Journal of Mathematical Physics

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Cite as: J. Math. Phys. **63**, 123304 (2022); https://doi.org/10.1063/5.0095042
Submitted: 07 April 2022 • Accepted: 21 November 2022 • Published Online: 09 December 2022

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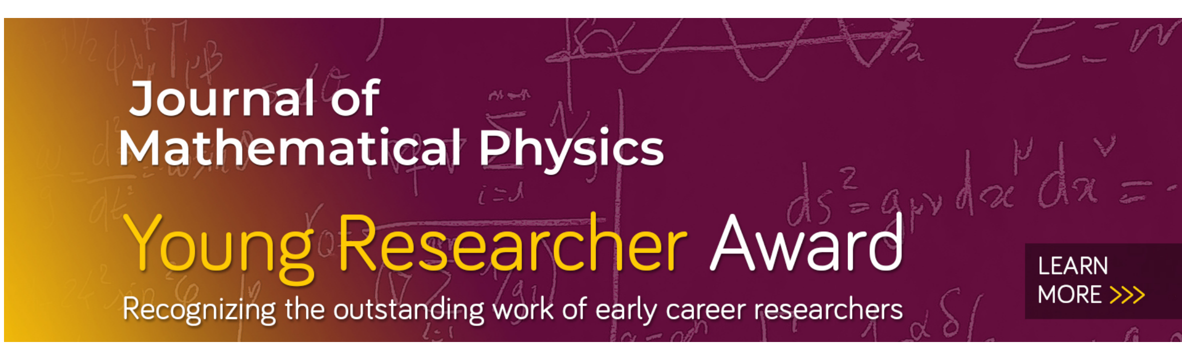

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Journal of Mathematical Physics ARTICLE scitation.org/journal/jmp

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Published Online: 9 December 2022

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ABSTRACT

In this paper, we obtain a moderate deviations principle (MDP) for a class of Langevin dynamic systems with a strong damping and fast Markovian switching. To facilitate our study, first, analysis of systems with bounded drifts is dealt with. To obtain the desired moderate deviations, the exponential tightness of the solution of the Langevin equation is proved. Then, the solution of its first-order approximation using local MDPs is examined. Finally, the MDPs are established. To enable the treatment of unbounded drifts, a reduction technique is presented near the end of the paper, which shows that Lipschitz continuous drifts can be dealt with.

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I. INTRODUCTION

This work is devoted to obtaining a moderate deviations principle for a class of Langevin equations with a strong damping and fast Markovian switching. Originally used for describing the motion of a system subject to combined deterministic and stochastic forces,[1] Langevin dynamic systems have been a basic mathematical physics model studied intensively in the literature; see, for example, applications to classical mechanics and thermodynamics,[2–4] stochastic chemical kinetics,[5] and statistical physics.[6–8] In mathematical physics, one often uses asymptotic analysis to reduce the computational complexity; see Refs. 3 and 6–11 and references therein. In this paper, we focus on the asymptotic properties of the Langevin dynamic systems through multi-scale formulation. Our aim is to fill in a gap between the range of normal deviations and large deviations. The use of the Markovian switching is to capture the features of random environments that are not fitting into the setting of usual stochastic differential equations. The essence is that in the overall systems, both continuous dynamics and discrete events coexist and interact. Such systems are often termed hybrid systems and used widely in many different applications.[12] To make the computation feasible, one often has to be contented with finding approximate solutions. A useful modeling and computational step is to use a multi-scale formulation. In the literature, Simon and Ando[13] used such an idea and introduced the so-called hierarchical decomposition and aggregation; Sethi and Zhang[14] initiated the study of nearly optimal controls for flexible manufacturing systems. In this paper, the fast Markov chain is along the aforementioned line of modeling, whereas the use of a strong damping follows from the motivations in the early works;[6,7] see also the application examples in Ref. 15.

For each $\varepsilon > 0$, considering the motion of a small particle with mass μ in the force field $b(q) + \sqrt{\varepsilon}\sigma(q)\dot{w}$ with variable friction proportional to the velocity, Newton's law gives

$$\begin{cases} \mu\ddot{q}^{\mu,\varepsilon}(t) = b(q^{\mu,\varepsilon}(t)) + \sqrt{\varepsilon}\sigma(q^{\mu,\varepsilon}(t))\dot{w}(t) - \alpha(q^{\mu,\varepsilon})\dot{q}^{\mu,\varepsilon}(t), \\ q^{\mu,\varepsilon}(0) = q,\ \dot{q}^{\mu,\varepsilon}(0) = p;\ p, q \in \mathbb{R}^d, \end{cases} \tag{1.1}$$

where $b(q)$ is the deterministic part of the force, $\dot{w}(t)$ is the standard Gaussian white noise in $\mathbb{R}^d$, and $\sigma(q)$ is a $d \times d$ matrix. The term $\alpha(q^{\mu,\varepsilon}(t))\dot{q}^{\mu,\varepsilon}(t)$ is the variable friction to the motion, and $\alpha(q^{\mu,\varepsilon}(t))$ is a scalar representing the friction coefficient.

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If the friction coefficient α is independent of q, it has been proven that the Smoluchowski–Kramers approximation holds.[10,16] That is, $q^{\mu,\varepsilon}(t)$ converges to $g^{\varepsilon}(t)$ in probability as $\mu \to 0$, where $g^{\varepsilon}(t)$ is the solution of the equation,

$$\dot{g}^{\varepsilon}(t) = \frac{b(g^{\varepsilon}(t))}{\alpha(g^{\varepsilon}(t))} + \sqrt{\varepsilon}\frac{\sigma(g^{\varepsilon}(t))}{\alpha(g^{\varepsilon}(t))}\dot{w}(t), \quad g^{\varepsilon}(0) = q \in \mathbb{R}^d. \tag{1.2}$$

It justifies the replacement of the motion of the particle by the first-order Eq. (1.2). If the friction coefficient is state-dependent, it was proved in Ref. 11 that $q^{\mu,\varepsilon}(t)$ converges to the solution of the first-order equation of the same type as (1.2), where an extra *noise-induced drift* term is added.

Dealing with the case of state-dependent friction coefficient and $\mu = \varepsilon^2$, Cerrai and Freidlin[6] considered the following Langevin equation with a strong damping:

$$\begin{cases} \varepsilon^2\ddot{q}_\varepsilon(t) = b(q_\varepsilon(t)) - \alpha(q_\varepsilon(t))\dot{q}_\varepsilon(t) + \sqrt{\varepsilon}\sigma(q_\varepsilon(t))\dot{w}(t), \\ q_\varepsilon(0) = q \in \mathbb{R}^d, \quad \dot{q}_\varepsilon(0) = \dfrac{p}{\varepsilon} \in \mathbb{R}^d. \end{cases} \tag{1.3}$$

They established large deviations principles (LDPs, for short) for the solution $\{q_\varepsilon(t)\}_{\varepsilon\geq 0}$ of (1.3) in $C([0,T],\mathbb{R}^d)$, the space of continuous functions defined on $[0,T]$ taking values in $\mathbb{R}^d$, and demonstrated that such LDPs have the same rate function (or action functional) I and the same speed function ε^{-1} with LDPs of the following first-order dynamic system:

$$\dot{g}_\varepsilon(t) = \frac{b(g_\varepsilon(t))}{\alpha(g_\varepsilon(t))} + \sqrt{\varepsilon}\frac{\sigma(g_\varepsilon(t))}{\alpha(g_\varepsilon(t))}\dot{w}(t), \quad g_\varepsilon(0) = q \in \mathbb{R}^d. \tag{1.4}$$

Recently, in our work,[17] we have extended the above results by considering the LDPs of the time-inhomogeneous Langevin equations with a strong damping in a random environment. More precisely, consider

$$\begin{cases} \varepsilon^2\ddot{q}_\varepsilon(t) = b(t, q_\varepsilon(t), \xi_{t/\varepsilon}) - \alpha_\varepsilon(t, q_\varepsilon(t))\dot{q}_\varepsilon(t) + \sqrt{\varepsilon}\sigma_\varepsilon(t, q_\varepsilon(t))\dot{w}(t), \\ q_\varepsilon(0) = q_0 \in \mathbb{R}^d, \quad \dot{q}_\varepsilon(0) = q_1 \in \mathbb{R}^d, \end{cases} \tag{1.5}$$

where ξ_t represents the random environment. It was shown that the solution $\{q_\varepsilon(t)\}_{\varepsilon\geq 0}$ of the second-order Eq. (1.5) and its corresponding first-order equation still possesses the same LDPs assuming that the corresponding first-order equation satisfies a local LDP.

Under a Markovian switching random environment setting in Ref. 15, LDPs were also established for Langevin equations by using appropriate H-functionals. In contrast to Ref. 15, we examine a different asymptotic range in this paper, which is somewhat closer to the asymptotic normality range. Nevertheless, the techniques used in Ref. 15 are no longer applicable and different approaches must be used. We establish the moderate deviations principles (MDPs, for short) of the following Langevin dynamic system with a Markovian switching:

$$\begin{cases} \varepsilon^2\ddot{q}_\varepsilon(t) = b(q_\varepsilon(t), r_\varepsilon(t)) - \alpha_\varepsilon(q_\varepsilon(t))\dot{q}_\varepsilon(t) + \sqrt{\varepsilon}\sigma(q_\varepsilon(t), r_\varepsilon(t))\dot{w}(t), \\ q_\varepsilon(0) = q \in \mathbb{R}^d, \quad \dot{q}_\varepsilon(0) = \dfrac{p}{\varepsilon} \in \mathbb{R}^d, \end{cases} \tag{1.6}$$

where $r_\varepsilon(t)$ is a fast-varying continuous-time Markov chain with a finite state space $\mathcal{M} = \{1, 2, \ldots, m\}$ generated by $Q(t)/\varepsilon$. The $Q(t) \in \mathbb{R}^{m\times m}$ is itself a generator of a Markov chain. The corresponding first-order equation of the Langevin Eq. (1.6) is given by

$$\dot{g}_\varepsilon(t) = \frac{b(g_\varepsilon(t), r_\varepsilon(t))}{\alpha_\varepsilon(g_\varepsilon(t))} + \sqrt{\varepsilon}\frac{\sigma(g_\varepsilon(t), r_\varepsilon(t))}{\alpha_\varepsilon(g_\varepsilon(t))}\dot{w}(t), \quad g_\varepsilon(0) = q \in \mathbb{R}^d. \tag{1.7}$$

We note that the above first-order equation is not of the exact form of Smoluchowski–Kramers approximation of (1.6) due to the state-dependent friction coefficient.[11] Note also that under irreducibility[18] (p. 23) of $Q(t)$, there is an averaged system in $\mathbb{R}^d$ when $\varepsilon \to 0$; i.e.,

$$\dot{q}_0(t) = \frac{\overline{b}(q_0(t), \nu(t))}{\alpha(q_0(t))}, \quad q_0(0) = q \in \mathbb{R}^d, \tag{1.8}$$

where $\alpha(\cdot)$ is the limit of $\{\alpha_\varepsilon(\cdot)\}_{\varepsilon>0}$. For convenience, we write $\alpha_0 = \alpha$. The $\alpha(\cdot)$ is a pointwise limit of $\alpha_\varepsilon(\cdot)$. However, we require more in assumption (A3), which is about the convergence rate of norm $\|\alpha_\varepsilon - \alpha\|$.

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For simplicity of presentation, we first treat the case that the drift is bounded. Near the end of the paper, we present how Lipschitz type of condition can be incorporated, leading to a reduction to the bounded coefficient case. We focus on the MDPs problems for the family $\{q_\varepsilon(t)\}_{\varepsilon>0}$ in (1.6) on the space $C^0([0,1];\mathbb{R}^d)$. That is, we are interested in the asymptotic behavior of the trajectory

$$X_\varepsilon(t) = \frac{1}{\sqrt{\varepsilon}h(\varepsilon)}(q_\varepsilon(t) - q_0(t)), \quad t \in [0,1], \tag{1.9}$$

where $h(\varepsilon)$ is the scale of deviations satisfying

$$h(\varepsilon) \to +\infty \text{ and } \sqrt{\varepsilon}h(\varepsilon) \to 0, \qquad \text{as } \varepsilon \to 0. \tag{1.10}$$

If $h(\varepsilon)$ is identically equal to 1, it is in the normal deviation range, leading to the central limit theorem (CLT). If $h(\varepsilon) = 1/\sqrt{\varepsilon}$, it is in the large deviations range, with the large deviations estimates provided in Ref. 19. To fill in the gap between the CLT scale and the large deviations scale, it is natural and necessary to study the moderate deviations where the scale satisfies condition (1.10). For the moderate deviations principle of Langevin dynamics (1.3), earlier work can be found in Ref. 7 and references therein. This paper demonstrates that not only do the solutions of the second-order Eq. (1.6) and those of the first-order Eq. (1.7) verify the same large deviations principle, but also they satisfy the same moderate deviations principle. Because of the ε-dependence of the drift and diffusion coefficients in (1.7), we first establish the exponential equivalence with respect to the MDPs between $g_\varepsilon(t)$ and $f_\varepsilon(t)$ that satisfy

$$\dot{f}_\varepsilon(t) = \frac{b(f_\varepsilon(t), r_\varepsilon(t))}{\alpha(f_\varepsilon(t))} + \sqrt{\varepsilon}\frac{\sigma(f_\varepsilon(t), r_\varepsilon(t))}{\alpha(f_\varepsilon(t))}\dot{w}(t). \tag{1.11}$$

More specifically, for any positive j,

$$\lim_{\varepsilon\to 0}\frac{1}{h^2(\varepsilon)}\log \mathbb{P}\left(\sup_{t\in[0,1]}|\eta_\varepsilon(t) - \hat{\eta}_\varepsilon(t)| > j\right) = -\infty,$$

where

$$\eta_\varepsilon(t) = \frac{g_\varepsilon(t) - q_0(t)}{\sqrt{\varepsilon}h(\varepsilon)} \quad \text{and} \quad \hat{\eta}_\varepsilon(t) = \frac{f_\varepsilon(t) - q_0(t)}{\sqrt{\varepsilon}h(\varepsilon)}. \tag{1.12}$$

In Ref. 20, Guillin established moderate deviations principles for stochastic differential equations with a small diffusion, where the random environment is an exponentially ergodic Markov process. In reference to his work, we are able to obtain that under suitable conditions, $\hat{\eta}_\varepsilon(t)$ satisfies an LDP in $C^0([0,1],\mathbb{R}^d)$ with speed $h^{-2}(\varepsilon)$ and a good rate function S given by

$$S(\gamma) = I\left(\gamma - \int_0^{\cdot}\overline{D}_1(q_0(s), \nu(s))\gamma(s)ds\right), \tag{1.13}$$

for any $\gamma \in C^0([0,1],\mathbb{R}^d)$, where $\overline{D}_1(\cdot,\cdot)$ is to be specified later. Our method is based on the explicit criteria for exponential tightness given by Liptser and Pukhalskii[21] and the equivalence of local moderate deviations principle between $X_\varepsilon(t)$ and $\eta_\varepsilon(t)$. Finally, the exponential equivalence with respect to the MDPs between $\eta_\varepsilon(t)$ and $\hat{\eta}_\varepsilon(t)$ and the MDPs of $\hat{\eta}_\varepsilon(t)$ yields the MDPs of $X_\varepsilon(t)$.

The rest of the paper is arranged as follows. In Sec. II, we first present some definitions and preliminary results. Then, assumptions on the Markov chain and coefficients in Langevin dynamics as well as the main result are presented. Section III is devoted to the proof of our main theorem. Section IV provides further discussions. Importantly, it proposes an alternative assumption, removes the boundedness condition of the drift, and indicates how to reduce such a case to the analysis under boundedness conditions. Finally, an Appendix is placed at the end of the paper to conclude the paper.

II. FORMULATION AND MAIN RESULTS

Denote by $|\cdot|$ the Euclidean norm of a vector in $\mathbb{R}^d$, $\langle\cdot,\cdot\rangle$ the inner product in $\mathbb{R}^d$, and $C^0([0,1],\mathbb{R}^d)$ the space of continuous functions from [0, 1] to $\mathbb{R}^d$ starting from 0 and equipped with the sup-norm $\|\cdot\|$. We use x' to denote the transpose of a vector $x \in \mathbb{R}^d$, and ∇ (resp., ∇_x) to represent the partial derivatives (gradient) (resp., partial derivatives with respect to first variable if more than one variable is involved). Furthermore, $[\cdot]_t$ denotes the quadratic variation of a stochastic process at time t. Recall that for a real-valued stochastic process X_t defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, the quadratic process $[X]_t$ is defined as

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$$[X]_t = \lim_{\|P\|\to 0}\sum_{k=1}^{n}(X_{t_k} - X_{t_{k-1}})^2,$$

where $\|P\|$ is the mesh of partitions of $[0,t]$ and the convergence is in probability. Throughout the paper, K is a generic positive constant independent of ε whose value may be different for different appearances.

A. Exponential tightness and local MDP

Let us start with some definitions and preliminary results. We first recall the definition of LDPs (large deviations principle); see Ref. 22. Let $Y^\varepsilon = \{Y^\varepsilon(t)\}_{t\in[0,1]}$ be a $C^0([0,1],\mathbb{R}^d)$ family.

Definition II.1. The family Y^ε obeys the LDPs in $C^0([0,1],\mathbb{R}^d)$ with speed $v(\varepsilon)$ and a good rate function I with respect to the supremum norm if

(a) there exists $I: C^0([0,1],\mathbb{R}^d)\to[0,\infty]$ such that I is inf-compact in that the level sets $\{I\le L\}$ for any $L\ge 0$ are compact;
(b) for any open set G in $C^0([0,1],\mathbb{R}^d)$,

$$\liminf_{\varepsilon\to 0} v(\varepsilon)\log\mathbb{P}(Y^\varepsilon(t)\in G) \ge -\inf_{y\in G} I(y);$$

(c) for any closed subset F in $C^0([0,1],\mathbb{R}^d)$,

$$\limsup_{\varepsilon\to 0} v(\varepsilon)\log\mathbb{P}(Y^\varepsilon(t)\in F) \le -\inf_{y\in F} I(y).$$

Next, we recall the definitions of exponential tightness and local LDP, which give sufficient conditions to a full LDP.

