Journal of Optimization Theory and Applications
https://doi.org/10.1007/s10957-022-02127-7

Harvesting of a Stochastic Population Under a Mixed Regular-Singular Control Formulation

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Received: 4 November 2021 / Accepted: 21 October 2022

Abstract
This work focuses on optimal harvesting-renewing for a stochastic population. A mixed regular-singular control formulation with a state constraint and regime switching is introduced. The decision-makers either harvest or renew with finite or infinite harvesting/renewing rates. The payoff functions depend on the harvesting/renewing rates. Several properties of the value function are established. The limiting value function as the white noise intensity approaches infinity is identified. The Markov chain approximation method is used to find numerical approximation of the value function and optimal strategies.

Keywords Harvesting problem · Controlled diffusion · Singular control · State constraint · Markovian switching

Mathematics Subject Classification 93E20 · 49N90 · 92D25

1 Introduction

This work focuses on harvesting and renewing strategies for stochastic ecosystems that are represented by controlled stochastic differential equations. Mathematically,

Communicated by Negash G. Medhin.

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Published online: 07 November 2022

the problem we consider belongs to a class of singular stochastic control problems, namely harvesting-type problems. Such problems have been studied extensively in various settings for different domain of applications; see [1, 2, 8–12, 17, 19, 24]. To mention just a few of the recent developments, we refer to [25, 26] for single species and interacting population systems with regime switching. The paper [3] focuses on sustainable harvesting policies under long-run average criteria, which are further studied in [18]; see also [21] for a related work on a predator–prey system and [14] for an ergodic two-sided singular control formulation. In [6, 7], the authors study ecosystems in which both renewing and harvesting actions are included. Optimal exploitation problems of renewable natural resources, which are harvesting-type problems, are studied in [13, 27]. Related works on a general one-dimensional diffusion that is reflected at zero can be found in [4] and references therein. Intensive treatment of stochastic population systems and hybrid diffusions can be found in [20, 28], while applications in various areas of singular stochastic control can be found in [5, 23] and many references therein.

In this work, we propose a generalized harvesting model for a stochastic population. The controller can perform either a regular control or a singular control to harvest and renew the species. Moreover, we also consider the control objective associated with renewing and harvesting. It should be noted that in optimal harvesting formulations in [1, 2, 6, 7, 25], the control cost is simply combined with the price functions. As a result, it is frequently seen that when the manager decides to harvest (resp., renew), she should do that with the maximal possible harvesting rates (resp. renewing rates); see [1, 6, 7]. In [13, 27], interesting phenomena appear when the control objective is taken into account. For instance, the maximal possible rates are no longer optimal for harvesting and renewing in certain cases. With the generalized formulation proposed in this work, the distinctions are even more pronounced. In addition, as a new twist, state constraint is considered in this work. In particular, the time horizon of the control problem is $[0, \tau]$ where τ is the first time the population process is below a predetermined level. Another important issue of interest is the impact of a white noise with large intensity. For Kolmogorov-type ecosystems, it is known that very large white noise make the species extinct, and have a major impact on harvesting actions; see [1, 20, 26]. A novelty of the paper is the identification of the limiting value function as the white noise intensity approaches infinity for a general stochastic population.

In contrast to the existing results, our new contributions in this paper are as follows. (i) We formulate a harvesting problem with renewing and the consideration of control objective, a state constraint, and regime switching. Both bounded and unbounded harvesting-renewing rates are allowed. (ii) We establish the finiteness and the continuity of the value function. We show that in common cases, it is optimal to keep the population size in a compact set. (iii) We study the impact of a white noise with large intensity on harvesting. (iv) Based on the Markov chain approximation method, we construct a controlled Markov chain to approximate the given controlled population. It enables us to approximate the value function and near-optimal strategies.

The rest of our work is organized as follows. Section 2 begins with the problem formulation. Section 3 focuses on properties of the value function and the impact of a white noise with large intensity. In Sect. 4, we construct a controlled Markov chain to approximate the given controlled population system. Finally, the paper is concluded

with several numerical examples for illustration together with additional remarks in the last section.

2 Formulation

We work with a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\})$ with the filtration $\{\mathcal{F}_t\}$ satisfying the usual condition (i.e., it is right-continuous and $\mathcal{F}_0$ contains all the null sets). Let $\mathbb{R}_+ = [0, \infty)$ and $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$. For a real number x, we denote $x^+ = \max\{x, 0\}$ and $x^- = \max\{-x, 0\}$. Thus, $x = x^+ - x^-$ and $|x| = x^+ + x^-$. Suppose that the population size $\xi(t)$ of a species at time t is given by

$$\mathrm{d}\xi(t) = b\big(\xi(t), \Lambda(t)\big)\mathrm{d}t + \sigma\big(\xi(t), \Lambda(t)\big)\mathrm{d}w(t). \tag{2.1}$$

In the model, $w(\cdot)$ is a one-dimensional standard Brownian motion and $\Lambda(\cdot)$ is a continuous-time Markov chain. Moreover, $\Lambda(\cdot)$ and $w(\cdot)$ are independent and $\{\mathcal{F}_t\}$-adapted. Suppose $\Lambda(\cdot)$ takes values in $\mathcal{M} = \{1, 2, \dots, m_0\}$ with generator $\Gamma = (\Gamma_{\alpha\ell})_{m_0\times m_0}$ and m_0 being a positive integer. The coefficients $b(\cdot,\cdot)$ and $\sigma(\cdot,\cdot)$ are real-valued functions defined on $\mathbb{R}_+ \times \mathcal{M}$. The transition probabilities of $\Lambda(\cdot)$ are described by

$$\mathbb{P}\{\Lambda(t+\Delta t) = \ell | \Lambda(t) = \alpha\} = \begin{cases} \Gamma_{\alpha\ell}\Delta t + o(\Delta t) & \text{if } \alpha \neq \ell, \\ 1 + \Gamma_{\alpha\alpha}\Delta t + o(\Delta t) & \text{if } \alpha = \ell. \end{cases} \tag{2.2}$$

Note that $\Gamma_{\alpha\ell} \geq 0$ if $\alpha \neq \ell$ and $\sum_{\ell\in\mathcal{M}} \Gamma_{\alpha\ell} = 0$ for any $\alpha \in \mathcal{M}$. To illustrate, consider the stochastic logistic population growth model given by

$$\mathrm{d}X(t) = X(t)\Big(\kappa_1(\Lambda(t)) - \kappa_2(\Lambda(t))X(t)\Big)\mathrm{d}t + \kappa_3(\Lambda(t))X(t)\mathrm{d}w(t), \tag{2.3}$$

where $\kappa_1(\cdot), \kappa_2(\cdot), \kappa_3(\cdot)$ are real-valued functions defined on the state space $\mathcal{M}$ of the Markov chain $\Lambda(\cdot)$. The switching component $\Lambda(\cdot)$ is introduced to capture the major environmental shifts (daily or seasonal changes or catastrophes) leading to the changes in the carrying capacities and interactions in different environments. To visualize the dynamics of (2.3), without loss of generality, assume that $\Lambda(0) = \alpha$. Then, the Markov chain rests in state α for an exponentially distributed random duration, in this time interval, the model given by (2.3) obeys

$$\mathrm{d}X(t) = X(t)\Big(\kappa_1(\alpha) - \kappa_2(\alpha)X(t)\Big)\mathrm{d}t + \kappa_3(\alpha)X(t)\mathrm{d}w(t),$$

until the Markov chain $\Lambda(\cdot)$ jumps to another state ℓ. Then, the model (2.3) obeys

$$\mathrm{d}X(t) = X(t)\Big(\kappa_1(\ell) - \kappa_2(\ell)X(t)\Big)\mathrm{d}t + \kappa_3(\ell)X(t)\mathrm{d}w(t)$$

for an exponentially distributed random time until the Markov chain $\Lambda(\cdot)$ jumps to a new state again and so on.

To proceed, we introduce the generator of the process $\big(\xi(t), \Lambda(t)\big)$. For a function $\Phi(\cdot,\cdot) : \mathbb{R} \times \mathcal{M} \mapsto \mathbb{R}$ satisfying $\Phi(\cdot,\alpha) \in C^2(\mathbb{R})$ for each $\alpha \in \mathcal{M}$, we define

$$\mathcal{L}\Phi(x,\alpha) = b(x,\alpha)\Phi'(x,\alpha) + \frac{1}{2}\sigma^2(x,\alpha)\Phi''(x,\alpha) + \sum_{\ell\in\mathcal{M}} \Gamma_{\alpha\ell}\Phi(x,\ell),$$

where $\Phi'(\cdot,\alpha)$ and $\Phi''(\cdot,\alpha)$ denote the first and second derivatives (w.r.t. x) of $\Phi(\cdot,\alpha)$, respectively.

Next, we suppose that the population can be instantaneously harvested, instantaneously renewed, harvested with bounded rates, or renewed with bounded rates. In order to harvest or renew instantaneously (that is, harvest or renew with infinite rates), the controller needs to exercise an impulsive control. Meanwhile, to harvest or renew with bounded rates, the controller performs a regular control. Specifically, we assume the dynamics of the species is given by

$$\begin{aligned} X(t) = x &+ \int_0^t b\big(X(s),\Lambda(s)\big)\mathrm{d}s + \int_0^t \sigma\big(X(s),\Lambda(s)\big)\mathrm{d}w(s) \\ &- \int_0^t f\big(X(s),C(s)\big)\mathrm{d}s - Y(t) + Z(t), \end{aligned} \tag{2.4}$$

where $x \in \mathbb{R}_+$, $X(t)$ is the population size at time $t \geq 0$, $f : \mathbb{R}_+ \times \mathcal{U} \mapsto \mathbb{R}$ is the harvesting-renewing rate corresponding to the control $C(\cdot)$ taking values in a nonempty compact set $\mathcal{U}$ in $\mathbb{R}$, while $Y(\cdot)$ and $Z(\cdot)$ are impulsive controls. In particular, $Y(t)$ denotes the amount of the species that has been instantaneously harvested up to time t, while $Z(t)$ denotes the amount of the species that has been instantaneously renewed up to time t.

