hydrodynamics*, Commun. Pure Appl. Anal. **20**, 2885 (2021).

[108] F. S. Bemfica, M. M. Disconzi, C. Rodriguez, and Y. Shao, *Local Existence and Uniqueness in Sobolev Spaces for First-Order Conformal Causal Relativistic Viscous Hydrodynamics*, Commun. Pure Appl. Anal. **20**, 2279 (2021).

[109] H. Freistühler, *A Class of Hadamard Well-Posed Five-Field Theories of Dissipative Relativistic Fluid Dynamics*, J. Math. Phys. (N.Y.) **61**, 033101 (2020).

[110] F. Taghinavaz, *Causality and Stability Conditions of a Conformal Charged Fluid*, arXiv:2004.01897.

[111] A. Das, W. Florkowski, and R. Ryblewski, *Correspondence between Israel-Stewart and First-Order Casual and Stable Hydrodynamics for the Boost-Invariant Massive Case with Zero Baryon Density*, Phys. Rev. D **102**, 031501 (R) (2020).

[112] E. Romenski, I. Peshkov, M. Dumbser, and F. Fambri, *A New Continuum Model for General Relativistic Viscous Heat-Conducting Media*, Phil. Trans. R. Soc. A **378**, 20190175 (2020).

[113] T. Dore, E. McLaughlin, and J. Noronha-Hostler, *Far-from-Equilibrium Search for the QCD Critical Point*, Phys. Rev. D **102**, 074017 (2020).

[114] N. Andersson and A. Schmitt, *Dissipation Triggers Dynamical Two-Stream Instability*, Particles **2**, 457 (2019).

[115] N. Poovuttikul and W. Sybesma, *First Order Non-Lorentzian Fluids, Entropy Production, and Linear Instabilities*, Phys. Rev. D **102**, 065007 (2020).

[116] A. Das, W. Florkowski, J. Noronha, and R. Ryblewski, *Equivalence between First-Order Causal and Stable Hydrodynamics and Israel-Stewart Theory for Boost-Invariant Systems with a Constant Relaxation Time*, Phys. Lett. B **806**, 135525 (2020).

[117] M. Shokri and F. Taghinavaz, *Conformal Bjorken Flow in the General Frame and Its Attractor: Similarities and Discrepancies with the MIS Formalism*, Phys. Rev. D **102**, 036022 (2020).

[118] A. Erschfeld, S. Floerchinger, and M. Rupprecht, *General Relativistic Nonideal Fluid Equations for Dark Matter from a Truncated Cumulant Expansion*, Phys. Rev. D **102**, 063520 (2020).

[119] T. Celora, N. Andersson, and G. L. Comer, *Linearizing a Non-Linear Formulation for General Relativistic Dissipative Fluids*, Classical Quantum Gravity **38,** 065009 (2021).

[120] J. M. Stewart, *Non-Equilibrium Relativistic Kinetic Theory*, Lect. Notes Phys. **10,** 117 (1971).

[121] A. Lichnerowicz, *Théories Relativistes de la Gravitation et de l'Électromagnétism* (Masson et Cie, Paris, 1955).

[122] M. M. Disconzi, *On the Well-Posedness of Relativistic Viscous Fluids*, Nonlinearity **27,** 1915 (2014).

[123] M. Czubak and M. M. Disconzi, *On the Well-Posedness of Relativistic Viscous Fluids with Non-Zero Vorticity*, J. Math. Phys. (N.Y.) **57,** 042501 (2016).

[124] M. M. Disconzi, T. W. Kephart, and R. J. Scherrer, *A New Approach to Cosmological Bulk Viscosity*, Phys. Rev. D **91,** 043532 (2015).

[125] M. M. Disconzi, T. W. Kephart, and R. J. Scherrer, *On a Viable First Order Formulation of Relativistic Viscous Fluids and Its Applications to Cosmology*, Int. J. Mod. Phys. **26,** 1750146 (2017).

[126] G. Acquaviva and A. Beesham, *Dynamical Analysis of a First Order Theory of Bulk Viscosity*, Classical Quantum Gravity **35,** 195011 (2018).

[127] G. Montani and M. Venanzi, *Bianchi I Cosmology in the Presence of a Causally Regularized Viscous Fluid*, Eur. Phys. J. C **77,** 486 (2017).