Definition II.2. The family $\{Y^\varepsilon\}$ is said to be exponentially tight with speed $v(\varepsilon)\to 0$ in the space $C^0([0,1],\mathbb{R}^d)$ if there exists an increasing sequence of compact sets $\{\mathcal{O}_j\}_{j\ge 1}$ of $C^0([0,1],\mathbb{R}^d)$ such that

$$\lim_{j\to\infty}\limsup_{\varepsilon\to 0} v(\varepsilon)\log\mathbb{P}(Y^\varepsilon(t)\notin\mathcal{O}_j) = -\infty. \tag{2.1}$$

Sufficient conditions for exponential tightness in the space of continuous trajectory can be found in Liptser and Pukhalskii[21] (Theorem 3.1); see also Feng and Kurtz[23] (Remark 4.2). It requires us to prove

$$\lim_{j\to\infty}\limsup_{\varepsilon\to 0}\frac{1}{h^2(\varepsilon)}\log\mathbb{P}\left(\sup_{t\in[0,1]}|X_\varepsilon(t)| > j\right) = -\infty, \tag{2.2}$$

$$\lim_{\delta\to 0}\limsup_{\varepsilon\to 0}\sup_{s\in[0,1]}\frac{1}{h^2(\varepsilon)}\log\mathbb{P}\left(\sup_{s\le t\le s+\delta}|X_\varepsilon(t)-X_\varepsilon(s)| > j\right) = -\infty,\quad \forall j>0. \tag{2.3}$$

Definition II.3. The family $\{Y^\varepsilon\}$ is said to satisfy a local LDP with speed $v(\varepsilon)\to 0$ in $C^0([0,1],\mathbb{R}^d)$ with the rate function $\widehat{I}$ if for any $y\in C^0([0,1],\mathbb{R}^d)$,

$$\begin{aligned}&\lim_{\delta\to 0}\limsup_{\varepsilon\to 0} v(\varepsilon)\log\mathbb{P}(Y^\varepsilon\in B(y,\delta))\\ &\quad=\lim_{\delta\to 0}\liminf_{\varepsilon\to 0} v(\varepsilon)\log\mathbb{P}(Y^\varepsilon\in B(y,\delta))\\ &\quad=-\widehat{I}(y),\end{aligned} \tag{2.4}$$

where $B(y,\delta)$ is the ball of radius δ centered at y.

The LDP is guaranteed by the following well-known theorem; see Refs. 22 and 24.

Proposition II.4. If the family $\{Y^\varepsilon\}$ is exponentially tight and satisfies a local LDP with the rate function $\widehat{I}$ in $C^0([0,1],\mathbb{R}^d)$, then it satisfies the full LDP with the rate function $I(y)\equiv\widehat{I}(y)$, which is inf-compact.

B. Assumptions

Let $r_\varepsilon(t)$ be a continuous-time and time-inhomogeneous Markov chain with a finite state space $\mathcal{M} = \{1, \dots, m\}$ and a generator $Q(t)/\varepsilon$, where $Q(t)$ is a generator and ε is a small parameter as given at the beginning of the paper. For a Markov chain with time-dependent generator $Q(t)$, we refer the reader to the definition of Ref. 18 (Sec. 2.3). Recall that a generator $Q(t)$ (or its associated Markov chain) is irreducible for $t \ge 0$ if the system of equations

$$\begin{cases} \nu(t)Q(t) = 0, \\ \sum_{i=1}^{m} \nu_i(t) = 1 \end{cases} \tag{2.5}$$

has a unique solution such that $\nu_i(t) > 0$ for each $i \in \mathcal{M}$. Throughout the paper, we assume the irreducibility of $Q(t)$ for each $t \in [0, 1]$. We impose the following assumptions on the coefficients $b(\cdot,\cdot)$, $\sigma(\cdot,\cdot)$, and $\alpha_\varepsilon(\cdot)$ in dynamics (1.6).

Assumption II.5. We assume the following conditions:

(A1) For each $i \in \mathcal{M}$, $b(\cdot, i) : \mathbb{R}^d \to \mathbb{R}^d$ is a bounded and continuous function with bounded first-order partial derivatives.

(A2) For each $i \in \mathcal{M}$, $\sigma(\cdot, i) : \mathbb{R}^d \to \mathbb{R}^{d\times n}$ is a bounded and continuous function with bounded first- and second-order partial derivatives.

(A3) The function $\alpha_\varepsilon(\cdot) : \mathbb{R}^d \to \mathbb{R}$ satisfying that $\alpha_\varepsilon(\cdot) \in C_b^1(\mathbb{R}^d)$ (the class of bounded continuously differentiable functions on $\mathbb{R}^d$) and that there exist some constants $0 < \ell_0 \le \ell_1 < \infty$ and $K > 0$ such that

$$\ell_0 = \liminf_{\varepsilon\to 0} \inf_{x\in\mathbb{R}^d} \alpha_\varepsilon(x), \;\; \ell_1 = \limsup_{\varepsilon\to 0} \sup_{x\in\mathbb{R}^d} \alpha_\varepsilon(x), \;\; \sup_{x\in\mathbb{R}^d} |\nabla\alpha_\varepsilon(x)| \le K\varepsilon^2,$$

and

$$\limsup_{\varepsilon\to 0} \sup_{x\in\mathbb{R}^d} \frac{|\alpha_\varepsilon(x) - \alpha(x)|}{\sqrt{\varepsilon}h(\varepsilon)} = 0.$$

(A4) $w(\cdot)$ is a Wiener process in $\mathbb{R}^n$ independent of the Markov chain $r_\varepsilon(\cdot)$.

Remark II.6. Assumptions (A1) and (A2) are concerned with the functions $b(\cdot,\cdot)$ and $\sigma(\cdot,\cdot)$. The boundedness of function b can be weakened to the Lipschitz continuity, which will be addressed in Sec. IV. Assumption (A3) is a technical condition, which is necessary to ensure $e_2(\varepsilon)$ defined later in (3.30) approaching 0 in order to establish the exponential tightness. One of the examples is that $\alpha_\varepsilon(x) = \varepsilon^2 \sin(x) + c_0$ and the limit $\alpha(x) = c_0$, a constant.

Throughout the paper, both r_ε and w are defined on the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$. We introduce the following notation. For each $i \in \mathcal{M}$,

$$b_1(x,i) = \frac{b(x,i)}{\alpha(x)} \quad \text{and} \quad \sigma_1(x,i) = \frac{\sigma(x,i)}{\alpha(x)}. \tag{2.6}$$

Under the irreducibility of r_ε, the averaged coefficients

$$\overline{b}(x,\nu) = \sum_{i=1}^{m} b(x,i)\nu_i \quad \text{and} \quad \overline{b}_1(x,\nu) = \sum_{i=1}^{m} b_1(x,i)\nu_i = \frac{\overline{b}(x,\nu)}{\alpha(x)} \tag{2.7}$$

are well-defined for $x \in \mathbb{R}^d$. Moreover, we denote

$$\begin{aligned} &(D_1)_k^j(x,i) = \frac{\partial}{\partial x_k} b_1^j(x,i) \text{ and } D_1(x,i) = ((D_1)_k^j(x,i))_{1\le j,k\le d}, \\ &\overline{D}_1(x,\nu) = \sum_{i=1}^{m} D_1(x,i)\nu_i. \end{aligned} \tag{2.8}$$

Define

$$\lambda^\varepsilon(t) = \frac{1}{\sqrt{\varepsilon}h(\varepsilon)} \int_0^t b(q_0(s), r_\varepsilon(s)) - \overline{b}(q_0(s), \nu(s))ds, \tag{2.9}$$

$$\lambda_1^\varepsilon(t) = \frac{1}{\sqrt{\varepsilon}h(\varepsilon)} \int_0^t b_1(q_0(s), r_\varepsilon(s)) - \overline{b}_1(q_0(s), \nu(s))ds, \tag{2.10}$$

$$\widehat{M}_t^\varepsilon = \frac{1}{h(\varepsilon)} \int_0^t \sigma_1(q_0(s), r_\varepsilon(s))dw(s), \tag{2.11}$$

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with q_0 and r_ε being given earlier. To proceed, we recall some preliminary results to be used in the rest of the paper.

Lemma II.7 (Ref. 18, Sec. 5.3.3). For each $i \in \mathcal{M}$, let $\beta_i(\cdot)$ be a bounded measurable deterministic function and

$$\widetilde{n}_i^\varepsilon(t) = \frac{1}{\sqrt{\varepsilon}} \int_0^t \big(I_{\{r_\varepsilon(s)=i\}} - \nu_i(s)\big)\beta_i(s)ds,$$

with $\widetilde{n}^\varepsilon(t) = (\widetilde{n}_1^\varepsilon(t), \ldots, \widetilde{n}_m^\varepsilon(t))'$, where z' denotes the transpose of z. Then, $\widetilde{n}^\varepsilon(\cdot)$ converges weakly to a Gaussian process $\widetilde{n}(\cdot)$ such that

$$\mathbb{E}\widetilde{n}(t) = 0, \quad \mathbb{E}[\widetilde{n}_i(t)\widetilde{n}_j(t)] = \int_0^t \beta_i(s)\beta_j(s)A_{ij}(s)ds,$$

and

$$A_{ij}(t) = \nu_i(t)\int_0^\infty \psi_{ij}(u,t)du + \nu_j(t)\int_0^\infty \psi_{ji}(u,t)du$$

where $\Psi(u,t) = (\psi_{ij}(u,t))$ satisfies

$$\frac{\Psi(t,t_0)}{dt} = \Psi(t,t_0)Q(t_0), \quad t \geq 0, \quad \Psi(0,t_0) = I - P^{(0)}(t_0)$$

and

$$P^{(0)}(t) = (\nu(t), \ldots, \nu(t))'.$$

Denote $A(t) := (A_{ij}(t))_{1\leq i,j\leq m}$, and define the matrix $B(t) = (b^i(q_0(t),j))_{1\leq i\leq d, 1\leq j\leq m}$. Let $\tilde{C}(t) = B(t)A(t)B'(t)$, and define a function $\tilde{B} : [0,1] \times \mathcal{M} \to \mathbb{R}^d$ as

$$\tilde{B}(s,i) = b(q_0(s),i) - \overline{b}(q_0(s),\nu(s)).$$

Similarly, we could also define the matrix $B_1 = ((b_1^i(q_0(t),j))_{ij}$, $\tilde{C}_1(t)$, and the function $\tilde{B}_1$. Then, recalling Theorem 4.1 in Ref. 25, we have the following lemma.

Lemma II.8. Under Assumption II.5, $\lambda^\varepsilon(t)$ (resp. $(\lambda_1^\varepsilon(t))$ satisfies an LDP in $C^0([0,1],\mathbb{R}^d)$ with speed $h^{-2}(\varepsilon)$ and a good rate function $I_r^{\tilde{B}}$ (resp. *$I_r^{\tilde{B}_1}$), where*

$$I_r^{\tilde{B}}(\gamma) = \begin{cases} \displaystyle\int_0^1 \sup_{\beta\in\mathbb{R}^d}\Big[\langle\dot{\gamma}(s),\beta\rangle - \frac{1}{2}\langle\tilde{C}(s)\beta,\beta\rangle\Big]ds, & \text{if } d\gamma(s) = \dot{\gamma}(s)ds,\ \gamma(0) = 0, \\ +\infty, & \text{otherwise} \end{cases}$$

*[*resp.*, $I_r^{\tilde{B}_1}$, where $I_r^{\tilde{B}_1}$ is defined as above with $\tilde{C}(s)$ replaced by $\tilde{C}_1(s)$].*

Corollary II.9. Under Assumption II.5, $\sup_{t\in[0,1]}|\lambda^\varepsilon(t)|$ satisfies an LDP in $\mathbb{R}$ with speed $h^{-2}(\varepsilon)$ and a good rate function J. In particular,

$$\lim_{x\to\infty} J(x) = +\infty,$$

$$\lim_{j\to\infty}\lim_{\varepsilon\to0}\frac{1}{h^2(\varepsilon)}\log\mathbb{P}\Bigg(\sup_{t\in[0,1]}|\lambda^\varepsilon(t)| \geq j\Bigg) = -\infty. \tag{2.12}$$

Proof. The result is a consequence of contraction principle and is similar to Ref. 20 (Corollary 1), so the verbatim is omitted here. □

Lemma II.10. Under Assumption II.5, $\widehat{M}_t^\varepsilon$ satisfies an LDP in $C^0([0,1],\mathbb{R}^d)$ with speed $h^{-2}(\varepsilon)$ and rate function I_w given by

$$I_w(\gamma) = \begin{cases} \displaystyle\int_0^1 \sup_{\beta\in\mathbb{R}^d}\Big(\beta'\dot{\gamma}(s) - \frac{1}{2}\beta'\overline{\Sigma}_s^2\beta\Big)ds, & \text{if } d\gamma(s) = \dot{\gamma}(s)ds,\ \gamma(0) = 0, \\ +\infty, & \text{otherwise,} \end{cases} \tag{2.13}$$

where

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$$\overline{\Sigma}_s^2 = \sum_{i=1}^m \sigma_1(q_0(s),i)\sigma_1'(q_0(s),i)\nu_i(s).$$

Proof. Under assumption (A2) and (A3), $\sigma_1(\cdot,\cdot)$ is bounded and Lipschitz with respect to the first variable, following the proof of Ref. 20 (Proposition 1) [see also Ref. 21 (Theorems 2.1 and 3)], and leads to the desired result. □

We conclude this section by stating our main result. The proof is postponed until Sec. III.

Theorem II.11. *Suppose that the $r_\varepsilon(t)$ is an inhomogeneous irreducible Markov chain with generator $Q(t)/\varepsilon$, where ε is a small parameter. Furthermore, suppose that conditions (A1)–(A5) hold. Then, $X_\varepsilon(t)$ satisfies an LDP in $C^0([0,1],\mathbb{R}^d)$ with speed $h^{-2}(\varepsilon)$ and a good rate function S given for $\gamma \in C^0([0,1],\mathbb{R}^d)$ by*

$$S(\gamma) = I\Big(\gamma - \int_0^{\cdot} \overline{D}_1(q_0(s),\nu(s))\gamma(s)ds\Big), \tag{2.14}$$

where I is defined by

$$I(\gamma) = \inf\{I_\xi^{\tilde{B}_1}(\gamma-\varphi) + I_w(\varphi); \varphi \in C^0([0,1],\mathbb{R}^d)\}. \tag{2.15}$$

Furthermore, assuming that $\tilde{C}_1(s)$ is invertible, $I(\gamma)$ can be explicitly expressed as

$$I(\gamma) = \begin{cases} \frac{1}{2}\int_0^1 \|(\tilde{C}_1(s) + \overline{\Sigma}_s^2)^{-1/2}\dot{\gamma}(s)\|ds, & \text{if } d\gamma(s) = \dot{\gamma}(s)ds,\ \gamma(0)=0, \\ +\infty, & \text{otherwise.} \end{cases} \tag{2.16}$$

III. PROOF OF MDP

This section is devoted to the Proof of Theorem II.11. Note that (1.6) can be rewritten as a first-order system,

$$\begin{cases} \dot{q}_\varepsilon(t) = p_\varepsilon(t), \\ \varepsilon^2\dot{p}_\varepsilon(t) = b(q_\varepsilon(t),r_\varepsilon(t)) - \alpha_\varepsilon(q_\varepsilon(t))p_\varepsilon(t) + \sqrt{\varepsilon}\sigma(q_\varepsilon(t),r_\varepsilon(t))\dot{w}(t), \\ q_\varepsilon(0) = q \in \mathbb{R}^d, \quad p_\varepsilon(0) = \frac{p}{\varepsilon} \in \mathbb{R}^d. \end{cases}$$

From the variation of parameters formula, we have

$$p_\varepsilon(t) = \frac{p}{\varepsilon}e^{-A_\varepsilon(t)} + \frac{1}{\varepsilon^2}\int_0^t e^{-A_\varepsilon(t,s)}b(q_\varepsilon(s),r_\varepsilon(s))ds + \frac{1}{\varepsilon^2}H_\varepsilon(t), \tag{3.1}$$

where for any $0 \le s \le t \le 1, 0 < \varepsilon < 1$,

$$A_\varepsilon(t,s) := \frac{1}{\varepsilon^2}\int_s^t \alpha_\varepsilon(q_\varepsilon(u))du, \quad A_\varepsilon(t) = A_\varepsilon(t,0),$$
$$H_\varepsilon(t) := \sqrt{\varepsilon}e^{-A_\varepsilon(t)}\int_0^t e^{A_\varepsilon(s)}\sigma(q_\varepsilon(s),r_\varepsilon(s))dw(s).$$

Then, the solution $q_\varepsilon(t)$ of (1.6) can be expressed as

$$q_\varepsilon(t) = q + \frac{p}{\varepsilon}\int_0^t e^{-A_\varepsilon(s)}ds + \frac{1}{\varepsilon^2}\int_0^t\int_0^s e^{-A_\varepsilon(s,u)}b(q_\varepsilon(u),r_\varepsilon(u))duds + \frac{1}{\varepsilon^2}\int_0^t H_\varepsilon(s)ds. \tag{3.2}$$

Employing integration by parts formula gives

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$$q_\varepsilon(t) = q + \int_0^t \frac{b(q_\varepsilon(s), r_\varepsilon(s))}{\alpha_\varepsilon(q_\varepsilon(s))} ds + \sqrt{\varepsilon} \int_0^t \frac{\sigma(q_\varepsilon(s), r_\varepsilon(s))}{\alpha_\varepsilon(q_\varepsilon(s))} dw(s) + R_\varepsilon(t), \tag{3.3}$$

where

$$\begin{aligned} R_\varepsilon(t) :=& \frac{p}{\varepsilon}\int_0^t e^{-A_\varepsilon(s)} ds - \frac{1}{\alpha_\varepsilon(q_\varepsilon(t))}\int_0^t e^{-A_\varepsilon(t,s)} b(q_\varepsilon(s), r_\varepsilon(s)) ds \\ & - \int_0^t \left(\int_0^s e^{-A_\varepsilon(s,u)} b(q_\varepsilon(u), r_\varepsilon(u)) du \right) \frac{1}{\alpha_\varepsilon^2(q_\varepsilon(s))} \langle \nabla \alpha_\varepsilon(q_\varepsilon(s)), \dot{q}_\varepsilon(s) \rangle ds \\ & - \frac{1}{\alpha_\varepsilon(q_\varepsilon(t))} H_\varepsilon(t) - \int_0^t \frac{1}{\alpha_\varepsilon^2(q_\varepsilon(s))} H_\varepsilon(s) \langle \nabla \alpha_\varepsilon(q_\varepsilon(s)), \dot{q}_\varepsilon(s) \rangle ds \\ :=& \sum_{k=1}^5 R_\varepsilon^k(t) \end{aligned} \tag{3.4}$$

In view of the definition of $X_\varepsilon(t)$ in (1.9), we have that for any $t \in [0,1]$,

$$\begin{aligned} X_\varepsilon(t) &= \frac{1}{\sqrt{\varepsilon}h(\varepsilon)}(q_\varepsilon(t) - q_0(t)) \\ &= \frac{1}{\sqrt{\varepsilon}h(\varepsilon)} \int_0^t \frac{b(q_\varepsilon(s), r_\varepsilon(s))}{\alpha_\varepsilon(q_\varepsilon(s))} - \frac{\bar{b}(q_0(s), \nu(s))}{\alpha(q_0(s))} ds \\ &\quad + \frac{1}{h(\varepsilon)} \int_0^t \frac{\sigma(q_\varepsilon(s), r_\varepsilon(s))}{\alpha_\varepsilon(q_\varepsilon(s))} dw(s) + \frac{1}{\sqrt{\varepsilon}h(\varepsilon)} R_\varepsilon(t) \\ &:= 1_1^\varepsilon(t) + 1_2^\varepsilon(t) + 1_3^\varepsilon(t). \end{aligned} \tag{3.5}$$

A. Exponential tightness

By virtue of the sufficient conditions, to prove the exponential tightness of $X_\varepsilon(t)$, we need only verify (2.2) and (2.3). To proceed, we first obtain *a priori* estimates for $q_\varepsilon(t)$ and $p_\varepsilon(t)$.

