

LOWER BOUNDS ON THE F -PURE THRESHOLD AND EXTREMAL SINGULARITIES

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ABSTRACT. We prove that if f is a reduced homogeneous polynomial of degree d , then its F -pure threshold at the unique homogeneous maximal ideal is at least $\frac{1}{d-1}$. We show, furthermore, that its F -pure threshold equals $\frac{1}{d-1}$ if and only if $f \in \mathfrak{m}^{[q]}$ and $d = q + 1$, where q is a power of p . Up to linear changes of coordinates (over a fixed algebraically closed field), we classify such “extremal singularities”, and show that there is at most one with isolated singularity. Finally, we indicate several ways in which the projective hypersurfaces defined by such forms are “extremal”, for example, in terms of the configurations of lines they can contain.

1. INTRODUCTION

Fix an algebraically closed field k of positive characteristic p . What is the most singular possible hypersurface singularity over k ?

The *multiplicity* is the first crude measurement of singularity—roughly speaking higher multiplicity singularities are more singular. But we want to identify which singularities are the most singular, even taking multiplicity into account. Among multiplicity two singularities, for example, the cusp $y^2 - x^3$ is more singular than the normal crossing xy , but also the cusp is more singular in some characteristics than others. For example, the derivative with respect to x vanishes to all orders when k has characteristic three, but only to order two in any other characteristic.

The *F -pure threshold* is a more refined numerical invariant for comparing singularities of hypersurfaces in positive characteristic. Analogous to—but much more subtle than—the log canonical threshold for a complex hypersurface, the F -pure threshold is equal to one at each smooth point (or more generally, at each *F -pure* point), with “worse singularities” having *smaller* F -pure thresholds. Using the F -pure threshold, can we identify a class of singularities that is “maximally singular”? What are the properties of such “extremal singularities”?

This paper proves an elegant lower bound on the F -pure threshold of a reduced homogeneous form in terms of its degree—that is, in terms of the multiplicity at the origin. We show that our lower bound is sharp, and characterize precisely what

Received by the editors November 1, 2021.

2020 *Mathematics Subject Classification*. Primary 13A35; Secondary 14B05.

This project began at the AWM-sponsored workshop “Women in Commutative Algebra” at the Banff International Research Station. Partial funding for participants was supplied by NSF grant numbers 193439 and NSF-HRD 150048. In addition, partial funding was provided by SERB(DST) grant number MTR/2021/000879 (for Jyoti Singh), NSF grant number 1801697, 1952399, and 2101075 (for Karen Smith), and NSF CAREER grant 1945611 (for Emily Witt).

forms achieve it. One can argue that such hypersurfaces—those with minimal F -pure threshold—are “maximally singular”, and indeed, we prove they have several interesting extremal algebraic and geometric properties. We also prove a complete classification of these *extremal singularities* up to linear change of coordinates.

The main theorem of the first half of the paper is the following sharp bound on the F -pure threshold and characterization of the forms achieving it:

Theorem 1.1. *Fix any field k of positive characteristic p . Let $f \in k[x_1, \dots, x_n]$ be a homogeneous polynomial of degree $d = \deg(f) \geq 2$, reduced over the algebraic closure of k . Then*

$$(1) \quad \text{fpt}(f) \geq \frac{1}{d-1}.$$

Furthermore, equality holds in (1) if and only if $d = q + 1$, where q is a power of p and $f \in \langle x_1^q, \dots, x_n^q \rangle$.

Theorem 1.1 is a combination of Theorem 3.1, in which we prove the general bound in terms of degree and find the degrees that can achieve it, and Theorem 4.3, in which we characterize the *forms* achieving the bound. The most difficult step is showing that the bound can be sharp only for certain d , which uses delicate analytic estimates in Section 2. While there has been much research into computing the F -pure threshold in specific settings (see, for example, [BS15], [Her16], [HNWZ16], [HT17]), we are not aware of any prior research into lower bounds on the F -pure threshold.

The forms achieving the lower bound (1) are extremal in many ways unrelated to F -pure threshold. In Section 8, we discuss several extremal geometric properties of the corresponding projective hypersurfaces. For example, when the form is in at least four variables and defines a smooth hypersurface, these are the only projective hypersurfaces (of degree larger than two) with the property that all smooth hyperplane sections are isomorphic; this follows from Corollary 8.3 and a theorem of Beauville (see Remark 8.4). They have inseparable Gauss maps and are isomorphic to their Gauss duals; see Proposition 8.5 for a precise statement and Remark 8.6 for a discussion of how such behavior is extremal. Their extremal nature is also reflected in the very special configurations of lines on them: despite containing many coplanar lines, no plane section can contain a triangle (Proposition 8.10). This triangle-free property characterizes smooth cubic surfaces defined by forms of minimal F -pure threshold [KKP⁺21].

There is no analog for such “extremely singular” singularities in characteristic zero. For example, regardless of the characteristic, a smooth cubic surface can contain at most eighteen Eckardt points—unless its defining equation achieves the minimal F -pure threshold of bound (1), in which case this *extremal* cubic surface contains exactly *forty-five* Eckardt points—one in each and every of the forty-five tri-tangent planes [KKP⁺21]. As another example, in characteristic zero (or when $p > d$), the maximal number of lines on a smooth projective surface of degree d is bounded above by a quadratic function in d [Seg43, RS15, BR20]; however, the number of lines on a smooth projective *extremal* surface—one whose defining polynomial achieves the lower bound (1)—is *quartic* in its degree [BPRS21, 3.2.3]. These examples confirm that singularities can be *much worse* in characteristic p than in characteristic zero, reflecting the fact that our sharp lower bound on F -pure

threshold is much smaller than corresponding bounds on log canonical threshold; see paragraph (1.2).

In the second half of the paper, we tackle the classification of the extremal forms achieving the lower bound (1). Theorem 1.1 says that they are in high “Frobenius powers” of the unique homogeneous maximal ideal—more precisely, we can represent them by

$$x_1^q L_1 + \cdots + x_n^q L_n,$$

where the L_i are linear forms. We call such polynomials *Frobenius forms*. Frobenius forms have a convenient matrix representation, not unlike the matrix representation for a quadratic form, that we can exploit to prove:

Theorem 1.2. *Over any algebraically closed field k of positive characteristic p , there is a bijection between the projective equivalence classes of non-degenerate Frobenius forms in n variables (of fixed degree $q+1$) and partitions of n . Only one of these defines an isolated singularity, namely the one represented by the “diagonal” form $x_1^{q+1} + x_2^{q+1} + \cdots + x_n^{q+1}$.*

In three variables, for example, we have three distinct projective equivalence classes of non-degenerate extremal singularities (over each allowable field and of each allowable degree), corresponding to the three partitions 3 , $2+1$ and $1+1+1$. For example, Theorem 1.2 implies that there are precisely five extremal *cubic* surfaces up to change of coordinates, not including the degenerate ones (which are cones over extremal curves); these extremal cubic surfaces turn out to be exactly the non-Frobenius split cubic surfaces, a class studied in [KKP⁺21]. The case where $n = 5$ was handled in [KPS⁺21].

Theorem 1.2 is a combination of Theorems 6.1 and 7.1 proved in Sections 6 and 7, respectively. We believe that the classification of Frobenius forms by partitions, Theorem 7.1, is new, though it resembles a *different* classification problem considered by Hasse and Witt [HW36] (see Remark 7.5). Theorem 6.1, that there is only one (up to change of coordinates) with isolated singularity, follows also from a theorem of Beauville when there are at least four variables [Bea90]; we give a straightforward argument using Hilbert’s Nullstellensatz.

1.1. Comparison to Hermitian forms in prime characteristic. Frobenius forms should not be confused with characteristic p *Hermitian forms*, which form a very special subclass of Frobenius forms. Hermitian forms have long been known to be extremal with respect to the number of rational points the corresponding projective hypersurface contains; see, for example, [Seg65], [BC66], [HK16]. The existence of many rational points is a consequence of the many linear spaces contained in the projective hypersurface defined by a Frobenius form; see the discussion in [Kol15, §35] for example. The classification of Hermitian forms is much simpler than general Frobenius forms: there is only one of each rank [BC66], [HW36]. For the definition of Hermitian form, see Remark 5.5.

1.2. Comparison to lower bounds for log canonical threshold. Lower bounds on log canonical thresholds have been studied in [CP02], [dFEM03], and [DP14]. Our lower bound on the F -pure threshold immediately implies an analogous bound on the log canonical threshold of a complex hypersurface by reduction to characteristic p ; see Section 2.1. However, while sharp in prime characteristics, the corresponding bound for the log canonical threshold is far from sharp. This is to

be expected: singularities in prime characteristic can be “bad” in ways not possible over \mathbb{C} .

For example, de Fernex, Ein, and Mustață [dFEM03] prove that for a complex homogeneous polynomial of degree d in n variables, the log canonical threshold is bounded below by

$$\min\left(\frac{n-r}{d}, 1\right),$$

where r is the dimension of the singular locus of the corresponding affine hypersurface. The corresponding statement in positive characteristic, however, is spectacularly false. For example, the polynomial in characteristic p defined by

$$x_1^{p^e+1} + x_2^{p^e+1} + \cdots + x_n^{p^e+1}$$

has F -pure threshold $\frac{1}{p^e}$ (a simple case of Theorem 1.1), which is much smaller, for $n \gg p$, than the value provided by the de Fernex-Ein-Mustață bound.

2. BACKGROUND ON THE F -PURE THRESHOLD

Fix a field k of positive characteristic p . Given a homogeneous form $f \in k[x_1, \dots, x_n]$, we consider the singularity, at the origin, of the affine hypersurface defined by the vanishing of f . Algebraically stated, we consider the singularity of f at \mathfrak{m} , where \mathfrak{m} denotes the maximal ideal $\langle x_1, \dots, x_n \rangle$.

Definition 2.1. The F -pure threshold of $f \in k[x_1, \dots, x_n]$ (at the maximal ideal \mathfrak{m}) is the real number

$$\text{fpt}(f) = \sup \left\{ \frac{N}{p^e} \mid f^N \notin \mathfrak{m}^{[p^e]} \right\} = \inf \left\{ \frac{N}{p^e} \mid f^N \in \mathfrak{m}^{[p^e]} \right\},$$

where $\mathfrak{m}^{[p^e]}$ denotes the Frobenius power $\langle x_1^{p^e}, \dots, x_n^{p^e} \rangle$ of \mathfrak{m} .

While first explicitly defined (in a more general setting) by Takagi and Watanabe [TW04] as the “threshold c ” beyond which the pair (S, f^c) fails to be F -pure (see [HY03]), the definition above is a reformulation that has evolved through the work of many authors (e.g., see [MTW05], [BMS08]). A gentle introduction can be found in the survey [BFS13].