Notation. For each time t, $X(t-)$ represents the state before harvesting or renewing starts at time t, while $X(t)$ is the state immediately after. We assume the initial population size to be $X(0-) = x$ and initial regime to be $\Lambda(0) = \alpha$, respectively. Hence, $X(0)$ may not be equal to $X(0-)$ due to an impulsive harvesting $Y(0)$ or an impulsive renewing $Z(0)$. Throughout the paper, we use the convention that $Y(0-) = Z(0-) = 0$. The jump sizes of $Y(\cdot)$ and $Z(\cdot)$ at time t are denoted by $\Delta Y(t) = Y(t) - Y(t-)$ and $\Delta Z(t) = Z(t) - Z(t-)$, respectively. Thus,

$$Y(t) = \sum_{0\leq s\leq t} \Delta Y(s), \quad Z(t) = \sum_{0\leq s\leq t} \Delta Z(s).$$

Suppose $\lambda \in \mathbb{R}_+$ and denote

$$\mathcal{S} = \{x \in \mathbb{R}_+ : x \geq \lambda\}.$$

In this work, we consider the harvesting problem on the time horizon $[0, \tau]$, where

$$\tau = \inf\{t \geq 0 : X(t) \notin \mathcal{S}\}.$$

The price per unit of the species is a positive constant a_1. The harvesting and renewing control is costly. Consider a function $g : \mathbb{R}_+ \times \mathcal{M} \times \mathcal{U} \mapsto \mathbb{R}_+$. The accumulated cost of the regular control $C(\cdot)$ is

$$\int_0^\tau e^{-\delta s} g\big(X(s), \Lambda(s), C(s)\big)\mathrm{d}s,$$

where $\delta > 0$ is the discount factor. Note that we separate cost and income here. Later, we will take into account of both and set up the payoff. Regarding the regular control $C(\cdot)$, the accumulated income of selling the harvested amount is $\int_0^\tau e^{-\delta s} a_1 f^+\big(X(s), C(s)\big)\mathrm{d}s$ and the accumulated expense of the renewed amount is $\int_0^\tau e^{-\delta s} a_1 f^-\big(X(s), C(s)\big)\mathrm{d}s$. For notational simplicity, we define the price-cost function $p : \mathbb{R}_+ \times \mathcal{M} \times \mathcal{U} \to \mathbb{R}$ given by

$$p(x, \alpha, c) = a_1 f(x, c) - g(x, \alpha, c) \text{ for } (x, \alpha, c) \in \mathbb{R}_+ \times \mathcal{M} \times \mathcal{U}. \tag{2.5}$$

Then, the payoff functional associated with the regular control $C(\cdot)$ is

$$\mathbb{E}_{x,\alpha}\Big[\int_0^\tau e^{-\delta s} p\big(X(s), \Lambda(s), C(s)\big)\mathrm{d}s\Big],$$

where $\mathbb{E}_{x,\alpha}$ denotes the expectation with $X(0-) = x$ and $\Lambda(0) = \alpha$. Let a_2 and a_3 be two positive constants. Suppose that the cost of instantaneous harvesting a unit of the species is a_2, while the cost of instantaneous renewing a unit of the species is a_3. Then, the accumulated cost of exercising the impulsive control $Y(\cdot)$ is $\int_0^\tau e^{-\delta s} a_2 \mathrm{d}Y(s)$, while the accumulated income of selling the harvested amount by $Y(\cdot)$ is $\int_0^\tau e^{-\delta s} a_1 \mathrm{d}Y(s)$. Meanwhile, the accumulated cost of exercising the impulsive control $Z(\cdot)$ is $\int_0^\tau e^{-\delta s} a_3 \mathrm{d}Z(s)$ and the accumulated expense of the renewed amount by $Z(\cdot)$ is $\int_0^\tau e^{-\delta s} a_1 \mathrm{d}Z(s)$. Hence, the payoff functional associated with the impulsive controls $Y(\cdot)$ and $Z(\cdot)$ is

$$\mathbb{E}_{x,\alpha}\Big[\int_0^\tau e^{-\delta s}(a_1 - a_2)\mathrm{d}Y(s) - \int_0^\tau e^{-\delta s}(a_1 + a_3)\mathrm{d}Z(s)\Big].$$

We define

$$q = a_1 - a_2, \quad r = a_1 + a_3.$$

Then, for a harvesting-renewing strategy $\Psi \equiv (C, Y, Z)$, we define the *payoff functional* as

$$\begin{aligned} J(x, \alpha, \Psi) = \mathbb{E}_{x,\alpha}\Big[&\int_0^\tau e^{-\delta s} p\big(X(s), \Lambda(s), C(s)\big)\mathrm{d}s \\ &+ \int_0^\tau e^{-\delta s} q \mathrm{d}Y(s) - \int_0^\tau e^{-\delta s} r \mathrm{d}Z(s)\Big]. \end{aligned} \tag{2.6}$$

Control strategy Let $\mathcal{A}_{x,\alpha}$ denote the collection of all admissible controls with initial condition $X(0-) = x$, $\Lambda(0) = \alpha$. A harvesting-renewing strategy $\Psi = (C, Y, Z)$ will be in $\mathcal{A}_{x,\alpha}$ if it satisfies the following conditions.

(a) The processes $C(\cdot)$, $Y(\cdot)$, and $Z(\cdot)$ are adapted to $\sigma\{w(s), \Lambda(s) : 0 \le s \le t\}$; $C(\cdot)$ takes values in $\mathcal{U}$, $Y(\cdot)$ and $Z(\cdot)$ are impulsive controls with non-decreasing, non-negative, piecewise constant, and right-continuous sample paths; $\Delta Y(s)\Delta Z(s) = 0$ for any $s \in \mathbb{R}_+$; $\Delta Y(s) = \Delta Z(s) = 0$ for any $s \ge \tau$;
(b) System (2.4) has a unique solution $X(\cdot)$ with $X(t) \in \mathcal{S}$ for any $t \in [0, \tau]$;
(c) $0 \le J(x, \alpha, \Psi) < \infty$, where $J(\cdot)$ is the functional defined in (2.6).

The problem we are interested in is to maximize the payoff functional and find an optimal strategy $\Psi^* = (C^*, Y^*, Z^*) \in \mathcal{A}_{x,\alpha}$ such that

$$J(x, \alpha, \Psi^*) = V(x, \alpha) := \sup_{\Psi \in \mathcal{A}_{x,\alpha}} J(x, \alpha, \Psi). \tag{2.7}$$

The function $V(\cdot)$ is called the *value function*.

The standing assumptions are given below.

(A) (a) For any $n \in \mathbb{Z}_+$, there exists a positive constant K_n such that for any $x, y \in \mathbb{R}_+$ with $|x| \le n$, $|y| \le n$ and any $\alpha \in \mathcal{M}$,

$$|b(x, \alpha) - b(y, \alpha)| + |\sigma(x, \alpha) - \sigma(y, \alpha)| \le K_n |x - y|.$$

Moreover, for each initial condition $(x, \alpha) \in \mathbb{R}_+ \times \mathcal{M}$, the population system (2.1) has a unique global solution.

(b) The control set $\mathcal{U}$ is a nonempty compact set of real numbers, $0 \in \mathcal{U}$, a_1, a_2, a_3 are positive constants and $a_1 > a_2$. The function $g(\cdot, \cdot, \cdot)$ is continuous and bounded on $\mathbb{R}_+ \times \mathcal{M} \times \mathcal{U}$. The function $f(\cdot, \cdot)$ is continuous and bounded on $\mathbb{R}_+ \times \mathcal{U}$ and $f(\cdot, 0) = 0$.

Remark 2.1 For simplicity, we require the systems of equations having a global solution. In fact, the existence and uniqueness of global solutions for a large class of Kolmogorov systems are guaranteed under suitable conditions; see the recent work [22] and the references therein, and also [20] for various sufficient conditions so that the population system (2.1) has a unique global solution. Of course, here we are dealing with a controlled system so some modifications are needed. Nevertheless, in order to concentrate on our main task, we choose to simply assume this condition.

The mixed regular-singular control formulation together with the consideration of a state constraint, cost function allows us to take into account various aspects of harvesting-type problems that have not been considered to date. For instance, depending on the costs, one can choose to apply either an impulsive harvesting/renewing or harvest and renew through the regular control.

Remark 2.2 The consideration of λ is motivated by sustainability. To the best of our knowledge, the available literature focuses on the case $\lambda = 0$ in which the optimal or near-optimal harvesting strategies might drive the population process to a very low level or extinction; see [1, 6, 25]. Because of the state constraint, one needs to treat carefully control actions when the population size is close to λ.

The function $f : \mathbb{R}_+ \times \mathcal{U} \mapsto \mathbb{R}$ is the harvesting-renewing rate corresponding to the control $C(\cdot)$. In [7], the authors studied the case $f(x, c) = c$ and $g(x, \alpha, c) = 0$

for any $(x, \alpha, c) \in \mathbb{R}_+ \times \mathcal{M} \times \mathcal{U}$. Motivated by the observations in [1, Section 3] and [13], one can also take $f(x, c) = \min\{cx, \kappa\}$ and $g(x, \alpha, c) = 0$ for any $(x, \alpha, c) \in \mathbb{R}_+ \times \mathcal{M} \times \mathcal{U}$, where κ is a positive constant. By considering the general form $f : \mathbb{R}_+ \times \mathcal{U} \mapsto \mathbb{R}$, our formulation is much more general than the aforementioned cases.

3 Properties of the Value Function

This section is devoted to several properties of the value function. We begin with a lemma that allows us to establish the finiteness and construct upper bounds of the value function.

Lemma 3.1 *Let* (A) *be satisfied. Suppose that there exists a function* $\Phi : \mathcal{S} \times \mathcal{M} \mapsto \mathbb{R}_+$ *such that* $\Phi(\cdot, \alpha) \in C^2(\mathcal{S})$ *for each* $\alpha \in \mathcal{M}$ *and* $\Phi(\cdot, \cdot)$ *solves the following coupled system of quasi-variational inequalities*

$$\max\left\{\mathcal{G}\Phi(x, \alpha), q - \Phi'(x, \alpha), \Phi'(x, \alpha) - r\right\} \leq 0 \ \text{for} \ (x, \alpha) \in \mathcal{S} \times \mathcal{M}, \quad (3.1)$$

where

$$\mathcal{G}\Phi(x, \alpha) = (\mathcal{L} - \delta)\Phi(x, \alpha) + \max_{c \in \mathcal{U}}\Big[p(x, \alpha, c) - \Phi'(x, \alpha) f(x, c)\Big].$$

Recall that δ *is the discount factor given in the payoff functional* (2.6)*. Then, we have*

$$V(x, \alpha) \leq \Phi(x, \alpha) \ \text{for} \ (x, \alpha) \in \mathcal{S} \times \mathcal{M}.$$

Proof For a fixed $(x, \alpha) \in \mathcal{S} \times \mathcal{M}$ and $\Psi = (C, Y, Z) \in \mathcal{A}_{x,\alpha}$, let X denote the corresponding harvested process. Choose N sufficiently large so that $|x| < N$. Define

$$\tau_N = \inf\{t \geq 0 : X(t) \geq N\}, \quad T_N = N \wedge \tau_N \wedge \tau.$$

