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Costello's pushforward formula: errata and generalization

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Abstract. Costello's pushforward formula relates virtual fundamental classes of virtually birational algebraic stacks. Its original formulation omits a necessary hypothesis, whose addition is not sufficient to correct the proof. We supply a substitute for Costello's notion of pure degree and prove the pushforward formula with this definition. We also show the hypotheses of the corrected pushforward formula are satisfied in a variety of its applications. Some adjustments to the original proofs are required in several cases, including the original one.

1. Introduction

If $f : X' \to X$ is a proper, birational morphism of varieties, then the fundamental class of X' pushes forward to the fundamental class of X. Birationality can be relaxed to generic finiteness of degree d, in which case $f_*[X'] = d[X]$. Costello's pushforward formula asserts the same holds for virtual fundamental classes in a situation that might be called "virtual birationality".

Theorem 1.1. (Costello's pushforward formula) *Suppose there is a cartesian diagram*

$$\begin{array}{ccc} X' & \xrightarrow{f} & X \\ {\scriptstyle p'}\downarrow & \ulcorner & \downarrow{\scriptstyle p} \\ Y' & \xrightarrow[g]{} & Y \end{array} \tag{1}$$

such that

(1) *X' and X are Deligne–Mumford stacks;*
(2) *Y' and Y are Artin stacks of the same pure dimension;*
(3) *g is a morphism of Deligne–Mumford type and* pure degree *d;*
(4) *f is proper;*

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(5) *p has a perfect relative obstruction theory E inducing a perfect relative obstruction theory E' for p' by pullback.*

Then $f_*[X'/Y']^{\mathrm{vir}} = d[X/Y]^{\mathrm{vir}}$.

At issue is the definition of pure degree. Costello defines a morphism $g : Y' \to Y$ of Deligne–Mumford type to be of pure degree d if both Y' and Y have the same pure dimension and all of the generic fibers of Y' over Y are finite of degree d. With this definition of pure degree, Theorem 1.1 is false: see Examples 2.1 and 2.2 . However, Manolache shows that the formula is true if either g is projective or g is proper and Y is a Deligne–Mumford stack [1, Proposition 5.29 and Remark 5.30].

In Sect. 2 we prove that Costello's original statement is valid, provided pure degree is defined as in Definition 2.3. Our definition includes all diagrams (1) in which g is proper, which is easier to verify in practice than projectivity.

We found almost twenty papers that used Costello's pushforward formula, including several by the second author. The bulk of the present paper is devoted to checking that the relevant maps are indeed proper to ensure the formula was used correctly. This list is not meant to be exhaustive, but representative of techniques used to remedy the situation.

2. Pure degree and the pushforward formula

The following two examples of squares (1) show some properness assumption is necessary for the pushforward formula to hold.

Example 2.1. Let Y be the affine line, Y' the affine line with a doubled origin, and $Y' \to Y$ the projection that is the identity on each copy. Let X be the origin of Y. The morphisms p and p' are local complete intersection embeddings, so their canonical obstruction theories induce virtual fundamental classes that are the ordinary fundamental classes. The map g has pure degree 1 but f has degree 2.

Example 2.2. Let Y be the affine line and X its origin. Let Y' be the disjoint union of $\mathbb{A}^1$ and $\mathbb{A}^1 - \{0\}$. Then $Y' \to Y$ has pure degree 2 but $X' \to X$ is an isomorphism, hence has pure degree 1.

On the other hand, it would be too much to insist that g actually be proper. For example, Y' might have a component that is not proper over Y but is sufficiently far away from the image of p so as not to affect $[X/Y]^{\mathrm{vir}}$. We propose the following definition of pure degree:

Definition 2.3. We say that a Deligne–Mumford type morphism of locally noetherian Artin stacks $g : Y' \to Y$ is *pure* along $p : X \to Y$ if, whenever S is the spectrum of a discrete valuation ring with closed point s, and $f : S \to Y$ is a morphism such that $f(s)$ lies in the image of X, the base change $Y'_S \to S$ is proper.