Although not obvious, the F -pure threshold is in fact a *rational number* [BMS08]. Further basic properties, including some immediate upper and lower bounds well-known to experts, are summarized below in the setting we will need them:

Proposition 2.2. *Let f be a homogeneous form of degree $d > 0$ over a field k of characteristic $p > 0$. Then*

- (1) $\text{fpt}(f) \leq 1$.
- (2) *For any $r \geq 1$, we have $\text{fpt}(f^r) = \frac{\text{fpt}(f)}{r}$.*
- (3) $\text{fpt}(f) \geq \frac{1}{d}$, with equality when f is a power of a linear form; [TW04, 4.1].
- (4) *If f is in n variables, then $\text{fpt}(f) \leq \frac{n}{d}$.*

Proof. For (1), since $f^1 \in \mathfrak{m}^{[p^0]} = \mathfrak{m}$, $1 \in \left\{ \frac{N}{p^e} \mid f^N \in \mathfrak{m}^{[p^e]} \right\}$. So the infimum of this set is at most one, always. The second statement similarly follows easily from the definition.

For (3), we argue as follows. Let $r_e = \lfloor p^e/d \rfloor - 1$. Then f^{r_e} has degree less than p^e , so is not in $\mathfrak{m}^{[p^e]}$. So $\frac{\lfloor p^e/d \rfloor - 1}{p^e} \in \left\{ \frac{N}{p^e} \mid f^N \notin \mathfrak{m}^{[p^e]} \right\}$ for all e , and so the supremum is at least $\frac{\lfloor p^e/d \rfloor - 1}{p^e}$ for all e . Since these converge from below to $\frac{1}{d}$ as e goes to infinity, the F -pure threshold is at least $\frac{1}{d}$.

For (4), observe that $\mathfrak{m}^m \subset \mathfrak{m}^{[p^e]}$ for any $m \geq np^e - n + 1$. So if f has degree d , then for any natural number r_e such that $r_e \geq \frac{np^e - n + 1}{d}$, we have

$$f^{r_e} \in \mathfrak{m}^{[p^e]}.$$

In particular, this is true for

$$r_e = \left\lceil \left(\frac{np^e - n + 1}{d} \right) \right\rceil = \left\lceil p^e \left(\frac{n}{d} - \frac{n}{dp^e} + \frac{1}{dp^e} \right) \right\rceil.$$

This means that

$$\frac{r_e}{p^e} \in \left\{ \frac{N}{p^e} \mid f^N \in \mathfrak{m}^{[p^e]} \right\}$$

for all e . Since $\frac{r_e}{p^e}$ converges to $\frac{n}{d}$ from above, the infimum of this set is at most $\frac{n}{d}$. \square

The idea is that “worse” singularities have smaller F -pure threshold, just as smaller log canonical thresholds for a complex singularity indicate “worse singularities”

2.1. Connection with log canonical threshold. The *log canonical threshold* is an invariant of a complex singularity that can be defined using either integrability or Hironaka’s resolution of singularities. For a \mathbb{Q} -divisor D on a smooth complex variety X , it is the threshold beyond which the pair (X, cD) fails to be log canonical; see, for example, [Kol97].

A form f over the complex numbers determines a collection of forms over fields of characteristic p , for varying p , as follows. Let A be the finitely generated subring of \mathbb{C} obtained by adjoining all the complex coefficients of f to \mathbb{Z} . Interpreting the form f as an element of $A[x_1, \dots, x_n]$, for each $\mu \in \text{maxSpec } A$, we define f_μ to be the image of f in the quotient ring $A/\mu[x_1, \dots, x_n]$, a polynomial ring over the finite field A/μ .

The work of Hara and Yoshida [HY03] and Takagi and Watanabe [TW04] implies that the log canonical threshold of f is

$$(2) \quad \sup \{ \text{fpt}(f_\mu) \mid \mu \in \text{maxSpec } A \}.$$

In particular, any lower bound on the F -pure threshold (independent of p) implies one for the log canonical threshold. In particular, our Theorem 3.1 implies that the log canonical threshold of a reduced complex form of degree d is bounded below by $\frac{1}{d-1}$. However, this bound is far from sharp; see paragraph 1.2.

On the other hand, (2) implies that upper bounds for the log canonical threshold suggest tight upper bounds for the F -pure threshold. For example, it is easy to compute that a degree d form over \mathbb{C} defining an isolated singularity has log canonical threshold $\min(\frac{n}{d}, 1)$, and indeed, the proof of Proposition 2.2 (4) above shows that $\min(\frac{n}{d}, 1)$ is what we might expect for the F -pure threshold of most general forms f of degree d in n variables. This intuition is made precise in [Her16].

One research thread in the literature is concerned with understanding the extent to which the F -pure threshold pushes up against these theoretic *upper* bounds. A

long-standing open conjecture predicts that when f is obtained by reduction to characteristic p from a polynomial over \mathbb{C} , then for infinitely many p , the F -pure threshold will be equal to the log canonical threshold of the corresponding complex singularity; see, for example, [MTW05, 3.6].

In this paper, we tackle the opposite question: find general *lower bounds* on the F -pure threshold in terms of degree and investigate the extent to which we push up against these bounds.

3. LOWER BOUNDS ON THE F -PURE THRESHOLD

In this section, we establish the lower bound of our main theorem, Theorem 1.1, by proving the following:

Theorem 3.1. *Fix an arbitrary field k of characteristic $p > 0$. Let $f \in k[x_1, \dots, x_n]$ be a homogeneous geometrically reduced polynomial of degree $d = \deg(f) \geq 2$. Then*

$$(3) \quad \text{fpt}(f) \geq \frac{1}{d-1}.$$

Furthermore, if equality holds in (3), then $d = q + 1$, where $q \geq 1$ is a power of p .

Having degree $p^e + 1$ is necessary but not sufficient to achieve the lower bound (3) on the F -pure threshold: some forms of degree $p^e + 1$ are more singular than others. Characterizing these “maximally singular” forms where the lower bound is achieved is the task of the next section; see Theorem 4.3.

Remark 3.2. For the bound (3), a careful reading of the proof shows that our hypothesis can be weakened from “reduced” to “not a power of a linear form”. But for powers of linear forms, of course, the lower bound is false: the F -pure threshold of x^d is $\frac{1}{d}$.

For the subsequent statement about what happens when the lower bound is achieved, however, we need the “reduced” hypothesis. For example, x^6y has F -pure threshold $\frac{1}{6} = \frac{1}{d-1}$ in every characteristic, but 6 is not a power of p .

Remark 3.3. The main theorem of [BS15] can be interpreted to give a lower bound of $1 - \frac{d-2}{p}$ on the F -pure threshold in the very special case where the degree d of the form f is equal to the number of variables n , the projective hypersurface defined by f is smooth, and the characteristic $p > n = d$. (See also [Mül18] for the quasi-homogeneous case.) However, because of the restriction that $p > d$, the Bhatt-Singh bound never applies in our extremal case.

Before beginning the proof of Theorem 3.1, we need a few lemmas. The first allows us to assume the ground field is algebraically closed.

Lemma 3.4. *Let $k \subset k'$ be any field extension of characteristic $p > 0$. For any $f \in k[x_1, \dots, x_n]$, the F -pure threshold is independent of whether we view f as a polynomial over k or over k' .*

Proof. Let $A = k[x_1, \dots, x_n]$ have homogeneous maximal ideal \mathfrak{m}_A and $B = k'[x_1, \dots, x_n]$ have homogeneous maximal ideal \mathfrak{m}_B . Note that $A \subset B$ is faithfully flat so that $IB \cap A = I$ for any ideal I of A .

Now, if $f^N \in \mathfrak{m}_A^{[p^E]}$, then the same is true in B . So the F -pure threshold over A is at least the F -pure threshold over B . But conversely, if $f^N \in \mathfrak{m}_B^{[p^E]} = \mathfrak{m}_A^{[p^E]}B$,

then intersecting with A , we have $f^N \in \mathfrak{m}_A^{[p^E]} B \cap A = \mathfrak{m}_A^{[p^E]}$. This gives the reverse inequality. \square

The next is a codification of a well-known trick we use many times, whose statement we make explicit for the convenience of the reader. The proof we leave as an exercise.

Lemma 3.5. *Let y_1, \dots, y_n be a regular sequence in a commutative ring. Suppose there exists an element g , and natural numbers $a_i \leq N_i$ such that*

$$(y_1^{a_1} y_2^{a_2} \dots y_n^{a_n})g \in \langle y_1^{N_1}, y_2^{N_2}, \dots, y_n^{N_n} \rangle.$$

Then $g \in \langle y_1^{N_1-a_1}, y_2^{N_2-a_2}, \dots, y_n^{N_n-a_n} \rangle$.

The next two lemmas will be used to reduce Theorem 3.1 to the case of two variables.

Lemma 3.6. *Let f be a homogeneous polynomial in $k[x_1, \dots, x_n]$. For any linear form L not dividing f , let \bar{f} denote the image of f in $k[x_1, \dots, x_n]/\langle L \rangle \cong k[x_1, \dots, x_{n-1}]$. Then*

$$\text{fpt}(f) \geq \text{fpt}(\bar{f}).$$

Proof. Suppose $f^N \in \mathfrak{m}^{[p^e]}$. Then also $\bar{f}^N \in \bar{\mathfrak{m}}^{[p^e]}$. So we have an inclusion of sets

$$\left\{ \frac{N}{p^e} \mid f^N \in \mathfrak{m}^{[p^e]} \right\} \subseteq \left\{ \frac{N}{p^e} \mid \bar{f}^N \in \bar{\mathfrak{m}}^{[p^e]} \right\}.$$

So the infimum of the left-hand set is at least the infimum of the right-hand set. That is, $\text{fpt}(f) \geq \text{fpt}(\bar{f})$. \square

Remark 3.7. Lemma 3.6 can be used to give a quick proof of the easy lower bound $\text{fpt}(f) \geq \frac{1}{d}$, where $d = \deg(f)$, shown in Proposition 2.2. Namely, by modding out $n-1$ linearly independent one-forms, we eventually reduce to the one-variable case, and have $\text{fpt}(f) \geq \text{fpt}(x^d) = \frac{1}{d}$.