Then,

$$\tau_N \to \infty \ \text{and} \ T_N \to \tau \ \text{almost surely as} \ N \to \infty, \quad (3.2)$$

where $\tau = \inf\{t \geq 0 : X(t) \notin \mathcal{S}\}$. Then, Dynkin's formula leads to

$$\begin{aligned}\mathbb{E}\left[e^{-\delta T_N}\Phi\big(X(T_N), \Lambda(T_N)\big)\right] = \Phi(x, \alpha) &+ \mathbb{E}\int_0^{T_N} e^{-\delta s}(\mathcal{L} - \delta)\Phi\left(X(s), \Lambda(s)\right) \mathrm{d}s \\ &- \mathbb{E}\int_0^{T_N} e^{-\delta s}\Phi'\left(X(s), \Lambda(s)\right) f\big(X(s), C(s)\big)\mathrm{d}s \\ &+ \mathbb{E}\sum_{0 \leq s \leq T_N} e^{-\delta s}\Big[\Phi\left(X(s), \Lambda(s-)\right) - \Phi\left(X(s-), \Lambda(s-)\right)\Big].\end{aligned} \quad (3.3)$$

It follows from (3.1) that

$$\begin{aligned}&\mathbb{E}\int_0^{T_N} e^{-\delta s}(\mathcal{L}-\delta)\Phi\left(X(s),\Lambda(s)\right)\mathrm{d}s - \mathbb{E}\int_0^{T_N} e^{-\delta s}\Phi'\left(X(s),\Lambda(s)\right) f\big(X(s),C(s)\big)\mathrm{d}s\\ &\le -\mathbb{E}\int_0^{T_N} e^{-\delta s} p\left(X(s),C(s),\Lambda(s)\right)\mathrm{d}s.\end{aligned} \tag{3.4}$$

For each $s \in [0, T_N]$, by the mean value theorem, there exists a point $\widetilde{X}(s)$ between $X(s)$ and $X(s-)$ such that

$$\begin{aligned}\Phi\left(X(s),\Lambda(s-)\right) - \Phi\big(X(s-),\Lambda(s-)\big) \;&= -\Delta Y(s)\Phi'\big(\widetilde{X}(s),\Lambda(s-)\big)\\ &\quad +\Delta Z(s)\Phi'\big(\widetilde{X}(s),\Lambda(s-)\big).\end{aligned}$$

By (3.1), we have

$$\Phi\big(X(s),\Lambda(s-)\big) - \Phi\big(X(s-),\Lambda(s-)\big) \le -q\Delta Y(s) + r\Delta Z(s). \tag{3.5}$$

It follows from (3.3), (3.4), (3.5), and the nonnegativity of $\Phi(\cdot,\cdot)$ that

$$\begin{aligned}\Phi(x,\alpha) \;&\ge \mathbb{E}\int_0^{T_N} e^{-\delta s} p\big(X(s),\Lambda(s),C(s)\big)\mathrm{d}s\\ &\quad +\mathbb{E}\int_0^{T_N} e^{-\delta s} q\,\mathrm{d}Y(s) - \mathbb{E}\int_0^{T_N} e^{-\delta s} r\,\mathrm{d}Z(s).\end{aligned}$$

Letting $N \to \infty$, it follows from (3.2) and the bounded convergence theorem that $\Phi(x,\alpha) \ge J(x,\alpha,\Psi)$. Taking supremum over all $\Psi \in \mathcal{A}_{x,\alpha}$, we obtain $\Phi(x,\alpha) \ge V(x,\alpha)$. The conclusion follows. □

Using Lemma 3.1, we proceed to present an easily verifiable condition for the finiteness of the value function.

Theorem 3.1 *Let* (A) *be satisfied. Moreover, suppose that there is a positive constant* K *such that*

$$b(x,\alpha) \le \delta x + K \;\; \textit{for} \;\; (x,\alpha) \in \mathcal{S}\times\mathcal{M}. \tag{3.6}$$

Then, there exists a positive constant M *such that*

$$V(x,\alpha) \le qx + M \;\; \textit{for} \;\; (x,\alpha) \in \mathcal{S}\times\mathcal{M}.$$

Proof Define

$$\Phi(x,\alpha) = qx + \frac{Kq+\kappa_0}{\delta} \;\text{ for }\; (x,\alpha) \in \mathcal{S}\times\mathcal{M},$$

where

$$\kappa_0 = \sup_{(y,\alpha,c)\in\mathbb{R}_+\times\mathcal{M}\times\mathcal{U}} \Big(p(y,\alpha,c) - qf(y,c)\Big).$$

By assumption (A)(b), κ_0 is finite. Since $q = a_1 - a_2 < r = a_1 + a_3$ and $\Phi'(x,\alpha) = q$, it is clear that

$$q - \Phi'(x,\alpha) = 0, \quad \Phi'(x,\alpha) - r < 0 \ \text{ for } \ (x,\alpha) \in \mathcal{S} \times \mathcal{M}. \tag{3.7}$$

By (3.6), we have

$$\begin{aligned} \mathcal{G}\Phi(x,\alpha) &= b(x,\alpha)q - \delta\Big(qx + \tfrac{Kq+\kappa_0}{\delta}\Big) + \max_{c\in\mathcal{U}}\Big[p(x,\alpha,c) - qf(x,c)\Big] \\ &\le (\delta x + K)q - (\delta q x + Kq + \kappa_0) + \kappa_0 \\ &= 0 \ \text{ for } \ (x,\alpha) \in \mathcal{S} \times \mathcal{M}. \end{aligned} \tag{3.8}$$

It follows from (3.7) and (3.8) that

$$\max\Big\{\mathcal{G}\Phi(x,\alpha), q - \Phi'(x,\alpha), \Phi'(x,\alpha) - r\Big\} \le 0 \ \text{ for } \ (x,\alpha) \in \mathcal{S} \times \mathcal{M}.$$

That is, $\Phi(\cdot,\cdot)$ solves the system of inequalities (3.1). By virtue of Lemma 3.1, $V(x,\alpha) \le \Phi(x,\alpha)$ for any $(x,\alpha) \in \mathcal{S} \times \mathcal{M}$. This completes the proof. □

Next, we establish the continuity of the value function.

Theorem 3.2 *Let* (A) *be satisfied. Then, the following assertions hold.*

(a) *For any $x, y \in \mathcal{S}$ and $\alpha \in \mathcal{M}$,*

$$V(x,\alpha) \ge q(x-y)^+ - r(y-x)^+ + V(y,\alpha). \tag{3.9}$$

(b) *$V(\cdot)$ is Lipschitz continuous on $\mathcal{S} \times \mathcal{M}$.*

Proof (a) Fix $\Psi = (C, Y, Z) \in \mathcal{A}_{y,\alpha}$. Define

$$\widetilde{C}(t) = C(t), \quad \widetilde{Y}(t) = Y(t) + (x-y)^+, \quad \widetilde{Z}(t) = Z(t) + (y-x)^+, \quad t \ge 0,$$

and $\widetilde{\Psi} = (\widetilde{C}, \widetilde{Y}, \widetilde{Z})$. Then, $\widetilde{\Psi} \in \mathcal{A}_{x,\alpha}$ and

$$J(x,\alpha,\widetilde{\Psi}) = q(x-y)^+ - r(y-x)^+ + J(y,\alpha,\Psi).$$

Since $V(x,\alpha) \ge J(x,\alpha,\widetilde{\Psi})$, we have

$$V(x,\alpha) \ge q(x-y)^+ - r(y-x)^+ + J(y,\alpha,\Psi),$$

from which, (3.9) follows by taking supremum over $\Psi \in \mathcal{A}_{y,\alpha}$.

(b) Similar to (3.9), we have

$$V(y,\alpha) \ge q(y-x)^+ - r(x-y)^+ + V(x,\alpha). \tag{3.10}$$

In view of (3.9), (3.10), for any $x, y \in \mathcal{S}$ and $\alpha \in \mathcal{M}$,

$$|V(x,\alpha) - V(y,\alpha)| \le \big(|q| + |r|\big)|x - y|.$$

Thus, $V(\cdot)$ is Lipschitz continuous on $\mathcal{S} \times \mathcal{M}$. □

For population models given by diffusion processes, under optimal or near-optimal harvesting strategies, one should keep the population sizes in a bounded set. In other words, if the initial population is too high, an impulsive harvesting should be performed instantaneously; see [1, 6, 7]. We proceed to provide a proof of that result for a general setting of a controlled regime-switching diffusions with a mixed regular-singular control.

Theorem 3.3 *Suppose that there exists a number $U > \lambda$ such that*

$$q\Big(b(x,\alpha) - \delta(x-U)\Big) + \sup_{c\in\mathcal{U}}\Big(a_2 f(x,c) - g(x,\alpha,c)\Big) < 0, \quad (x,\alpha) \in (U,\infty)\times\mathcal{M}. \tag{3.11}$$

Then, for each $(x,\alpha) \in (U,\infty)\times\mathcal{M}$,

$$V(x,\alpha) = V(U,\alpha) + q(x-U).$$

Remark 3.1 By assumption (A)(b),

$$\kappa_1 := \sup_{(x,\alpha,c)\in\mathbb{R}_+\times\mathcal{M}\times\mathcal{U}}\Big(a_2 f(x,c) - g(x,\alpha,c)\Big) < \infty.$$

If $\limsup_{x\to\infty} b(x,\alpha) < -\kappa_1/q$ for each $\alpha \in \mathcal{M}$, then we can take

$$U = \sup\big\{x > \lambda + 1 : \sup_{\alpha\in\mathcal{M}} b(x,\alpha) \ge -\kappa_1/q\big\}.$$

Proof Fix some $(x,\alpha) \in (U,\infty)\times\mathcal{M}$, $\Psi = (C,Y,Z) \in \mathcal{A}_{x,\alpha}$, and let X be the corresponding harvested process. Let $\varepsilon \in (0,1)$ be a constant and define

$$\Phi(y,\alpha) = q(y-U) + \varepsilon \ \text{ for } \ (y,\alpha) \in [U,\infty)\times\mathcal{M}. \tag{3.12}$$

We can extend $\Phi(\cdot,\cdot)$ to the entire $\mathcal{S}\times\mathcal{M}$ so that $\Phi(\cdot,\alpha)$ is a C^2 function for each $\alpha \in \mathcal{M}$, and $\Phi(y,\alpha) > 0$ for all $(y,\alpha) \in \mathcal{S}\times\mathcal{M}$. By assumption (3.11), we can check that

$$(\mathcal{L} - \delta)\Phi(y,\alpha) + a_2 f(y,c) - g(y,\alpha,c) < 0, \quad (y,\alpha,c)\times[U,\infty)\times\mathcal{M}\times\mathcal{U}.$$

Thus,

$$\begin{aligned}(\mathcal{L}-\delta)\Phi(y,\alpha)-qf(y,c) &< -\big[a_2 f(y,c)-g(y,\alpha,c)\big]-qf(y,c)\\ &= -\big[(a_2+q)f(y,c)-g(y,\alpha,c)\big]\\ &= -p(y,\alpha,c),\quad (y,\alpha,c)\times[U,\infty)\times\mathcal{M}\times\mathcal{U}.\end{aligned} \tag{3.13}$$

For an integer N satisfying $N > U$, we define

$$\tau_N = \inf\{t\geq 0 : X(t)\geq N\},\quad \widetilde{\gamma}_U = \inf\{t\geq 0 : X(t)\leq U\},\quad T_N = N\wedge\tau_N\wedge\widetilde{\gamma}_U.$$