We elaborate on the phrase "of degree d over" in Appendix A.

Remark 2.4. Definition 2.3 is related, but not equivalent, to the definition of Raynaud and Gruson [5, Définition (3.3)]. The map $Y' \to Y$ in Example 2.2 fails to be pure by either definition, while Example 2.2 is pure according to Raynaud and Gruson [5, Définition (3.3)] but not according to Definition 2.3.

Remark 2.5. We record some consequences of Definition 2.3.

- If a map $Y' \to Y$ is proper, it is pure along any morphism $X \to Y$.
- If $\widetilde{X} \to X$ is surjective and $Y' \to Y$ is pure along $\widetilde{X} \to X \to Y$, then it's also pure along $X \to Y$.
- If $Y' \to Y$ is of pure degree along a map $X \to Y$, then it is of pure degree along any map $Z \to X \to Y$.
- Suppose in diagram (1) that p, p' are open immersions. If f is proper, then $g : Y' \to Y$ is of pure degree along p for topological reasons.
- Purity is stable under base change in Y.
- Purity of g in diagram (1) implies properness of f, so the assumption that f be proper is redundant.

Proposition 2.6. *With notation as in the statement of Theorem 1.1, the map of relative intrinsic normal cones $C_{X'/Y'} \to C_{X/Y}$ is of degree d over each generic point of $C_{X/Y}$.*

Proof. This assertion is local in $C_{X/Y}$, so it is also local in X and Y. Replace both by smooth covers to assume X, Y are affine schemes.

The morphism $p : X \to Y$ can be factored as a closed embedding followed by a smooth morphism: $X \to \widetilde{Y} \to Y$. Then $C_{X/Y}$ is the stack quotient of $C_{X/\widetilde{Y}}$ by $T_{\widetilde{Y}/Y} \times_{\widetilde{Y}} X$; likewise $C_{X'/Y'}$ is the quotient of $C_{X'/\widetilde{Y}'}$ by $T_{\widetilde{Y}/Y} \times_{\widetilde{Y}} X'$ (where $\widetilde{Y}'$ is the base change of $\widetilde{Y}$ to Y'). The generic fibers of $C_{X'/Y'}$ over $C_{X/Y}$ have the same degrees as the generic fibers of $C_{X'/\widetilde{Y}'}$ over $C_{X/\widetilde{Y}}$. We may therefore replace Y by $\widetilde{Y}$ and assume that $p : X \to Y$ is a closed embedding.

Let $M' \to M$ be the morphism of deformations to the normal cone induced by the commutative diagram (1). Recall that M is the complement of the strict transform of $Y \times \{0\}$ in the blowup of $Y \times \mathbb{A}^1$ along $X \times \{0\}$. The normal cone $C = C_{X/Y}$ is the fiber of M over $0 \in \mathbb{A}^1$.

Let ξ be a generic point of C. Let R be the integral closure of $\mathcal{O}_{M,\xi}$. Then R is a 1-dimensional, integrally closed, noetherian local ring, hence is a discrete valuation ring. By construction, the composition of $\operatorname{Spec} R \to M \to Y$ sends the closed point of $\operatorname{Spec} R$ to $X \subset Y$ and its open point to the generic point of Y as in Definition 2.3.