Finally, we need the following Bertini-type theorem for reduced varieties:

Lemma 3.8. *Consider a polynomial ring $k[x_1, \dots, x_n]$ in at least three variables over an infinite field. Fix a reduced form f in $k[x_1, \dots, x_n]$, and for any linear form L , let \bar{f} denote the image of f in the quotient ring $k[x_1, \dots, x_n]/\langle L \rangle \cong k[x_1, \dots, x_{n-1}]$. If L is sufficiently general, then the polynomial \bar{f} is also reduced in the polynomial ring $k[x_1, \dots, x_{n-1}]$.*

Proof. This is essentially a restatement of [FOV99, 3.4.14], which implies that (over an infinite field) a general hyperplane section of a reduced scheme $X \subset \mathbb{P}^{n-1}$ is reduced: because f is reduced, the projective variety defined by f in \mathbb{P}^{n-1} is reduced, so also the general hyperplane section—namely, the variety defined by f and L —is reduced. Because the homogeneous coordinate ring of this hyperplane section is isomorphic to $k[x_1, \dots, x_n]/\langle L, f \rangle \cong k[x_1, \dots, x_{n-1}]/\langle \bar{f} \rangle$, we see that also \bar{f} is reduced. \square

Proof of Theorem 3.1. By Lemma 3.4, we may assume the ground field is algebraically closed.

We first reduce to the case of two variables. Given a form f of degree d in n variables, we can successively mod out a sequence of $n-2$ independent linear forms until we have a form \bar{f} of degree d in two variables. Now Lemma 3.6 implies that

any lower bound on the F -pure threshold of \bar{f} is also a lower bound for f . Likewise, if this lower bound is achieved for f , then it is also achieved for \bar{f} . So because d and p are the same for f and \bar{f} , it suffices to prove the required implication for \bar{f} . Finally, if f is reduced, Lemma 3.8 ensures that \bar{f} is reduced, by choosing a sufficiently general sequence of linear forms. Thus the proof of Theorem 3.1 reduces to the case of two variables.

We next dispose of the case $d = 2$. Any reduced polynomial of degree 2 in two variables factors into two distinct linear factors, so without loss of generality, $f = xy$. By direct computation, the F -pure threshold is 1 in every characteristic. So equality holds in (3) and $d = p^0 + 1$.

Now assume $d \geq 3$. Given a form f of degree d in two variables, we can factor as $f = xyg$ where g is a form of degree $d - 2$. To prove the lower bound (3), it suffices to show that for any p^E such that $f^N \in \mathfrak{m}^{[p^E]}$, we have

$$(4) \quad \frac{N}{p^E} \geq \frac{1}{d-1}.$$

But if $f^N \in \mathfrak{m}^{[p^E]} = \langle x^{p^E}, y^{p^E} \rangle$, then writing

$$(xyg)^N = Ax^{p^E} + By^{p^E}$$

for some homogeneous A and B , we can use the fact that $\{x, y\}$ is a regular sequence (Lemma 3.5) to see that

$$g^N \in \langle x^{p^E - N}, y^{p^E - N} \rangle.$$

By comparing degrees, it follows that $N(d - 2) \geq p^E - N$, which is equivalent to the desired inequality (4). This shows the F -pure threshold is at least $\frac{1}{d-1}$.

We now investigate what happens when equality holds in (3). Assume that $\text{fpt}(f) = \frac{1}{d-1}$. For all $e \geq 1$, we have

$$\frac{\lceil \frac{p^e+1}{d-1} \rceil}{p^e} \geq \frac{p^e+1}{p^e(d-1)} > \frac{1}{d-1} = \text{fpt}(f).$$

Since $\text{fpt}(f)$ is the supremum of the set $\{\frac{N}{p^E} \mid f^N \notin \mathfrak{m}^{[p^E]}\}$, and $\frac{\lceil \frac{p^e+1}{d-1} \rceil}{p^e}$ is strictly bigger than this supremum, it must be that

$$f^{\lceil \frac{p^e+1}{d-1} \rceil} \in \mathfrak{m}^{[p^e]}.$$

To ease notation, we set $K_e = \lceil \frac{p^e+1}{d-1} \rceil$. We have $f^{K_e} \in \langle x^{p^e}, y^{p^e} \rangle$, so we write

$$(5) \quad f^{K_e} = Ax^{p^e} + By^{p^e},$$

where A and B are forms of degree $dK_e - p^e$.

Also using the strict inequality $\frac{K_e}{p^e} > \frac{1}{d-1}$, we have $\frac{K_e}{p^e} - \frac{1}{p^E} > \frac{1}{d-1}$ for all sufficiently large E . So similarly,

$$f^{p^{E-e}K_e-1} \in \langle x^{p^E}, y^{p^E} \rangle,$$

and we can write

$$(6) \quad f^{p^{E-e}K_e-1} = Cx^{p^E} + Dy^{p^E}$$

for some forms C and D .

Now raising (5) to the power p^{E-e} and multiplying (6) by f , we have two different expressions for the form $f^{p^{E-e}K_e}$. Subtracting them, we have

$$(7) \quad (A^{p^{E-e}} - fC)x^{p^E} + (B^{p^{E-e}} - fD)y^{p^E} = 0.$$

Again using the fact that x, y is a regular sequence, we conclude that

$$(8) \quad (A^{p^{E-e}} - fC) \in \langle y^{p^E} \rangle.$$

But we claim that

$$(9) \quad \deg(A^{p^{E-e}} - fC) = p^{E-e} \deg A < p^E,$$

which implies that $A^{p^{E-e}} - fC = 0$.

To check claim (9), recall that the degree of A is $dK_e - p^e$, so that (9) is equivalent to $dK_e < 2p^e$. In turn, we have

$$dK_e = d \left\lceil \frac{p^e + 1}{d-1} \right\rceil \leq d \left(\frac{p^e + 1}{d-1} + 1 \right),$$

which is less than $2p^e$ for all large e .

Having established the veracity of claim (9) for $E \gg e \gg 0$, we can conclude using the inclusion in (8) that $A^{p^{E-e}} - fC = 0$, so that

$$A^{p^{E-e}} = fC.$$

We now invoke the fact that f is a product of *distinct* irreducible polynomials: the unique factorization property of the polynomial ring implies that f must divide the form A . Similarly, f divides B .

Now, because f divides both A and B , we can divide f out of both sides of equation (5) above, to get

$$(10) \quad f^{K_e-1} \in \langle x^{p^e}, y^{p^e} \rangle.$$

Remembering that $f = xyg$, where g has degree $d-2$, we can again use the fact that x, y is a regular sequence (Lemma 3.5) to deduce that

$$(11) \quad g^{K_e-1} \in \langle x^{p^e-K_e+1}, y^{p^e-K_e+1} \rangle.$$

Looking at degrees, this says that

$$(K_e - 1)(d - 2) \geq p^e - K_e + 1,$$

which is equivalent to

$$(d - 1)K_e \geq p^e + d - 1,$$

or equivalently,

$$(12) \quad K_e \geq \frac{p^e + d - 1}{d - 1} = \frac{(p^e + 1) + (d - 2)}{d - 1} = \frac{p^e + 1}{d - 1} + \frac{d - 2}{d - 1}.$$

Remembering that $K_e = \lceil \frac{p^e + 1}{d - 1} \rceil$, we see that inequality (12) can hold only if $\frac{p^e + 1}{d - 1}$ is *not* an integer and

$$\frac{p^e + 1}{d - 1} \quad \text{and} \quad \frac{p^e + 1}{d - 1} + \frac{d - 2}{d - 1}$$

round up to the same integer. This means that $\frac{p^e + 1}{d-1}$ must be equal to $\left\lfloor \frac{p^e + 1}{d-1} \right\rfloor + \frac{1}{d-1}$; put differently, the remainder when we divide $p^e + 1$ by $d-1$ is 1. So $d-1$ divides p^e . In this case, $d-1$ is a power of p (as desired). The proof is complete. \square

4. EXTREMAL SINGULARITIES

In this section, we complete the proof of Theorem 1.1 by characterizing those reduced homogeneous polynomials for which the lower bound

$$\text{fpt}(f) \geq \frac{1}{\deg f - 1}$$

is achieved.

Definition 4.1. A geometrically reduced form f of degree $d > 2$ is called an *extremal singularity* if its F -pure threshold is equal to $\frac{1}{d-1}$.

Remark 4.2. We exclude the case $d = 2$ from Definition 4.1 because the F -pure threshold of a geometrically reduced quadratic is one in every characteristic. So not much interesting is happening: quadratic forms achieve both the theoretical lower bound and the theoretic upper bound on the F -pure threshold (Proposition 2.2) in every case—no finer gradation of singularities exists.

Theorem 4.3 characterizes extremal singularities in terms of their equations:

Theorem 4.3. *Let f be a geometrically reduced form of degree d over a field of positive characteristic p . Then the F -pure threshold of f is $\frac{1}{d-1}$ if and only if f can be written*

$$(13) \quad x_1^{p^e} L_1 + x_2^{p^e} L_2 + \cdots + x_n^{p^e} L_n$$

for some $e \geq 0$, where the L_i are linear forms.

The proof of Theorem 4.3 relies on Lemma 4.4 justifying the intuition that polynomials in “Frobenius powers” $\mathfrak{m}^{[p^e]} = \langle x_1^{p^e}, \dots, x_n^{p^e} \rangle$ are “more singular” than polynomials not in $\mathfrak{m}^{[p^e]}$:

Lemma 4.4. *The F -pure threshold of f is less than or equal to $\frac{1}{p^e}$ if and only if $f \in \mathfrak{m}^{[p^e]}$. In fact, if $f \notin \mathfrak{m}^{[p^e]}$, then $\text{fpt}(f) \geq \frac{1}{p^e} + \frac{1}{p^{2e}}$.*

Proof. First assume $f \in \mathfrak{m}^{[p^e]}$. Then $\frac{1}{p^e}$ is in the set $\left\{ \frac{N}{p^E} \mid f^N \in \mathfrak{m}^{[p^E]} \right\}$. So the F -pure threshold, which is the *infimum* of this set, is bounded above by $\frac{1}{p^e}$.

For the converse statement, first observe that $S = k[x_1, \dots, x_n]$ can be viewed as a free module over the subring $S^{p^{e'}}$ for all e' . Indeed, we can take $\{\lambda x^m\}$ as a basis, where λ ranges over a basis for k over $k^{p^{e'}}$ and x^m ranges through all monomials in the x_i . Moreover, if $f \notin \mathfrak{m}^{[p^{e'}]}$, then it can be taken to be a part of a free basis for S over $S^{p^{e'}}$ —for example, taking any monomial $x^{m'}$ that appears in f with all exponents less than $p^{e'}$, we can replace $x^{m'}$ in the basis $\{\lambda x^m\}$ by f to get another basis. In particular, projection onto the $S^{p^{e'}}$ -submodule generated by f gives us an $S^{p^{e'}}$ -linear map $\pi : S \rightarrow S^{p^{e'}}$ sending f to 1.