We have $T_N \to \widetilde{\gamma}_U$ almost surely as $N\to\infty$. Note that $\widetilde{\gamma}_U \leq \tau$. By Dynkin's formula,

$$\begin{aligned}&\mathbb{E}\left[e^{-\delta T_N}\Phi\left(X(T_N),\Lambda(T_N)\right)\right]-\Phi(x,\alpha)\\ &\quad=\mathbb{E}\int_0^{T_N} e^{-\delta s}(\mathcal{L}-\delta)\Phi\left(X(s),\Lambda(s)\right)\mathrm{d}s\\ &\qquad-\mathbb{E}\int_0^{T_N} e^{-\delta s}\Phi'\left(X(s),\Lambda(s)\right)f\big(X(s),C(s)\big)\mathrm{d}s\\ &\qquad+\mathbb{E}\sum_{0\leq s\leq T_N} e^{-\delta s}\Big[\Phi\left(X(s),\Lambda(s-)\right)-\Phi\left(X(s-),\Lambda(s-)\right)\Big].\end{aligned} \tag{3.14}$$

For each $s\in[0,T_N]$, we have

$$\begin{aligned}\Phi\left(X(s),\Lambda(s-)\right)-\Phi\left(X(s-),\Lambda(s-)\right) &= -q\Delta Y(s)+q\Delta Z(s)\\ &\leq -q\Delta Y(s)+r\Delta Z(s).\end{aligned} \tag{3.15}$$

We obtain from (3.13), (3.14), and (3.15) that

$$\begin{aligned}&\mathbb{E}\left[e^{-\delta T_N}\Phi\left(X(T_N),\Lambda(T_N)\right)\right]-\Phi(x,\alpha)\\ &\quad\leq -\mathbb{E}\int_{s_0}^{T_N} e^{-\delta s}p\big(X(s),\Lambda(s),C(s)\big)\mathrm{d}s\\ &\qquad-\mathbb{E}\sum_{0\leq s\leq T_N} e^{-\delta s}q\Delta Y(s)+\mathbb{E}\sum_{0\leq s\leq T_N} e^{-\delta s}r\Delta Z(s).\end{aligned} \tag{3.16}$$

Since $\Phi(y,\alpha)>0$ for any $(y,\alpha)\in\mathcal{S}\times\mathcal{M}$, it follows from (3.16) that

$$\begin{aligned}&\mathbb{E}\int_0^{T_N} e^{-\delta s}p\big(X(s),\Lambda(s),C(s)\big)\mathrm{d}s\\ &\qquad+\mathbb{E}\sum_{0\leq s\leq T_N} e^{-\delta s}q\Delta Y(s)-\mathbb{E}\sum_{0\leq s\leq T_N} e^{-\delta s}r\Delta Z(s)\\ &\quad\leq\Phi(x,\alpha).\end{aligned}$$

Letting $N \to \infty$, we obtain

$$\begin{aligned}\mathbb{E}\int_0^{\widetilde{\gamma}_U} e^{-\delta s} p\big(X(s), \Lambda(s), C(s)\big)\mathrm{d}s \\ +\mathbb{E}\sum_{0\le s\le\widetilde{\gamma}_U} e^{-\delta s} q\Delta Y(s) - \mathbb{E}\sum_{0\le s\le\widetilde{\gamma}_U} e^{-\delta s} r\Delta Z(s) \\ \le \Phi(x,\alpha).\end{aligned} \tag{3.17}$$

As a result,

$$\begin{aligned} J(x,\alpha,\Psi) &\le V(U,\alpha)+\Phi(x,\alpha) \\ &= V(U,\alpha)+q(x-U)+\varepsilon. \end{aligned}$$

Letting $\varepsilon \to 0$ yields

$$J(x,\alpha,\Psi) \le V(U,\alpha)+q(x-U). \tag{3.18}$$

Since (3.18) holds for any $\Psi \in \mathcal{A}_{x,\alpha}$,

$$V(x,\alpha) \le V(U,\alpha)+q(x-U). \tag{3.19}$$

On the other hand, it is obvious (by harvesting instantaneously $x-U$ at time $t=0$) that

$$V(x,\alpha) \ge V(U,\alpha)+q(x-U). \tag{3.20}$$

In view of (3.19) and (3.20),

$$V(x,\alpha) = V(U,\alpha)+q(x-U).$$

The conclusion follows. □

We proceed to discuss the impact of large white noise. By using Lemma 3.1, we construct an upper bound for the value function. Then, we show that the value function approaches the upper bound as the white noise intensity approaches infinity. The following result is motivated by the presence of multiplicative noise.

Theorem 3.4 *Suppose that $\lambda > 0$. Moreover, there exist positive constants $\beta \in (0,1)$, K, and N such that*

$$xb(x,\alpha) \le K(1+x^2), \quad b(x,\alpha)q - \delta q(x-\lambda) \le K(x^\beta+1), \tag{3.21}$$

and

$$|\sigma(x,\alpha)| \ge Nx \ \text{ for } \ (x,\alpha) \in \mathcal{S}\times\mathcal{M}.$$

Then,

$$\lim_{N\to\infty} V(x,\alpha) = q(x-\lambda),$$

uniformly on $[\lambda, M] \times \mathcal{M}$ *for any positive constant* $M > \lambda$.

Proof Let $M > \lambda$. For a fixed $\varepsilon \in (0, r-q)$, let $K_\varepsilon > 0$ be sufficiently large such that

$$\frac{(M+1)^\beta}{K_\varepsilon} \le \varepsilon, \quad \frac{\beta}{K_\varepsilon} \le r - q - \varepsilon.$$

Define

$$\Phi(x,\alpha) = q(x-\lambda) + \frac{(x+1)^\beta}{K_\varepsilon}, \quad (x,\alpha) \in \mathcal{S} \times \mathcal{M}.$$

Detailed computations lead to

$$\begin{aligned} \Phi'(x,\alpha) &= q + \frac{\beta}{K_\varepsilon (x+1)^{1-\beta}}, \\ \Phi''(x,\alpha) &= -\frac{\beta(1-\beta)}{K_\varepsilon (x+1)^{2-\beta}}, \quad (x,\alpha) \in \mathcal{S} \times \mathcal{M}. \end{aligned}$$

It is clear that

$$q - \Phi'(x,\alpha) \le 0, \quad \Phi'(x,\alpha) - r \le 0 \ \text{ for } \ (x,\alpha) \in \mathcal{S} \times \mathcal{M}. \tag{3.22}$$

By assumption (A)(b),

$$\kappa_2 := \sup_{(x,\alpha,c) \in \mathbb{R}_+ \times \mathcal{M} \times \mathcal{U}} \Big[p(x,\alpha,c) - \Phi'(x,\alpha) f(x,c) \Big] < \infty.$$

Hence,

$$\begin{aligned} \mathcal{G}\Phi(x,\alpha) \ &\le b(x,\alpha) q + \frac{\beta}{K_\varepsilon (x+1)^{1-\beta}} b(x,\alpha) \\ &\quad - \frac{\beta(1-\beta) N^2 x^2}{2K_\varepsilon (x+1)^{2-\beta}} - \delta q (x-\lambda) - \frac{\delta (x+1)^\beta}{K_\varepsilon} + \kappa_2. \end{aligned} \tag{3.23}$$

For $x \ge \lambda > 0$, by the first inequality in (3.21),

$$\begin{aligned} \frac{\beta}{K_\varepsilon (x+1)^{1-\beta}} b(x,\alpha) \ &\le \frac{\beta}{K_\varepsilon x (x+1)^{1-\beta}} x b(x,\alpha) \\ &\le \frac{\beta K (1+x^2)}{K_\varepsilon x^{2-\beta}} \\ &\le \frac{\beta K}{K_\varepsilon} \Big(x^\beta + \frac{1}{\lambda^{2-\beta}} \Big). \end{aligned}$$

It follows from (3.21) and (3.23) that for $x \geq \lambda$,

$$\begin{aligned}\mathcal{G}\Phi(x,\alpha) &\leq K(x^\beta+1)+\frac{\beta}{K_\varepsilon(x+1)^{1-\beta}}b(x,\alpha)-\frac{\beta(1-\beta)N^2x^2}{K_\varepsilon(x+1)^{2-\beta}}+\kappa_2\\ &\leq \Big[K+\frac{\beta K}{K_\varepsilon}-\frac{\beta(1-\beta)N^2x^{2-\beta}}{2K_\varepsilon(x+1)^{2-\beta}}\Big]x^\beta+K+\frac{\beta K}{K_\varepsilon\lambda^{2-\beta}}+\kappa_2.\end{aligned}$$

Hence, there exists a positive number N_0 such that for any $N \geq N_0$,

$$\mathcal{G}\Phi(x,\alpha) \leq 0 \ \text{ for } \ (x,\alpha) \in \mathcal{S}\times\mathcal{M}. \tag{3.24}$$

By (3.22), (3.24), and Lemma 3.1, we obtain

$$V(x,\alpha) \leq q(x-\lambda)+\frac{(x+1)^\beta}{K_\varepsilon}, \ \ (x,\alpha)\in\mathcal{S}\times\mathcal{M}$$

provided that $N \geq N_0$. In particular, since $\dfrac{(M+1)^\beta}{K_\varepsilon} \leq \varepsilon$, then for $N \geq N_0$, we have

$$V(x,\alpha) \leq q(x-\lambda)+\varepsilon \ \text{ for } \ (x,\alpha)\in[\lambda,M]\times\mathcal{M}. \tag{3.25}$$

Let $\Psi = (C, Y, Z) \in \mathcal{A}_{x,\alpha}$ given by

$$C(t)=0, \ \ Y(t)=x-\lambda, \ \ Z(t)=0 \ \text{ for } \ t\geq 0.$$

Then, $J(x,\alpha,\Psi) = q(x-\lambda)$. It follows that $V(x,\alpha) \geq q(x-\lambda)$. This together with (3.25) yields that

$$0 \leq V(x,\alpha)-q(x-\lambda) \leq \varepsilon \ \text{ for } \ (x,\alpha)\in[\lambda,M]\times\mathcal{M},$$

which leads to the desired conclusion. □

We have a similar result regarding to the case of additive noise.

Theorem 3.5 *Suppose that $\lambda > 0$. Moreover, there exist positive constants K and N such that*

$$xb(x,\alpha) \leq K(1+x^2), \ \ b(x,\alpha)q-\delta q(x-\lambda) \leq K, \tag{3.26}$$

and

$$|\sigma(x,\alpha)| \geq N \ \text{ for } \ (x,\alpha)\in\mathcal{S}\times\mathcal{M}.$$

Then,

$$\lim_{N\to\infty} V(x,\alpha) = q(x-\lambda),$$

uniformly on $[\lambda, M]\times\mathcal{M}$ for any positive constant $M > \lambda$.