By assumption, this implies the base change $Y'_R := Y' \times_Y \operatorname{Spec} R \to \operatorname{Spec} R$ is proper. Then $M' \times_M \operatorname{Spec} R \to \operatorname{Spec} R$ is also proper: the valuative criterion requires a unique lift for a commutative diagram

$$\begin{array}{ccccccc}
\operatorname{Spec} K' & \longrightarrow & & \longrightarrow & M' & \longrightarrow & Y' \\
\downarrow & & & & \downarrow & & \downarrow \\
\operatorname{Spec} R' & \longrightarrow & \operatorname{Spec} R & \longrightarrow & M & \longrightarrow & Y
\end{array}$$

after a finite extension R' of R, but we get the lift $\operatorname{Spec} R' \to Y'$ by the properness of $Y'_R \to \operatorname{Spec} R$. This induces a map $\operatorname{Spec} R' \to Y' \times \mathbb{A}^1$ that factors through the blowup $\overline{M}'$ of $Y' \times \mathbb{A}^1$ along $X' \times \{0\}$. This map lies in $M' \subseteq \overline{M}'$ because the generic point of $\operatorname{Spec} R'$ maps a point of $Y' \times \mathbb{A}^1$ over the generic point of $\mathbb{A}^1$. In particular, the closed point cannot lie in the strict transform of $Y \times \{0\}$, so the image of $\operatorname{Spec} R' \to \overline{M}'$ is contained in M.

Let us write $M'_R = \operatorname{Spec} R \times_M M'$ (note that the fiber product is over M, not over Y). We have just seen that $M'_R \to \operatorname{Spec} R$ is proper. We argue that it is also flat. It suffices to show M'_R is torsion free. But under the map $\operatorname{Spec} R \to M \to \mathbb{A}^1$, a uniformizer t of $\mathbb{A}^1$ at the origin pulls back to a nonzero element of R, which is a power of the maximal ideal of R, since R is a discrete valuation ring. By construction of M', it has no t-torsion, so M'_R must be torsion-free over R.

Now M'_R is a proper and flat Deligne–Mumford stack over $\operatorname{Spec} R$. It remains only to show that the fibers of M'_R have the same degree over $\operatorname{Spec} R$. We can replace R with a flat cover, so we assume R is complete. Let $U \to M'_R$ be an étale cover with U affine. Then U is 1-dimensional, flat, of finite type over $\operatorname{Spec} R$. Therefore it is quasifinite over R. Since R is complete, $U = U_0 \sqcup V$ where U_0 is finite over R and the closed fiber of V is empty. We can replace U by U_0 and then $U \to M'_R$ and $U \to \operatorname{Spec} R$ are both finite and flat. Note $U \to M'_R$ is finite because M'_R is proper over $\operatorname{Spec} R$.

Assume without loss of generality that M'_R is connected. If d is the degree of M'_R over the generic fiber of $\operatorname{Spec} R$ then $d = \operatorname{rank}_R \mathcal{O}_U / \operatorname{rank}_{\mathcal{O}_{M'_R}} \mathcal{O}_U$, which is the same whether evaluated at the generic or the special point of $\operatorname{Spec} R$. On the special fiber, this ratio is the multiplicity of the pushforward of $C_{X'/Y'}$ at the point ξ. On the generic fiber, it is the pure degree of Y' over Y, as required. □

Proof of Theorem 1.1. Let $E^\vee$ and $E'^\vee$ denote the vector bundle stacks dual to the obstruction theories E and E'. As relative obstruction theories, there are closed embeddings $C_{X/Y} \subset E^\vee$ and $C_{X'/Y'} \subset E'^\vee$. Their compatibility entails a commutative diagram whose lower square is cartesian:

$$\begin{array}{ccc} C_{X'/Y'} & \longrightarrow & C_{X/Y} \\ \downarrow & & \downarrow \\ E'^\vee & \xrightarrow{h} & E^\vee \\ {\scriptstyle q'}\downarrow & \ulcorner & \downarrow{\scriptstyle q} \\ X' & \xrightarrow{f} & X. \end{array}$$

Using compatibility of proper pushforward and flat pullback $q^* f_* = h_* q'^*$, we have

$$q^* f_* [X'/Y']^{\text{vir}} = h_* q'^* [X'/Y']^{\text{vir}} = h_*[C_{X'/Y'}] = d[C_{X/Y}] = q^*\big(d[X/Y]^{\text{vir}}\big) \tag{2}$$

But $[X/Y]^{\text{vir}}$ is the unique cycle class on X such that $q^*[X/Y]^{\text{vir}} = [C_{X/Y}]$, so we conclude that $f_*[X'/Y']^{\text{vir}} = [X/Y]^{\text{vir}}$, as required. □

Remark 2.7. In this paper, we verify the hypotheses of Theorem 1.1 by showing g is proper. We did find applications of Costello's theorem in the literature where g was pure but not proper, but we found it easier to replace Y' by a smaller stack that was proper than to verify purity directly.