Now, assume $f \notin \mathfrak{m}^{[p^e]}$. Then for all e' , the flatness of Frobenius [Kun69] implies that also $f^{p^{e'}} \notin \mathfrak{m}^{[p^{e+e'}]}$. Furthermore, for all $e' \geq e$, we have

$$f^{p^{e'}+1} = f \cdot f^{p^{e'}} \notin \mathfrak{m}^{[p^{e+e'}]},$$

for otherwise, we could apply π from the previous paragraph to see that $f^{p^{e'}} \in \mathfrak{m}^{[p^{e+e'}]}$. This means that the rational number $\frac{p^{e'}+1}{p^{e+e'}}$ is in the set $\left\{ \frac{N}{p^e} \mid f^N \notin \mathfrak{m}^{[p^e]} \right\}$ for all $e' \geq e$. So the supremum of this set is at least $\frac{1}{p^e} + \frac{1}{p^{e+e'}}$ for all $e' \geq e$. The largest of these is when $e = e'$, so the supremum is at least $\frac{1}{p^e} + \frac{1}{p^{2e}}$. This completes the proof. \square

Proof of Theorem 4.3. Both directions follow from facts we have already established.

First, we claim that the form $f = x_1^{p^e} L_1 + x_2^{p^e} L_2 + \cdots + x_n^{p^e} L_n$ has F -pure threshold $\frac{1}{d-1} = \frac{1}{p^e}$. Indeed, $\text{fpt}(f) \geq \frac{1}{p^e}$ by Theorem 3.1, whereas $\text{fpt}(f) \leq \frac{1}{p^e}$ by Lemma 4.4.

For the other direction, suppose that $\text{fpt}(f) = \frac{1}{d-1}$ for some reduced form f . Now Theorem 3.1 tells us that $d = p^e + 1$ for some e , which means that $\text{fpt}(f) = \frac{1}{d-1} = \frac{1}{p^e}$ for some e . Now Lemma 4.4 guarantees that $f \in \mathfrak{m}^{[p^e]}$. Thinking about degrees, we see that f must be of the form (13). \square

5. MATRIX REPRESENTATION OF FROBENIUS FORMS

Our next goal is to study the forms which achieve the minimal possible F -pure threshold according to the bounds proved in the previous sections. Theorem 4.3 tells us that such polynomials have the special form (13), which warrants a name:

Definition 5.1. A *Frobenius form* is a form of degree $q + 1$ in the ideal $\mathfrak{m}^{[q]} = \langle x_1^q, \dots, x_n^q \rangle$, where q is a positive power of the characteristic, p . Equivalently, a Frobenius form is a polynomial that can be written

$$(14) \quad \sum x_i^{p^e} L_i,$$

where the L_i are linear forms and $e > 0$.

The formula (14) implies that a Frobenius form can be uniquely factored as

$$(15) \quad h = \begin{bmatrix} x_1^{p^e} & x_2^{p^e} & \cdots & x_n^{p^e} \end{bmatrix} A \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},$$

where A is the $n \times n$ matrix whose i -th row is made up of the coefficients of the linear form L_i . This allows us to use linear algebra to conveniently study Frobenius forms.

Consider how changing coordinates affects the matrix representing a Frobenius form. For a matrix B of any size, we denote by $B^{[p^e]}$ the matrix obtained by raising all entries to the p^e -th power. If g is a change of coordinates represented by an

invertible $n \times n$ matrix, then

$$g \cdot \begin{bmatrix} x_1^{p^e} \\ x_2^{p^e} \\ \vdots \\ x_n^{p^e} \end{bmatrix} = g^{[p^e]} \begin{bmatrix} x_1^{p^e} \\ x_2^{p^e} \\ \vdots \\ x_n^{p^e} \end{bmatrix} = [g \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}]^{[p^e]}.$$

Here the notation \cdot indicates the ring automorphism induced by the linear change of coordinates, and all other adjacent symbols are usual matrix product.

So our change of coordinates formula for g acting on h is

$$g \cdot \begin{bmatrix} x_1^{p^e} & x_2^{p^e} & \dots & x_n^{p^e} \end{bmatrix} A \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1^{p^e} & x_2^{p^e} & \dots & x_n^{p^e} \end{bmatrix} \begin{bmatrix} g^{[p^e]} \end{bmatrix}^{tr} A g \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},$$

where the superscript “ tr ” indicates the transpose. We can write this in the compact form

$$g \cdot [(\vec{x}^{[p^e]})^{tr} A \vec{x}] = (\vec{x}^{[p^e]})^{tr} \begin{bmatrix} g^{[p^e]} \end{bmatrix}^{tr} A g \vec{x}.$$

That is, if h is a Frobenius form represented by the matrix A , then the Frobenius form $g \cdot h$, where g is any linear change of coordinates, is represented by the matrix $\begin{bmatrix} g^{[p^e]} \end{bmatrix}^{tr} A g$. This action is *different* than some others that may be familiar to the reader; see Remark 7.5.

It is worth recording, for future reference, how each elementary coordinate operation affects the matrix A representing a Frobenius form.

Lemma 5.2. *Let*

$$(16) \quad h = \begin{bmatrix} x_1^{p^e} & x_2^{p^e} & \dots & x_n^{p^e} \end{bmatrix} A \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

be a Frobenius form. Then elementary linear changes of coordinates are reflected in A as follows:

- *Swapping two variables ($x_i \leftrightarrow x_j$), fixing the others changes A by swapping columns C_i and C_j and rows R_i and R_j , fixing the others.*
- *Multiplying coordinate x_i by a non-zero scalar λ ($x_i \mapsto \lambda x_i$), fixing the others changes A by multiplying row R_i by λ^{p^e} and column C_i by λ .*
- *Replacing x_i by $x_i + \lambda x_j$ for some $j \neq i$, fixing the others changes A by replacing column C_j by column $C_j + \lambda C_i$ and row R_j by row $R_j + \lambda^{p^e} R_i$.*

5.1. Embedding dimension, rank and the singular locus. A form $f \in k[x_1, \dots, x_n]$ is *non-degenerate* if it can't be written as a polynomial in fewer variables after any linear change of coordinates. In this case, the singularity defined by f has *embedding dimension n* , meaning that the dimension of its Zariski cotangent space $\mathfrak{m}/\mathfrak{m}^2$ is n .

The *rank* of a Frobenius form is the rank of the representing matrix. The Proposition 5.3 implies that the rank is the same as the co-dimension of the singular locus of the corresponding extremal singularity.

Proposition 5.3. *The singular locus of an extremal singularity defined by the Frobenius form*

$$\begin{bmatrix} x_1^{p^e} & x_2^{p^e} & \cdots & x_n^{p^e} \end{bmatrix} A \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

is the p^e -fold linear subvariety defined by the equations

$$A^{tr} \begin{bmatrix} x_1^{p^e} \\ x_2^{p^e} \\ \vdots \\ x_n^{p^e} \end{bmatrix} = 0.$$

Put differentially, the (reduced) singular set is the linear space defined as the kernel of the matrix $(A^{[1/p^e]})^{tr}$, where $(A^{[1/p^e]})^{tr}$ is the transpose of the matrix whose entries are the p^e -th roots of the entries of A .

Proof. Write

$$h = x_1^{p^e} L_1 + x_2^{p^e} L_2 + \cdots + x_n^{p^e} L_n$$

where the coefficients of the linear forms L_i are given by the rows of $A = [a_{ij}]$. The singular locus is defined by the vanishing of the partial derivatives $\frac{\partial h}{\partial x_j}$. But for each j ,

$$\frac{\partial h}{\partial x_j} = x_1^{p^e} a_{1j} + \cdots + x_n^{p^e} a_{nj} = [a_{1j} \ \cdots \ a_{nj}] \begin{bmatrix} x_1^{p^e} \\ \vdots \\ x_n^{p^e} \end{bmatrix} = \left[\begin{bmatrix} a_{1j}^{\frac{1}{p^e}} & \cdots & a_{nj}^{\frac{1}{p^e}} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \right]^{p^e},$$

so the proposition follows. \square

Finally, we record a simple lemma which gives a nice form for a Frobenius form in terms of its rank and the minimal number of variables in which it can be written.

Lemma 5.4. *A Frobenius form of rank r can be written, in suitable coordinates, as*

$$h = x_1^{p^e} L_1 + x_2^{p^e} L_2 + \cdots + x_r^{p^e} L_r,$$

where the L_i are linearly independent linear forms. In this case, if h is non-degenerate, then its embedding dimension n is equal to the dimension of the space spanned by the forms $x_1, x_2, \dots, x_r, L_1, \dots, L_r$. In particular, $2r \geq n$.

Proof. Let h be a non-degenerate Frobenius form in n variables, and let A denote its matrix. Swapping variables, assume the first r rows of A are linearly independent. Because the rows beyond the r -th are all dependent on the first r , a suitable sequence of row operations can be used to transform these bottom rows into zero rows; the corresponding column operations (Lemma 5.2) do not affect these zero rows. Thus without loss of generality, we can assume the bottom $n - r$ rows of A are zero rows. This implies that h can be written as $x_1^{p^e} L_1 + x_2^{p^e} L_2 + \cdots + x_r^{p^e} L_r$ for some linear forms L_i . The L_i are linearly independent because their coefficient vectors span the row space of the matrix A , which has rank r . For the final statement, note that if $x_1, x_2, \dots, x_r, L_1, \dots, L_r$ span a space of dimension less than n , then h can be written in fewer than n variables, so it is degenerate. \square

Remark 5.5. A very special kind of Frobenius form is a *Hermitian form* of characteristic p . These are Frobenius forms in which the matrix A representing the form satisfies $a_{ij} = a_{ji}^q$, where q is a power of p for all i, j . In particular, since this implies $a_{ij}^{q^2} = a_{ij}$ for all i, j , a Hermitian form is defined over the finite field \mathbb{F}_{q^2} and the Frobenius map (q -th power map) is an involution that plays a role analogous to complex conjugation. Hermitian hypersurfaces—projective hypersurfaces defined by Hermitian forms—have well-studied “extremal” properties, such as an abundance of rational points; see [BC66], [Seg65], and [HK16].

6. EXTREMAL SINGULARITIES OF FULL RANK

In this section, we prove the following characterization of isolated extremal singularities.

Theorem 6.1. *Every full rank extremal singularity over an algebraically closed field k of characteristic $p > 0$ is represented, in suitable linear coordinates, by the diagonal form $x_1^{q+1} + \dots + x_n^{q+1}$, where q is some power of p .*

We prove Theorem 6.1 using only Hilbert’s Nullstellensatz. In the special case of a *Hermitian form*—that is, where the matrix satisfies $a_{ij} = a_{ji}^p$ for all i, j —Theorem 6.1 can be found¹ in [HW36] or [BC66]. When the embedding dimension is at least four, Theorem 6.1 follows from the main theorem of [Bea90].

Proof. We will prove this by induction on n . The case where $n = 1$ is trivial.