Proof Let $M > \lambda$. For a fixed $\varepsilon \in (0, r-q)$, let $\beta \in (0,1)$ be such that $\beta K < \delta$ and let $K_\varepsilon > 0$ be sufficiently large such that

$$\frac{\beta}{K_\varepsilon} \le r - q - \varepsilon, \quad \frac{(M+1)^\beta}{K_\varepsilon} \le \varepsilon.$$

Define

$$\Phi(x,\alpha) = q(x-\lambda) + \frac{(x+1)^\beta}{K_\varepsilon} \text{ for } (x,\alpha) \in \mathcal{S} \times \mathcal{M}.$$

We have

$$\begin{aligned} \Phi'(x,\alpha) &= q + \frac{\beta}{K_\varepsilon (x+1)^{1-\beta}}, \\ \Phi''(x,\alpha) &= -\frac{\beta(1-\beta)}{K_\varepsilon (x+1)^{2-\beta}}, \quad (x,\alpha) \in \mathcal{S} \times \mathcal{M}. \end{aligned}$$

We can check that

$$q - \Phi'(x,\alpha) \le 0, \quad \Phi'(x,\alpha) - r \le 0 \text{ for } (x,\alpha) \in \mathcal{S} \times \mathcal{M}. \tag{3.27}$$

By assumption (A)(b),

$$\kappa_3 := \sup_{(x,\alpha,c) \in \mathbb{R}_+ \times \mathcal{M} \times \mathcal{U}} \Big[p(x,\alpha,c) - \Phi'(x,\alpha) f(x,c) \Big] < \infty.$$

Hence,

$$\begin{aligned} \mathcal{G}\Phi(x,\alpha) \ &\le b(x,\alpha) q + \frac{\beta}{K_\varepsilon (x+1)^{1-\beta}} b(x,\alpha) \\ &\quad - \frac{\beta(1-\beta)N^2}{2K_\varepsilon (x+1)^{2-\beta}} - \delta q (x-\lambda) - \frac{\delta (x+1)^\beta}{K_\varepsilon} + \kappa_3. \end{aligned} \tag{3.28}$$

For $x \ge \lambda > 0$, by the first inequality in (3.26),

$$\begin{aligned} \frac{\beta}{K_\varepsilon (x+1)^{1-\beta}} b(x,\alpha) \ &\le \frac{\beta}{K_\varepsilon x (x+1)^{1-\beta}} x b(x,\alpha) \\ &\le \frac{\beta K (1+x^2)}{K_\varepsilon x^{2-\beta}} \\ &\le \frac{\beta K}{K_\varepsilon} \Big(x^\beta + \frac{1}{\lambda^{2-\beta}} \Big). \end{aligned}$$

It follows from the second inequality in (3.26) and (3.28) that for $x \geq \lambda$,

$$\begin{aligned}\mathcal{G}\Phi(x,\alpha) &\leq K + \frac{\beta K}{K_\varepsilon}\Big(x^\beta + \frac{1}{\lambda^{2-\beta}}\Big) - \frac{\beta(1-\beta)N^2}{2K_\varepsilon(x+1)^{2-\beta}} - \frac{\delta x^\beta}{K_\varepsilon} + \kappa_3 \\ &\leq K + \frac{\beta K}{K_\varepsilon \lambda^{2-\beta}} + \kappa_3 - \frac{\delta - \beta K}{K_\varepsilon} x^\beta - \frac{\beta(1-\beta)N^2}{2K_\varepsilon(x+1)^{2-\beta}}.\end{aligned}$$

Since $\beta K < \delta$, we can choose a constant $\lambda_1 > \lambda$ such that

$$K + \frac{\beta K}{K_\varepsilon \lambda^{2-\beta}} + \kappa_3 - \frac{\delta - \beta K}{K_\varepsilon} x^\beta \leq 0 \ \text{ for } \ (x,\alpha) \in [\lambda_1, \infty) \times \mathcal{M}.$$

There exists a positive number N_0 such that for any $N \geq N_0$,

$$K + \frac{\beta K}{K_\varepsilon \lambda^{2-\beta}} + \kappa_3 - \frac{\beta(1-\beta)N^2}{2K_\varepsilon(x+1)^{2-\beta}} \leq 0 \ \text{ for } \ (x,\alpha) \in [\lambda, \lambda_1].$$

Hence, for $N \geq N_0$,

$$\mathcal{G}\Phi(x,\alpha) \leq 0 \ \text{ for } \ (x,\alpha) \in \mathcal{S} \times \mathcal{M}. \tag{3.29}$$

By (3.27), (3.29), and Lemma 3.1, we have

$$V(x,\alpha) \leq q(x-\lambda) + \frac{(x+1)^\beta}{K_\varepsilon} \ \text{ for } \ (x,\alpha) \in \mathcal{S} \times \mathcal{M}$$

provided that $N \geq N_0$. In particular, since $\dfrac{(M+1)^\beta}{K_\varepsilon} \leq \varepsilon$, for $N \geq N_0$,

$$V(x,\alpha) \leq q(x-\lambda) + \varepsilon \ \text{ for } \ (x,\alpha) \in [\lambda, M] \times \mathcal{M}. \tag{3.30}$$

Let $\Psi = (C, Y, Z) \in \mathcal{A}_{x,\alpha}$ given by

$$C(t) = 0, \quad Y(t) = x - \lambda, \quad Z(t) = 0 \ \text{ for } \ t \geq 0.$$

Then, $J(x,\alpha,\Psi) = q(x-\lambda)$. It follows that $V(x,\alpha) \geq q(x-\lambda)$. This together with (3.30) gives us that

$$0 \leq V(x,\alpha) - q(x-\lambda) \leq \varepsilon \ \text{ for } \ (x,\alpha) \in [\lambda, M] \times \mathcal{M},$$

which leads to the desired conclusion. □

4 Numerical Approximation

Formally, the associated Hamilton–Jacobi–Bellman equation of the underlying problem is given by

$$\begin{aligned} &\max\left\{\mathcal{G}\Phi(x,\alpha), q-\Phi'(x,\alpha), \Phi'(x,\alpha)-r\right\}=0 \ \text{ for } \ (x,\alpha)\in\mathcal{S}\times\mathcal{M},\\ &\Phi(x,\alpha)=0 \ \text{ for } \ (x,\alpha)\notin\mathcal{S}\times\mathcal{M}, \end{aligned} \tag{4.1}$$

where

$$\mathcal{G}\Phi(x,\alpha)=(\mathcal{L}-\delta)\Phi(x,\alpha)+\max_{c\in\mathcal{U}}\Big[p(x,\alpha,c)-\Phi'(x,\alpha)f(x,c)\Big].$$

A closed-form solution to (4.1) is virtually impossible to obtain. Thus, we proceed with the Markov chain approximation method; see [10, 15, 16]. That is, we construct a discrete time, finite-state, controlled Markov chain to approximate the controlled switching diffusions. In [10], the authors applied that method to solve a harvesting-type problem for a regime-switching diffusion with a regular control and a singular control. In this paper, we need to adopt and modify the approximation to fit the combination of a two-sided singular control and regular control formulation. In view of Theorem 3.3, we only need to choose a large positive integer U and compute the value function on $\mathcal{S}\cap[0,U]=[\lambda,U]$. With $x\in[\lambda,U]$, we can rewrite (2.4) as

$$\begin{aligned} X(t) \ &= x+\int_0^t\big(b(X(s),\Lambda(s))-f\big(X(s),C(s)\big)\mathrm{d}s+\int_0^t\sigma\big(X(s),\Lambda(s)\big)\mathrm{d}w(s)\\ &\quad -Y(t)+Z(t). \end{aligned} \tag{4.2}$$

The payoff functional is

$$\begin{aligned} J(x,\alpha,\Psi)=\mathbb{E}\Big[&\int_0^\tau e^{-\delta s}p(X(s),\Lambda(s),C(s))\mathrm{d}s\\ &+\int_0^\tau e^{-\delta s}q\mathrm{d}Y(s)-\int_0^\tau e^{-\delta s}r\mathrm{d}Z(s)\Big]. \end{aligned} \tag{4.3}$$

4.1 Approximating Markov Chains

Let h be a discretization parameter for $X(\cdot)$. Assume without loss of generality that U is a multiple of h. Define

$$S_h:=[0,U]\cap\{x\in[\lambda-h,U]: x=kh, k\in\mathbb{Z}_+\}.$$

Let $\{(X_n^h,\Lambda_n^h): n\in\mathbb{Z}_+\}$ be a discrete-time controlled Markov chain with state space $S_h\times\mathcal{M}$. For each n, the increments of the chain $\Delta X_n^h=X_{n+1}^h-X_n^h$ approximate exactly one of the following quantities dynamically.

– Diffusion step: $\big(b(X(t),\Lambda(t))-f(X(t),C(t)\big)\mathrm{d}t+\sigma(X(t),\Lambda(t))\mathrm{d}w(t)$.

– Impulsive harvesting step: $-\mathrm{d}Y(t)$.
– Impulsive renewing step: $\mathrm{d}Z(t)$.

We use π_n^h to denote the type of step n. We set $\pi_n^h = 0$ if step n is a diffusion step, $\pi_n^h = 1$ if step n is an impulsive harvesting step, and $\pi_n^h = -1$ if step n is an impulsive renewing step. Each of these steps is described precisely in what follows.

(a) If $\pi_n^h = 0$, a regular control C_n^h is exercised so that ΔX_n^h is to behave like an increment of $\int(b - f)\mathrm{d}t + \sigma\,\mathrm{d}w$ over a small time interval. (b) If $\pi_n^h = 1$, a regular control C_n^h and an impulsive harvesting $\Delta Y_n^h = h$ are applied. (c) If $\pi_n^h = -1$, a regular control C_n^h and an impulsive renewing $\Delta Z_n^h = h$ are applied.

Thus, the amounts of impulsive harvesting and impulsive renewing at any step n are $\Delta Y_n^h = hI_{\{\pi_n^h=1\}}$ and $\Delta Z_n^h = hI_{\{\pi_n^h=-1\}}$, respectively. Let $C^h = \{C_n^h\}$ be a sequence of regular controls. The space of regular controls is given by $\mathcal{U}$. Let $\pi^h = \{\pi_n^h\}_{n\geq 0}$ denote the sequence of control types. We denote by $p^h((x,\alpha),(y,\ell)|\pi,c)$ the transition probability from state (x,α) to another state (y,ℓ) under the control type π and regular control c. Denote $\mathcal{F}_n^h = \sigma\{X_k^h, \Lambda_k^h, \pi_k^h, C_k^h, k \leq n\}$.