Proposition III.1. There is a constant K independent of ε such that

$$|q_\varepsilon(t)| \le K\left(1 + \frac{1}{\varepsilon^2}\int_0^t |H_\varepsilon(s)| ds\right), \tag{3.6}$$

$$|p_\varepsilon(t)| \le K\left(1 + \frac{1}{\varepsilon} + \frac{1}{\varepsilon^2} \sup_{t\in[0,1]} |H_\varepsilon(t)|\right). \tag{3.7}$$

The Proof of Proposition III.1 is postponed to Appendix for a better flow of presentation. In order to achieve the exponential tightness of $X_\varepsilon(t)$, we first give the exponential tightness of $H_\varepsilon(t)$ with respect to MDPs, which is the main ingredient for that of $X_\varepsilon(t)$.

Lemma III.2. Under assumptions of Theorem II.11,

$$\lim_{j\to\infty} \limsup_{\varepsilon\to 0} \frac{1}{h^2(\varepsilon)} \log \mathbb{P}\left(\frac{1}{\sqrt{\varepsilon}h(\varepsilon)} \sup_{t\in[0,1]} |H_\varepsilon(t)| > j \right) = -\infty. \tag{3.8}$$

Proof. If $f \in C^1([0,t])$ and $g \in C([0,t])$, then the following Stieltjes integral

$$\int_0^t f(s) dg(s), \quad t \ge 0,$$

is well defined. By integration by parts, it follows

$$\int_{t_1}^{t_2} f(s) dg(s) = f(t_2)g(t_2) - f(t_1)g(t_1) - \int_{t_1}^{t_2} g(s) df(s), \quad 0 \le t_1 \le t_2 \le t. \tag{3.9}$$

In particular, if $g(0) = 0$, then

$$\int_0^t f(s) dg(s) = g(t)f(0) + \int_0^t (g(t) - g(s)) df(s), \quad t \ge 0. \tag{3.10}$$

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The above formula will be used frequently implicitly in the sequel. Since the trajectory of the Markov chain $r_\varepsilon(\cdot)$ has piecewise constant sample paths, let

$$0 = \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_k < \cdots$$

be the jump times. Denote by $n(t)$ the random counting process representing the number of jumps up to time t. Noting that $r_\varepsilon(t) = r_\varepsilon(\tau_k)$ for $t \in [\tau_k, \tau_{k+1})$, denoting $i_k = r_\varepsilon(\tau_k)$, and using integration by parts formula (3.9) with $f(s) = e^{A_\varepsilon(s)}\sigma(q_\varepsilon(s), i_k)$ and $g(s) = w(s)$ on each interval $[\tau_k, \tau_{k+1})$ yields

$$\begin{aligned}\int_0^t e^{A_\varepsilon(s)}\sigma(q_\varepsilon(s),\alpha_\varepsilon(s))dw(s) &= \sum_{k=0}^{n(t)-1}\int_{\tau_k}^{\tau_{k+1}} e^{A_\varepsilon(s)}\sigma(q_\varepsilon(s),i_k)dw(s) + \int_{\tau_{n(t)}}^t f(s)dg(s)\\
&= \sum_{k=0}^{n(t)-1}\Big(f(\tau_{k+1})g(\tau_{k+1}) - f(\tau_k)g(\tau_k) - \int_{\tau_k}^{\tau_{k+1}} g(s)df(s)\Big)\\
&\quad + f(t)g(t) - f(\tau_{n(t)})g(\tau_{n(t)}) - \int_{\tau_{n(t)}}^t g(s)df(s)\\
&= f(t)g(t) - f(0)g(0) - \int_0^t g(s)df(s)\\
&= w(t)\sigma(q, r_\varepsilon(0))\\
&\quad + \int_0^t (w(t)-w(s))e^{A_\varepsilon(s)}\Big[\frac{\alpha_\varepsilon(q_\varepsilon(s))}{\varepsilon^2}\sigma(q_\varepsilon(s), r_\varepsilon(s)) + \nabla_x\sigma(q_\varepsilon(s), r_\varepsilon(s))p_\varepsilon(s)\Big]ds. \end{aligned} \tag{3.11}$$

Using estimates (3.7) and assumptions (A1) and (A3) leads to

$$\begin{aligned}\frac{1}{\sqrt{\varepsilon}h(\varepsilon)}\sup_{t\in[0,1]}|H_\varepsilon(t)| &\le K\frac{1}{h(\varepsilon)}\|w(t)\| + K\frac{1}{h(\varepsilon)}\|w(t)\|\Big(\varepsilon^2 + \varepsilon + \sup_{t\in[0,1]}|H_\varepsilon(t)|\Big)\\
&\le K\frac{(\varepsilon+1)}{h(\varepsilon)}\|w(t)\| + K\sqrt{\varepsilon}\|w(t)\|\Big(\frac{1}{\sqrt{\varepsilon}h(\varepsilon)}\sup_{t\in[0,1]}|H_\varepsilon(t)|\Big).\end{aligned}$$

It follows

$$\begin{aligned}\mathbb{P}\Big(\frac{1}{\sqrt{\varepsilon}h(\varepsilon)}\sup_{t\in[0,1]}|H_\varepsilon(t)| > j\Big) &= \mathbb{P}\Big(\frac{1}{\sqrt{\varepsilon}h(\varepsilon)}\sup_{t\in[0,1]}|H_\varepsilon(t)| > j; 1 - K\sqrt{\varepsilon}\|w(t)\| > 0\Big)\\
&\quad + \mathbb{P}\Big(\frac{1}{\sqrt{\varepsilon}h(\varepsilon)}\sup_{t\in[0,1]}|H_\varepsilon(t)| > j; 1 - K\sqrt{\varepsilon}\|w(t)\| \le 0\Big)\\
&\le \mathbb{P}\Big(\frac{1}{\sqrt{\varepsilon}h(\varepsilon)}\sup_{t\in[0,1]}|H_\varepsilon(t)| > j; 1 - K\sqrt{\varepsilon}\|w(t)\| > 0\Big) + \mathbb{P}\big(1 - K\sqrt{\varepsilon}\|w(t)\| \le 0\big)\\
&\le \mathbb{P}\Big(\|w(t)\| > \frac{jh(\varepsilon)}{K(\varepsilon+1) + j\sqrt{\varepsilon}h(\varepsilon)}\Big) + \mathbb{P}\Big(\|w(t)\| \ge \frac{1}{K\sqrt{\varepsilon}}\Big).\end{aligned}$$

Denote

$$e_1(\varepsilon) := K(\varepsilon+1) + j\sqrt{\varepsilon}h(\varepsilon) \to K \text{ a constant, as } \varepsilon \to 0.$$

Then, Bernstein's inequality[26] (pp. 153–154) yields

$$\mathbb{P}\Big(\|w(t)\| > \frac{jh(\varepsilon)}{e_1(\varepsilon)}\Big) = \mathbb{P}\Big(\|w(t)\| > \frac{jh(\varepsilon)}{e_1(\varepsilon)}; [w(t)]_1 \le 1\Big) \le \exp\Big(\frac{-j^2h^2(\varepsilon)}{e_1^2(\varepsilon)}\Big)$$

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and

$$\mathbb{P}\bigg(\|w(t)\| \geq \frac{1}{K\sqrt{\varepsilon}}\bigg) \leq \exp\bigg(-\frac{1}{K\varepsilon}\bigg).$$

Thus, combining above estimates gives

$$\begin{aligned}\mathbb{P}\bigg(\frac{1}{\sqrt{\varepsilon}h(\varepsilon)}\sup_{t\in[0,1]}|H_\varepsilon(t)| > j\bigg) &\leq \exp\bigg(\frac{-j^2h^2(\varepsilon)}{e_1^2(\varepsilon)}\bigg) + \exp\bigg(-\frac{1}{K\varepsilon}\bigg)\\ &\leq 2\bigg(\exp\bigg(-\frac{j^2h^2(\varepsilon)}{e_1^2(\varepsilon)}\bigg) \vee \exp\bigg(-\frac{1}{K\varepsilon}\bigg)\bigg),\end{aligned} \tag{3.12}$$

which implies (3.8). The proof is complete. □

Using the integration by parts formula (3.11), we have

$$\begin{aligned}|H_\varepsilon(t) - H_\varepsilon(s)| &= \sqrt{\varepsilon}\Big|e^{-A_\varepsilon(t)}w(t)\sigma(q, r_\varepsilon(0)) - e^{-A_\varepsilon(s)}w(s)\sigma(q, r_\varepsilon(0))\Big|\\ &\quad+\sqrt{\varepsilon}\bigg|\int_0^t (w(t)-w(u))e^{-A_\varepsilon(t,u)}B(u)du - \int_0^s (w(s)-w(u))e^{-A_\varepsilon(s,u)}B(u)du\bigg|\\ &\leq K\sqrt{\varepsilon}|e^{-A_\varepsilon(t)} - e^{-A_\varepsilon(s)}|\,|w(t)| + K\sqrt{\varepsilon}e^{-A_\varepsilon(s)}|w(t)-w(s)|\\ &\quad+\sqrt{\varepsilon}\bigg|\int_0^s (w(t)-w(s))e^{-A_\varepsilon(t,u)}B(u)du\bigg|\\ &\quad+\sqrt{\varepsilon}\bigg|\int_0^s (w(s)-w(u))\Big[e^{-A_\varepsilon(t,u)} - e^{-A_\varepsilon(s,u)}\Big]B(u)du\bigg|\\ &\quad+\sqrt{\varepsilon}\bigg|\int_s^t (w(t)-w(s))e^{-A_\varepsilon(t,u)}B(u)du\bigg| =: \sum_{k=1}^5 2_k^\varepsilon(t,s),\end{aligned}$$

where

$$B(u) := \frac{\alpha_\varepsilon(q_\varepsilon(u))}{\varepsilon^2}\sigma(q_\varepsilon(u), r_\varepsilon(u)) + \nabla_x\sigma(q_\varepsilon(u), r_\varepsilon(u))p_\varepsilon(u).$$

Lemma III.3. Under the assumption of Theorem II.11, for any $j > 0$*,*

$$\lim_{\delta\to 0}\limsup_{\varepsilon\to 0}\sup_{s\in[0,1]}\frac{1}{h^2(\varepsilon)}\log\mathbb{P}\bigg(\frac{1}{\sqrt{\varepsilon}h(\varepsilon)}\sup_{s\leq t\leq s+\delta}|H_\varepsilon(t) - H_\varepsilon(s)| > j\bigg) = -\infty. \tag{3.13}$$

Proof. Since

$$\begin{aligned}\mathbb{P}\bigg(\frac{1}{\sqrt{\varepsilon}h(\varepsilon)}\sup_{s\leq t\leq s+\delta}|H_\varepsilon(t) - H_\varepsilon(s)| > j\bigg) &\leq \sum_{i=1}^5 \mathbb{P}\bigg(\frac{1}{\sqrt{\varepsilon}h(\varepsilon)}\sup_{s\leq t\leq s+\delta}|\mathrm{II}_i^\varepsilon(t,s)| > j/5\bigg)\\ &\leq 5\bigg(\max_{1\leq i\leq 5}\mathbb{P}\bigg(\frac{1}{\sqrt{\varepsilon}h(\varepsilon)}\sup_{s\leq t\leq s+\delta}|\mathrm{II}_i^\varepsilon(t,s)| > j/5\bigg)\bigg),\end{aligned} \tag{3.14}$$

to prove (3.13), it suffices to show

$$\lim_{\delta\to 0}\limsup_{\varepsilon\to 0}\sup_{s\in[0,1]}\frac{1}{h^2(\varepsilon)}\log\mathbb{P}\bigg(\frac{1}{\sqrt{\varepsilon}h(\varepsilon)}\sup_{s\leq t\leq s+\delta}|\mathrm{II}_i^\varepsilon(t,s)| > j\bigg) = -\infty, \quad i = 1,2,3,4,5.$$

Step 1: Note that

$$\mathrm{II}_1^\varepsilon(t,s) = K\sqrt{\varepsilon}|e^{-A_\varepsilon(t)} - e^{-A_\varepsilon(s)}|\,|w(t)| \leq 2K\sqrt{\varepsilon}|w(t)|.$$

To proceed, we need an exponential inequality of the form

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$$\mathbb{P}\left[\sup_{s\le t}|B_s|\ge\delta\right]\le 2d\,\exp(-\delta^2/(2dt)), \tag{3.15}$$

for a d-dimensional Brownian motion B. This can be seen as follows:

$$\mathbb{P}\left[\sup_{s\le t}|B_s|\ge\delta\right]=\mathbb{P}\left[\sup_{s\le t}\sup_{|\theta|=1}\langle\theta,B_s\rangle|\ge\delta\right]=\mathbb{P}\left[\sup_{s\le t}\langle\theta_1,B_s\rangle\ge\delta\right]\le\sum_{i=1}^d\mathbb{P}\left[\sup_{s\le t}\theta_1^iB_s^i\ge\frac{\delta}{d}\right]$$
$$\le\sum_{i=1}^d\left\{\mathbb{P}\left[\sup_{s\le t}B_s^i\ge\frac{\delta}{\theta_1^id}\right]+\mathbb{P}\left[\inf_{s\le t}B_s^i\le\frac{\delta}{\theta_1^id}\right]\right\},$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product, $\theta_1=B_s/|B_s|$ is a unit vector in $\mathbb{R}^d$, and θ_1^i and B_s^i are ith coordinates of θ_1 and B_s, respectively. Then, applying Proposition 1.8 in Ref. 26, we have

$$\mathbb{P}\left[\sup_{s\le t}|B_s|\ge\delta\right]\le 2\sum_{i=1}^d\exp\left\{-\frac{\delta^2}{2d^2(\theta_1^i)^2t}\right\}.$$

Noting $|\theta_1|=1$ implies $\inf_{1\le i\le d}(\theta_1^i)^2\ge\frac{1}{d}$, and taking infimum on the right-hand side with $|\theta_1|=1$ yields (3.15). Consequently,

$$\mathbb{P}\left(\sup_{s\le t\le s+\delta}\frac{1}{\sqrt{\varepsilon}h(\varepsilon)}|\,\mathrm{II}_1^\varepsilon(t,s)|>j\right)\le\mathbb{P}\left(\sup_{s\le t\le s+\delta}|w(t)|>\frac{jh(\varepsilon)}{2K}\right)$$
$$\le\mathbb{P}\left(\sup_{0\le t\le s+\delta}|w(t)|>\frac{jh(\varepsilon)}{2K}\right)\le 2n\,\exp\left(-\frac{j^2h^2(\varepsilon)}{8K^2n(s+\delta)}\right)\le 2n\,\exp\left(-\frac{j^2h^2(\varepsilon)}{8K^2n(1+\delta)}\right). \tag{3.16}$$

Thus, we have

$$\lim_{\delta\to0}\limsup_{\varepsilon\to0}\sup_{s\in[0,1]}\frac{1}{h^2(\varepsilon)}\log\mathbb{P}\left(\frac{1}{\sqrt{\varepsilon}h(\varepsilon)}\sup_{s\le t\le s+\delta}|\,\mathrm{II}_1^\varepsilon(t,s)|>j\right)=-\infty.$$

Step 2: Using the same argument to (3.16), it has

$$\mathbb{P}\left(\sup_{s\le t\le s+\delta}\frac{1}{\sqrt{\varepsilon}h(\varepsilon)}|\,\mathrm{II}_2^\varepsilon(t,s)|>j\right)\le\mathbb{P}\left(\sup_{s\le t\le s+\delta}|w(t)|>\frac{jh(\varepsilon)}{2K}\right)\le 2\,\exp\left(-\frac{j^2h^2(\varepsilon)}{8K^2(1+\delta)}\right),$$

which implies

$$\lim_{\delta\to0}\limsup_{\varepsilon\to0}\sup_{s\in[0,1]}\frac{1}{h^2(\varepsilon)}\log\mathbb{P}\left(\sup_{s\le t\le s+\delta}\frac{1}{\sqrt{\varepsilon}h(\varepsilon)}|\,\mathrm{II}_2^\varepsilon(t,s)|>j\right)=-\infty. \tag{3.17}$$