3. Higher genus stable maps and genus zero orbifold stable maps

Let X be a smooth, projective scheme and work over Spec $\mathbb{C}$. The original application of the pushforward formula was to the following fiber square in the proof of [6, Lemma 8.0.2]:

$$\begin{array}{ccc} \overline{M}_\eta(X) & \longrightarrow & \overline{M}_v(X) \\ \downarrow & \ulcorner & \downarrow \\ \mathfrak{M}_\eta & \longrightarrow & \mathfrak{M}_v. \end{array} \tag{3}$$

The stack $\mathfrak{M}_\eta$ parametrizes finite, étale, d-sheeted covers $C' \to C$ with fixed numerical data η. The spaces $\overline{M}_v(X)$, $\mathfrak{M}_v$ parameterize stable maps and prestable curves with fixed data v. The horizontal arrow sends such a cover to its source curve C'. The stack $\overline{M}_\eta(X)$ is defined to make this square cartesian [6, pp. 575, 591, 593].[1] We replace this square in Sect. 3.1 and defer to Costello [6] for details because we will not need it later.

The next example illustrates that the horizontal arrows in diagram (3) are not proper, and therefore that $\overline{M}_\eta(X)$ is not proper. Since stable maps to $[\mathrm{Sym}^d X]$ do form a proper Deligne–Mumford stack, $\overline{M}_\eta(X)$ cannot be one of its components, as claimed in [6, Lemma 2.4.2]. The pushforward formula cannot be applied to $\overline{M}_\eta(X)$ because even its statement requires proper pushforward along the upper horizontal arrow $\overline{M}_\eta(X) \to \overline{M}_v(X)$.

Example 3.1. Let $X = \mathbb{P}^2$ and $U = \mathbb{A}^1_\lambda \setminus \{0\}$. Consider the family of plane cubics $C'_\lambda \subseteq X$ indexed by $\lambda \in U$ given by the projective closure of

$$y^2 = x(x+\lambda)(x+1).$$

We will describe a modification of $\mathfrak{M}_\eta$ that makes the horizontal arrows proper and revives [6, Lemma 2.4.2]. Assume the graph η has a single vertex and eliminate the graphs from the notation for simplicity. We leave it to the reader to adapt the method to more complicated graphs and deduce Costello's main theorem in its original form.

The closure of the projection $[x : y : z] \mapsto [x : z]$ gives a map of curves $C'_\lambda \to \mathbb{P}^1 \times U$ over U as in Fig. 1. Mark the four sections of $\mathbb{P}^1 \times U$ given by the branch locus and endow them with $B\mathbb{Z}/2$-stack structure. This yields a family of stacky projective lines over U which we call C_λ. The map $C'_\lambda \to C_\lambda$ is a proper,

[1] In the fiber products on pp. 575 and 591, $\overline{\mathcal{M}}_{s(\eta)}$ and $\overline{\mathcal{M}}_{r(s(\eta))}$ were presumably meant to be $\overline{\mathcal{M}}_{s(\eta)}(X)$ and $\overline{\mathcal{M}}_{r(s(\eta))}(X)$, respectively.