The sequence $\{(\pi_n^h, C_n^h)\}$ is said to be admissible if it satisfies the following conditions:

(a) π_n^h, C_n^h are $\sigma\{X_0^h, \ldots, X_n^h, \Lambda_0^h, \ldots, \Lambda_n^h, \pi_0^h, \ldots, \pi_{n-1}^h, C_1^h, \ldots, C_{n-1}^h\}-$adapted,
(b) For any $(x,\alpha) \in S_h \times \mathcal{M}$, we have

$$\begin{aligned}\mathbb{P}\{X_{n+1}^h = x, \Lambda_{n+1}^h = \alpha|\mathcal{F}_n^h\} &= \mathbb{P}\{X_{n+1}^h = x, \Lambda_{n+1}^h = \alpha|X_n^h, \Lambda_n^h, \pi_n^h, C_n^h\}\\ &= p^h\big((X_n^h, \Lambda_n^h), (x,\alpha)|\pi_n^h, C_n^h\big),\end{aligned}$$

(c) $\pi_n^h \in \{0, \pm 1\}$, $C_n^h \in \mathcal{U}$, $X_n^h \in S_h$, $\Lambda_n^h \in \mathcal{M}$ for all $n \in \mathbb{Z}_+$.

The class of all admissible control sequences (π^h, C^h) for initial state (x,α) will be denoted by $\mathcal{A}_{x,\alpha}^h$.

For each $(x,\alpha,i,c) \in S_h \times \mathcal{M} \times \{0, \pm 1\} \times \mathcal{U}$, we define a family of the interpolation intervals $\Delta t^h(x,\alpha,i,c)$. The values of $\Delta t^h(x,\alpha,i,c)$ will be specified later. Then, we define

$$t_0^h = 0, \quad \Delta t_k^h = \Delta t^h(X_k^h, \Lambda_k^h, \pi_k^h, C_k^h), \quad t_n^h = \sum_{k=0}^{n-1} \Delta t_k^h. \tag{4.4}$$

For $(x,\alpha) \in S_h \times \mathcal{M}$ and $(\pi^h, C^h) \in \mathcal{A}_{x,\alpha}^h$, the payoff functional for the controlled Markov chain is defined as

$$J^h(x,\alpha,\pi^h,C^h) = \mathbb{E}\sum_{k=0}^{\eta_h-1} e^{-\delta t_k^h}\left\{p(X_k^h, \Lambda_k^h, C_k^h)\Delta t_k^h + q\,\Delta Y_k^h - r\,\Delta Z_k^h\right\}, \tag{4.5}$$

with $\eta_h = \inf\{n \geq 0 : X_n^h \notin \mathcal{S}\}$. The value function of the controlled Markov chain is

$$V^h(x,\alpha) = \sup_{(\pi^h,C^h)\in\mathcal{A}_{x,\alpha}^h} J^h(x,\alpha,\pi^h,C^h). \tag{4.6}$$

The corresponding dynamic programming equation for the discrete approximation is given by $V^h(x,\alpha) = \max\{V_{i,c}^h(x,\alpha) : i \in \{-1,0,1\}, c \in \mathcal{U}\}$ for any $(x,\alpha) \in$

$S_h \times \mathcal{M}$, where

$$
\begin{aligned}
V^h_{1,c}(x,\alpha) &= \big(V^h(x-h,\alpha)+qh\big)I_{\{x>\lambda\}},\\
V^h_{-1,c}(x,\alpha) &= \big(V^h(x+h,\alpha)-rh\big)I_{\{x<U\}},\\
V^h_{0,c}(x,\alpha) &= e^{-\delta\Delta t^h(x,\alpha,0,c)}\sum_{(y,\ell)\in S_h} V^h(y,\ell)p^h\big((x,\alpha),(y,\ell)|0,c\big)I_{\{x<U\}}\\
&\qquad +p(x,\alpha,c)\Delta t^h(x,\alpha,0,c)I_{\{x<U\}}.
\end{aligned}
$$

Note that $V^h(x,\alpha)=0$ for $(x,\alpha)\notin S_h\times\mathcal{M}$.

4.2 Transition Probabilities

Let $\mathbb{E}^{h,\pi,c}_{x,\alpha,n}$, $\mathbb{C}ov^{h,\pi,c}_{x,\alpha,n}$ denote the conditional expectation and covariance given by

$$
\{X^h_k,\Lambda^h_k,\pi^h_k,C^h_k,k\le n-1,X^h_n=x,\Lambda^h_n=\alpha,\pi^h_n=\pi,C^h_n=c\},
$$

respectively. We proceed to define transition probabilities $p^h\big((x,\alpha),(y,\ell)|\pi,c\big)$ so that the controlled Markov chain $\{(X^h_n,\Lambda^h_n)\}$ is locally consistent with respect to $(X(\cdot),\Lambda(\cdot))$ in the sense that the following hold.

$$
\begin{aligned}
&\mathbb{E}^{h,0,c}_{x,\alpha,n}\Delta X^h_n = b(x,\alpha)\Delta t^h(x,\alpha,0,c)+o(\Delta t^h(x,\alpha,0,c)),\\
&Cov^{h,0,c}_{x,\alpha,n}\Delta X^h_n = \sigma^2(x,\alpha)\Delta t^h(x,\alpha,0,c)+o(\Delta t^h(x,\alpha,0,c)),\\
&\sup_{n,\ \omega}|\Delta X^h_n|\to 0 \ \text{ as } \ h\to 0.
\end{aligned}
\tag{4.7}
$$

To this end, motivated by the procedure in [10, 16], we construct the transition probabilities below. For $(x,\alpha)\in S_h\times\mathcal{M}$, $\lambda\le x<U$ and $c\in\mathcal{U}$, we define

$$
Q_h(x,\alpha,0,c)=\sigma^2(x,\alpha)+h|b(x,\alpha)-f(x,c)|-h^2\Gamma_{\alpha\alpha}+\zeta(h).
$$

We set $\zeta(h)=h$. If, in addition, $\sigma^2(x,\alpha)>0$ for any $(x,\alpha)\in\mathcal{S}\times\mathcal{M}$, we can simply take $\zeta(h)=0$. Then, we define

$$
\begin{aligned}
p^h\left((x,\alpha),(x+h,\alpha)|0,c\right) &= \frac{\sigma^2(x,\alpha)/2+\big(b(x,\alpha)-f(x,c)\big)^+h}{Q_h(x,\alpha,0,c)},\\
p^h\left((x,\alpha),(x-h,\alpha)|0,c\right) &= \frac{\sigma^2(x,\alpha)/2+\big(b(x,\alpha)-f(x,c)\big)^-h}{Q_h(x,\alpha,0,c)},\\
p^h\left((x,\alpha),(x,\ell)|0,c\right) &= \frac{h^2\Gamma_{\alpha\ell}}{Q_h(x,\alpha,0,c)} \text{ for } \alpha\neq\ell,\\
p^h\left((x,\alpha),(x,\alpha)|0,c\right) &= \frac{\zeta(h)}{Q_h(x,\alpha,0,c)},\quad \Delta t^h(x,\alpha,0,c)=\frac{h^2}{Q_h(x,\alpha,0,c)},\\
p^h\left((x,\alpha),(y,\ell)|0,c\right) &= 0 \quad \text{for all not listed values of } (y,\ell)\in S_h\times\mathcal{M}.
\end{aligned}
\tag{4.8}
$$

At impulsive harvesting and renewing steps, we define

$$\begin{aligned} &p^h\left((x,\alpha),(x-h,\alpha)|1,c\right)=1, \quad \Delta t^h(x,\alpha,1,c)=0,\\ &p^h\left((x,\alpha),(x+h,\alpha)|-1,c\right)=1, \quad \Delta t^h(x,\alpha,-1,c)=0. \end{aligned} \tag{4.9}$$

Thus, $p^h\left((x,\alpha),(y,\ell)|\pm 1,c\right)=0$ for all nonlisted values of $(y,\ell)\in S_h\times\mathcal{M}$. The definition of $p^h((x,\alpha),\cdot|i,c)$ for $(x,\alpha)\notin\mathcal{S}\times\mathcal{M}$ is not important since in the analysis of the control problem, the chain will be stopped at the first time $\{X_n^h\}$ exits $\mathcal{S}$. Using the above transition probabilities, we can check that the local consistency conditions of $\{(X_n^h,\Lambda_n^h)\}$ in (4.7) are satisfied.

The convergence result is based on a continuous-time interpolation of the chain and relaxed control representations. In addition, a "stretched-out" timescale is introduced to overcome the possible non-tightness of the piecewise constant interpolations associated with $Y(\cdot)$ and $Z(\cdot)$; see [10] and [15]. Using weak convergence methods, we can obtain the convergence of the value function. The main convergence result is given below. The proof is a modification of that in [10], we omit it for brevity.

Theorem 4.1 *Let $V(x,\alpha)$ and $V^h(x,\alpha)$ be value functions defined in (2.7) and (4.6), respectively. Then, $V^h(x,\alpha)\to V(x,\alpha)$ as $h\to 0$.*

5 Numerical Examples

We consider a stochastic population with harvesting and renewing given by

$$\mathrm{d}X(t)=b(X(t),\Lambda(t))\mathrm{d}t+\sigma(X(t),\Lambda(t))\mathrm{d}w(t)-C(t)\mathrm{d}t-\mathrm{d}Y(t)+\mathrm{d}Z(t), \tag{5.1}$$

where

$$b(x,\alpha)=x(\alpha-1.5x), \quad \sigma(x,\alpha)=(\alpha/2)x.$$

Suppose that $\mathcal{M}=\{1,2\}$ and the generator $\Gamma=(\Gamma_{\alpha\ell})_{2\times 2}$ of the Markov chain $\Lambda(\cdot)$ is given by

$$\Gamma_{11}=-1, \quad \Gamma_{12}=1, \quad \Gamma_{21}=1.5, \quad \Gamma_{22}=-1.5.$$

We also assume that

$$\begin{aligned} &q=1, \quad r=2.25, \quad \delta=0.05,\\ &p(x,\alpha,c)=\frac{3c}{2}-\frac{\alpha c^2}{8(1+x)} \ \text{ for } \ (x,\alpha,c)\in\mathbb{R}_+\times\mathcal{M}\times\mathcal{U},\\ &f(x,c)=c \ \text{ for } \ (x,c)\in\mathbb{R}_+\times\mathcal{U}. \end{aligned}$$

The original price and costs are $a_1=1.5$, $a_2=0.5$, $a_3=0.75$ and the regular control cost function is $g(x,\alpha,c)=\frac{\alpha c^2}{8(1+x)}$. Thus, for each $x\in\mathbb{R}_+$ and $\alpha\in\mathcal{M}$, $g(x,\alpha,c)$ has a quadratic form. For a fixed regular control c and regime $\alpha\in\mathcal{M}$, the cost function is decreasing with respect to the population size. This is motivated by an observation in harvesting problems that when the species is rare, it is more difficult to harvest leading to the higher harvesting cost; see [2]. By the state constraint, the time horizon of the control problem is $[0,\tau]$, where $\tau=\inf\{t\geq 0: X(t)<\lambda\}$. The value of λ and