Step 3: Invoking estimate (3.7), it yields

$$\left|\int_0^s(w(t)-w(s))e^{-A_\varepsilon(t,u)}B(u)du\right|\le|w(t)-w(s)|\sup_{u\in[0,1]}|B(u)|\int_0^se^{-A_\varepsilon(t,u)}du$$
$$\le K(1+\varepsilon)|w(t)-w(s)|+K|w(t)-w(s)|\sup_{t\in[0,1]}|H_\varepsilon(t)|.$$

Thus, for any $L>0$,

$$\mathbb{P}\left(\sup_{s\le t\le s+\delta}\frac{1}{\sqrt{\varepsilon}h(\varepsilon)}|\,\mathrm{II}_3^\varepsilon(t,s)|>j\right)\le\mathbb{P}\left(\sup_{s\le t\le s+\delta}\frac{K(1+\varepsilon)}{h(\varepsilon)}|w(t)-w(s)|>\frac{j}{2}\right)$$
$$+\mathbb{P}\left(\sup_{s\le t\le s+\delta}|w(t)-w(s)|\sup_{t\in[0,1]}|H_\varepsilon(t)|\ge jh(\varepsilon)/2K;\ \sup_{s\le t\le s+\delta}|w(t)-w(s)|\le Lh(\varepsilon)\right)$$
$$+\mathbb{P}\left(\sup_{s\le t\le s+\delta}|w(t)-w(s)|\sup_{t\in[0,1]}|H_\varepsilon(t)|\ge jh(\varepsilon)/2K;\ \sup_{s\le t\le s+\delta}|w(t)-w(s)|>Lh(\varepsilon)\right)$$
$$\le\mathbb{P}\left(\sup_{s\le t\le s+\delta}|w(t)-w(s)|>\frac{jh(\varepsilon)}{K(1+\varepsilon)}\right)+\mathbb{P}\left(\frac{1}{\sqrt{\varepsilon}h(\varepsilon)}\sup_{t\in[0,1]}|H_\varepsilon(t)|>\frac{j}{2KL\sqrt{\varepsilon}h(\varepsilon)}\right)$$
$$+\mathbb{P}\left(\sup_{s\le t\le s+\delta}|w(t)-w(s)|>Lh(\varepsilon)\right). \tag{3.18}$$

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In view of (3.12), it is readily to see that

$$\lim_{\delta\to 0}\limsup_{\varepsilon\to 0}\sup_{s\in[0,1]}\frac{1}{h^2(\varepsilon)}\log\mathbb{P}\left(\frac{1}{\sqrt{\varepsilon}h(\varepsilon)}\sup_{t\in[0,1]}|H_\varepsilon(t)|>\frac{j}{2KL\sqrt{\varepsilon}h(\varepsilon)}\right)=-\infty. \tag{3.19}$$

In addition, a similar argument to (3.16) yields

$$\lim_{\delta\to 0}\limsup_{\varepsilon\to 0}\sup_{s\in[0,1]}\frac{1}{h^2(\varepsilon)}\log\mathbb{P}\left(\sup_{s\le t\le s+\delta}|w(t)-w(s)|>\frac{jh(\varepsilon)}{K(1+\varepsilon)}\right)=-\infty \tag{3.20}$$

and

$$\lim_{\delta\to 0}\limsup_{\varepsilon\to 0}\sup_{s\in[0,1]}\frac{1}{h^2(\varepsilon)}\log\mathbb{P}\left(\sup_{s\le t\le s+\delta}|w(t)-w(s)|>Lh(\varepsilon)\right)=-\infty. \tag{3.21}$$

Combining (3.19)–(3.21), we have

$$\lim_{\delta\to 0}\limsup_{\varepsilon\to 0}\sup_{s\in[0,1]}\frac{1}{h^2(\varepsilon)}\log\mathbb{P}\left(\frac{1}{\sqrt{\varepsilon}h(\varepsilon)}\sup_{s\le t\le s+\delta}|\,\mathrm{II}_3^\varepsilon(t,s)|>j\right)=-\infty. \tag{3.22}$$

Step 4: Note that

$$\begin{aligned}\mathrm{II}_4^\varepsilon(t,s)&\le 2\sqrt{\varepsilon}\sup_{0\le u\le s+\delta}|w(u)|\sup_{u\in[0,1]}|B(u)|\left|\int_0^s e^{-A_\varepsilon(t,u)}-e^{-A_\varepsilon(s,u)}du\right|\\&\le K\sqrt{\varepsilon}\sup_{0\le u\le s+\delta}|w(u)|\varepsilon^2\left(1+\frac{1}{\varepsilon^2}+\frac{1}{\varepsilon}+\frac{1}{\varepsilon^2}\sup_{t\in[0,1]}|H_\varepsilon(t)|\right)\\&\le K\sqrt{\varepsilon}(1+\varepsilon)\sup_{0\le u\le s+\delta}|w(u)|+K\sqrt{\varepsilon}\sup_{0\le u\le s+\delta}|w(u)|\sup_{t\in[0,1]}|H_\varepsilon(t)|.\end{aligned}$$

Applying a similar argument to (3.18) and recalling estimate (3.16), one obtains

$$\lim_{\delta\to 0}\limsup_{\varepsilon\to 0}\sup_{s\in[0,1]}\frac{1}{h^2(\varepsilon)}\log\mathbb{P}\left(\sup_{s\le t\le s+\delta}\frac{1}{\sqrt{\varepsilon}h(\varepsilon)}|\,\mathrm{II}_4^\varepsilon(t,s)|\ge j\right)=-\infty. \tag{3.23}$$

Step 5: Since

$$\begin{aligned}\mathrm{II}_5^\varepsilon(t,s)&=\sqrt{\varepsilon}\left|\int_s^t(w(t)-w(s))e^{-A_\varepsilon(t,u)}B(u)du\right|\\&\le 2\sqrt{\varepsilon}\sup_{s\le t\le s+\delta}|w(t)|\sup_{u\in[0,1]}|B(u)|\left|\int_s^t e^{-A_\varepsilon(t,u)}du\right|\\&\le K\sqrt{\varepsilon}(1+\varepsilon)\sup_{s\le t\le s+\delta}|w(t)|+K\sqrt{\varepsilon}\sup_{s\le t\le s+\delta}|w(t)|\sup_{t\in[0,1]}|H_\varepsilon(t)|,\end{aligned}$$

following Step 4 gives

$$\lim_{\delta\to 0}\limsup_{\varepsilon\to 0}\sup_{s\in[0,1]}\frac{1}{h^2(\varepsilon)}\log\mathbb{P}\left(\sup_{s\le t\le s+\delta}\frac{1}{\sqrt{\varepsilon}h(\varepsilon)}|\,\mathrm{II}_5^\varepsilon(t,s)|>j\right)=-\infty. \tag{3.24}$$

Finally, combining all of the above steps completes the proof. □

1. Proof of Eq. (2.2)

By virtue of the Lipschitz property of $b(\cdot,\cdot)$ and the boundedness of $\overline{b}(\cdot,\cdot)$ and $\alpha_\varepsilon(\cdot)$, we have

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$$\begin{aligned}|1_1^\varepsilon(t)| &= \left|\frac{1}{\sqrt{\varepsilon}h(\varepsilon)}\int_0^t \frac{b(q_\varepsilon(s),r_\varepsilon(s))}{\alpha_\varepsilon(q_\varepsilon(s))} - \frac{\overline{b}(q_0(s),v(s))}{\alpha(q_0(s))}ds\right| \\ &= \left|\frac{1}{\sqrt{\varepsilon}h(\varepsilon)}\int_0^t \frac{b(q_\varepsilon(s),r_\varepsilon(s)) - b(q_0(s),r_\varepsilon(s))}{\alpha_\varepsilon(q_\varepsilon(s))} + \frac{b(q_0(s),r_\varepsilon(s)) - \overline{b}(q_0(s),v(s))}{\alpha_\varepsilon(q_\varepsilon(s))}\right. \\ &\qquad \left. + \frac{\overline{b}(q_0(s),v(s))}{\alpha_\varepsilon(q_\varepsilon(s))} - \frac{\overline{b}(q_0(s),v(s))}{\alpha_\varepsilon(q_0(s))} + \frac{\overline{b}(q_0(s),v(s))}{\alpha_\varepsilon(q_0(s))} - \frac{\overline{b}(q_0(s),v(s))}{\alpha(q_0(s))}ds\right| \\ &\le \frac{K}{\ell_0}\int_0^t |X_\varepsilon(s)|ds + K|\lambda^\varepsilon(t)| + K\sup_{x\in\mathbb{R}^d}\frac{|\alpha_\varepsilon(x)-\alpha(x)|}{\sqrt{\varepsilon}h(\varepsilon)}. \end{aligned} \tag{3.25}$$

In addition, due to $e^{-A_\varepsilon(s)} \le e^{-\ell_0 s/\varepsilon^2}$, it yields

$$\left|\frac{1}{\sqrt{\varepsilon}h(\varepsilon)}R_\varepsilon^1(t)\right| = \left|\frac{1}{\sqrt{\varepsilon}h(\varepsilon)}\frac{p}{\varepsilon}\int_0^t e^{-A_\varepsilon(s)}ds\right| \le \frac{|p|\sqrt{\varepsilon}}{h(\varepsilon)\ell_0}. \tag{3.26}$$

Using the boundedness of $b(\cdot,\cdot)$ and $\alpha_\varepsilon(\cdot)$ to estimate $R_\varepsilon^2(\cdot)$ yields

$$\left|\frac{1}{\sqrt{\varepsilon}h(\varepsilon)}R_\varepsilon^2(t)\right| = \left|\frac{1}{\sqrt{\varepsilon}h(\varepsilon)}\frac{1}{\alpha_\varepsilon(q_\varepsilon(t))}\int_0^t e^{-A_\varepsilon(t,s)}b(q_\varepsilon(s),r_\varepsilon(s))ds\right| \le \frac{K\varepsilon^2}{\ell_0^2\sqrt{\varepsilon}h(\varepsilon)}. \tag{3.27}$$

Furthermore, (3.7) yields

$$\begin{aligned}\left|\frac{1}{\sqrt{\varepsilon}h(\varepsilon)}R_\varepsilon^3(t)\right| &\le \frac{K\varepsilon^2}{\ell_0^2\sqrt{\varepsilon}h(\varepsilon)}\int_0^t\left|\int_0^s e^{-A_\varepsilon(s,u)}b(q_\varepsilon(u),r_\varepsilon(u))du\right||p_\varepsilon(s)|ds \\ &\le \frac{K\varepsilon^2}{\sqrt{\varepsilon}h(\varepsilon)}\left(1+\frac{1}{\varepsilon}+\frac{1}{\varepsilon^2}\sup_{t\in[0,1]}|H_\varepsilon(t)|\right)\int_0^t\left|\int_0^s e^{-A_\varepsilon(s,u)}b(q_\varepsilon(u),r_\varepsilon(u))du\right|ds \\ &\le \frac{K(\varepsilon^2+\varepsilon)}{\sqrt{\varepsilon}h(\varepsilon)} + K\left(\frac{1}{\sqrt{\varepsilon}h(\varepsilon)}\sup_{t\in[0,1]}|H_\varepsilon(t)|\right), \end{aligned} \tag{3.28}$$

$$\begin{aligned}\left|\frac{1}{\sqrt{\varepsilon}h(\varepsilon)}R_\varepsilon^5(t)\right| &\le \frac{K\varepsilon^2}{\ell_0^2\sqrt{\varepsilon}h(\varepsilon)}\left(1+\frac{1}{\varepsilon}+\frac{1}{\varepsilon^2}\sup_{t\in[0,1]}|H_\varepsilon(t)|\right)\int_0^t|H_\varepsilon(s)|ds \\ &\le \frac{K(\varepsilon^2+\varepsilon)}{\sqrt{\varepsilon}h(\varepsilon)}\sup_{t\in[0,1]}|H_\varepsilon(t)| + K\sqrt{\varepsilon}h(\varepsilon)\left(\frac{1}{\sqrt{\varepsilon}h(\varepsilon)}\sup_{t\in[0,1]}|H_\varepsilon(t)|\right)^2 \\ &\le K\varepsilon\left(\frac{1}{\sqrt{\varepsilon}h(\varepsilon)}\sup_{t\in[0,1]}|H_\varepsilon(t)|\right) + K\sqrt{\varepsilon}h(\varepsilon)\left(\frac{1}{\sqrt{\varepsilon}h(\varepsilon)}\sup_{t\in[0,1]}|H_\varepsilon(t)|\right)^2. \end{aligned} \tag{3.29}$$

Combining estimates (3.25)–(3.29) and applying Grönwall's inequality lead to

$$\begin{aligned}|X_\varepsilon(t)| \le K\left(\frac{1}{\sqrt{\varepsilon}h(\varepsilon)}\sup_{t\in[0,1]}|H_\varepsilon(t)|\right) + K\sqrt{\varepsilon}h(\varepsilon)\left(\frac{1}{\sqrt{\varepsilon}h(\varepsilon)}\sup_{t\in[0,1]}|H_\varepsilon(t)|\right)^2 \\ +K|\lambda^\varepsilon(t)| + \frac{K}{h(\varepsilon)}\left|\int_0^t\frac{\sigma(q_\varepsilon(s),r_\varepsilon(s))}{\alpha_\varepsilon(q_\varepsilon(s))}dw(s)\right| + e_2(\varepsilon),\end{aligned}$$

where

$$e_2(\varepsilon) := \frac{K|p|\sqrt{\varepsilon}}{h(\varepsilon)\ell_0} + \frac{K\varepsilon^2}{\sqrt{\varepsilon}h(\varepsilon)} + \frac{K(\varepsilon^2+\varepsilon)}{\sqrt{\varepsilon}h(\varepsilon)} + K\sup_{x\in\mathbb{R}^d}\frac{|\alpha_\varepsilon(x)-\alpha(x)|}{\sqrt{\varepsilon}h(\varepsilon)} \to 0, \qquad \text{as } \varepsilon\to 0. \tag{3.30}$$

First, the boundedness of functions $\sigma(\cdot,\cdot)$ and $\alpha_\varepsilon(\cdot)$ and Bernstein's inequality imply

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$$\lim_{j\to\infty}\limsup_{\varepsilon\to 0}\frac{1}{h^2(\varepsilon)}\log\mathbb{P}\left(\sup_{t\in[0,1]}\frac{K}{h(\varepsilon)}\left|\int_0^t\frac{\sigma(q_\varepsilon(s),r_\varepsilon(s))}{\alpha_\varepsilon(q_\varepsilon(s))}dw(s)\right|>j\right)=-\infty.$$

Second, Lemma III.2 suggests

$$\begin{aligned}&\lim_{j\to\infty}\limsup_{\varepsilon\to 0}\frac{1}{h^2(\varepsilon)}\log\mathbb{P}\left(K\sqrt{\varepsilon}h(\varepsilon)\left(\frac{1}{\sqrt{\varepsilon}h(\varepsilon)}\sup_{t\in[0,1]}|H_\varepsilon(t)|\right)^2>j\right)\\&=\lim_{j\to\infty}\limsup_{\varepsilon\to 0}\frac{1}{h^2(\varepsilon)}\log\mathbb{P}\left(\frac{1}{\sqrt{\varepsilon}h(\varepsilon)}\sup_{t\in[0,1]}|H_\varepsilon(t)|>\sqrt{j/(K\sqrt{\varepsilon}h(\varepsilon))}\right)=-\infty.\end{aligned}$$

Finally, using Corollary II.9 and Lemma III.2 yields

$$\lim_{j\to\infty}\limsup_{\varepsilon\to 0}\frac{1}{h^2(\varepsilon)}\log\mathbb{P}\left(\sup_{t\in[0,1]}|X_\varepsilon(t)|>j\right)=-\infty.$$

Thus, (2.2) is proved.