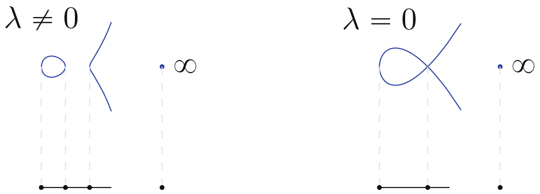

Fig. 1. A family of maps to $\mathbb{P}^1$ with general fiber a ramified cover and special fiber that is not

étale, 2-sheeted cover of stacky curves. Degenerating the source to $\lambda = 0$, we see there's no way to add stack structure to the base $\mathbb{P}^1$ to complete this family to an étale $\mathbb{Z}/2$-torsor.

The map $C'_\lambda \to C_\lambda$ is classified by a map $C_\lambda \to BS_2$, which is classified in turn by a map $U \to \overline{M}_{0,4}(BS_2)$. We will instead take the limit in the sense of twisted stable maps (equivalently, admissible covers), which allows C_λ to degenerate into two copies of $\mathbb{P}^1$ joined at a node.

Write S_d for the symmetric group on d letters and $\langle d\rangle := \{1, 2, \ldots, d\}$. Our solution is to use twisted stable maps to the stack $[\mathrm{Sym}^d X] = [X^d/S_d]$. A map $C \to [\mathrm{Sym}^d X]$ may be interpreted equivalently as a torsor $S_d \circlearrowright P \to C$ with an equivariant map $P \to X^d$ or a d-sheeted finite étale cover $C' \to C$ given by $C' = \langle d\rangle \times^{S_d} P$ with a map $C' \to X$ [6, Lemma 2.2.1]. We will apply Costello's pushforward formula to the cartesian diagram of moduli stacks of twisted curves after comparing conventions for torsors and twisted stable maps.

The map $S_{d-1} \to S_d$ including those permutations which fix the last element induces a group action $S_{d-1} \circlearrowright S_d$. Consider the d-element set $\langle d\rangle \simeq S_{d-1}\backslash S_d$ as a set-theoretic quotient with *right*-action by S_d. We may view $BS_{d-1} \simeq \langle d\rangle/S_d$ similarly. This latter identification does not depend on which $(d-1)$-element subset S_{d-1} is allowed to act.

Lemma 3.2. *If $T \to BS_d$ classifies an S_d-torsor $P \to T$, then the contracted product and fiber product are the same:*

$$\langle d\rangle \times^{S_d} P \simeq T \times_{BS_d} BS_{d-1}.$$

Applying this lemma to the torsor $X^d \to [\mathrm{Sym}^d X]$, the associated d-sheeted cover is

$$X \times [\mathrm{Sym}^{d-1} X] \simeq [X^d/S_{d-1}] \simeq [\mathrm{Sym}^d X] \times_{BS_d} BS_{d-1}.$$

This comes with a canonical projection to X.

Remark 3.3. Our description of the equivalence between the categories of S_d-torsors and d-sheeted covers over T is the opposite of Costello [6, Lemma 2.2.1].

Remark 3.4. The space of twisted stable maps $\mathcal{K}_{0,n}(V)$ to a smooth projective target V [7] allow marked points to have nontrivial gerbe structure. We instead use a stack $\overline{M}^{tw}_{0,n}(V)$ that requires those gerbes to be trivialized by sections at each marked point, as in [6, 2]. Our marked points are *globally* of the form $B\mu_r$ for some "ramification order" $r \in \mathbb{Z}_{\geqslant 1}$. We demand similarly that the gerbes of relative twisted stable maps $\overline{M}^{tw}_{0,n}(V/W)$ [8, 8.3] for a map $V \to W$ be trivialized.

The map $\overline{M}^{tw}_{0,n}(V) \to \mathcal{K}_{0,n}(V)$ is the universal gerbe, of degree $\dfrac{1}{r_1 \cdot r_2 \cdot \cdots \cdot r_n}$ over the locus where the gerbes are $B\mu_{r_i}$.