[128] R. Geroch and L. Lindblom, *Dissipative Relativistic Fluid Theories of Divergence Type*, Phys. Rev. D **41,** 1855 (1990).

[129] R. Geroch, *Relativistic Theories of Dissipative Fluids*, J. Math. Phys. (N.Y.) **36,** 4226 (1995).

[130] H.-O. Kreiss, G. B. Nagy, O. E. Ortiz, and O. A. Reula, *Global Existence and Exponential Decay for Hyperbolic Dissipative Relativistic Fluid Theories*, J. Math. Phys. (N.Y.) **38,** 5272 (1997).

[131] G. B. Nagy, O. E. Ortiz, and O. A. Reula, *The Behavior of Hyperbolic Heat Equations' Solutions Near Their Parabolic Limits*, J. Math. Phys. (N.Y.) **35,** 4334 (1994).

[132] J. Peralta-Ramos and E. Calzetta, *Divergence-Type Nonlinear Conformal Hydrodynamics*, Phys. Rev. D **80,** 126002 (2009).

[133] J. Peralta-Ramos and E. Calzetta, *Divergence-Type $2+1$ Dissipative Hydrodynamics Applied to Heavy-Ion Collisions*, Phys. Rev. C **82,** 054905 (2010).

[134] L. Lehner, O. A. Reula, and M. E. Rubio, *Hyperbolic Theory of Relativistic Conformal Dissipative Fluids*, Phys. Rev. D **97,** 024013 (2018).

[135] N. Andersson and G. Comer, *A Covariant Action Principle for Dissipative Fluid Dynamics: From Formalism to Fundamental Physics*, Classical Quantum Gravity **32,** 075008 (2015).

[136] L. Gavassino, M. Antonelli, and B. Haskell, *Bulk Viscosity in Relativistic Fluids: From Thermodynamics to Hydrodynamics*, Classical Quantum Gravity **38,** 075001 (2021).

[137] J. Noronha, M. Spaliński, and E. Speranza, *Transient Relativistic Fluid Dynamics in a General Hydrodynamic Frame*, arXiv:2105.01034.

[138] G. S. Rocha and G. S. Denicol, *Transient Fluid Dynamics with General Matching Conditions: A First Study from the Method of Moments*, Phys. Rev. D **104,** 096016 (2021).

[139] J. Polchinski, *Effective Field Theory and the Fermi Surface*, arXiv:hep-th/9210046.

[140] A. V. Manohar, *Effective Field Theories*, Lect. Notes Phys. **479,** 311 (1997).

[141] P. F. Bedaque and U. van Kolck, *Effective Field Theory for Few Nucleon Systems*, Annu. Rev. Nucl. Part. Sci. **52,** 339 (2002).

[142] D. Forster, *Hydrodynamic Fluctuations, Broken Symmetry, and Correlation Functions*, Advanced Books Classics (CRC Press, Taylor & Francis Group, Boca Raton, FL, 1995).

[143] Note that in this paper we have $u_\mu u_\nu T^{\mu\nu} = \varepsilon + \mathcal{A}$, where $\mathcal{A}$ is given in Eq. (11c). It is worth mentioning that this choice gives $\mathcal{A} = \mathcal{O}(\partial^2)$ on shell, as can be seen from Eqs. (13), which means that the on-shell out-of-equilibrium energy $u_\mu u_\nu T^{\mu\nu} = \varepsilon + \mathcal{O}(\partial^2)$ is a positive defined quantity up to the order of validity of the theory. This is in accordance with the weak energy condition.

[144] C. Courant and D. Hilbert, *Methods of Mathematical Physics*, 1st ed. (John Wiley & Sons, Inc., New York, 1991), Vol. 2, p. 852.

[145] M. M. Disconzi and J. Speck, *The Relativistic Euler Equations: Remarkable Null Structures and Regularity Properties*, Ann. Inst Henri Poincaré **20,** 2173 (2019).

[146] M. E. Taylor, *Partial Differential Equations III: Nonlinear Equation* (Spring, New York, 2010).

[147] L. Gavassino, *Can We Make Sense of Dissipation without Causality?*, arXiv:2111.05254.

[148] R. E. Hoult and P. Kovtun, *Causal First-Order Hydrodynamics from Kinetic Theory and Holography*, arXiv:2112.14042.