Let h be a full rank Frobenius form in n variables with $n \geq 2$. Write h as

$$h = \begin{bmatrix} x_1^{p^e} & x_2^{p^e} & \dots & x_n^{p^e} \end{bmatrix} A \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},$$

where A is an $n \times n$ matrix over k . The induction happens by showing that we can change coordinates to put A into the block form

$$(17) \quad \begin{bmatrix} * & * & \dots & * & 0 \\ * & * & \dots & * & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & \dots & * & 0 \\ 0 & 0 & \dots & 0 & 1 \end{bmatrix}.$$

Equivalently, this says we can write the Frobenius form as

$$h = x_n^{p^e+1} + g(x_1, \dots, x_{n-1}),$$

where g is full rank Frobenius form in the first $n - 1$ variables. So if we know by induction that g can be put into the desired form by a linear change of coordinates involving only the variables x_1, \dots, x_{n-1} , then it follows that h is in this form as well.

We will use Lemma 6.2:

¹There appears to be some confusion in the literature interpreting the 1936 paper of Hasse and Witt (see, e.g., the “warning and request” in [AH19]).

Lemma 6.2. *If a full rank Frobenius form*

$$(18) \quad h = \begin{bmatrix} x_1^{p^e} & x_2^{p^e} & \cdots & x_{n-1}^{p^e} & x_n^{p^e} \end{bmatrix} \begin{bmatrix} * & * & \cdots & * & a_{1n} \\ * & * & \cdots & * & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & \cdots & * & a_{n-1,n} \\ a_{n1} & a_{n2} & \cdots & a_{n,n-1} & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}$$

satisfies

$$(19) \quad a_{in}a_{nn}^{p^e-1} = a_{ni}^{p^e} \quad \text{for all } i = 1, \dots, n-1,$$

then we can change coordinates to put h in the block form (17). That is, we can change coordinates to get h in the form

$$x_n^{p^e+1} + g,$$

where g is a Frobenius form in x_1, x_2, \dots, x_{n-1} .

Proof. Note that if $a_{nn} = 0$ is zero, then the condition (19) implies that the last row is zero, contrary to the full rank assumption on A . So $a_{nn} \neq 0$. We can therefore assume, without loss of generality, that $a_{nn} = 1$. Indeed, scaling x_n by a (p^e+1) -th root of a_{nn}^{-1} (call it c) changes the matrix A by multiplying row n by c^{p^e} and column n by c (see Lemma 5.2). This allows us to assume that $a_{nn} = 1$ without destroying condition (19).

Now, assuming $a_{nn} = 1$, the change of coordinates that sends

$$x_n \mapsto x_n - a_{n1}x_1 - a_{n2}x_2 - \cdots - a_{n,n-1}x_{n-1}$$

and fixes x_i for $1 \leq i \leq n-1$ gives us the desired form. One simply checks that substituting $x_n - a_{n1}x_1 - a_{n2}x_2 - \cdots - a_{n,n-1}x_{n-1}$ for x_n into h produces a polynomial of the form $x_n^{p^e+1} + g(x_1, \dots, x_{n-1})$. Alternatively, invoking Lemma 5.2, because of the special form of A , we see that subtracting a_{ni} times column n from column i will place a zero in the i -th column of the final row, while the corresponding row operation also makes the i -th row of the final column zero. \square

Continuing the proof of Theorem 6.1 now armed with Lemma 6.2, we note that it suffices to show that any full rank Frobenius form can be put in the form (18). Let $h = [\vec{x}^{[p^e]}]^{tr} A [\vec{x}]$ be an arbitrary Frobenius form. Suppose g is a change of coordinate matrix with indeterminate entries. Changing coordinates, the new matrix of $g \cdot h$ is

$$(20) \quad \tilde{A} = (g^{[p^e]})^{tr} A g.$$

We need to show that there is a choice of g such that the entries of \tilde{A} satisfy the hypothesis of Lemma 6.2.

Thinking of the entries of g as indeterminates Y_{ij} , the matrix product (20) has entries

$$(21) \quad \tilde{A}_{ij} = \sum_{1 \leq k, \ell \leq n} a_{\ell k} Y_{\ell i}^{p^e} Y_{kj},$$

which are homogeneous polynomials in the Y_{ij} . It suffices to prove that there exist values of the Y_{ij} that satisfy the equations

$$(22) \quad \tilde{A}_{in} \tilde{A}_{nn}^{p^e-1} = \tilde{A}_{ni}^{p^e} \quad \text{for all } i = 1, 2, \dots, n-1,$$

and for which the matrix g has a non-zero determinant.

Plugging in the expressions (21), the equations (22) become

$$(23) \quad \tilde{A}_{nn}^{p^e-1} \left(\sum_{1 \leq k, \ell \leq n} a_{\ell k} Y_{\ell i}^{p^e} Y_{kn} \right) = \left(\sum_{1 \leq k, \ell \leq n} a_{\ell k} Y_{\ell n}^{p^e} Y_{ki} \right)^{p^e} \quad i = 1, 2, \dots, n-1,$$

each of which can be rearranged into a linear equation in $Y_{1i}^{p^e}, Y_{2i}^{p^e}, \dots, Y_{ni}^{p^e}$:

$$(24) \quad F_1 Y_{1i}^{p^e} + F_2 Y_{2i}^{p^e} + \dots + F_n Y_{ni}^{p^e} = 0 \quad i = 1, \dots, n-1,$$

where the coefficients F_j of $Y_{ji}^{p^e}$ are

$$F_j = \left(\tilde{A}_{nn}^{p^e-1} \sum_{k=1}^n a_{jk} Y_{kn} - \sum_{\ell=1}^n a_{\ell j}^{p^e} Y_{\ell n}^{p^e} \right) \quad j = 1, \dots, n.$$

The key things to notice here are that the coefficient F_j of $Y_{ji}^{p^e}$ in the equations (24) is the *same* for each $i = 1, \dots, n$, and that it is a polynomial in *only* Y_{1n}, \dots, Y_{nn} , the entries of the last column of the matrix g . Thus the F_1, \dots, F_n form a homogenous system of polynomials in the n indeterminates Y_{1n}, \dots, Y_{nn} of degree p^{2e} .

We claim that there is a choice of values for Y_{1n}, \dots, Y_{nn} , not all zero, for which all F_1, \dots, F_n vanish. In this case, we can take g to be the matrix that has these values as its final column, with any choice of the first $n-1$ columns that makes g invertible. For this choice of g , we will have proved that changing coordinates by g , the form h can be put into the desired form of Lemma 6.2. Thus the proof is complete once we have found a non-zero solution to the system $\{F_1 = F_2 = \dots = F_n = 0\}$.

To prove this claim, we invoke Hilbert's Nullstellensatz: provided the ideal generated by F_1, F_2, \dots, F_n in $k[Y_{1n}, Y_{12}, \dots, Y_{nn}]$ is not $\langle Y_{1n}, Y_{12}, \dots, Y_{nn} \rangle$ -primary, the Nullstellensatz provides the needed non-zero solution. But expanding out the vacuously true expression $\tilde{A}_{nn} \tilde{A}_{nn}^{p^e-1} = \tilde{A}_{nn}^{p^e}$ produces the following relation:

$$Y_{1n}^{p^e} F_1 + Y_{2n}^{p^e} F_2 + \dots + Y_{nn}^{p^e} F_n = 0.$$

Since $Y_{nn}^{p^e}$ has degree p^e , it cannot be in the ideal generated by the elements F_1, F_2, \dots, F_{n-1} , which have degree p^{2e} , showing that $\{F_1, F_2, \dots, F_n\}$ is *not* a regular sequence. Thus the ideal $\langle F_1, F_2, \dots, F_n \rangle$ has height strictly less than n . Thus the Nullstellensatz gives the needed non-zero solution to the system $F_1 = F_2 = \dots = F_n = 0$. The theorem is proved. \square

Remark 6.3. János Kollár suggested, in personal communication[Kol20], a different argument for Theorem 6.1 based on showing the stabilizer of the GL_n -action on the space of Frobenius forms is zero dimensional.

7. ISOMORPHISM TYPES OF EXTREMAL SINGULARITIES

In this section, we classify Frobenius forms up to linear changes of coordinates, working always over an algebraically closed field k of characteristic $p > 0$. Throughout, q denotes a fixed power of p .

Theorem 7.1. *There is a bijection between partitions of n and non-degenerate Frobenius forms in n variables (with fixed degree), up to change of coordinates.*

The bijection sends a partition $n = \sum s$ to the Frobenius form class represented by a block diagonal matrix with exactly one $s \times s$ block J_s along the diagonal for each summand s in the partition; the blocks J_s are

$$(25) \quad J_1 = [1] \quad \text{and} \quad J_s = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix} \quad \text{for } s \geq 2,$$

where the matrix J_s for $s \geq 2$ has 1's on the super-diagonal and zeros elsewhere.

Example 7.2. There are three partitions of three: $3 = 1+1+1$, $3 = 1+2$ and $3 = 3$. So Theorem 7.1 says there are three equivalence classes of non-degenerate Frobenius forms in three variables, corresponding, respectively, to the three matrices (with the blocks J_s shaded)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

These determine, respectively, the forms $x^{q+1} + y^{q+1} + z^{q+1}$, $x^{q+1} + y^q z$ and $x^q y + y^q z$. (These were classified already in [KKP⁺21].)

Definition 7.3. A Frobenius form (or its matrix) constructed from a partition in this way will be said to be in *standard form*.

Theorem 7.1 says that every Frobenius form can be brought into one and only one standard form, up to permuting the blocks, by a linear change of coordinates. The full rank n case is the content of Theorem 6.1 and the rank $n - 1$ was proved by Hoang [HH16].

Remark 7.4. The special class of Hermitian forms (see definition in Remark 5.5) are uniquely determined by their rank: up to changing coordinates, we have only $x_1^{q+1} + \dots + x_r^{q+1}$ [BC66, 4.1]. In particular, there are many more Frobenius forms than Hermitian forms.

Remark 7.5. The standard forms of Theorem 7.1 look similar to those in [HW36, Satz 11], but those are normal forms for a *different* action of $\mathrm{GL}(n)$ on $n \times n$ matrices over a field of characteristic p defined by $A \mapsto gAg^{[-p]}$ (where $g^{[-p]}$ means the inverse of $g^{[p]}$). Indeed, while the *number* of distinct orbits is the same as for the action $A \mapsto (g^{[p]})^{tr} Ag$, the orbits themselves are *different*. For example, $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ are equivalent under the action in [HW36, Satz 11], but these matrices do not define equivalent Frobenius forms.

7.1. The proof. Theorem 7.1 will be proved in two steps. We first show that every Frobenius form can be put into standard form by a linear change of coordinates (Theorem 7.13). We then show that if two Frobenius forms f_α and f_β in standard form are equivalent, then their partitions α and β are the same (Lemma 7.14).

Towards the first step, it is helpful to consider a broader class of Frobenius forms:

Definition 7.6. A matrix is *sparse* if it has at most one non-zero entry in each column and in each row, and these non-zero entries are all 1. A Frobenius form is *sparse* if its matrix is sparse.