Remark 3.5. A d-fold cover of orbifold curves $C' \to C$ over S has discrete invariants including the genera, the stack structures at marked points, and the maps $B\mu_{r_i} \to BS_d$ from each ith marked point of C encoding its fiber in $C' \to C$. All are locally constant in S. If the fiber over the ith marked point is denoted $\langle \ell_i \rangle$, there is a function $\tau : \langle \ell_i \rangle \to \mathbb{Z}_{\geqslant 1}$ sending each point to its ramification order. The function depends in a locally constant fashion on the cover $C' \to C$.

Let Ξ be a monodromy profile, specified by n maps $B\mu_{r_i} \to BS_d$ parametrizing the fiber over each marked point as a d-sheeted cover of $B\mu_{r_i}$ (specifying, in other words, the monodromy of the cover around the ith parked point of the base). The substack $\overline{M}^{tw}_{\Xi}([\mathrm{Sym}^d X]) \subseteq \overline{M}^{tw}_{0,n}([\mathrm{Sym}^d X])$ of stable maps from covers $C' \to C$ with monodromy profile Ξ is open and closed. The fiber over the ith marked point consists of ℓ_i ramified points. Choosing an ordering of each fiber among S_{ℓ_i} choices makes the source C' into a $\ell = \sum \ell_i$-marked curve. There is an open and closed substack inside $\overline{M}^{tw}_{\Xi}([\mathrm{Sym}^d X])$ that also fixes τ. If τ is increasing for example, $r_i > r_j$ implies $i > j$ and ramified points come later in the ordering.

The monodromy profile Ξ is a component of the cyclotomic intertia stack $I_\mu(BS_d)$ [9, Definition 3.2.1], recipient of evaluation maps

$$\overline{M}^{tw}_{0,n}([\mathrm{Sym}^d X]) \to (I_\mu([\mathrm{Sym}^d X]))^n \to (I_\mu(BS_d))^n.$$

3.1. Applying the pushforward formula to the new diagram

Fix nonnegative integers $g, n, d, \ell = \sum \ell_i$ with $n \leqslant \ell \leqslant dn$ and monodromy profile Ξ as in Remark 3.5. We are ready to reinterpret (3):

$$\begin{array}{ccc} \widetilde{M}_\Xi([\mathrm{Sym}^d X]) & \xrightarrow{q} & \overline{M}_{g,\ell}(X) \\ \downarrow{\scriptstyle \pi'} & \ulcorner & \downarrow{\scriptstyle \pi} \\ \widetilde{\mathfrak{M}}_\Xi(BS_d) & \xrightarrow{p} & \mathfrak{M}_{g,\ell}. \end{array} \tag{4}$$

The stacks $\overline{M}_{g,\ell}(X)$, $\mathfrak{M}_{g,\ell}$ parametrize ordinary stable maps to X and prestable curves with ℓ marked points, while the map π forgets all but the source curve of the stable map. The stack $\widetilde{\mathfrak{M}}_\Xi(BS_d)$ carries more data than prestable maps $\mathfrak{M}_\Xi(BS_d)$. We now introduce p and $\widetilde{\mathfrak{M}}_\Xi(BS_d)$ in three steps.

Step 1: Relative maps

Let $u : \mathfrak{D} \to \mathfrak{M}_{g,\ell}$ be the universal curve and $\overline{M}^{tw}_{0,n}(u) = \overline{M}^{tw}_{0,n}([\mathrm{Sym}^d \mathfrak{D}]/\mathfrak{M}_{g,\ell})$ be the stack of relative twisted stable maps with trivialized marked gerbes, as described in Remark 3.4. If $S \to \mathfrak{M}_{g,\ell}$ classifies a curve $D \to S$, points of this stack are given by:

$$
\begin{array}{ccc} & & \overline{M}^{tw}_{0,n}(u) \\ & \nearrow & \downarrow \\ S & \xrightarrow[D]{} & \mathfrak{M}_{g,\ell} \end{array}
:= \left\{ \begin{array}{ccc} C & \xrightarrow{\widehat{f}} & [\mathrm{Sym}^d\, D] \\ & \searrow \quad \swarrow & \\ & S & \end{array} \;\middle|\; \begin{array}{l} C \text{ is a connected, nodal} \\ \text{orbifold curve with trivialized} \\ \text{marked gerbes, and} \\ \widehat{f} \text{ is representable and stable} \end{array} \right\}.
$$