2. Proof of Eq. (2.3)

In what follows, we consider arbitrary but fixed $s\in[0,1]$, $\delta>0$, and let $0<s\le t\le s+\delta<1$. Note that

$$\begin{aligned}X_\varepsilon(t)-X_\varepsilon(s)&=\frac{1}{\sqrt{\varepsilon}h(\varepsilon)}\int_s^t\frac{b(q_\varepsilon(u),r_\varepsilon(u))}{\alpha_\varepsilon(q_\varepsilon(u))}-\frac{\bar{b}(q_0(u),v(u))}{\alpha(q_0(u))}du\\&\quad+\frac{1}{h(\varepsilon)}\int_s^t\frac{\sigma(q_\varepsilon(u),r_\varepsilon(u))}{\alpha_\varepsilon(q_\varepsilon(u))}dw(u)+\frac{1}{\sqrt{\varepsilon}h(\varepsilon)}(R_\varepsilon(t)-R_\varepsilon(s))\\&:=\mathrm{III}_1^\varepsilon(t,s)+\mathrm{III}_2^\varepsilon(t,s)+\mathrm{III}_3^\varepsilon(t,s).\end{aligned}\tag{3.31}$$

Effectively, we consider $X_\varepsilon(t)-X_\varepsilon(s)$ in a small interval shrinking to zero and divide the difference into three parts to estimate their difference. Similar to (3.14), it is sufficient to show that for any positive j,

$$\lim_{\delta\to 0}\limsup_{\varepsilon\to 0}\sup_{s\in[0,1]}\frac{1}{h^2(\varepsilon)}\log\mathbb{P}\left(\sup_{s\le t\le s+\delta}|\mathrm{III}_i^\varepsilon(t,s)|>j\right)=-\infty,\quad i=1,2,3.$$

In view of (3.4), the explicit form of the term $\mathrm{III}_3^\varepsilon(t)$ can be written as

$$\begin{aligned}\mathrm{III}_3^\varepsilon(t,s)=&\frac{1}{\sqrt{\varepsilon}h(\varepsilon)}\frac{p}{\varepsilon}\int_s^t e^{-A_\varepsilon(u)}du\\&-\frac{1}{\sqrt{\varepsilon}h(\varepsilon)}\left(\frac{1}{\alpha_\varepsilon(q_\varepsilon(t))}\int_0^t e^{-A_\varepsilon(t,u)}b(q_\varepsilon(u),r_\varepsilon(u))du-\frac{1}{\alpha_\varepsilon(q_\varepsilon(s))}\int_0^s e^{-A_\varepsilon(s,u)}b(q_\varepsilon(u),r_\varepsilon(u))du\right)\\&-\frac{1}{\sqrt{\varepsilon}h(\varepsilon)}\int_s^t\left(\int_0^u e^{-A_\varepsilon(u,v)}b(q_\varepsilon(v),r_\varepsilon(v))dv\right)\frac{1}{\alpha_\varepsilon^2(q_\varepsilon(u))}\langle\nabla\alpha_\varepsilon(q_\varepsilon(u)),\dot{q}_\varepsilon(u)\rangle du\\&-\frac{1}{\sqrt{\varepsilon}h(\varepsilon)}\left(\frac{1}{\alpha_\varepsilon(q_\varepsilon(t))}H_\varepsilon(t)-\frac{1}{\alpha_\varepsilon(q_\varepsilon(s))}H_\varepsilon(s)\right)\\&-\frac{1}{\sqrt{\varepsilon}h(\varepsilon)}\int_s^t\frac{1}{\alpha_\varepsilon^2(q_\varepsilon(u))}H_\varepsilon(u)\langle\nabla\alpha_\varepsilon(q_\varepsilon(u)),\dot{q}_\varepsilon(u)\rangle du\\:=&\sum_{k=1}^5(R_\varepsilon^k(t)-R_\varepsilon^k(s)).\end{aligned}\tag{3.32}$$

Observe that

$$\begin{aligned}|R_\varepsilon^1(t)-R_\varepsilon^1(s)|&=\left|\frac{1}{\sqrt{\varepsilon}h(\varepsilon)}\frac{p}{\varepsilon}\int_s^t e^{-A_\varepsilon(u)}du\right|=\frac{|p|}{\sqrt{\varepsilon}h(\varepsilon)\varepsilon}\left|\frac{\varepsilon^2}{\ell_0}e^{-\ell_0 s/\varepsilon^2}-\frac{\varepsilon^2}{\ell_0}e^{-\ell_0 t/\varepsilon^2}\right|\\&\le\frac{2|p|\sqrt{\varepsilon}}{\ell_0 h(\varepsilon)}=:e_3(\varepsilon)\to 0,\qquad\text{as }\varepsilon\to 0.\end{aligned}$$

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Step 1: For the term $R^2_\varepsilon(t) - R^2_\varepsilon(s)$, note that

$$\begin{aligned}\left|\frac{1}{\alpha_\varepsilon(q_\varepsilon(t))} - \frac{1}{\alpha_\varepsilon(q_\varepsilon(s))}\right| &\leq \frac{1}{\ell_0^2}|\alpha_\varepsilon(q_\varepsilon(t)) - \alpha_\varepsilon(q_\varepsilon(s))| = \frac{1}{\ell_0^2}|\nabla\alpha_\varepsilon(q_\varepsilon(s^+))p_\varepsilon(s^+)\|t-s| \\ &\leq K\varepsilon^2\left(1 + \frac{1}{\varepsilon} + \frac{1}{\varepsilon^2}\sup_{t\in[0,1]}|H_\varepsilon(t)|\right)|t-s| \\ &\leq K\varepsilon\delta + K\delta\sup_{t\in[0,1]}|H_\varepsilon(t)|,\end{aligned} \tag{3.33}$$

where $s^+ = \theta s + (1-\theta t)$ for some constant $\theta \in [0,1]$. It follows

$$\begin{aligned}|R^2_\varepsilon(t) - R^2_\varepsilon(s)| &\leq \frac{1}{\sqrt{\varepsilon}h(\varepsilon)}\left|\frac{1}{\alpha_\varepsilon(q_\varepsilon(t))} - \frac{1}{\alpha_\varepsilon(q_\varepsilon(s))}\right|\left|\int_0^t e^{-A_\varepsilon(t,u)}b(q_\varepsilon(u), r_\varepsilon(u))du\right| \\ &\quad + \frac{1}{\sqrt{\varepsilon}h(\varepsilon)\ell_0}\left|\int_0^s\left[e^{-A_\varepsilon(t,u)} - e^{-A_\varepsilon(s,u)}\right]b(q_\varepsilon(u), r_\varepsilon(u))du\right| \\ &\quad + \frac{1}{\sqrt{\varepsilon}h(\varepsilon)\ell_0}\left|\int_s^t e^{-A_\varepsilon(t,u)}b(q_\varepsilon(u), r_\varepsilon(u))du\right| \\ &\leq \frac{K\varepsilon\delta}{\sqrt{\varepsilon}h(\varepsilon)} + K\delta\left(\frac{1}{\sqrt{\varepsilon}h(\varepsilon)}\sup_{t\in[0,1]}|H_\varepsilon(t)|\right) \\ &\quad + \frac{K}{\sqrt{\varepsilon}h(\varepsilon)}\left|e^{-A_\varepsilon(t,s)} - 1\right|\int_0^s|e^{-A_\varepsilon(s,u)}|du \\ &\quad + \frac{K}{\sqrt{\varepsilon}h(\varepsilon)}\left|\int_s^t e^{-A_\varepsilon(t,u)}b(q_\varepsilon(u), r_\varepsilon(u))du\right| \\ &\leq \frac{K\sqrt{\varepsilon}\delta}{h(\varepsilon)} + K\delta\left(\frac{1}{\sqrt{\varepsilon}h(\varepsilon)}\sup_{t\in[0,1]}|H_\varepsilon(t)|\right) + \frac{K\varepsilon^2}{\sqrt{\varepsilon}h(\varepsilon)} \\ &= e_4(\varepsilon,\delta) + K\delta\left(\frac{1}{\sqrt{\varepsilon}h(\varepsilon)}\sup_{t\in[0,1]}|H_\varepsilon(t)|\right),\end{aligned}$$

where $e_4(\varepsilon,\delta) := K\sqrt{\varepsilon}\delta/h(\varepsilon) + K\varepsilon^2/(\sqrt{\varepsilon}h(\varepsilon)) \to 0$ as $\varepsilon \to 0$. Then, (3.12) yields

$$\begin{aligned}\mathbb{P}\left(\sup_{s\leq t\leq s+\delta}|R^2_\varepsilon(t) - R^2_\varepsilon(s)| > j\right) &\leq \mathbb{P}\left(\frac{1}{\sqrt{\varepsilon}h(\varepsilon)}\sup_{t\in[0,1]}|H_\varepsilon(t)| > \frac{j - e_3(\varepsilon,\delta)}{K\delta}\right) \\ &\leq 2\left(\exp\left(-\frac{(j - e_3(\varepsilon,\delta))^2h^2(\varepsilon)}{K^2\delta^2e_1^2(\varepsilon)}\right) \vee \exp\left(-\frac{1}{K\varepsilon}\right)\right).\end{aligned}$$

It follows

$$\lim_{\delta\to 0}\limsup_{\varepsilon\to 0}\sup_{s\in[0,1]}\frac{1}{h^2(\varepsilon)}\log\mathbb{P}\left(\sup_{s\leq t\leq s+\delta}|R^2_\varepsilon(t) - R^2_\varepsilon(s)| > j\right) = -\infty.$$

Step 2: For the term $R^3_\varepsilon(t) - R^3_\varepsilon(s)$, the boundedness of function $b(\cdot,\cdot)$ and $\alpha_\varepsilon(\cdot)$ implies

$$\begin{aligned}|R^3_\varepsilon(t) - R^3_\varepsilon(s)| &\leq \frac{K}{\sqrt{\varepsilon}h(\varepsilon)}\int_s^t\left(\int_0^u e^{-A_\varepsilon(u,v)}dv\right)|\langle\nabla\alpha_\varepsilon(q_\varepsilon(u)), \dot{q}_\varepsilon(u)\rangle|du \\ &\leq \frac{K\varepsilon^2}{\sqrt{\varepsilon}h(\varepsilon)}|t-s|\left(1 + \frac{1}{\varepsilon} + \frac{1}{\varepsilon^2}\sup_{t\in[0,1]}|H_\varepsilon(t)|\right) \\ &\leq \frac{K\sqrt{\varepsilon}\delta}{h(\varepsilon)} + \frac{K\delta}{\sqrt{\varepsilon}h(\varepsilon)}\sup_{t\in[0,1]}|H_\varepsilon(t)|.\end{aligned}$$

Then, (3.12) yields, for any $j > 0$,

$$\begin{aligned}\mathbb{P}\left(\sup_{s\leq t\leq s+\delta}|R^3_\varepsilon(t) - R^3_\varepsilon(s)| > j\right) &\leq \mathbb{P}\left(\frac{1}{\sqrt{\varepsilon}h(\varepsilon)}\sup_{t\in[0,1]}|H_\varepsilon(t)| > \frac{j - K\sqrt{\varepsilon}\delta/h(\varepsilon)}{K\delta}\right) \\ &\leq 2\left(\exp\left(-\frac{(j - K\delta\sqrt{\varepsilon}/h(\varepsilon))^2}{K^2\delta^2e_1^2(\varepsilon)}\right) \vee \exp\left(-\frac{1}{K\varepsilon}\right)\right).\end{aligned}$$

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It is readily to see that

$$\lim_{\delta\to 0}\limsup_{\varepsilon\to 0}\sup_{s\in[0,1]}\frac{1}{h^2(\varepsilon)}\log\mathbb{P}\left(\sup_{s\le t\le s+\delta}|R_\varepsilon^3(t)-R_\varepsilon^3(s)|>j\right)=-\infty.$$

Step 3: We then deal with the term $R_\varepsilon^4(t)-R_\varepsilon^4(s)$ in (3.32). Observe that

$$\begin{aligned}&\frac{1}{\alpha_\varepsilon(q_\varepsilon(t))}H_\varepsilon(t)-\frac{1}{\alpha_\varepsilon(q_\varepsilon(s))}H_\varepsilon(s)\\&\quad=\frac{1}{\alpha_\varepsilon(q_\varepsilon(t))}(H_\varepsilon(t)-H_\varepsilon(s))-\left[\frac{1}{\alpha_\varepsilon(q_\varepsilon(t))}-\frac{1}{\alpha_\varepsilon(q_\varepsilon(s))}\right]H_\varepsilon(s).\end{aligned}$$

By virtue of (3.12) and (3.33), one has, for any $j>0$,

$$\begin{aligned}&\mathbb{P}\left(\frac{1}{\sqrt{\varepsilon}h(\varepsilon)}\sup_{s\le t\le s+\delta}\left|\left[\frac{1}{\alpha_\varepsilon(q_\varepsilon(s))}-\frac{1}{\alpha_\varepsilon(q_\varepsilon(t))}\right]H_\varepsilon(s)\right|>\frac{j}{2}\right)\\&\le\mathbb{P}\left(\frac{1}{\sqrt{\varepsilon}h(\varepsilon)}\sup_{t\in[0,1]}|H_\varepsilon(t)|>\frac{j}{4K\varepsilon\delta}\right)+\mathbb{P}\left(\frac{1}{\sqrt{\varepsilon}h(\varepsilon)}\sup_{t\in[0,1]}|H_\varepsilon(t)|>\sqrt{\frac{j}{4K\delta\sqrt{\varepsilon}h(\varepsilon)}}\right)\\&\le\mathbb{P}\left(\frac{1}{\sqrt{\varepsilon}h(\varepsilon)}\sup_{t\in[0,1]}|H_\varepsilon(t)|>\frac{j}{4K\delta}\right)+\mathbb{P}\left(\frac{1}{\sqrt{\varepsilon}h(\varepsilon)}\sup_{t\in[0,1]}|H_\varepsilon(t)|>\sqrt{\frac{j}{4K\delta}}\right)\\&\le 3\left(\exp\left(-\frac{j^2h^2(\varepsilon)}{16K^2\delta^2e_1^2(\varepsilon)}\right)\vee\exp\left(-\frac{-jh^2(\varepsilon)}{4K\delta e_1^2(\varepsilon)}\right)\vee\exp\left(-\frac{1}{K\varepsilon}\right)\right).\end{aligned}\tag{3.34}$$

It follows

$$\lim_{\delta\to 0}\limsup_{\varepsilon\to 0}\sup_{s\in[0,1]}\frac{1}{h^2(\varepsilon)}\log\mathbb{P}\left(\frac{1}{\sqrt{\varepsilon}h(\varepsilon)}\sup_{s\le t\le s+\delta}\left|\left[\frac{1}{\alpha_\varepsilon(q_\varepsilon(s))}-\frac{1}{\alpha_\varepsilon(q_\varepsilon(t))}\right]H_\varepsilon(s)\right|>j\right)=-\infty.\tag{3.35}$$

Thus, Lemma III.3 and (3.35) give

$$\lim_{\delta\to 0}\limsup_{\varepsilon\to 0}\sup_{s\in[0,1]}\frac{1}{h^2(\varepsilon)}\log\mathbb{P}\left(\sup_{s\le t\le s+\delta}|R_\varepsilon^4(t)-R_\varepsilon^4(s)|>j\right)=-\infty.\tag{3.36}$$

Step 4: For the term $R_\varepsilon^5(t)-R_\varepsilon^5(s)$, we have

$$\begin{aligned}|R_\varepsilon^5(t)-R_\varepsilon^5(s)|&\le\frac{K\varepsilon^2|t-s|}{\sqrt{\varepsilon}h(\varepsilon)}\sup_{t\in[0,1]}|H_\varepsilon(t)|\left(1+\frac{1}{\varepsilon}+\frac{1}{\varepsilon^2}\sup_{t\in[0,1]}|H_\varepsilon(t)|\right)\\&\le\frac{K\varepsilon\delta}{\sqrt{\varepsilon}h(\varepsilon)}\sup_{t\in[0,1]}|H_\varepsilon(t)|+\frac{K\delta}{\sqrt{\varepsilon}h(\varepsilon)}\left(\sup_{t\in[0,1]}|H_\varepsilon(t)|\right)^2.\end{aligned}$$

Then, (3.34) implies

$$\lim_{\delta\to 0}\limsup_{\varepsilon\to 0}\sup_{s\in[0,1]}\frac{1}{h^2(\varepsilon)}\log\mathbb{P}\left(\sup_{s\le t\le s+\delta}|R_\varepsilon^5(t)-R_\varepsilon^5(s)|>j\right)=-\infty.$$

Finally, we deal with terms $\mathrm{III}_1^\varepsilon(t)$ and $\mathrm{III}_2^\varepsilon(t)$ in (3.31). For $\mathrm{III}_1^\varepsilon(t)$, owing to the boundedness of functions $b(\cdot,\cdot),\alpha_\varepsilon(\cdot),\overline{b}(\cdot,\cdot)$, and $\alpha(\cdot)$, it follows that

$$\lim_{\delta\to 0}\limsup_{\varepsilon\to 0}\sup_{s\in[0,1]}\frac{1}{h^2(\varepsilon)}\log\mathbb{P}\left(\sup_{s\le t\le s+\delta}|\mathrm{III}_1^\varepsilon(t)|>j\right)=-\infty.$$

Besides, Bernstein's inequality implies

$$\lim_{\delta\to 0}\limsup_{\varepsilon\to 0}\sup_{s\in[0,1]}\frac{1}{h^2(\varepsilon)}\log\mathbb{P}\left(\sup_{s\le t\le s+\delta}|\mathrm{III}_2^\varepsilon(t)|>j\right)=-\infty.$$

Consequently, combining all of the above steps, we obtain

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$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\sup\sup_{s\in[0,1]}\frac{1}{h^2(\varepsilon)}\log\mathbb{P}\bigg(\sup_{s\le t\le s+\delta}|X_\varepsilon(t)-X_\varepsilon(s)|>j\bigg)=-\infty.$$

This concludes the exponential tightness proof of $X_\varepsilon(t)$.

B. Local MDP

In this section, we consider the exponential equivalent property for $X_\varepsilon(t)$ and $\eta_\varepsilon(t)$ defined in (1.9) and (1.12) in a local sense. We more or less follow Ref. 17 (Proof of Proposition 4.4) with modifications to deal with moderate deviations.