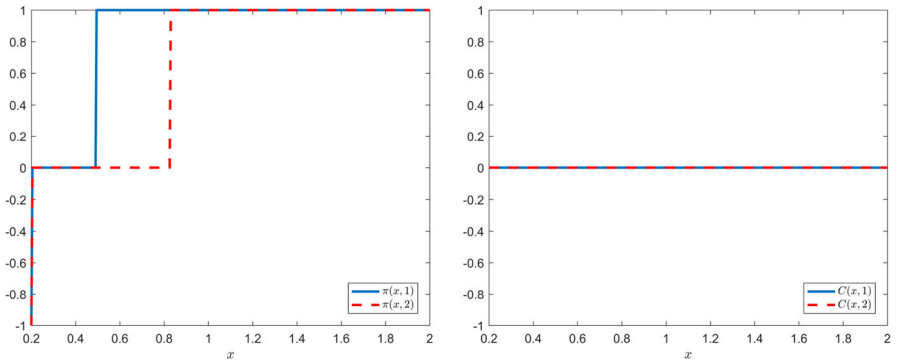

Fig. 1 Control type (left) and regular control (right) as functions of (x, α) for $\lambda = 0.2$ and $\mathcal{U} = \{0\}$ (Example 5.1)

the control set $\mathcal{U}$ are to be determined. Using the transition probabilities constructed in (4.8) and (4.9), we carry out the computation by using value iteration and policy iteration method. For each $(x, \alpha) \in S_h \times \mathcal{M}$, we use the following notations for the nth iteration: $C^{h,n}(x, \alpha)$ is the regular control, $\pi^{h,n}(x, \alpha)$ is the control type, and $V^{h,n}(x, \alpha)$ is the value function. Initially, we take

$$C^{h,0}(x, \alpha) = 0, \quad \pi^{h,0}(x, \alpha) = 1, \quad V^{h,0}(x, \alpha) = q(x - \lambda) \text{ for } (x, \alpha) \in S_h \times \mathcal{M}.$$

Note that $\left(\pi^{h,0}(x, \alpha), C^{h,0}(x, \alpha)\right)$ corresponds to the control $\Psi = (C, Y, Z) \in \mathcal{A}_{x,\alpha}$ given by

$$C(t) = 0, \quad Y(t) = x - \lambda, \quad Z(t) = 0 \text{ for } t \geq 0,$$

of the controlled switching diffusion (2.4). Based on our computation of the nth iteration, we define

$$\begin{aligned}
V_{1,c}^{h,n+1}(x, \alpha) &= \left(V^{h,n}(x - h, \alpha) + qh\right) I_{\{x > \lambda\}}, \\
V_{-1,c}^{h,n+1}(x, \alpha) &= \left(V^{h,n}(x + h, \alpha) - rh\right) I_{\{x < U\}}, \\
V_{0,c}^{h,n+1}(x, \alpha) &= e^{-\delta \Delta t^h(x,0,c,\alpha)} \big[V^{h,n}(x + h, \alpha) p^h\left((x, \alpha), (x + h, \alpha) | 0, c\right) \\
&\quad + V^{h,n}(x - h, \alpha) p^h\left((x, \alpha), (x - h, \alpha) | 0, c\right) \\
&\quad + V^{h,n}(x, \alpha) p^h\left((x, \alpha), (x, \alpha) | 0, c\right) \big] I_{\{x < U\}} + p(x, \alpha, c) \Delta t^h(x, \alpha, i, c) I_{\{x < U\}}.
\end{aligned}$$

We find an improved value $V^{h,n+1}(x, \alpha)$ and record the corresponding controls by

$$\begin{aligned}
\left(\pi^{h,n+1}(x, \alpha), C^{h,n+1}(x, \alpha)\right) &= \operatorname{argmax} \left\{ V_{i,c}^{h,n+1}(x, \alpha) : i \in \{-1, 0, 1\}, c \in \mathcal{U} \right\}, \\
V^{h,n+1}(x, \alpha) &= \max \left\{ V_{i,c}^{h,n+1}(x, \alpha) : i \in \{-1, 0, 1\}, c \in \mathcal{U} \right\}.
\end{aligned}$$

The iterations stop as soon as the increment $V^{h,n+1}(x, \alpha) - V^{h,n}(x, \alpha)$ reaches a pre-specified tolerance level. We set the error tolerance to be 10^{-6}.

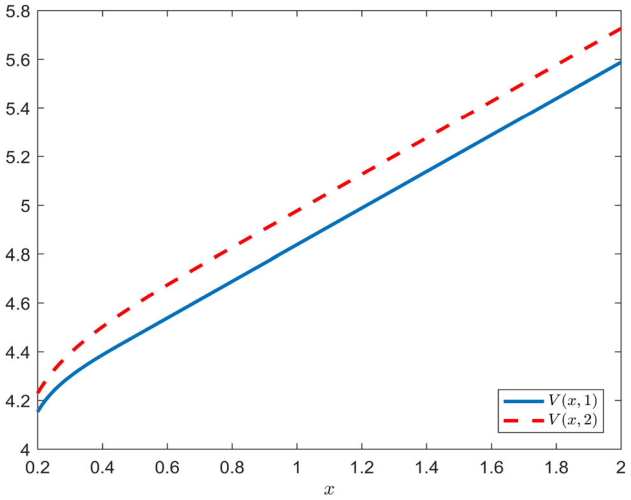

Fig. 2 Value function as a function of (x, α) for $\lambda = 0.2$ and $\mathcal{U} = \{0\}$ (Example 5.1)

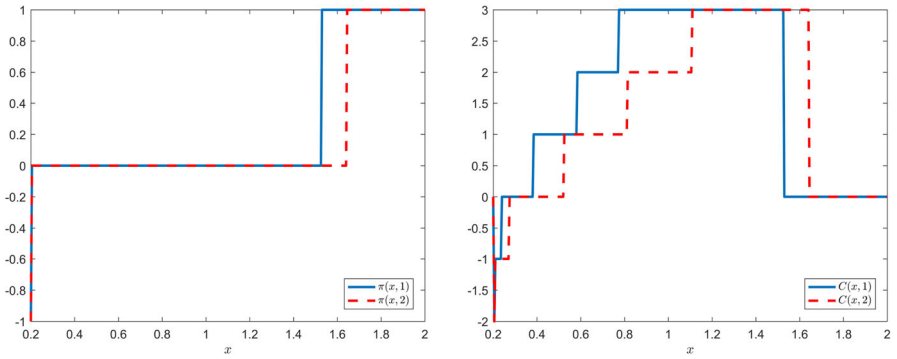

Fig. 3 Control type (left) and regular control (right) as functions of (x, α) for $\lambda = 0.2$ and $\mathcal{U} = \{k \in \mathbb{Z} : -2 \leq k \leq 3\}$ (Example 5.2)

Example 5.1 Let $\lambda = 0.2$ and $\mathcal{U} = \{0\}$. In view of Theorem 3.3, we can take $U = 2$. Figure 1 shows the control type, regular control rate as functions of population size x and regime α. The corresponding value function is shown in Fig. 2. Since $\mathcal{U} = \{0\}$, the regular control does not work. The controller can harvest or renew by exercising impulsive controls only. It appears that in both regimes, if $x = \lambda = 0.2$, an impulsive renewing is performed. There is a level U_α depending on α so that we should apply an impulsive harvesting whenever the population size is above U_α and should not harvest nor renew whenever the population size is in (λ, U_α). Thus, under this strategy, the population size is in $[\lambda, U_\alpha]$ for $t \in (0, \infty)$.

Example 5.2 Let $\lambda = 0.2$ and $\mathcal{U} = \{k \in \mathbb{Z} : -2 \leq k \leq 3\}$. In view of Theorem 3.3, we can take $U = 2$. Figure 3 shows the control type, regular control rate as functions of population size x and regime α. The corresponding value function is shown in Fig. 4.

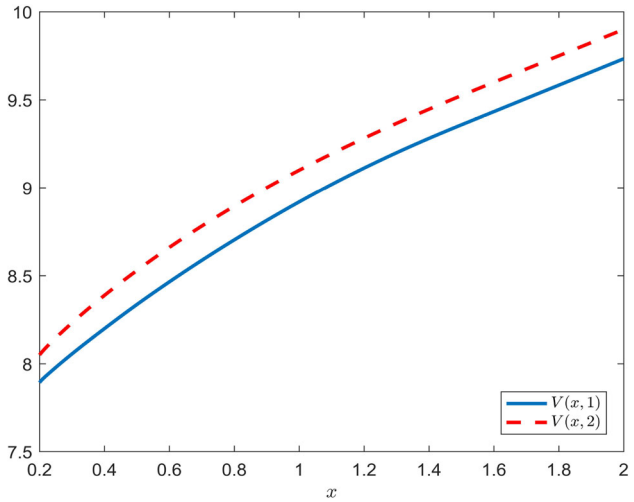

Fig. 4 Value function as a function of (x, α) for $\lambda = 0.2$ and $\mathcal{U} = \{k \in \mathbb{Z} : -2 \leq k \leq 3\}$ (Example 5.2)

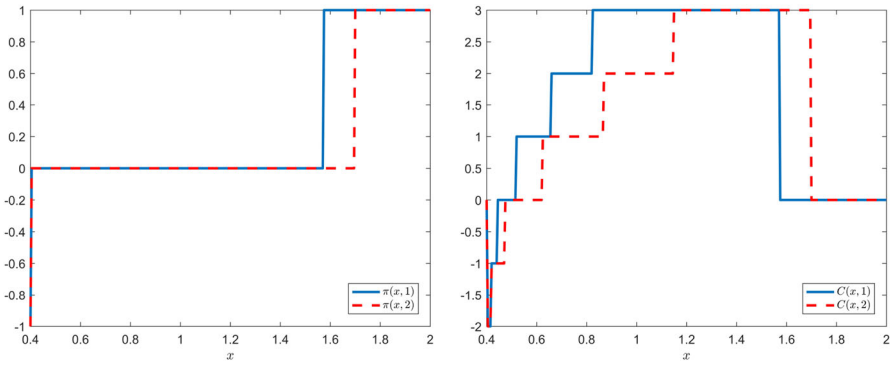

Fig. 5 Control type (left) and regular control (right) as functions of (x, α) for $\lambda = 0.2$ and $\mathcal{U} = \{k \in \mathbb{Z} : -2 \leq k \leq 3\}$ (Example 5.3)

Compared to the preceding example, the controller have more options to harvest or renew. Therefore, the value function in Fig. 4 is much larger than the preceding one. Moreover, as shown in Fig. 3, the larger the population is, the higher the harvesting rate is.