[149] A. Pandya and F. Pretorius, *Numerical Exploration of First-Order Relativistic Hydrodynamics*, Phys. Rev. D **104,** 023015 (2021).

[150] H. Freistühler, *Nonexistence and Existence of Shock Profiles in the Bemfica-Disconzi-Noronha Model*, Phys. Rev. D **103,** 124045 (2021).

[151] B. Schenke, P. Tribedy, and R. Venugopalan, *Fluctuating Glasma Initial Conditions and Flow in Heavy Ion Collisions*, Phys. Rev. Lett. **108,** 252301 (2012).

[152] H. Niemi and G. S. Denicol, *How Large Is the Knudsen Number Reached in Fluid Dynamical Simulations of Ultrarelativistic Heavy Ion Collisions?*, arXiv:1404.7327.

[153] C. Shen, Z. Qiu, H. Song, J. Bernhard, S. Bass, and U. Heinz, *The iEBE-VISHNU Code Package for Relativistic Heavy-Ion Collisions*, Comput. Phys. Commun. **199,** 61 (2016).

[154] J. Noronha-Hostler, J. Noronha, and M. Gyulassy, *Sensitivity of Flow Harmonics to Subnucleon Scale Fluctuations in Heavy Ion Collisions*, Phys. Rev. C **93,** 024909 (2016).

[155] M. P. Heller and M. Spalinski, *Hydrodynamics Beyond the Gradient Expansion: Resurgence and Resummation*, Phys. Rev. Lett. **115,** 072501 (2015).

[156] J. D. Bjorken, *Highly Relativistic Nucleus-Nucleus Collisions: The Central Rapidity Region*, Phys. Rev. D **27,** 140 (1983).

[157] S. S. Gubser, *Symmetry Constraints on Generalizations of Bjorken Flow*, Phys. Rev. D **82**, 085027 (2010).

[158] S. S. Gubser and A. Yarom, *Conformal Hydrodynamics in Minkowski and de Sitter Spacetimes*, Nucl. Phys. **B846**, 469 (2011).

[159] H. Marrochio, J. Noronha, G. S. Denicol, M. Luzum, S. Jeon, and C. Gale, *Solutions of Conformal Israel-Stewart Relativistic Viscous Fluid Dynamics*, Phys. Rev. C **91**, 014903 (2015).

[160] A. Pandya, E. R. Most, and F. Pretorius, *Conservative Finite Volume Scheme for First-Order Viscous Relativistic Hydrodynamics*, arXiv:2201.12317.

[161] H. Bantilan, Y. Bea, and P. Figueras, *Evolutions in First-Order Viscous Hydrodynamics*, arXiv:2201.13359.

[162] General constraints on the variables may be imposed by considering, for instance, energy conditions [20,22]. In fact, we note that the dominant energy condition [20] imposes that for all future-directed timelike vectors z^μ the vector $Y^\nu = -z_\mu T^{\mu\nu}$ must also be a future-directed timelike or null vector [22].

[163] A. Monnai, *Off-Equilibrium Corrections to Energy and Conserved Charge Densities in the Relativistic Fluid in Heavy-Ion Collisions*, Phys. Rev. C **98**, 034902 (2018).

[164] J. Leray, *Hyperbolic Differential Equations* (The Institute for Advanced Study, Princeton, NJ, 1953).

[165] See Ref. [27] for technical remarks on function spaces and local well posedness.

[166] K. Kasahara and M. Yamaguti, *Strongly Hyperbolic Systems of Linear Partial Differential Equations with Constant Coefficients*, Mem. Coll. Sci. Univ. Kyoto Ser. A **33**, 1 (1960).

[167] K. Kajitani, *Strongly Hyperbolic Systems with Variable Coefficients*, Publ. RIMS **9**, 597 (1973).

[168] G. Métivier and T. Nishitani, *Note on Strongly Hyperbolic Systems with Involutive Characteristics*, Kyoto J. Math. **58**, 569 (2018).

[169] G. Strang, *Necessary and Insufficient Conditions for Well-Posed Cauchy Problems*, J. Differ. Equations **2**, 107 (1966).

[170] A. Lichnerowicz, *Relativistic Hydrodynamics and Magnetohydrodynamics: Lectures on the Existence of Solutions* (W.A. Benjamin, New York, 1967).