Lemma III.4. Under the assumptions of Theorem II.11, for any $y\in C^0([0,1],\mathbb{R}^d)$ an absolutely continuous function and $\beta>0, N>0$, there exist $\beta_1>0,\beta_2>0$, and an $\widetilde{\varepsilon}$ such that for all $\varepsilon\le\widetilde{\varepsilon}$,

$$\mathbb{P}(\|X_\varepsilon(t)-y\|<\beta)\ge\mathbb{P}(\|\eta_\varepsilon(t)-y\|<\beta_1)-\exp(-Nh^2(\varepsilon)),\tag{3.37}$$

$$\mathbb{P}(\|\eta_\varepsilon(t)-y\|<\beta)\ge\mathbb{P}(\|X_\varepsilon(t)-y\|<\beta_2)-\exp(-Nh^2(\varepsilon)).\tag{3.38}$$

Proof. Letting $y(\cdot)\in C^0([0,1],\mathbb{R}^d)$ an absolutely continuous function and $\beta>0$ arbitrary but fixed, the definition of $X_\varepsilon(t)$ in (1.9) implies

$$\mathbb{P}(X_\varepsilon(t)\in B(y,\beta))=\mathbb{P}(q_\varepsilon(t)\in B(q_0(t)+\sqrt{\varepsilon}h(\varepsilon)y,\sqrt{\varepsilon}h(\varepsilon)\beta)).\tag{3.39}$$

Denoting $\varphi_t:=q_0(t)+\sqrt{\varepsilon}h(\varepsilon)y(t)$, then $\varphi(\cdot)\in C([0,1],\mathbb{R}^d)$ and is also absolutely continuous. Denote by $q_\varepsilon^\varphi(t)$ the solution of the following auxiliary equation:

$$\begin{cases}\varepsilon^2\ddot{q}_\varepsilon^\varphi(t)=b(\varphi_t,r_\varepsilon(t))-\alpha_\varepsilon(\varphi_t)\dot{q}_\varepsilon^\varphi(t)+\sqrt{\varepsilon}\sigma(\varphi_t,r_\varepsilon(t))\dot{w}(t),\\ q_\varepsilon^\varphi(0)=q\in\mathbb{R}^d,\quad \dot{q}_\varepsilon^\varphi(0)=\dfrac{p}{\varepsilon}\in\mathbb{R}^d,\end{cases}\tag{3.40}$$

and $g_\varepsilon^\varphi(t)$ the solution of

$$\dot{g}_\varepsilon^\varphi(t)=\frac{b(\varphi_t,r_\varepsilon(t))}{\alpha_\varepsilon(\varphi_t)}+\sqrt{\varepsilon}\frac{\sigma(\varphi_t,r_\varepsilon(t))}{\alpha_\varepsilon(\varphi_t)}\dot{w}(t),\quad g_\varepsilon^\varphi(0)=q\in\mathbb{R}^d.\tag{3.41}$$

Note that

$$|q_\varepsilon(t)-\varphi_t|\le|q_\varepsilon(t)-q_\varepsilon^\varphi(t)|+|q_\varepsilon^\varphi(t)-g_\varepsilon^\varphi(t)|+|g_\varepsilon^\varphi(t)-g_\varepsilon(t)|+|g_\varepsilon(t)-\varphi_t|.$$

Step 1: Estimate of $|q_\varepsilon(t)-q_\varepsilon^\varphi(t)|$. Similar to (3.1)–(3.4), it is readily seen that

$$q_\varepsilon^\varphi(t)=q+\frac{p}{\varepsilon}\int_0^t e^{-A_\varepsilon^\varphi(s)}ds+\frac{1}{\varepsilon^2}\int_0^t\int_0^s e^{-A_\varepsilon^\varphi(s,u)}b(\varphi_u,r_\varepsilon(u))duds+\frac{1}{\varepsilon^2}\int_0^t H_\varepsilon^\varphi(s)ds,\tag{3.42}$$

where

$$A_\varepsilon^\varphi(t,s):=\frac{1}{\varepsilon^2}\int_s^t\alpha_\varepsilon(\varphi_u)du,\quad A_\varepsilon^\varphi(t)=A_\varepsilon^\varphi(t,0),$$
$$H_\varepsilon^\varphi(t):=\sqrt{\varepsilon}\int_0^t e^{-A_\varepsilon^\varphi(t,s)}\sigma(\varphi_s,r_\varepsilon(s))dw(s).$$

Because of the boundedness of $\alpha_\varepsilon(\cdot)$, we have the following estimates that will be used frequently in the subsequent development:

$$A_\varepsilon^\varphi(t,s)\ge\frac{\ell_0}{\varepsilon^2}(t-s).\tag{3.43}$$

Employing the integration by parts yields

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$$q_\varepsilon^\varphi(t) = q + \int_0^t \frac{b(\varphi_s, r_\varepsilon(s))}{\alpha_\varepsilon(\varphi_s)} ds + \sqrt{\varepsilon}\int_0^t \frac{\sigma(\varphi_s, r_\varepsilon(s))}{\alpha_\varepsilon(\varphi_s)} dw(s) + R_\varepsilon^\varphi(t), \tag{3.44}$$

where

$$\begin{aligned} R_\varepsilon^\varphi(t) &:= \frac{p}{\varepsilon}\int_0^t e^{-A_\varepsilon^\varphi(s)} ds - \frac{1}{\alpha_\varepsilon(\varphi_t)}\int_0^t e^{-A_\varepsilon^\varphi(t,s)} b(\varphi_s, r_\varepsilon(s)) ds \\ &\quad - \int_0^t \left(\int_0^s e^{-A_\varepsilon^\varphi(s,u)} b(\varphi_u, r_\varepsilon(u)) du\right) \frac{1}{\alpha_\varepsilon^2(\varphi_s)} \langle \nabla \alpha_\varepsilon(\varphi_s), \dot{\varphi}_s\rangle ds \\ &\quad - \frac{1}{\alpha_\varepsilon(\varphi_t)} H_\varepsilon^\varphi(t) - \int_0^t \frac{1}{\alpha_\varepsilon^2(\varphi_s)} H_\varepsilon^\varphi(s)\langle \nabla\alpha_\varepsilon(\varphi_s), \dot{\varphi}_s\rangle ds \\ &:= \sum_{k=1}^5 R_\varepsilon^{k,\varphi}(t). \end{aligned}$$

Taking integration by parts only to stochastic integral in (3.42) yields

$$\begin{aligned} |q_\varepsilon(t) - q_\varepsilon^\varphi(t)| &\le \left|\frac{1}{\varepsilon^2}\int_0^t\int_0^s \left(e^{-A_\varepsilon(s,u)} b(q_\varepsilon(u), r_\varepsilon(u)) - e^{-A_\varepsilon^\varphi(s,u)} b(\varphi_u, r_\varepsilon(u))\right) du ds\right| \\ &\quad + \left|\sqrt{\varepsilon}\int_0^t \left(\frac{\sigma(q_\varepsilon(s), r_\varepsilon(s))}{\alpha_\varepsilon(q_\varepsilon(s))} - \frac{\sigma(\varphi_s, r_\varepsilon(s))}{\alpha_\varepsilon(\varphi_s)}\right) dw(s)\right| \\ &\quad + |R_\varepsilon^1(t) - R_\varepsilon^{1,\varphi}(t)| + |R_\varepsilon^4(t) - R_\varepsilon^{4,\varphi}(t)| + |R_\varepsilon^5(t) - R_\varepsilon^{5,\varphi}(t)|. \end{aligned} \tag{3.45}$$

Denoting

$$M_1^\varepsilon(t) := \sqrt{\varepsilon}\int_0^t \left(\frac{\sigma(q_\varepsilon(s), r_\varepsilon(s))}{\alpha_\varepsilon(q_\varepsilon(s))} - \frac{\sigma(\varphi_s, r_\varepsilon(s))}{\alpha_\varepsilon(\varphi_s)}\right) dw(s), \tag{3.46}$$

$M_1^\varepsilon(t)$ is a local martingale. In light of

$$\begin{aligned} &e^{-A_\varepsilon(s,u)} b(q_\varepsilon(u), r_\varepsilon(u)) - e^{-A_\varepsilon^\varphi(s,u)} b(\varphi_u, r_\varepsilon(u)) \\ &= (e^{-A_\varepsilon(s,u)} - e^{-A_\varepsilon^\varphi(s,u)}) b(q_\varepsilon(u), r_\varepsilon(u)) + e^{-A_\varepsilon^\varphi(s,u)}[b(q_\varepsilon(u), r_\varepsilon(u)) - b(\varphi_u, r_\varepsilon(u))], \end{aligned}$$

the mean value theorem and estimates (3.43) imply

$$\begin{aligned} |e^{-A_\varepsilon(s,u)} - e^{-A_\varepsilon^\varphi(s,u)}| &= e^{-\left(\theta A_\varepsilon(s,u) + (1-\theta)A_\varepsilon^\varphi(s,u)\right)} |A_\varepsilon(s,u) - A_\varepsilon^\varphi(s,u)| \\ &\le e^{-\ell_0(s-u)/\varepsilon^2}\int_u^s K|q_\varepsilon(v) - \varphi_v| dv \\ &\le K e^{-\ell_0(s-u)/\varepsilon^2}(s-u)\sup_{u\le v\le s}|q_\varepsilon(v) - \varphi_v|, \end{aligned} \tag{3.47}$$

where $\theta \in [0,1]$. Then, using the boundedness and the Lipschitz property of $b(\cdot,\cdot)$, we have

$$\begin{aligned} &\left|\frac{1}{\varepsilon^2}\int_0^t\int_0^s \left(e^{-A_\varepsilon(s,u)} b(q_\varepsilon(u), r_\varepsilon(u)) - e^{-A_\varepsilon^\varphi(s,u)} b(\varphi_u, r_\varepsilon(u))\right) du ds\right| \\ &\le \frac{1}{\varepsilon^2}\int_0^t\int_0^s K e^{-\ell_0(s-u)/\varepsilon^2}(s-u)\sup_{u\le v\le s}|q_\varepsilon(v) - \varphi_u| du ds \\ &\quad + \int_0^t\int_0^s K e^{-\ell_0(s-u)/\varepsilon^2}|q_\varepsilon(u) - \varphi_u| du ds \\ &\le K\varepsilon^2\int_0^t \sup_{0\le u\le s}|q_\varepsilon(u) - \varphi_u| ds. \end{aligned}$$

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For the term $R_\varepsilon^1(t) - R_\varepsilon^{1,\varphi}(t)$, estimate (3.47) and Young's inequality for convolution suggest

$$\begin{aligned}|R_\varepsilon^1(t) - R_\varepsilon^{1,\varphi}(t)| &\le \frac{|p|}{\varepsilon}\int_0^t Ke^{-\ell_0 s/\varepsilon^2}\int_0^s |q_\varepsilon(u) - \varphi_u| duds \\ &\le K\frac{|p|}{\varepsilon}\int_0^t\int_0^s e^{-\ell_0(s-u)/\varepsilon^2}|q_\varepsilon(u) - \varphi_u|duds \\ &\le K\frac{|p|}{\varepsilon}\int_0^t e^{-\ell_0 s/\varepsilon^2}ds\int_0^t |q_\varepsilon(s) - \varphi_s|ds \\ &\le K\varepsilon\int_0^t \sup_{0\le u\le s}|q_\varepsilon(u) - \varphi_u|ds.\end{aligned} \tag{3.48}$$

For the term $R_\varepsilon^4(t) - R_\varepsilon^{4,\varphi}(t)$, we have

$$|R_\varepsilon^4(t) - R_\varepsilon^{4,\varphi}(t)| = \left|\frac{1}{\alpha_\varepsilon(q_\varepsilon(s))}H_\varepsilon(t) - \frac{1}{\alpha_\varepsilon(\varphi_t)}H_\varepsilon^\varphi(t)\right| \le K(\|H_\varepsilon(t)\| + \|H_\varepsilon^\varphi(t)\|). \tag{3.49}$$

For the term $R_\varepsilon^5(t) - R_\varepsilon^{5,\varphi}(t)$, it gives

$$\begin{aligned}|R_\varepsilon^5(t) - R_\varepsilon^{5,\varphi}(t)| &= \left|\int_0^t \frac{1}{\alpha_\varepsilon^2(q_\varepsilon(s))}H_\varepsilon(s)\langle\nabla\alpha_\varepsilon(q_\varepsilon(s)),\dot q_\varepsilon(s)\rangle ds - \int_0^t\frac{1}{\alpha_\varepsilon^2(\varphi_s)}H_\varepsilon^\varphi(s)\langle\nabla\alpha_\varepsilon(\varphi_s),\dot\varphi_s\rangle ds\right| \\ &\le \left|\int_0^t\left[\frac{1}{\alpha_\varepsilon^2(q_\varepsilon(s))} - \frac{1}{\alpha_\varepsilon^2(\varphi_s)}\right]H_\varepsilon(s)\langle\nabla\alpha_\varepsilon(q_\varepsilon(s)),\dot q_\varepsilon(s)\rangle ds\right| \\ &\quad + \left|\int_0^t\frac{1}{\alpha_\varepsilon^2(\varphi_s)}[H_\varepsilon(s)\langle\nabla\alpha_\varepsilon(q_\varepsilon(s)),\dot q_\varepsilon(s)\rangle - H_\varepsilon^\varphi(s)\langle\nabla\alpha_\varepsilon(\varphi_s),\dot\varphi_s\rangle]ds\right| \\ &:= B_1 + B_2.\end{aligned} \tag{3.50}$$

The mean value theorem and estimate (3.7) then yield

$$B_1 \le K\varepsilon^2\|H_\varepsilon(t)\|(\varepsilon^2 + \varepsilon + \|H_\varepsilon(t)\|)\int_0^t \sup_{0\le u\le s}|q_\varepsilon(u) - \varphi_u|ds.$$

Due to assumption (A3), we obtain

$$\begin{aligned}B_2 &\le K\left|\int_0^t[H_\varepsilon(s) - H_\varepsilon^\varphi(s)]\langle\nabla\alpha_\varepsilon(q_\varepsilon(s)),\dot q_\varepsilon(s)\rangle ds\right| \\ &\quad + K\left|\int_0^t H_\varepsilon^\varphi(s)[\langle\nabla\alpha_\varepsilon(q_\varepsilon(s)),\dot q_\varepsilon(s)\rangle - \langle\nabla\alpha_\varepsilon(\varphi_s),\dot\varphi_s\rangle]ds\right| \\ &\le K\varepsilon^2\int_0^t|H_\varepsilon(s) - H_\varepsilon^\varphi(s)||p_\varepsilon(s)|ds + K\varepsilon^2\int_0^t|H_\varepsilon^\varphi(s)|(|p_\varepsilon(s)| + |\dot\varphi_s|)ds \\ &\le K(\varepsilon^2 + \varepsilon + \|H_\varepsilon(t)\|)\int_0^t|H_\varepsilon(s) - H_\varepsilon^\varphi(s)|ds \\ &\quad + K\left(\varepsilon^2 + \varepsilon + \|H_\varepsilon(t)\| + \varepsilon^2\int_0^t|\dot\varphi(s)|ds\right)\|H_\varepsilon^\varphi(t)\| \\ &\le K(\varepsilon^2 + \varepsilon + \|H_\varepsilon(t)\|)(\|H_\varepsilon(t)\| + \|H_\varepsilon^\varphi(t)\|) + K\varepsilon^2\|H_\varepsilon^\varphi(t)\|\int_0^t|\dot\varphi_s|ds.\end{aligned}$$

Combining all of the above estimates gives

$$\begin{aligned}|q_\varepsilon(t) - q_\varepsilon^\varphi(t)| &\le K(\varepsilon^2 + \varepsilon)(\|H_\varepsilon(t)\|^2 + \varepsilon^2\|H_\varepsilon(t)\| + 1)\int_0^t\sup_{0\le u\le s}|q_\varepsilon(u) - \varphi_u|ds + K|M_1^\varepsilon(t)| \\ &\quad + K(1 + \varepsilon^2 + \varepsilon + \|H_\varepsilon(t)\|)(\|H_\varepsilon(t)\| + \|H_\varepsilon^\varphi(t)\|) + K\varepsilon^2\|H_\varepsilon^\varphi(t)\|\int_0^t|\dot\varphi_s|ds.\end{aligned} \tag{3.51}$$

Step 2: Estimate of $|q_\varepsilon^\varphi(t) - g_\varepsilon^\varphi(t)|$. From (3.40) and (3.41), we have

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$$\varepsilon^2 \ddot{q}_\varepsilon^\varphi(t) = -\alpha_\varepsilon(\varphi_t)(\dot{q}_\varepsilon^\varphi(t) - \dot{g}_\varepsilon^\varphi(t)).$$

An integration gives

$$|q_\varepsilon^\varphi(t) - g_\varepsilon^\varphi(t)| = \left|\frac{\varepsilon^2}{\alpha_\varepsilon(\varphi_t)}(\dot{q}_\varepsilon^\varphi(t) - \dot{q}_\varepsilon^\varphi(0))\right| \le \frac{\varepsilon^2}{\ell_0}\left|p_\varepsilon^\varphi(t) - \frac{p}{\varepsilon}\right|,$$

where $p_\varepsilon^\varphi(t) := \dot{q}_\varepsilon^\varphi(t)$. Similar to the argument with Proposition III.1, it is readily to have

$$|q_\varepsilon^\varphi(t)| \le K\left(1 + \frac{1}{\varepsilon^2}\sup_{t\in[0,1]}|H_\varepsilon^\varphi(t)|\right),$$
$$|p_\varepsilon^\varphi(t)| \le K\left(1 + \frac{1}{\varepsilon} + \frac{1}{\varepsilon^2}\sup_{t\in[0,1]}|H_\varepsilon^\varphi(t)|\right).$$

Thus, estimate (3.7) yields

$$|q_\varepsilon^\varphi(t) - g_\varepsilon^\varphi(t)| \le K(\varepsilon^2 + \varepsilon + \|H_\varepsilon^\varphi(t)\|) + |p|\varepsilon/\ell_0 \le K(\varepsilon + \|H_\varepsilon^\varphi(t)\|). \tag{3.52}$$

Step 3: Estimate of $|g_\varepsilon^\varphi(t) - g_\varepsilon(t)|$. Note that

$$\begin{cases} \dot{g}_\varepsilon^\varphi(t) - \dot{g}_\varepsilon(t) = \left[\frac{b(\varphi_t, r_\varepsilon(t))}{\alpha_\varepsilon(\varphi_t)} - \frac{b(g_\varepsilon(t), r_\varepsilon(t))}{\alpha_\varepsilon(g_\varepsilon(t))}\right] + \sqrt{\varepsilon}\left[\frac{\sigma(\varphi_t, r_\varepsilon(t))}{\alpha_\varepsilon(\varphi_t)} - \frac{\sigma(g_\varepsilon(t), r_\varepsilon(t))}{\alpha_\varepsilon(g_\varepsilon(t))}\right]\dot{w}(t), \\ g_\varepsilon^\varphi(0) = g_\varepsilon(0) = q \in \mathbb{R}^d. \end{cases} \tag{3.53}$$

Taking integration and using the Lipschitz property of function b/α_ε, we have

$$|g_\varepsilon^\varphi(t) - g_\varepsilon(t)| \le K\|g_\varepsilon(t) - \varphi_t\| + |M_2^\varepsilon(t)|, \tag{3.54}$$

where $M_2^\varepsilon(t)$ is a local martingale defined as

$$M_2^\varepsilon(t) := \sqrt{\varepsilon}\int_0^t \frac{\sigma(\varphi_s, r_\varepsilon(s))}{\alpha_\varepsilon(\varphi_s)} - \frac{\sigma(g_\varepsilon(s), r_\varepsilon(s))}{\alpha_\varepsilon(g_\varepsilon(s))}dw(s).$$