Put differently, a sparse Frobenius form is one of the form

$$f = \sum_{k=1}^r x_{i_k}^q x_{j_k},$$

where the row indices $\{i_1, \dots, i_r\}$ are all distinct and the column indices $\{j_1, \dots, j_r\}$ are all distinct. Sparse forms are always non-degenerate (if we view them as forms in the variables that explicitly appear in them); this follows from Lemma 5.4 and is proved carefully in [KPS⁺21, 5.3].

The following crucial lemma reduces the proof of Theorem 7.1 to a combinatorial argument:

Lemma 7.7. *Every Frobenius form is projectively equivalent to a sparse Frobenius form.*

Proof. Consider a non-degenerate Frobenius form of embedding dimension n and rank r . We induce on n to show that after a change of coordinates, its matrix can be made sparse.

When $n \leq 2$, the form is projectively equivalent to x^{q+1} , $x^{q+1} + y^{q+1}$ or $x^q y$. The corresponding matrices are

$$[1], \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

which are sparse.

Now assume $n \geq 3$. By Theorem 6.1, we need only consider the case where $r < n$, since the diagonal form $\sum_i x_i^{q+1}$ is sparse.

We first claim that after a change of coordinates, the matrix A of the Frobenius form can be assumed to be in the block form

$$(26) \quad \begin{matrix} & 2r-n & n-r & n-r \\ & \begin{matrix} B & C & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{matrix} & & \end{matrix},$$

where I is an $(n-r) \times (n-r)$ identity matrix. This follows from Lemma 5.4: writing the form $x_1^q L_1 + \dots + x_r^q L_r$, the span of the linear forms $\{x_1, \dots, x_r, L_1, \dots, L_r\}$ is n dimensional, for otherwise the form is degenerate. So there is some set of $n-r$ forms L_i which, together with $\{x_1, \dots, x_r\}$, are linearly independent, and hence can be taken as our coordinates. Permuting the variables if needed, we can assume these are the last $n-r$ of the L_i . That is, renaming so that $L_r = x_n$, $L_{r-1} = x_{n-1}, \dots, L_{2r-n+1} = x_{r+1}$, the form can be assumed

$$x_1^q L'_1 + x_2^q L'_2 + \dots + x_{2r-n}^q L'_{2r-n} + \underbrace{x_{2r-n+1}^q x_{r+1} + \dots + x_{r-1}^q x_{n-1} + x_r^q x_n}_{\text{produces the identity block of size } n-r},$$

whose matrix nearly has our claimed form (26). The only issue is that the linear forms L'_i could involve the variables x_{r+1}, \dots, x_n which would place non-zero entries above the identity matrix block. However, in this case, we can add multiples of the middle $n-r$ rows upwards to clear out any offending non-zero entries; the

corresponding column operations may change the entries of B but not any of the relevant blocks of zero (or the identity block).

The submatrix B in (26) represents a Frobenius form in $2r - n$ variables, which can be written non-degenerately in, say, m variables. Let s be the rank of B . Thus after a change of coordinates, B is equivalent to a matrix with the block form

$$(27) \quad \begin{matrix} & m & 2r - n - m \\ m & \left[\begin{matrix} D & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{matrix} \right], \\ 2r - n - m & \end{matrix}$$

where D is non-degenerate of rank s . By induction, we may assume that D is sparse. As a sub-matrix of A , the transformations of B leading to this simplified form can be achieved by operations on A that preserve each block of zeros, as well as the identity sub-matrix in (26), though they may alter C . Relabelling the variables involved in D , we can assume its last $m - s$ rows are zero. Hence we can assume that A has form (26), where B is sparse, and its last $2r - n - s$ rows are zero.

The columns of B are the standard basis elements e_1, \dots, e_s , and so by adding multiples of them to middle block of $n - r$ columns of A in (26), we can clear out the first s rows of C . The corresponding row operations add multiples of rows $\{1, \dots, 2r - n\}$ to the middle block of $n - r$ rows of A in (26), possibly destroying those zero blocks. However, this can be corrected by adding multiples of the last $n - r$ columns of A to its first r columns, as the corresponding row operations have no further effect.

We have reduced to assuming that A has the form

$$(28) \quad \begin{matrix} & 2r - n & n - r & n - r \\ s & \left[\begin{matrix} E & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & F & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{matrix} \right], \\ 2r - n - s & \\ n - r & \\ n - r & \end{matrix}$$

where E is the sparse submatrix of B consisting of its first s rows, and F is the submatrix of C consisting of its last $2r - n - s$ rows. Next, we claim that if F can be transformed by column operations to a matrix G , then A is equivalent to a matrix of the form (28), with G replacing F , and without affecting any other blocks.

Indeed, column operations on F correspond to column operations on the middle block of $n - r$ columns of A . The corresponding row operations on A leave all zero blocks unchanged but may alter the block I in (28). However, any such alterations to I can be corrected by Gaussian elimination on the last $n - r$ columns, restoring the matrix I without disturbing anything else, since the corresponding rows are all zero-rows.

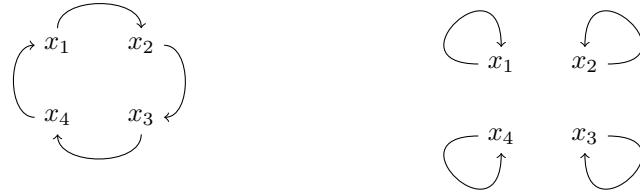
To conclude the proof, finally we show that F be transformed into sparse form via column operations. Indeed, using Gaussian column elimination, we can put F into a reduced column echelon form, which is sparse because F is a $(2r - n - s) \times n - r$ matrix of rank $2r - n - s$. This complete the proof of the lemma. \square

7.2. The directed graph of a sparse form. In light of Lemma 7.7, we have reduced the proof of Theorem 7.1 to the more combinatorial problem of classifying equivalence classes of sparse forms. For this, it is helpful to make the following definition.

Definition 7.8. The *labelled directed graph* of a sparse Frobenius form f is the unique graph Γ_f whose vertices are the variables x_1, \dots, x_n , with an edge from x_i to x_j whenever the term $x_i^q x_j$ appears with non-zero coefficient in f .

Put differently, the matrix A_f of a sparse Frobenius form f uniquely determines a labelled directed graph Γ_f whose adjacency matrix is A_f .

Remark 7.9. The connected components of Γ_f can be only loops (singleton vertices with one edge in and out), directed chains, and cycles: the sparseness of f means that each vertex has at most one edge entering and at most one edge leaving it.



The graph of $x_1^q x_2 + x_2^q x_3 + x_3^q x_4 + x_4^q x_1$ The graph of $x_1^{q+1} + x_2^{q+1} + x_3^{q+1} + x_4^{q+1}$

Remark 7.10. By Lemma 5.4, the graph of a sparse Frobenius form h satisfies:

- (a) The number of vertices is the embedding dimension of h ;
- (b) The number of edges is the rank of h .

The directed graph of a sparse Frobenius form is *not* invariant under change of coordinates, as the example above shows. Lemma 7.11 says that to understand projective equivalence, we can restrict our attention to graphs whose components are either loops or chains:

Lemma 7.11. *Every sparse Frobenius form is equivalent to a sparse Frobenius form whose directed graph has no cycles.*

Proof of Lemma 7.11. Let f be a sparse Frobenius form. If its graph Γ_f contains an ℓ -cycle (say $x_{i_1} \rightarrow x_{i_2} \rightarrow \dots \rightarrow x_{i_\ell} \rightarrow x_{i_1}$), then the form f can be written as

$$f = x_{i_1}^q x_{i_2} + x_{i_2}^q x_{i_3} + \dots + x_{i_{\ell-1}}^q x_{i_\ell} + x_{i_\ell}^q x_{i_1} + h,$$

where h does not involve the variables $x_{i_1}, \dots, x_{i_\ell}$. Now change coordinates involving only the variables $x_{i_1}, \dots, x_{i_\ell}$ to transform f into the equivalent form

$$\tilde{f} = x_{i_1}^{q+1} + x_{i_2}^{q+1} + \dots + x_{i_\ell}^{q+1} + h$$

(Theorem 6.1). This transforms Γ_f into the graph $\Gamma_{\tilde{f}}$ in which the ℓ -cycle has been broken into ℓ loops, but whose remaining components are the same as in Γ_f . Repeating the process on each cycle in the graph, we eventually arrive at a graph with no cycles. \square

Remark 7.12. An alternate way to prove Lemma 7.11 is to observe that we can strengthen the inductive hypothesis in the proof of Lemma 7.7 to show that every Frobenius form is equivalent to a sparse Frobenius form whose matrix is *upper triangular*. Then notice that the graph of such a sparse form can not contain any cycles.

A finite directed graph whose components are all loops or directed chains is essentially a partition—namely the partition of its vertices into components. We can put all these ideas together to complete the first step in our proof of Theorem 7.1:

Theorem 7.13. *Every Frobenius form is projectively equivalent to one in standard form.*

Proof. Using Lemma 7.7, we assume f is sparse, and using Lemma 7.11, we assume its graph Γ_f has no cycles. Thus Γ_f is a disjoint union of loops and directed chains (Remark 7.9). Relabel the vertices so all arrows point from x_i to x_{i+1} (or back to x_i itself). Focusing on one component, say of cardinality s , note that its adjacency matrix is the matrix J_s as defined in (25). So the full adjacency matrix A_f of Γ_f is a block diagonal matrix with blocks of the type J_s for different values of s , one for each component of Γ_f . In other words, A_f is a matrix in standard form (Definition 7.3). Thus the corresponding Frobenius form f is in standard form as well. \square

To complete the classification, we need to show that the Frobenius forms associated to *different* partitions are not equivalent. Therefore, it suffices to establish Lemma 7.14:

Lemma 7.14. *Let f and g be Frobenius forms in standard form. If f and g are equivalent, then the components of their associated graphs Γ_f and Γ_g determine the same partition.*

Proof. We induce on the embedding dimension n . The cases $n \leq 2$ are easy and were listed in the proof of Lemma 7.7. In addition, all full rank Frobenius forms are equivalent to $\sum_{i=1}^n x_i^{q+1}$ (Theorem 6.1), so correspond to the partition of all 1's. So we assume $n \geq 3$ and that the rank r satisfies $r < n$.