[171] If we work with uniformly local Sobolev spaces [21] instead of the ordinary Sobolev spaces, we can in fact assume $\overset{\circ}{n} \geq C > 0$, where C is a constant. The same is the case if Σ is compact. We remark that in view of the finite-speed propagation property assured by the causality of the equations, different asymptotic conditions can be considered for the initial data without significantly changing the conclusions of Theorem II.

[172] F. Becattini, *Covariant Statistical Mechanics and the Stress-Energy Tensor*, Phys. Rev. Lett. **108**, 244502 (2012).

[173] F. Becattini, *Thermodynamic Equilibrium in Relativity: Four-Temperature, Killing Vectors and Lie Derivatives*, Acta Phys. Pol. B **47**, 1819 (2016).

[174] T. G. Cowling, *The Non-Radial Oscillations of Polytropic Stars*, Mon. Not. R. Astron. Soc. **101**, 367 (1941).

[175] I. S. Gradshteyn and I. M. Ryzhik, in *Table of Integrals, Series, and Products*, 7th ed., Russian translation edited by A. Jeffrey and D. Zwillinger (Elsevier/Academic Press, Amsterdam, 2007), pp. xlviii, 1171; includes one CD-ROM (Windows, Macintosh, and UNIX).

[176] S. Collaboration, *Studying the Phase Diagram of QCD Matter at RHIC* (2014), https://drupal.star.bnl.gov/STAR/files/BES_WPII_ver6.9_Cover.pdf.

[177] T. Galatyuk (HADES), *HADES Overview*, Nucl. Phys. **A931**, 41 (2014).

[178] M. Durante et al., *All the Fun of the FAIR: Fundamental Physics at the Facility for Antiproton and Ion Research*, Phys. Scr. **94**, 033001 (2019).

[179] V. Kekelidze, A. Kovalenko, R. Lednicky, V. Matveev, I. Meshkov, A. Sorin, and G. Trubnikov, *Feasibility Study of Heavy-Ion Collision Physics at NICA JINR*, Nucl. Phys. **A967**, 884 (2017).

[180] J. A. Fotakis, M. Greif, C. Greiner, G. S. Denicol, and H. Niemi, *Diffusion Processes Involving Multiple Conserved Charges: A Study from Kinetic Theory and Implications to the Fluid-Dynamical Modeling of Heavy Ion Collisions*, Phys. Rev. D **101**, 076007 (2020).

[181] M. P. Heller, R. A. Janik, and P. Witaszczyk, *Hydrodynamic Gradient Expansion in Gauge Theory Plasmas*, Phys. Rev. Lett. **110**, 211602 (2013).

[182] G. Basar and G. V. Dunne, *Hydrodynamics, Resurgence, and Transasymptotics*, Phys. Rev. D **92**, 125011 (2015).

[183] A. Buchel, M. P. Heller, and J. Noronha, *Entropy Production, Hydrodynamics, and Resurgence in the Primordial Quark-Gluon Plasma from Holography*, Phys. Rev. D **94**, 106011 (2016).

[184] G. S. Denicol and J. Noronha, *Divergence of the Chapman-Enskog Expansion in Relativistic Kinetic Theory*, arXiv:1608.07869.

[185] M. P. Heller, A. Kurkela, M. Spalinski, and V. Svensson, *Hydrodynamization in Kinetic Theory: Transient Modes and the Gradient Expansion*, Phys. Rev. D **97**, 091503(R) (2018).

[186] J. Casalderrey-Solana, N. I. Gushterov, and B. Meiring, *Resurgence and Hydrodynamic Attractors in Gauss-Bonnet Holography*, J. High Energy Phys. 04 (2018) 042.

[187] G. S. Denicol and J. Noronha, *Analytical Attractor and the Divergence of the Slow-Roll Expansion in Relativistic Hydrodynamics*, Phys. Rev. D **97**, 056021 (2018).

[188] G. S. Denicol and J. Noronha, *Hydrodynamic Attractor and the Fate of Perturbative Expansions in Gubser Flow*, Phys. Rev. D **99**, 116004 (2019).

[189] B. Withers, *Short-Lived Modes from Hydrodynamic Dispersion Relations*, J. High Energy Phys. 06 (2018) 059.