Therefore, combining estimates (3.51), (3.52), and (3.54), and applying Grönwall's inequality, we obtain

$$\begin{aligned} \|q_\varepsilon(t) - \varphi_t\| \le{}& Ke^{Gt}\|g_\varepsilon(t) - \varphi_t\| + Ke^{Gt}(\|M_1^\varepsilon(t)\| + \|M_2^\varepsilon(t)\|) \\ &+ Ke^{Gt}\left[(1 + \varepsilon^2 + \varepsilon + \|H_\varepsilon(t)\|)(\|H_\varepsilon(t)\| + \|H_\varepsilon^\varphi(t)\|) + \varepsilon^2\|H_\varepsilon^\varphi(t)\|\int_0^t|\dot{\varphi}_s|ds\right] \\ &+ Ke^{Gt}(\varepsilon + \|H_\varepsilon^\varphi(t)\|), \end{aligned}$$

where

$$G := K(\varepsilon^2 + \varepsilon)(\|H_\varepsilon(t)\| + \varepsilon^2\|H_\varepsilon(t)\| + 1).$$

Step 4: For arbitrarily fixed $\beta, N > 0$, estimate (3.12) gives

$$\begin{aligned} \mathbb{P}(\|H_\varepsilon(t)\| > L_1) &= \mathbb{P}\left(\frac{1}{\sqrt{\varepsilon}h(\varepsilon)}\sup_{t\in[0,1]}|H_\varepsilon(t)| > \frac{L_1}{\sqrt{\varepsilon}h(\varepsilon)}\right) \\ &\le \exp\left(-\frac{L_1^2}{\varepsilon e_1^2(\varepsilon)}\right) + \exp\left(-\frac{1}{K\varepsilon}\right). \end{aligned}$$

Then, there exists a constant $L_1 = L_1(N)$ and $\varepsilon_1 = \varepsilon_1(K, N)$ such that the right hand side of the above inequality is less than $2\exp(-2Nh^2(\varepsilon))$. Denote $\Omega_{1,\varepsilon} = \{\omega : \|H_\varepsilon(t)\| > L_1\}$. Thus, under the event $\Omega_{1,\varepsilon}^c$,

$$G \le K(\varepsilon^2 + \varepsilon)(L^2 + \varepsilon^2 L + 1).$$

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Letting

$$\beta_1 = \frac{\beta}{2Ke^{K(\varepsilon^2+\varepsilon)(L^2+\varepsilon^2 L+1)}}, \tag{3.55}$$

the event $\{\|g_\varepsilon(t) - \varphi_t\| \le \beta_1 | \Omega_{1,\varepsilon}^c\}$ implies the event $\{Ke^{Gt}\|g_\varepsilon(t) - \varphi_t\| \le \beta/2\}$. That is,

$$\{(\|g_\varepsilon(t) - \varphi_t\| \le \beta_1)|\Omega_{1,\varepsilon}^c\} \subset \{Ke^{Gt}\|g_\varepsilon(t) - \varphi_t\| \le \beta/2\}.$$

In addition, due to the boundedness of the function $\sigma(\cdot,\cdot), \alpha_\varepsilon(\cdot)$, there is a constant $\widetilde{K} > 0$ such that the quadratic process

$$[M_i^\varepsilon]_1 \le \widetilde{K}\varepsilon, \quad i = 1, 2.$$

Then, Bernstein's inequality implies

$$\mathbb{P}(\|M_1^\varepsilon(t)\| + \|M_2^\varepsilon(t)\| > L_2) \le 2\exp(-L_2^2/4\widetilde{K}\varepsilon).$$

Consequently, there exists a positive constant $L_2 = L_2(N, \widetilde{K})$ such that the right hand side of the above inequality is less than $2\exp(-2Nh^2(\varepsilon))$. Denote

$$\Omega_{2,\varepsilon} = \{\omega : \|M_1^\varepsilon(t)\| + \|M_2^\varepsilon(t)\| > L_2\}.$$

Under $\Omega_{2,\varepsilon}^c \cap \Omega_{1,\varepsilon}^c$, we note that there exists an $\varepsilon_2 = \varepsilon_2(\widetilde{K}, L_1, K, L_2)$ such that for all $\varepsilon \le \varepsilon_2$,

$$Ke^{Gt}(\|M_1^\varepsilon(t)\| + \|M_2^\varepsilon(t)\|) \le Ke^{Gt}L_2 \le \beta/8.$$

Since the absolutely continuous function on a compact interval is differentiable almost everywhere, the integration by parts formula in (3.11) holds also for $H_\varepsilon^\varphi(t)$. Similar to Lemma III.2, we are able to obtain

$$\mathbb{P}(\|H_\varepsilon^\varphi(t)\| > L_3) \le \exp\left(-\frac{L_3^2}{\varepsilon e_1^2(\varepsilon)}\right) + \exp\left(-\frac{1}{K\varepsilon}\right). \tag{3.56}$$

Thus, there exists a constant $L_3 = L_3(K, N)$ and $\varepsilon_3 = \varepsilon_3(K, N)$ such that the right-hand side of the above inequality is less than $2\exp(-2Nh^2(\varepsilon))$ for all $\varepsilon \le \varepsilon_3$. Denote

$$\Omega_{3,\varepsilon} = \{\omega : \|H_\varepsilon^\varphi(t)\| > L_3\}.$$

Hence, under $\Omega_{1,\varepsilon}^c \cap \Omega_{2,\varepsilon}^c \cap \Omega_{3,\varepsilon}^c$, there exists an $\varepsilon_4 = \varepsilon_4(K, L_1, L_3)$ such that for all $\varepsilon \le \varepsilon_4$,

$$\begin{aligned}
&Ke^{Gt}[(\varepsilon^2 + \varepsilon + \|H_\varepsilon(t)\|)(\|H_\varepsilon(t)\| + \|H_\varepsilon^\varphi(t)\|)] \le \frac{\beta}{8},\\
&Ke^{Gt}\varepsilon^2\|H_\varepsilon^\varphi(t)\| \int_0^t |\dot{\varphi}_s| ds \le \frac{\beta}{8},\\
&Ke^{Gt}(\varepsilon + \|H_\varepsilon^\varphi(t)\|) \le \frac{\beta}{8}.
\end{aligned}$$

Consequently, for all $\varepsilon \le \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}$,

$$\{(\|g_\varepsilon(t) - \varphi_t\| < \beta_1)|(\cup_{i=1}^3 \Omega_{i,\varepsilon})^c\} \subset \{\|q_\varepsilon(t) - \varphi_t\| < \beta\}$$

implies

$$\begin{aligned}
&\mathbb{P}(\|g_\varepsilon(t) - \varphi_t\| < \beta_1) - 6\exp(-2Nh^2(\varepsilon))\\
&\le \mathbb{P}(\|g_\varepsilon(t) - \varphi_t\| < \beta_1) - \mathbb{P}(\cup_{i=1}^3 \Omega_{i,\varepsilon})\\
&\le \mathbb{P}\big(\|g_\varepsilon(t) - \varphi_t\| < \beta_1\,) \cap (\cup_{i=1}^3 \Omega_{i,\varepsilon})^c\big) \le \mathbb{P}(\|q_\varepsilon(t) - \varphi_t\| < \beta).
\end{aligned}$$

Finally, choose $\varepsilon_5 = \varepsilon_5(N)$ such that $6\exp(-2Nh^2(\varepsilon)) \le \exp(-Nh^2(\varepsilon))$ for all $\varepsilon \le \varepsilon_5$. Noting (3.39) and (3.55), we have for all $\varepsilon \le \widetilde{\varepsilon} := \min_{1\le i\le 5} \varepsilon_i$

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$$\mathbb{P}(\|X_\varepsilon(t)-y\|<\beta)\ge\mathbb{P}(\|\eta_\varepsilon(t)-y\|<\beta_1)-\exp(-Nh^2(\varepsilon)).$$

Likewise, observing

$$|g_\varepsilon(t)-\varphi_t|\le|g_\varepsilon(t)-g_\varepsilon^\varphi(t)|+|g_\varepsilon^\varphi(t)-q_\varepsilon^\varphi(t)|+|q_\varepsilon^\varphi(t)-q_\varepsilon(t)|+|q_\varepsilon(t)-\varphi_t|,$$

we obtain

$$\mathbb{P}(\|\eta_\varepsilon(t)-y\|<\beta)\ge\mathbb{P}(\|X_\varepsilon(t)-y\|<\beta_2)-\exp(-Nh^2(\varepsilon)).$$

Thus, the proof is complete. □

Theorem III.5. *Suppose that $\eta_\varepsilon(t)$ satisfies a local LDP with speed $h^{-2}(\varepsilon)$ and a rate function $S(\cdot)$ with $S(y)=\infty$ if y is not absolutely continuous. Then, under the assumptions in Theorem II.11, the sequence $X_\varepsilon(t)$ satisfies a local LDP with the same speed $h^{-2}(\varepsilon)$ and rate function $S(\cdot)$. That is, for any $y\in C^0([0,1],\mathbb{R}^d)$,*

$$\begin{aligned}&\lim_{\beta\to0}\limsup_{\varepsilon\to0}\frac{1}{h^2(\varepsilon)}\log\mathbb{P}(X_\varepsilon(t)\in B(y,\beta))\\&=\lim_{\beta\to0}\liminf_{\varepsilon\to0}\frac{1}{h^2(\varepsilon)}\log\mathbb{P}(X_\varepsilon(t)\in B(y,\beta))\\&=-S(y).\end{aligned}$$

Proof. By virtue of Lemma III.4, the proof is similar to Ref. 17 (Theorem 4.1). Thus, the detail is omitted here. □

Proof (Proof of Theorem II.11). From Theorem III.5, proving that $X_\varepsilon(t)$ satisfies an LDP with speed $h^{-2}(\varepsilon)$ and a good rate function $S(\cdot)$ only requires to show that $\eta_\varepsilon(t)$ satisfies a local LDP with speed $h^{-2}(\varepsilon)$ and a rate function $S(\cdot)$. In order to achieve the goal, we first establish the exponential equivalence between $\eta_\varepsilon(t)$ and $\hat{\eta}_\varepsilon(t)$ with respect to MDPs, and then, we claim that $\hat{\eta}_\varepsilon(t)$ satisfies the LDPs, which leads to the local LDPs with speed $h^{-2}(\varepsilon)$ and the rate function $S(\cdot)$ defined in (2.14).

Now, we proceed to prove that for any $j>0$,

$$\lim_{\varepsilon\to0}\frac{1}{h^2(\varepsilon)}\log\mathbb{P}\left(\sup_{t\in[0,1]}|\eta_\varepsilon(t)-\hat{\eta}_\varepsilon(t)|>j\right)=-\infty.$$

By the boundedness of $\alpha_\varepsilon(\cdot),\alpha(\cdot)$, and $b(\cdot,\cdot)$, and the Lipschitz property, we have

$$\begin{aligned}|\eta_\varepsilon(t)-\hat{\eta}_\varepsilon(t)|&=\frac{1}{\sqrt{\varepsilon}h(\varepsilon)}|g_\varepsilon(t)-f_\varepsilon(t)|\\&=\frac{1}{\sqrt{\varepsilon}h(\varepsilon)}\int_0^t\left|\frac{b(g_\varepsilon(s),r_\varepsilon(s))}{\alpha_\varepsilon(g_\varepsilon(s))}-\frac{b(f_\varepsilon(s),r_\varepsilon(s))}{\alpha(f_\varepsilon(s))}\right|ds\\&\quad+\left|\frac{1}{h(\varepsilon)}\int_0^t\frac{\sigma(g_\varepsilon(s),r_\varepsilon(s))}{\alpha_\varepsilon(g_\varepsilon(s))}-\frac{\sigma(f_\varepsilon(s),r_\varepsilon(s))}{\alpha(f_\varepsilon(s))}dw(s)\right|\\&\le K\int_0^t|\eta_\varepsilon(s)-\hat{\eta}_\varepsilon(s)|ds+\frac{K}{\sqrt{\varepsilon}h(\varepsilon)}\int_0^t|\alpha_\varepsilon(f_\varepsilon(s))-\alpha(f_\varepsilon(s))|ds\\&\quad+\left|\frac{1}{h(\varepsilon)}\int_0^t\frac{\sigma(g_\varepsilon(s),r_\varepsilon(s))}{\alpha_\varepsilon(g_\varepsilon(s))}-\frac{\sigma(f_\varepsilon(s),r_\varepsilon(s))}{\alpha(f_\varepsilon(s))}dw(s)\right|.\end{aligned}$$

The Grönwall inequality yields that

$$|\eta_\varepsilon(t)-\hat{\eta}_\varepsilon(t)|\le K\sup_{x\in\mathbb{R}^d}\frac{|\alpha_\varepsilon(x)-\alpha(x)|}{\sqrt{\varepsilon}h(\varepsilon)}+K|\tilde{M}_\varepsilon(t)|,$$

where

$$\tilde{M}_\varepsilon(t)=\frac{1}{h(\varepsilon)}\int_0^t\frac{\sigma(g_\varepsilon(s),r_\varepsilon(s))}{\alpha_\varepsilon(g_\varepsilon(s))}-\frac{\sigma(f_\varepsilon(s),r_\varepsilon(s))}{\alpha(f_\varepsilon(s))}dw(s).$$

Noting assumption (A3), it is sufficient to prove that

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$$\lim_{\varepsilon\to 0}\frac{1}{h^2(\varepsilon)}\log \mathbb{P}\Bigg(\sup_{t\in[0,1]}|\tilde{M}_\varepsilon(t)|>j\Bigg)=-\infty.$$

For any fixed arbitrary positive $\ell > 0$, observe

$$\mathbb{P}\Bigg(\sup_{t\in[0,1]}|\tilde{M}_\varepsilon(t)|>j\Bigg)\le \mathbb{P}\Bigg(\sup_{t\in[0,1]}|\tilde{M}_\varepsilon(t)|>j;[\tilde{M}_\varepsilon]_1\le \ell\varepsilon\Bigg)+\mathbb{P}\big([\tilde{M}_\varepsilon]_1>\ell\varepsilon\big).$$

Bernstein's inequality implies

$$\frac{1}{h^2(\varepsilon)}\log \mathbb{P}\Bigg(\sup_{t\in[0,1]}|\tilde{M}_\varepsilon(t)|>j;[\tilde{M}_\varepsilon]_1\le \ell\varepsilon\Bigg)\le -\frac{j^2}{2\ell\varepsilon h^2(\varepsilon)}.$$

Since we have $\varepsilon h^2(\varepsilon)\to 0$ as $\varepsilon\to 0$, it follows that for any positive j and ℓ,

$$\lim_{\varepsilon\to 0}\frac{1}{h^2(\varepsilon)}\log \mathbb{P}\Bigg(\sup_{t\in[0,1]}|\tilde{M}_\varepsilon(t)|>j;[\tilde{M}_\varepsilon]_1\le \ell\varepsilon\Bigg)=-\infty.$$

To conclude, we need only establish

$$\lim_{\ell\to\infty}\lim_{\varepsilon\to 0}\frac{1}{h^2(\varepsilon)}\log \mathbb{P}\bigg(\frac{1}{\varepsilon}[\tilde{M}_\varepsilon]_1>\ell\bigg)=-\infty.$$

Since

$$\begin{aligned}\frac{1}{\varepsilon}[\tilde{M}_\varepsilon]_1&=\frac{1}{\varepsilon h^2(\varepsilon)}\int_0^1\Bigg(\frac{\sigma(g_\varepsilon(s),r_\varepsilon(s))}{\alpha_\varepsilon(g_\varepsilon(s))}-\frac{\sigma(f_\varepsilon(s),r_\varepsilon(s))}{\alpha(f_\varepsilon(s))}\Bigg)^2ds\\&\le K\int_0^1|\eta_\varepsilon(t)-\hat{\eta}_\varepsilon(t)|^2dt+K\Bigg(\sup_{x\in\mathbb{R}^d}\frac{|\alpha_\varepsilon(x)-\alpha(x)|}{\sqrt{\varepsilon}h(\varepsilon)}\Bigg)^2\\&\le K\Bigg(\sup_{t\in[0,1]}|\eta_\varepsilon(t)|\Bigg)^2+K\Bigg(\sup_{t\in[0,1]}|\hat{\eta}_\varepsilon(t)|\Bigg)^2+K\Bigg(\sup_{x\in\mathbb{R}^d}\frac{|\alpha_\varepsilon(x)-\alpha(x)|}{\sqrt{\varepsilon}h(\varepsilon)}\Bigg)^2,\end{aligned}$$

the exponential equivalence holds if

$$\lim_{\ell\to 0}\lim_{\varepsilon\to 0}\frac{1}{h^2(\varepsilon)}\log \mathbb{P}\Bigg(\sup_{t\in[0,1]}|\eta_\varepsilon(t)|>\ell\Bigg)=-\infty, \tag{3.57}$$

$$\lim_{\ell\to 0}\lim_{\varepsilon\to 0}\frac{1}{h^2(\varepsilon)}\log \mathbb{P}\Bigg(\sup_{t\in[0,1]}|\hat{\eta}_\varepsilon(t)|>\ell\Bigg)=-\infty. \tag{3.58}$$

Similar to (3.25),

$$\begin{aligned}|\eta_\varepsilon(t)|&=\frac{1}{\sqrt{\varepsilon}h(\varepsilon)}|g_\varepsilon(t)-q_0(t)|\\&=\frac{1}{\sqrt{\varepsilon}h(\varepsilon)}\int_0^t\Bigg|\frac{b(g_\varepsilon(s),r_\varepsilon(s))}{\alpha_\varepsilon(g_\varepsilon(s))}-\frac{\bar{b}(q_0(s),v(s))}{\alpha(q_0(s))}\Bigg|ds+\Bigg|\frac{1}{h(\varepsilon)}\int_0^t\frac{\sigma(g_\varepsilon(s),r_\varepsilon(s))}{\alpha_\varepsilon(g_\varepsilon(s))}dw(s)\Bigg|\\&\le K\int_0^t|\eta_\varepsilon(s)|ds+\frac{K}{\sqrt{\varepsilon}h(\varepsilon)}\Bigg|\int_0^t b(q_0(s),r_\varepsilon(s))-\bar{b}(q_0(s),v(s))ds\Bigg|\\&\quad+K\sup_{x\in\mathbb{R}^d}\frac{|\alpha_\varepsilon(x)-\alpha(x)|}{\sqrt{\varepsilon}h(\varepsilon)}+\Bigg|\frac{1}{h(\varepsilon)}\int_0^t\frac{\sigma(g_\varepsilon(s),r_\varepsilon(s))}{\alpha_\varepsilon(g_\varepsilon(s))}dw(s)\Bigg|.\end{aligned}$$

Then, (2.9) and Grönwall's inequality lead to

$$\sup_{t\in[0,1]}|\eta_\varepsilon(t)|\le K|\lambda^\varepsilon(t)|+K\sup_{x\in\mathbb{R}^d}\frac{|\alpha_\varepsilon(x)-\alpha(x)|}{\sqrt{\varepsilon}h(\varepsilon)}+\frac{K}{h(\varepsilon)}\Bigg|\int_0^t\frac{\sigma(q_\varepsilon(s),r_\varepsilon(s))}{\alpha_\varepsilon(g_\varepsilon(s))}dw(s)\Bigg|.$$

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In light of the boundedness of functions $\sigma(\cdot,\cdot)$ and $\alpha_\varepsilon(\cdot)$, and Corollary II.9, it is readily seen that

$$\lim_{\ell\to 0}\lim_{\varepsilon\to 0}\frac{1}{h^2(\varepsilon)}\log\mathbb{P}\left(\sup_{t\in[0,1]}|\eta_\varepsilon(t)|>\ell\right)=-\infty.$$

Likewise, we could also obtain (3.58). Thus, the exponential equivalence with respect to MDPs between $\eta_\varepsilon(t)$ and $\hat{\eta}_\varepsilon(t)$ is established.