Example 5.3 Let $\lambda = 0.4$ and $\mathcal{U} = \{k \in \mathbb{Z} : -2 \leq k \leq 3\}$. In view of Theorem 3.3, we can take $U = 2$. Figure 5 shows the control type, regular control rate as functions of population size x and regime α. The corresponding value function is shown in Fig. 6. It can be seen that the value function is smaller than that in Example 5.2. This observation fits the fact that the one can harvest only if the population size is higher than $\lambda = 0.4$ compared to 0.2 in Example 5.2. In this and previous numerical experiments, we observe the same phenomena as follows: (a) if the population size hits λ, an impulsive renewing is performed to keep the species in $\mathcal{S} = [\lambda, \infty)$, which is the harvesting-renewing domain; (b) for each α, there are levels $L_\alpha^{(1)}$, $L_\alpha^{(2)}$, and $L_\alpha^{(3)}$

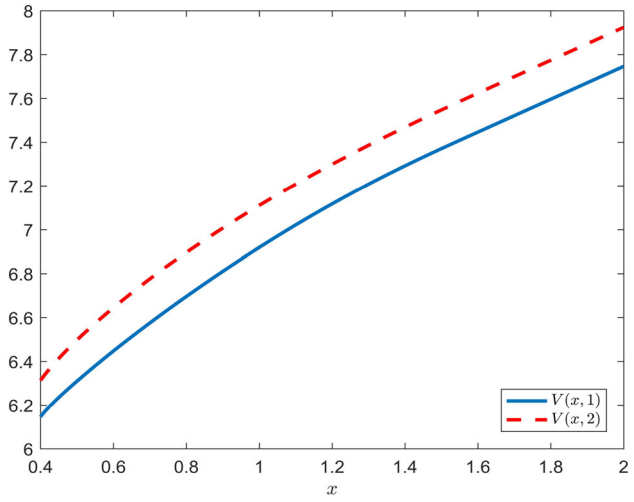

Fig. 6 Value function as a function of (x, α) for $\lambda = 0.2$ and $\mathcal{U} = \{k \in \mathbb{Z} : -2 \leq k \leq 3\}$ (Example 5.3)

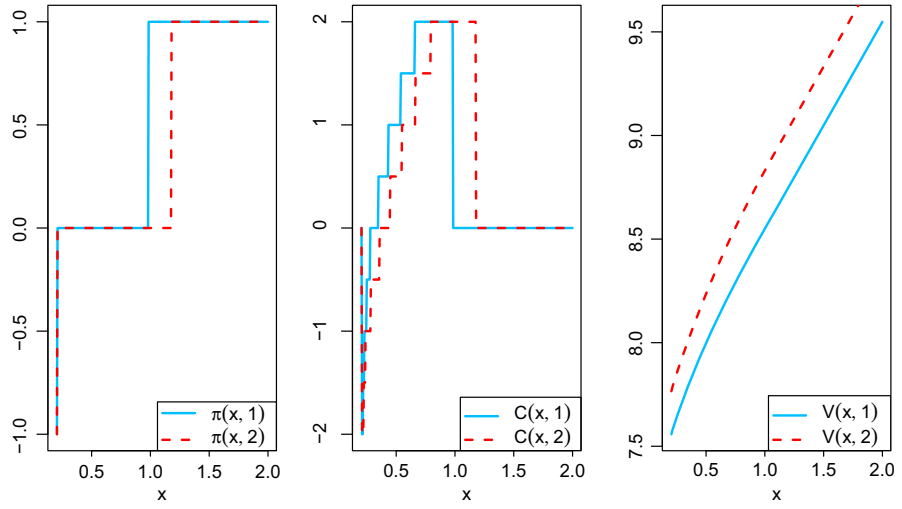

Fig. 7 Control type (left), regular control (middle), and the value function (right) as functions of (x, α) for $\lambda = 0.2, \mathcal{U} = \{k/2 : -4 \leq k \leq 4, k \in \mathbb{Z}\}$, $(\mu_1, \mu_2) = (1, 0)$ (Example 5.4)

so that we should renew with bounded rates if $x \in (\lambda, L_\alpha^{(1)})$, we should not harvest nor renew if $x \in [L_\alpha^{(1)}, L_\alpha^{(2)})$, we should harvest with bounded rates if $x \in [L_\alpha^{(2)}, L_\alpha^{(3)})$, and we should perform an impulsive harvesting if $x \in [L_\alpha^{(3)}, U]$.

It can be seen from the numerical experiments that we should keep the population size in the interval $[\lambda, U]$ for $t > 0$. In other words, if the initial population size $x > \lambda$, then $\tau = \infty$ almost surely. Thus, compared with the well-known formulations for harvesting-type problems, the proposed model offers an effective planning for the balance between economical aspect and the sustainable purpose.

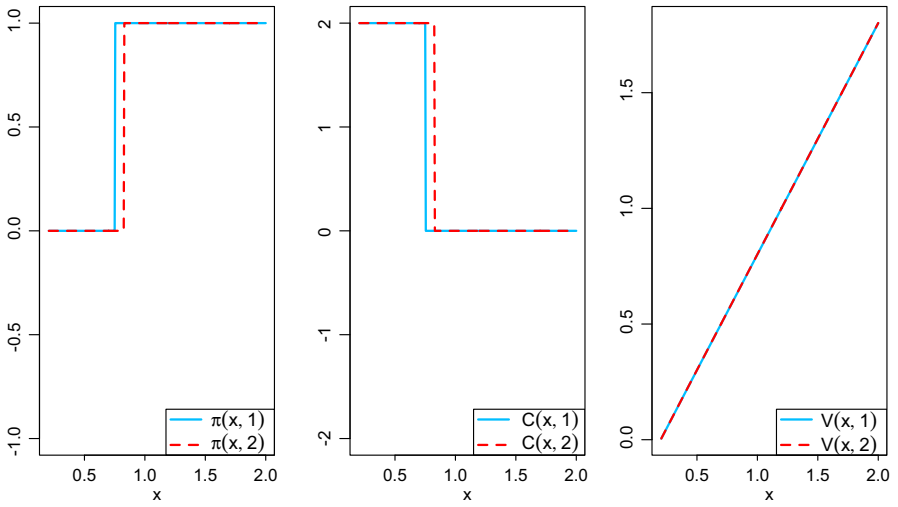

Fig. 8 Control type (left), regular control (middle), and the value function (right) as functions of (x, α) for $\lambda = 0.2, \mathcal{U} = \{k/2 : -4 \le k \le 4, k \in \mathbb{Z}\}$, $(\mu_1, \mu_2) = (30, 0)$ (Example 5.4)

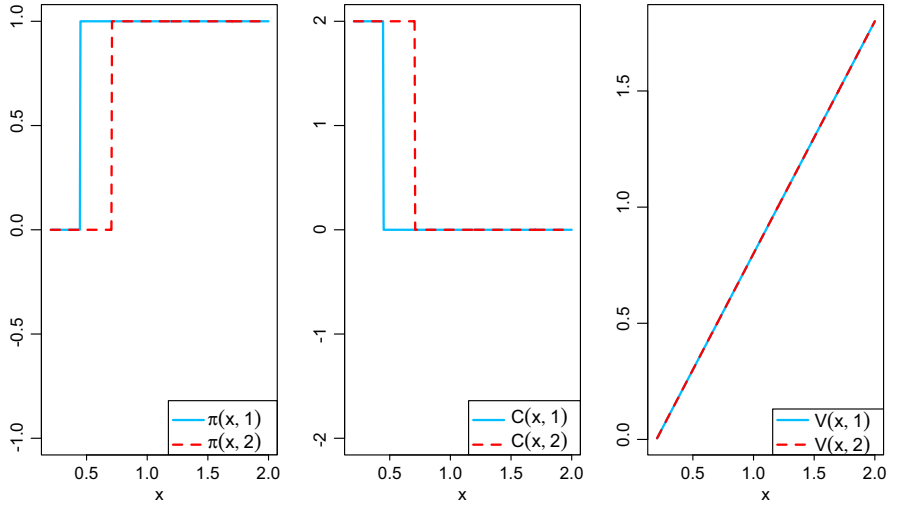

Fig. 9 Control type (left), regular control (middle), and the value function (right) as functions of (x, α) for $\lambda = 0.2, \mathcal{U} = \{k/2 : -4 \le k \le 4, k \in \mathbb{Z}\}$, $(\mu_1, \mu_2) = (0, 30)$ (Example 5.4)

Example 5.4 We consider Eq. (5.1) with

$$b(x, \alpha) = x(\alpha - 1.5x), \quad \sigma(x, \alpha) = \mu_1 x + \mu_2 \ \text{ for } \ (x, \alpha) \in \mathbb{R}_+ \times \mathcal{M},$$

where the constants μ_1 and μ_2 are to be determined. The regular cost function is $g(x, \alpha, c) = \frac{c^2}{10}$, the control set is $\mathcal{U} = \{k/2 : -4 \le k \le 4, k \in \mathbb{Z}\}$, and we keep the other data as in the preceding examples. To explore how noise impacts the problem,

we first choose $(\mu_1, \mu_2) = (1, 0)$ and record the results in Fig. 7. The results for large white noise intensities when $(\mu_1, \mu_2) = (30, 0)$ and $(\mu_1, \mu_2) = (0, 30)$ are presented in Figs. 8 and 9, respectively. It turns out, as stated in Theorems 3.4 and 3.5, when the white noise intensity is very large, the value function $V(x, \alpha)$ is close to $q(x - \lambda)$ for $(x, \alpha) \in [\lambda, \infty) \times \mathcal{M}$ (see Figs. 8 and 9). Also, as shown in Figs. 8 and 9, under a large white noise intensity, one should always harvest (either with the maximal bounded harvesting rate $C(x, \alpha) = 2$ or with an impulsive harvesting) and never renew. Meanwhile, as shown in Fig. 7, when the noise intensity is not large, the regular control takes a variety of values in the control set $\mathcal{U}$ and the value function is much higher than those in Figs. 8 and 9. We refer to [1, 6, 7] for more insight and numerical experiments regarding how noise can impact harvesting and renewing actions.

6 Further Remarks

This paper has been devoted to the study of a generalized harvesting problem for a stochastic population. We have established the finiteness and continuity of the value function. Moreover, we have shown that for common population systems, it is important to maintain the population size in a bounded set. We have also studied the impact of large white noise on harvesting. A numerical algorithm has been constructed by the Markov chain approximation methods. The numerical experiments have revealed the effect of Markovian switching and control costs associated with harvesting/renewing activities. It has been observed that under a mixed singular control formulation, the decision-makers have more options to harvest or renew the species, which is much more beneficial than the known models with no renewing, or with only one control component. Moreover, the state constraint provides an effective strategy for sustainable purpose.

In this paper, we focus on single species. The development in this paper can be carried over to the multidimensional case. As for single species models (Theorem 3.3), it appears that the controller should keep the population sizes in a bounded set. However, in order to identify such a bounded set, one needs to be careful to handle the interaction between species. Although multidimensional systems can be handled, the multidimensional structure added more difficulty to study the impact of large white noise. The computation of multidimensional ecosystems with many constraints and controls is another challenging task. In addition, one can also consider the problem under other constraints, random prices, and random cost functions.

Acknowledgements The research of Ky Q. Tran was supported by the National Research Foundation of Korea grant funded by the Korea Government (MIST) NRF-2021R1F1A1062361. The research of George Yin was supported in part by the National Science Foundation under Grant DMS-2204240.

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