[190] A. Behtash, S. Kamata, M. Martinez, and H. Shi, *Global Flow Structure and Exact Formal Transseries of the Gubser Flow in Kinetic Theory*, J. High Energy Phys. 07 (2020) 226.

[191] G. S. Denicol and J. Noronha, *Exact Hydrodynamic Attractor of an Ultrarelativistic Gas of Hard Spheres*, Phys. Rev. Lett. **124**, 152301 (2020).

[192] S. Grozdanov, P. K. Kovtun, A. O. Starinets, and P. Tadić, *Convergence of the Gradient Expansion in Hydrodynamics*, Phys. Rev. Lett. **122**, 251601 (2019).

[193] S. Grozdanov, P. K. Kovtun, A. O. Starinets, and P. Tadić, *The Complex Life of Hydrodynamic Modes*, J. High Energy Phys. 11 (2019) 097.

[194] M. P. Heller, A. Serantes, M. Spaliński, V. Svensson, and B. Withers, *The Hydrodynamic Gradient Expansion in Linear Response Theory*, Phys. Rev. D **104**, 066002 (2021).

[195] R. Ciolfi, *The Key Role of Magnetic Fields in Binary Neutron Star Mergers*, Gen. Relativ. Gravit. **52**, 59 (2020).

[196] V. Skokov, A. Illarionov, and V. Toneev, *Estimate of the Magnetic Field Strength in Heavy-Ion Collisions*, Int. J. Mod. Phys. A **24**, 5925 (2009).

[197] M. Chandra, C. F. Gammie, F. Foucart, and E. Quataert, *An Extended Magnetohydrodynamics Model for Relativistic Weakly Collisional Plasmas*, Astrophys. J. **810**, 162 (2015).

[198] J. Hernandez and P. Kovtun, *Relativistic Magnetohydrodynamics*, J. High Energy Phys. 05 (2017) 001.

[199] S. Grozdanov, D. M. Hofman, and N. Iqbal, *Generalized Global Symmetries and Dissipative Magnetohydrodynamics*, Phys. Rev. D **95**, 096003 (2017).

[200] G. S. Denicol, X.-G. Huang, E. Molnár, G. M. Monteiro, H. Niemi, J. Noronha, D. H. Rischke, and Q. Wang, *Nonresistive Dissipative Magnetohydrodynamics from the Boltzmann Equation in the 14-Moment Approximation*, Phys. Rev. D **98**, 076009 (2018).

[201] E. R. Most and J. Noronha, *Dissipative Magnetohydrodynamics for Nonresistive Relativistic Plasmas: An Implicit Second-Order Flux-Conservative Formulation with Stiff Relaxation*, Phys. Rev. D **104**, 103028 (2021).

[202] E. R. Most, J. Noronha, and A. A. Philippov, *Modeling General-Relativistic Plasmas with Collisionless Moments and Dissipative Two-Fluid Magnetohydrodynamics*, arXiv:2111.05752.

[203] J. Armas and F. Camilloni, *A Stable and Causal Model of Magnetohydrodynamics*, arXiv:2201.06847.

[204] D. Christodoulou, *The Formation of Shocks in 3-Dimensional Fluids*, EMS Monographs in Mathematics (European Mathematical Society, Zurich, 2007), pp. viii, 992.

[205] D. Christodoulou, *The Shock Development Problem*, EMS Monographs in Mathematics (European Mathematical Society, Zurich, 2019), p. 932.

[206] J. Speck, *Shock Formation in Small-Data Solutions to 3D Quasilinear Wave Equations*, Mathematical Surveys and Monographs Vol. 214 (American Mathematical Society, Providence, 2016), pp. xxiii, 515.

[207] J. Casalderrey-Solana, E. Shuryak, and D. Teaney, *Conical Flow Induced by Quenched QCD Jets*, J. Phys. Conf. Ser. **27**, 22 (2005).

[208] L. Satarov, H. Stoecker, and I. Mishustin, *Mach Shocks Induced by Partonic Jets in Expanding Quark-Gluon Plasma*, Phys. Lett. B **627**, 64 (2005).

[209] S. S. Gubser, S. S. Pufu, and A. Yarom, *Sonic Booms and Diffusion Wakes Generated by a Heavy Quark in Thermal AdS/CFT*, Phys. Rev. Lett. **100**, 012301 (2008).