The last thing we need to verify is the MDPs of $\hat{\eta}_\varepsilon(t)$. We provide a sketch of proof leaving details to Ref. 20. Let us recall (1.11) and (1.12), and the notation introduced in (2.6), (2.7), (2.10), and (2.11). For $t\in[0,1]$,

$$\begin{aligned}\hat{\eta}_\varepsilon(t)&=\frac{f_\varepsilon(t)-q_0(t)}{\sqrt{\varepsilon}h(\varepsilon)}\\&=\frac{1}{\sqrt{\varepsilon}h(\varepsilon)}\int_0^t b_1(f_\varepsilon(s),r_\varepsilon(s))-\overline{b}_1(q_0(s),\nu(s))ds+\frac{1}{h(\varepsilon)}\int_0^t\sigma_1(f_\varepsilon(s),r_\varepsilon(s))dw(s).\end{aligned}$$

Introduce $\tilde{\eta}_\varepsilon(t)$, defined as

$$\begin{aligned}\tilde{\eta}_\varepsilon(t)&=\frac{1}{\sqrt{\varepsilon}h(\varepsilon)}\int_0^t b_1(q_0(s),r_\varepsilon(s))-\overline{b}(q_0(s),\nu(s))ds\\&\quad+\frac{1}{h(\varepsilon)}\int_0^t\sigma_1(q_0(s),r_\varepsilon(s))dw(s)+\int_0^t\overline{D}_1(q_0(s),\nu(s))\tilde{\eta}_\varepsilon(s)ds.\end{aligned}$$

We will establish that $\tilde{\eta}_\varepsilon(t)$ satisfies an LDP of Theorem II.11 and then the LDPs of $\hat{\eta}_\varepsilon(t)$ are guaranteed by the exponential equivalence between $\hat{\eta}_\varepsilon(t)$ and $\tilde{\eta}_\varepsilon(t)$; see details in Ref. 20 (Sec. 4.2) and Ref. 25 (Theorem 5.6). By contraction principle, the LDP of $\tilde{\eta}_\varepsilon(t)$ follows from the LDP of $\tilde{R}_\varepsilon(t)$,

$$\tilde{R}_\varepsilon(t):=\lambda_1^\varepsilon(t)+\widehat{M}_t^\varepsilon.$$

See Ref. 20 (Proposition 2). The last LDPs can be obtained by showing exponential tightness, a local LDP, and the explicit expression of the rate function. The first two properties can be found in Ref. 20 (Secs. 4.1.1 and 4.1.2). For the explicit representation of the rate function, note that if $\tilde{C}_1(t)$ is invertible for each $t\in[0,1]$, we have

$$I_r^{\tilde{B}_1}(\gamma)=\int_0^1\langle\tilde{C}^{-1}(s)\dot{\gamma}(s),\dot{\gamma}(s)\rangle ds,\qquad\text{if } d\gamma(s)=\dot{\gamma}(s)ds,\ \gamma(0)=0,$$

and otherwise, $I_r^{\tilde{B}_1}(\gamma)=\infty$. Thus, assuming the invertibility of $\tilde{C}_1(s)$, the identification of the explicit form of the rate function is obtained by perturbing $\overline{\Sigma}_s^2$ as $\overline{\Sigma}_s^2+\rho I_d$, where ρ is a positive number and I_d is the d-dimensional identity matrix. Taking $\rho\to 0$ completes the proof; see details in Ref. 20 (Sec. 4.1.3). □

C. Applications

For simplicity, we consider the one-dimensional Brownian motion of a particle in a gas or fluid studied in Ref. 3,

$$m\ddot{x}=-\gamma\dot{x}+F(x)+\sqrt{2\gamma k_BT}\dot{w}(t),$$

where m is the mass of the particle, γ is the friction coefficient, $F(x)$ is the external forces, k_B is the Boltzmann constant, T denotes the temperature, and $w(t)$ is the standard Brownian motion. Using the same setting as in Cerrai and Freidlin,[6] we mainly focus on the motion of small particles. Thus, let $m=\varepsilon^2$, where $\varepsilon\ll 1$ is a small parameter. In addition, we also assume that the temperature T is small with $T=\varepsilon$. That is, we are in a low-temperature physics setting. Thus, we arrive at

$$\varepsilon^2\ddot{x}_\varepsilon=-\gamma\dot{x}_\varepsilon+F(x_\varepsilon)+\sqrt{\varepsilon}\sigma\dot{w}(t),\tag{3.59}$$

where $\sigma:=\sqrt{2\gamma k_B}$. In this application, we are considering the constant friction coefficient related to the medium (gas or fluid, etc.) in which the particle is located. The external force $F(x)+\sqrt{\varepsilon}\sigma\dot{w}(t)$ is under a finite number of configurations. For a certain configuration i, the external force is $F(x,i)+\sqrt{\varepsilon}\sigma(i)\dot{w}(t)$. In addition, those configurations are changing at a random time, which gives a stochastic process associated with it. Consequently, we generalize (3.59) as

$$\varepsilon^2\ddot{x}_\varepsilon=-\gamma\dot{x}_\varepsilon+F(x_\varepsilon,r_\varepsilon(t))+\sqrt{\varepsilon}\sigma(r_\varepsilon(t))\dot{w}(t),\tag{3.60}$$

where $r_\varepsilon(t)$ is a fast-varying continuous-time Markov chain used to describe the change of configurations. The state space of the Markov chain $r_\varepsilon(t)$ is finite with values in $\mathcal{M}=\{1,\ldots,m\}$, and the generator is $Q(t)/\varepsilon$, where $Q(t)$ is irreducible with a quasi-invariant measure $\nu(t)$.

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Consider the first order-equation (3.60)

$$\dot{v}_\varepsilon = \frac{F(v_\varepsilon, r_\varepsilon(t))}{\gamma} + \sqrt{\varepsilon}\frac{\sigma(r_\varepsilon(t))}{\gamma}\dot{w}(t)$$

and the averaged deterministic equation

$$\dot{x}_0 = \frac{\overline{F}(x_0, v(t))}{\gamma}.$$

For each $i \in \mathcal{M}$, suppose $F(\cdot, i)$ is bounded and continuous with a bounded first-order partial derivative. Then, Theorem II.11 gives that $(x_\varepsilon - x_0)/(\sqrt{\varepsilon}h(\varepsilon))$ satisfies an LDP in $C^0([0,1], \mathbb{R})$ with speed $h^{-2}(\varepsilon)$ and a good rate function $S(\cdot)$. Roughly speaking, the MDP result shows that the asymptotic probability of $\mathbb{P}(|x_\varepsilon - x_0| \geq \delta\sqrt{\varepsilon}h(\varepsilon))$ converges exponentially to 0 as $\varepsilon \to 0$ for any $\delta > 0$. For more applications of moderate deviations principle, we refer the reader to Refs. 27 and 28 for constructing asymptotic confidence interval and Ref. 29 for obtaining an asymptotic evaluation for the exit time.

IV. DISCUSSION AND REMARKS

A. Discussion on non-homogeneity

This paper investigated the moderate deviations principles of the Langevin dynamics with a strong damping and rapid Markovian switching. We only consider the situation when the Langevin dynamics are time-independent. In fact, motivated by Refs. 25 and 30, one can extend to the case when the dynamics are time-dependent, i.e., considering

$$\begin{cases} \varepsilon^2 \ddot{q}_\varepsilon(t) = b(t, q_\varepsilon(t), r_\varepsilon(t)) - \alpha_\varepsilon(t, q_\varepsilon(t))\dot{q}_\varepsilon(t) + \sqrt{\varepsilon}\sigma(t, q_\varepsilon(t), r_\varepsilon(t))\dot{w}(t), \\ q_\varepsilon(0) = q \in \mathbb{R}^d, \quad \dot{q}_\varepsilon(t) = \dfrac{p}{\varepsilon} \in \mathbb{R}^d, \end{cases}$$

where $r_\varepsilon(t)$ is a time-inhomogeneous irreducible Markov chain generated by $Q(t)/\varepsilon$. An analog of Theorem II.11 can be obtained by examining the time-inhomogeneity.

B. Unbounded $b(\cdot, i)$

We can replace the boundedness condition on the function $b(\cdot, i)$ by the Lipschitz continuity. To this end, we assume that the following condition holds.

(A1′) For each $i \in \mathcal{M}$, $b(\cdot, i) : \mathbb{R}^d \to \mathbb{R}^d$ is continuously differentiable with bounded derivatives.

By virtue of (A1′), $b(\cdot, i)$ is Lipschitz continuous; that is, there exists a constant $c_1 > 0$ such that for all $x, y \in \mathbb{R}^d$,

$$|b(x, i) - b(y, i)| \leq c_1|x - y|.$$

Under Assumption II.5 with (A1) replaced by (A1′), one can obtain [see Ref. 15 (Theorem 2.2)]

$$\limsup_{L\to 0}\limsup_{\varepsilon\to 0} \varepsilon \log \mathbb{P}(\|q_\varepsilon\| > L) = -\infty.$$

Thus, there exist some positive constants R and C such that

$$\limsup_{\varepsilon\to 0} \varepsilon \log \mathbb{P}(\|q_\varepsilon\| \geq R) \leq -C.$$

Consequently, (1.10) implies

$$\limsup_{\varepsilon\to 0} \frac{1}{h^2(\varepsilon)} \log \mathbb{P}(\|q_\varepsilon\| \geq R) = -\infty. \tag{4.1}$$

For any fixed $M > R$, let $S_M = \{x : |x| \leq M\}$ be the sphere with center at the origin and radius M. We define the truncated version of function b as b^M, where for each $i \in \mathcal{M}$,

$$b^M(x, i) = b(x, i)\mu^M(x),$$

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with $\mu^M(\cdot)$ being a smooth function satisfying

$$\mu^M(x) = \begin{cases} 1 \text{ if } x \in S_M, \\ 0 \text{ if } x \in \mathbb{R}^d - S_{M+1}. \end{cases} \tag{4.2}$$

Denote by $q_\varepsilon^M(\cdot)$ the solution of (1.6) with b replaced by b^M, and q_0^M the solution of (1.8) with $\bar{b}$ replaced by $\bar{b}^M$. Because of the continuity of b and α, $\|q_0\|$ is finite. Thus, choosing M large enough, we have $q_0(t) = q_0^M(t)$ for all $t \in [0, 1]$. Defining $X_\varepsilon^M(t)$ similar to (1.9), we establish the exponential equivalence with respect to MDPs between X_ε and X_ε^M as follows. For any $j > 0$,

$$\begin{aligned} &\limsup_{\varepsilon\to 0} \frac{1}{h^2(\varepsilon)} \log \mathbb{P}(\|X_\varepsilon - X_\varepsilon^M\| > j) \\ &= \limsup_{\varepsilon\to 0} \frac{1}{h^2(\varepsilon)} \log \mathbb{P}\left(\left\|\frac{q_0 - q_0^M}{\sqrt{\varepsilon}h(\varepsilon)}\right\| > j\right) \\ &\le \limsup_{\varepsilon\to 0} \frac{1}{h^2(\varepsilon)} \log \mathbb{P}(\|q_\varepsilon - q_\varepsilon^M\| > 0) \\ &= \limsup_{\varepsilon\to 0} \frac{1}{h^2(\varepsilon)} \log \mathbb{P}(\|q_\varepsilon - q_\varepsilon^M\| > 0; \|q_\varepsilon\| \le M) \\ &\quad + \limsup_{\varepsilon\to 0} \frac{1}{h^2(\varepsilon)} \log \mathbb{P}(\|q_\varepsilon - q_\varepsilon^M\| > 0; \|q_\varepsilon\| > M) \\ &\le \limsup_{\varepsilon\to 0} \frac{1}{h^2(\varepsilon)} \log \mathbb{P}(\|q_\varepsilon\| > M) = -\infty, \end{aligned}$$

where the last inequality is because $q_\varepsilon = q_\varepsilon^M$ when $\|q_\varepsilon\| \le M$ and (4.1). Thus, establishing the LDPs of $\{X_\varepsilon(t)\}$ under Assumption II.5 with (A1) replaced by (A1′) is equivalent to establishing that for $\{X_\varepsilon^M(t)\}$, which is the situation considered in Theorem II.11.

ACKNOWLEDGMENTS

This research was supported in part by the National Science Foundation under Grant No. DMS-2204240.

AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Hongjiang Qian: Writing – review & editing (equal). **George Yin**: Writing – review & editing (equal).

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

APPENDIX: TECHNICAL ESTIMATES

Proof of Proposition III.1. From (3.2), due to assumption (A3),

$$\begin{aligned} |q_\varepsilon(t)| &\le |q| + \varepsilon|p| + \frac{1}{\varepsilon^2}\left|\int_0^t \int_0^s e^{-A_\varepsilon(s,u)} b(q_\varepsilon(u), r_\varepsilon(u)) du ds\right| + \frac{1}{\varepsilon^2}\int_0^t |H_\varepsilon(s)| ds \\ &\le |q| + \varepsilon|p| + \frac{1}{\varepsilon^2}\left|\int_0^t \int_0^s e^{-\ell_0(s-u)/\varepsilon^2} b(q_\varepsilon(r), r_\varepsilon(u)) du ds\right| + \frac{1}{\varepsilon^2}\int_0^t |H_\varepsilon(s)| ds \\ &= |q| + \varepsilon|p| + \frac{1}{\varepsilon^2}\left|\int_0^t e^{-\ell_0 s/\varepsilon^2} * b(q_\varepsilon(s), r_\varepsilon(s)) ds\right| + \frac{1}{\varepsilon^2}\int_0^t |H_\varepsilon(s)| ds, \end{aligned}$$

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where $*$ represents convolution. By Young's inequality for convolution and the boundedness and Lipschitz property of $b(\cdot,\cdot)$, we obtain

$$\begin{aligned}\frac{1}{\varepsilon^2}\left|\int_0^t e^{-\ell_0 s/\varepsilon^2} * b(q_\varepsilon(s), r_\varepsilon(s))ds\right| &\le \frac{1}{\varepsilon^2}\int_0^t e^{-\ell_0 s/\varepsilon^2}ds\int_0^t |b(q_\varepsilon(s), r_\varepsilon(s))|ds\\ &\le \frac{1}{\ell_0}\int_0^t |b(q_\varepsilon(s), r_\varepsilon(s))|ds = \frac{1}{\ell_0}\int_0^t |b(q_\varepsilon(s), r_\varepsilon(s)) - b(0, r_\varepsilon(s)) + b(0, r_\varepsilon(s))|ds\\ &\le \frac{K}{\ell_0}\int_0^t |q_\varepsilon(s)|ds + \frac{K}{\ell_0}.\end{aligned} \tag{A1}$$

Thus,

$$|q_\varepsilon(t)| \le K + |q| + \varepsilon|p| + \frac{K}{\ell_0}\int_0^t |q_\varepsilon(s)|ds + \frac{1}{\varepsilon^2}\int_0^t |H_\varepsilon(s)|ds, \tag{A2}$$

and then, Grönwall's inequality implies

$$|q_\varepsilon(t)| \le K(|q| + |p|) + \frac{K}{\varepsilon^2}\int_0^t |H_\varepsilon(s)|ds \le K\left(1 + \frac{1}{\varepsilon^2}\int_0^t |H_\varepsilon(s)|ds\right).$$

Furthermore, (3.1) implies

$$|p_\varepsilon(t)| \le \frac{|p|}{\varepsilon}e^{-A_\varepsilon(t)} + \frac{K}{\varepsilon^2}\int_0^t e^{-\ell_0(t-s)/\varepsilon^2}(1 + |q_\varepsilon(s)|)ds + \frac{1}{\varepsilon^2}|H_\varepsilon(t)|,$$

and thanks to (A2), we obtain

$$\begin{aligned}|p_\varepsilon(t)| &\le \frac{|p|}{\varepsilon}e^{-\ell_0 t/\varepsilon^2} + K(1 + |p| + |q|) + \frac{K}{\varepsilon^2}\sup_{t\in[0,1]}|H_\varepsilon(t)|\\ &\le K\left(1 + \frac{1}{\varepsilon} + \frac{1}{\varepsilon^2}\sup_{t\in[0,1]}|H_\varepsilon(t)|\right).\end{aligned}$$

□

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