The graphs Γ_f and Γ_g partition the vertices into components, determining partitions γ_f and γ_g , respectively, of n . The partition γ_f can be written

$$n = \underbrace{1 + \cdots + 1}_{d_1} + \underbrace{2 + \cdots + 2}_{d_2} + \underbrace{3 + \cdots + 3}_{d_3} + \cdots + \underbrace{t + \cdots + t}_{d_t},$$

where d_i is the number of times the integer i appears in the partition (note that some d_i can be zero). Likewise, the partition γ_g can be written

$$n = \underbrace{1 + \cdots + 1}_{e_1} + \underbrace{2 + \cdots + 2}_{e_2} + \underbrace{3 + \cdots + 3}_{e_3} + \cdots + \underbrace{t + \cdots + t}_{e_t}.$$

Formula 7.15. Using Remark 7.10, we count vertices and edges to get the following formulas for the embedding dimension and rank of f and g in terms of the partitions:

- (i) $n = \sum_{i \geq 1} id_i = \sum_{i \geq 1} ie_i.$
- (ii) $r = d_1 + \sum_{i \geq 2} (i-1)d_i = d_1 + \sum_{i \geq 2} (i-1)e_i.$
- (iii) $n - r = d_2 + \cdots + d_t = e_2 + \cdots + e_t.$

Say that a vertex of a directed graph is *terminal* if there is no edge emanating from it. The number of terminal vertices in the graph of a sparse form is $n - r$, by Formula 7.15. Let $\{y_1, \dots, y_{n-r}\}$ denote the terminal variables of f ; relabeling the variables of g , we can assume $\{y_1, \dots, y_{n-r}\}$ are also the terminal variables of g . Denote the remaining variables—the non-terminal variables—by $\{x_1, \dots, x_r\}$. These are the ones appearing in f and g with exponent q .

Now define the *pre-terminal* variables of f to be those connected to a terminal variable of Γ_f by an edge, and note that there are exactly $n - r$ pre-terminal variables. Call them $\{x_{i_1}, \dots, x_{i_{n-r}}\}$. Relabeling the variables of g , we can assume that these are also the pre-terminal variables of g . So we can write

$$(29) \quad \begin{aligned} f &= f_1(x_1, \dots, x_r) + x_{i_1}^q y_1 + x_{i_2}^q y_2 + \dots + x_{i_{n-r}}^q y_{n-r} \\ g &= g_1(x_1, \dots, x_r) + x_{i_1}^q y_1 + x_{i_2}^q y_2 + \dots + x_{i_{n-r}}^q y_{n-r} \end{aligned}$$

where x_{i_ℓ} is the pre-terminal variable corresponding to terminal variable y_ℓ .

Claim 7.16. With notation as above, any linear change of coordinates ϕ such that $\phi(f) = g$ must preserve the ideal $\langle x_{i_1}, \dots, x_{i_{n-r}} \rangle$ generated by the pre-terminal variables.

To justify Claim 7.16, note that the defining ideal of singular locus of both f and g is generated by $\{x_1, \dots, x_r\}$ (Proposition 5.3). In particular, ϕ must preserve the ideal $\langle x_1, \dots, x_r \rangle$. So if L_i denotes the image $\phi(x_i)$, then L_i must be a linear form in $\{x_1, \dots, x_r\}$.

Let M_1, \dots, M_{n-r} be the images of the terminal variables y_1, \dots, y_{n-r} under ϕ . Applying ϕ to the first line of (29), we have

$$(30) \quad \begin{aligned} \phi(f) &= f_1(L_1, \dots, L_r) + L_{i_1}^q M_1 + L_{i_2}^q M_2 + \dots + L_{i_{n-r}}^q M_{n-r} \\ &= g_1(x_1, \dots, x_r) + x_{i_1}^q y_1 + x_{i_2}^q y_2 + \dots + x_{i_{n-r}}^q y_{n-r}. \end{aligned}$$

Writing

$$M_i = c_{i1}x_1 + \dots + c_{ir}x_r + b_{i1}y_1 + \dots + b_{in-r}y_{n-r},$$

and comparing the coefficient of y_m in the equal expressions (30), we see that

$$x_{i_m}^q = \sum_{\ell=1}^{n-r} b_{\ell m} L_{i_\ell}^q = \left(\sum_{\ell=1}^{n-r} b_{\ell m}^{1/q} L_{i_\ell} \right)^q.$$

In particular,

$$\langle x_{i_1}, \dots, x_{i_{n-r}} \rangle \subset \langle L_{i_1}, \dots, L_{i_{n-r}} \rangle = \langle \phi(x_{i_1}), \dots, \phi(x_{i_{n-r}}) \rangle.$$

Since both ideals are generated by $n - r$ linearly independent linear forms, they must be the same. This says that ϕ preserves the ideal of pre-terminal variables. The claim is proved.

Given Claim 7.16, it follows that ϕ induces an isomorphism

$$(31) \quad \bar{\phi} : \frac{k[x_1, \dots, x_n]}{\langle x_{i_1}, \dots, x_{i_{n-r}} \rangle} \longrightarrow \frac{k[x_1, \dots, x_n]}{\langle x_{i_1}, \dots, x_{i_{n-r}} \rangle},$$

which we view as a change of coordinates for the quotient polynomial ring. Letting \bar{f} and \bar{g} denote the images of f and g , respectively, in the quotient ring, we see that $\bar{\phi}$ defines a equivalence between the Frobenius forms \bar{f} and \bar{g} .

Now let us examine the graph $\Gamma_{\bar{f}}$ of \bar{f} . Killing the pre-terminal variables zeros out the monomials of f in which the pre-terminal variables appear. This removes the pre-terminal vertices from each directed chain of Γ_f , as well as any edge connected to them and any vertices left isolated by the process. So the graph $\Gamma_{\bar{f}}$ of \bar{f} is obtained by removing chains of length 2 and 3 from Γ_f , and turning chains of length $\ell \geq 4$ into chains of length $\ell - 2$.

For example, the figure below shows a directed graph Γ_f of a Frobenius form f , highlighting the pre-terminal vertices and every edge connected to a pre-terminal vertex:

$$\begin{array}{ccccccc}
\text{Graph } \Gamma_f: & \bullet \xrightarrow{\text{red}} \bullet & \bullet \xrightarrow{\text{red}} \bullet & \bullet \xrightarrow{\text{red}} \bullet \xrightarrow{\text{red}} \bullet & \bullet \xrightarrow{\text{red}} \bullet \xrightarrow{\text{red}} \bullet \xrightarrow{\text{red}} \bullet & \bullet \xrightarrow{\text{red}} \bullet & \bullet \xrightarrow{\text{red}} \bullet \\
x_1^{q+1} + & (x_2^q y_1) + & (x_3^q y_2) + & (x_4^q x_5 + x_5^q y_3) + & (x_6^q x_7 + x_7^q x_8 + x_8^q x_9 + x_9^q y_4) & &
\end{array}$$

Modding out the pre-terminal variables $\{x_2, x_3, x_5, x_9\}$ produces the Frobenius form \bar{f} with graph $\Gamma_{\bar{f}}$:

$$\begin{array}{ccccccc}
\text{Graph } \Gamma_{\bar{f}}: & \bullet \xrightarrow{\text{red}} \bullet \xrightarrow{\text{red}} \bullet & & & & & \\
x_1^{q+1} + & (x_6^q x_7 + x_7^q x_8) & & & & &
\end{array}$$

In particular, the partition $\gamma_{\bar{f}}$ is

$$r = \underbrace{1 + \cdots + 1}_{d_1} + \underbrace{2 + \cdots + 2}_{d_4} + \underbrace{3 + \cdots + 3}_{d_5} + \cdots + \underbrace{(t-2) + \cdots + (t-2)}_{d_t},$$

and the partition $\gamma_{\bar{g}}$ is

$$r = \underbrace{1 + \cdots + 1}_{e_1} + \underbrace{2 + \cdots + 2}_{e_4} + \underbrace{3 + \cdots + 3}_{e_5} + \cdots + \underbrace{(t-2) + \cdots + (t-2)}_{e_t}.$$

By induction, because \bar{f} and \bar{g} are equivalent and of embedding dimension less than n , their partitions are the same, so that $d_i = e_i$ for all $i \neq 2, 3$.

Since γ_f and γ_g are both partitions of n , Formula 7.15(i) now implies that $2d_2 + 3d_3 = 2e_2 + 3e_3$. But Formula 7.15(iii) also gives $d_2 + d_3 = e_2 + e_3$. Together, these two equations imply finally that $d_2 = e_2$ and $d_3 = e_3$. Thus f and g had the same partition to start. \square

8. GEOMETRIC PROPERTIES OF EXTREMAL SINGULARITIES

Smooth projective varieties defined by (certain special) Frobenius forms have long been understood to be extremal in various ways, going at least back to Beniamino Segre [Seg65]. It is easy to see that they contain many linear subspaces, for example, which can be used to show that they are extremal from the point of view of containing rational points; see [Kol15], [BC66] and [HK16].

In this section, we collect a few interesting properties of extremal hypersurfaces. By *extremal hypersurface*, we mean a *projective* hypersurface defined by a (not necessarily reduced) Frobenius form; in the reduced case, an extremal hypersurface is a projective hypersurface for which the affine cone over it is an extremal singularity.

8.1. Hyperplane sections. It is easy to see that every hyperplane section of an extremal hypersurface is extremal. Somewhat surprisingly, the converse is also true:

Theorem 8.1. *If X is an extremal hypersurface, then so is every hyperplane section (which is not just a component of X). Conversely, if the ground field is algebraically closed and $n \geq 3$, then any hypersurface $X \subset \mathbb{P}^n$ with the property that all its hyperplane sections are extremal must itself be extremal.*

Example 8.2. The second claim of Theorem 8.1 is false for $n < 3$. For example, the plane curve of characteristic two defined by the vanishing of $x^3 + y^3 + z^3 + xyz$ is not extremal, but every hyperplane section is extremal. Indeed, every polynomial of degree three in two variables x, y is contained in $\langle x^2, y^2 \rangle$.

Proof. Since both degree and inclusion in $\mathfrak{m}^{[p^e]}$ are preserved under taking the quotient by a linear form, the first statement is clear.

For the converse, we set up some notation. For a form f and a linear form L , let \bar{f} denote the form $f \bmod L$ in the polynomial ring $k[x_0, x_1, \dots, x_n]/\langle L \rangle$.

Now suppose that f is the defining equation of the hypersurface X with the property that every hyperplane section is extremal. Then \bar{f} is a Frobenius form (for all choices of L) so f has degree $p^e + 1$ for some e .

Write f uniquely as $f = \sum_{i=0}^n x_i^{p^e} L_i + g$, where the L_i are linear forms, and g is some form *none of whose monomials are divisible by any $x_i^{p^e}$* . We need to show that g is zero. For this, it suffices to show that g is divisible by infinitely many distinct (up to scalar multiple) linear forms.

Fix any linear form L . By hypothesis, \bar{f} is a Frobenius form. Since the set of Frobenius forms is closed under addition, also \bar{g} is a Frobenius form. Now if $L = x_i$, the restriction on the monomials of g implies that $\bar{g} = 0$ —that is, that x_i divides g for each i . So without loss of generality

$$g = (x_0 x_1 \dots x_n)h,$$

where h is a form of degree $p^e + 1 - (n + 1)$.