[210] B. Betz, M. Gyulassy, J. Noronha, and G. Torrieri, *Anomalous Conical Di-Jet Correlations in pQCD vs AdS/CFT*, Phys. Lett. B **675**, 340 (2009).

[211] B. Betz, J. Noronha, G. Torrieri, M. Gyulassy, I. Mishustin, and D. H. Rischke, *Universality of the Diffusion Wake from Stopped and Punch-Through Jets in Heavy-Ion Collisions*, Phys. Rev. C **79**, 034902 (2009).

[212] I. Bouras, E. Molnar, H. Niemi, Z. Xu, A. El, O. Fochler, C. Greiner, and D. H. Rischke, *Relativistic Shock Waves in Viscous Gluon Matter*, Phys. Rev. Lett. **103**, 032301 (2009).

[213] I. Bouras, E. Molnar, H. Niemi, Z. Xu, A. El, O. Fochler, C. Greiner, and D. H. Rischke, *Investigation of Shock Waves in the Relativistic Riemann Problem: A Comparison of Viscous Fluid Dynamics to Kinetic Theory*, Phys. Rev. C **82**, 024910 (2010).

[214] B. Betz, J. Noronha, G. Torrieri, M. Gyulassy, and D. H. Rischke, *Universal Flow-Driven Conical Emission in Ultrarelativistic Heavy-Ion Collisions*, Phys. Rev. Lett. **105**, 222301 (2010).

[215] I. Bouras, B. Betz, Z. Xu, and C. Greiner, *Mach Cones in Viscous Heavy-Ion Collisions*, Phys. Rev. C **90**, 024904 (2014).

[216] Y. Tachibana and T. Hirano, *Momentum Transport Away from a Jet in an Expanding Nuclear Medium*, Phys. Rev. C **90**, 021902(R) (2014).

[217] L. Yan, S. Jeon, and C. Gale, *Jet-Medium Interaction and Conformal Relativistic Fluid Dynamics*, Phys. Rev. C **97**, 034914 (2018).

[218] Y. Tachibana, N.-B. Chang, and G.-Y. Qin, *Full Jet in Quark-Gluon Plasma with Hydrodynamic Medium Response*, Phys. Rev. C **95**, 044909 (2017).

[219] Y. Tachibana, *Medium Response to Jet-Induced Excitation: Theory Overview*, Nucl. Phys. **A982**, 156 (2019).

[220] E. Calzetta, *Fully Developed Relativistic Turbulence*, Phys. Rev. D **103**, 056018 (2021).

[221] Mathematically, $c_a = 1$, which corresponds to $\xi_\mu \xi^\mu = 0$, is allowed for causality. But this would correspond to information propagating at the speed of light in the matter sector, which physically is not expected. Thus, our results are in fact a bit stronger than the statement of the theorem in that we obtain causality with the characteristic speeds of the matter sector strictly less than the speed of light. In particular, it is because we demand the strict inequality $c_a < 1$ that we have strict inequalities in Eqs. (21a), (21c), and (21d).

[222] We recall that we are interested in the diagonalization because it allows us to invoke known techniques to prove local well-posedness. If the system is not diagonalizable, it remains possible that different techniques would lead to local well-posedness.

[223] M. Hadžić and J. Speck, *The Global Future Stability of the FLRW Solutions to the Dust-Einstein System with a Positive Cosmological Constant*, J. Hyperbol. Differ. Equations **12**, 87 (2015).

[224] J. Leray and Y. Ohya, *Équations et Systèmes Non Linéaires, Hyperboliques Non-Stricts*, in *Hyperbolic Equations and Waves (Rencontres, Battelle Res. Inst., Seattle, WA, 1968)* (Springer, Berlin, 1970), pp. 331–369.

[225] Y. Choquet-Bruhat, *Diagonalisation des Systèmes Quasi-Linéaires et Hyperbolicité Non Stricte*, J. Math. Pure Appl. **45**, 371 (1966).

[226] R. S. Palais, *Seminar on the Atiyah-Singer Index Theorem*, Annals of Mathematics Studies No. 57 (Princeton University Press, Princeton, NJ, 1965), p. x+366, with contributions by M. F. Atiyah, A. Borel, E. E. Floyd, R. T. Seeley, W. Shih, and R. Solovay.