Next, we consider what happens when $L = x_0 - cx_1$ for some $c \in k$. Using the isomorphism

$$k[x_0, x_1, \dots, x_n]/\langle x_0 - cx_1 \rangle \longrightarrow k[y_1, \dots, y_n] \quad \begin{cases} x_0 \mapsto cy_1 \\ x_i \mapsto y_i & i \geq 1 \end{cases}$$

we see that because $g \bmod L$ is a Frobenius form, also

$$y_1^{p^e} y_2 \dots y_n \tilde{h} \in \langle y_1^{p^e}, \dots, y_n^{p^e} \rangle,$$

where \tilde{h} denotes the image of h in the polynomial ring $k[y_1, \dots, y_n]$. Because y_1, \dots, y_n form a regular sequence, this yields (see Lemma 3.5)

$$\tilde{h} \in \langle y_1^{p^e-2}, y_2^{p^e-1}, \dots, y_n^{p^e-1} \rangle.$$

But the degree of \tilde{h} is $p^e - n$ which is strictly less than $p^e - 2$. So $\tilde{h} = 0$. In other words, $x_0 - cx_1$ divides h . Since c was an arbitrary element of k , h must have at least $|k|$ distinct linear factors. Since k is infinite, the proof is complete. \square

Corollary 8.3. *If X is a smooth extremal hypersurface over an algebraically closed field, then all smooth hyperplane sections are isomorphic.*

Proof. The hyperplane sections of X are extremal by Theorem 8.1. So the smooth hyperplane sections are cones over full rank extremal singularities, and hence all projectively equivalent to the diagonal hypersurface $\sum_{i=1}^n x_i^{p^e+1}$ by Theorem 6.1. \square

Remark 8.4. The converse of Corollary 8.3 is a theorem of Beauville [Bea90]; restated in our language, it says that if a smooth projective hypersurface X has the property that all its smooth hyperplane sections are isomorphic to each other, then X is an extremal hypersurface.

8.2. Gauss map. Fix an algebraically closed field. Consider a reduced closed subscheme $X \subset \mathbb{P}^n$ of equi-dimension d . The *Gauss map* of X is the rational map

$$X \dashrightarrow G(d, \mathbb{P}^n) \quad x \mapsto T_x X$$

sending each smooth point x to its embedded projective tangent space $T_x X$, considered as a point in the Grassmannian of d -dimensional linear subspaces of \mathbb{P}^n . For a hypersurface $X = \mathbb{V}(f) \subset \mathbb{P}^n$ defined by a reduced form f , the Gauss map can be described explicitly as

$$X \dashrightarrow (\mathbb{P}^n)^* \quad x \mapsto \left[\frac{\partial f}{\partial x_0} : \frac{\partial f}{\partial x_1} : \cdots : \frac{\partial f}{\partial x_n} \right].$$

This is undetermined along the singular locus of X .

It is not hard to see that the Gauss map is finite when X is smooth (without linear components, which would contract to points under the Gauss map). More generally, the (closure of the) image of the Gauss map has dimension $\dim X - \dim \text{Sing}(X) - 1$ [Zak93, 2.8].

In characteristic zero, the Gauss map of a smooth projective variety is birational, but this can fail in characteristic p . Many authors have studied the question of precisely *how* this failure happens, eventually realizing that (at least for hypersurfaces), the issue appears to be only the *inseparability* of the Gauss map; see [Wal56], [KP91], or [Kaj89] for example.

A smooth extremal hypersurface has the property that its Gauss map is highly inseparable—purely inseparable of maximal degree—and its dual hypersurface is also extremal. The following straightforward statement may be folklore among experts, but we have not found it simply stated in the literature:

Proposition 8.5. *A smooth extremal hypersurface of degree $q+1$ and dimension d has a purely inseparable Gauss map of degree q^d . The dual hypersurface (that is, the image under the Gauss map) is also a smooth extremal hypersurface of the same degree.*

By *purely inseparable*, here, we mean that the induced map on generic stalks is a purely inseparable field extension.

Proof. Suppose $X = \mathbb{V}(h)$ is an extremal hypersurface in \mathbb{P}^n . Write

$$h = x_0^{p^e} L_0 + x_1^{p^e} L_1 + \cdots + x_n^{p^e} L_n = (\vec{x}^{[p^e]})^{tr} A \vec{x}.$$

The Gauss map is

$$\begin{aligned} x &\mapsto \left[\frac{\partial h}{\partial x_0} : \frac{\partial h}{\partial x_1} : \cdots : \frac{\partial h}{\partial x_n} \right] \\ &= \left[\sum_{i=0}^n a_{i0} x_i^{p^e} : \sum_{i=0}^n a_{i1} x_i^{p^e} : \cdots : \sum_{i=0}^n a_{in} x_i^{p^e} \right] \\ &= [x_0^{p^e} : x_1^{p^e} : \cdots : x_n^{p^e}] A, \end{aligned}$$

where A is the (invertible) matrix representing the Frobenius form h . So the Gauss map factors as

$$[x_0 : x_1 : \cdots : x_n] \mapsto [x_0^{p^e} : x_1^{p^e} : \cdots : x_n^{p^e}] \mapsto [x_0^{p^e} : x_1^{p^e} : \cdots : x_n^{p^e}] A.$$

Since A is just a linear change of coordinates, we can analyze the induced map on the generic stalk for the map $[x_0 : x_1 : \cdots : x_n] \mapsto [x_0^{p^e} : x_1^{p^e} : \cdots : x_n^{p^e}]$ only. Without

loss of generality, the generic stalk is the fraction field of $k\left[\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right] / \left\langle \frac{h}{x_0^{p^e+1}} \right\rangle$, which is a purely transcendental extension of k of transcendence degree $n-1$ generated by the rational functions $\frac{x_1}{x_0}, \dots, \frac{x_{n-1}}{x_0}$. So the Gauss map on stalks can be viewed as simply the inclusion $k\left(\left(\frac{x_1}{x_0}\right)^{p^e}, \dots, \left(\frac{x_{n-1}}{x_0}\right)^{p^e}\right) \subset k\left(\frac{x_1}{x_0}, \dots, \frac{x_{n-1}}{x_0}\right)$, which is purely inseparable of degree $(p^e)^{n-1}$ where $n-1$ is the dimension of the hypersurface.

To see that the image is extremal, note because the matrix A is invertible, it suffices to show the p^e -th power map on the homogeneous coordinates has extremal image. But the relation $h = (\vec{x}^{[p^e]})^{tr} A \vec{x}$ on the homogeneous coordinates of X implies the relation $((x^{[p^e]})^{[p^e]})^{tr} A^{[p^e]} \vec{x}^{[p^e]}$ on the coordinates $[x_0^{p^e} : x_1^{p^e} : \dots : x_n^{p^e}]$ of the image. So the image is isomorphic to the extremal singularity defined by the Frobenius form represented by $A^{[p^e]}$. \square

Remark 8.6. Conjecture 2 in [KP91] can be interpreted as predicting that any smooth hypersurface of degree $d \geq 3$ with the property that its dual hypersurface is smooth *must be* defined by a Frobenius form. (This is known for curves [Hom87, 6.1, 6.7], [Hef89, 7.8] and surfaces [KP91, 14].) Thus, in light of Proposition 8.5, we should expect a smooth hypersurface is extremal if and only if its dual hypersurface is smooth.

Remark 8.7. If the extremal hypersurface X is not smooth, the proof of Proposition 8.5 shows that its Gauss map is the p^e -th power map followed by a linear projection.

8.3. Lines on extremal hypersurfaces. Extremal hypersurfaces are extremal also in the behavior of the linear subspaces they contain; see, for example, the discussion in [Kol15, §35]. One simple way to describe this is by looking at the special configurations of intersecting lines on them.

Definition 8.8. A configuration of lines in the projective plane is *perfect star* of degree $d \geq 3$ if it projectively equivalent to d reduced concurrent lines with slopes ranging through the d -th roots of unity. Equivalently, a perfect star of degree d is defined by an equation $x^d - y^d$, where the characteristic of the ground field does not divide d .

Remark 8.9. We could include $d = 1, 2$ in Definition 8.8, but then every configuration of d lines forms a perfect star. When $d = 3$, a configuration of lines is a perfect star if and only if the three lines are concurrent. The condition becomes more restrictive as d gets larger.

Perfect stars are clearly very special configurations of lines—we don't expect most hypersurfaces to contain *any*, unless the hypersurface contains an entire plane. So the following result emphasizes that extremal hypersurfaces really have extremal behavior in terms of the configuration of lines they contain:

Proposition 8.10. *Let $X \subset \mathbb{P}^n$ be an extremal hypersurface of degree $q+1$, where q is a power of the characteristic $p > 0$. Suppose ℓ_1 and ℓ_2 are intersecting lines contained in X , and let Λ be the plane they span. If Λ is not contained in X , then the plane section $\Lambda \cap X$ is either a perfect star of degree $q+1$ or the union of a q -fold line and a reduced line.*

Proof. If Λ does not lie on X , then $\Lambda \cap X \subset \Lambda \cong \mathbb{P}^2$ is an extremal curve by Theorem 8.1. Choose coordinates $\{x, y, z\}$ for \mathbb{P}^2 so that ℓ_1 and ℓ_2 are given by

the vanishing of x and y in $\Lambda \cong \mathbb{P}^2$, and let \bar{h} be the equation of the plane section $\Lambda \cap X$. Since this curve is extremal,

$$\bar{h} = x^q L_1 + y^q L_2 + z^q L_3,$$

for some linear forms L_i . Because ℓ_1 and ℓ_2 lie on this curve, we know both x and y divide \bar{h} . This forces $y \mid L_1, x \mid L_2$, and $xy \mid L_3$. In particular, $L_3 = 0$, since its degree is one. So

$$\bar{h} = ax^q y + by^q x = xy(ax^{q-1} + by^{q-1})$$

for some scalars a, b . So \bar{h} factors into $q + 1$ linear forms, all distinct unless one of a or b is zero. In the former case, we can scale x and y to assume $a = 1$ and $b = -1$ to get a perfect star and in the latter case, we have the non-reduced line configuration defined by $x^q y$. \square

ACKNOWLEDGMENTS

The authors thank Karl Schwede, Mircea Mustaţă, Daniel Hernández, and János Kollár for their interest, and especially Kollár for directing us to the references [Kol15] and [HW36], Manuel Blickle for help translating [HW36], and Daniel Hernández for help streamlining the proof of Lemma 7.7. The fifth author also gratefully acknowledges lively discussions with Damiano Testa, where she first learned of some of the extremal properties of certain forms, such as the inseparability of their Gauss maps.

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