Maps $C \to [\mathrm{Sym}^d\, D]$ are equivalent to d-sheeted finite étale covers $C' \to C$ of twisted curves with a representable map $C' \to D \times [\mathrm{Sym}^{d-1}\, D]$. Stability requires that the sheaf of automorphisms of the trio $(C \leftarrow C' \to D)$ that restrict to the identity on D be finite at geometric points.

Step 2: Marked points

Endow C' with the marked points pulled back from those of C. These preimages are *unordered*, so C' is not yet a marked curve.

If $C' \to C$ were an *untwisted* finite étale cover, order the d preimages of each marked point of C. Then $\ell_i = d$, $\ell = dn$, and ordering amounts to a $(S_d)^n$-torsor on moduli spaces in the untwisted case. Globally fixing a lexicographic ordering of $\langle n \rangle \times \langle d \rangle$ then equates a $\langle n \rangle \times \langle d \rangle$-marked curve with a dn-marked curve.

For twisted/ramified covers $C' \to C$, the sizes ℓ_i of the fibers over the marked points vary according to Ξ. Ordering them nevertheless entails a $\prod S_{\ell_i}$-torsor $\widetilde{M}_{0,n}(u)$ over $\overline{M}^{tw}_{0,n}(u)$, as in Remark 3.5.

Now $\widetilde{M}_{0,n}(u)$ parametrizes trios of curves $(C \leftarrow C' \to D)$ where C' is a marked curve, but $C' \to D$ needn't be a map of marked curves. This condition cuts out a closed substack $\widetilde{M}'_{0,n}(u) \subseteq \widetilde{M}_{0,n}(u)$ where $C' \to D$ maps marked points to marked points in order.

Step 3: Partial stabilization

Think of the map $f : C' \to D$ as a prestable twisted map to D and take its stabilization $\overline{f} : \overline{C}' \to D$. If $\overline{f}$ identifies D with the coarse moduli space of $\overline{C}'$ then f is called a *partial stabilization*. Define

$$\widetilde{\mathfrak{M}}_{0,n}(BS_d) \subseteq \widetilde{M}'_{0,n}(u)$$

to be the substack where f is a partial stabilization; in particular, C' is connected. This substack is open and closed:

Lemma 3.6. *Let $f : C' \to D$ be a morphism of untwisted prestable curves over a base S. The locus in S where f is a partial stabilization is both open and closed.*

Proof. The stabilization $\overline{f} : \overline{C}' \to D$ is an isomorphism over an open locus [10, Tag 05XD]. This locus is also stable under specialization by the uniqueness of stable limits of stable maps. □

The map p in diagram (4) is the composite of all the proper maps above:

$$p : \widetilde{\mathfrak{M}}_{0,n}(BS_d) \subseteq \widetilde{M}'_{0,n}(u) \subseteq \widetilde{M}_{0,n}(u) \to \overline{M}^{tw}_{0,n}(u) \to \mathfrak{M}_{g,\ell}.$$

Corollary 3.7. *The map $p : \widetilde{\mathfrak{M}}_{0,n}(BS_d) \to \mathfrak{M}_{g,\ell}$ is proper, thus pure.*

Proof. It was constructed as a closed substack of a finite cover of the space of stable maps to a target that is proper over $\mathfrak{M}_{g,\ell}$. Since stable maps to a proper target form a proper space, this implies that p is proper. □

An example due to Costello [6, Lemma 6.0.1] of (g, n, d, Ξ) where p is generically finite is worked out in Sect. 3.2.

The stack of twisted stable maps $\overline{M}_{0,n}([\mathrm{Sym}^d X])$ parametrizes $(C \leftarrow C' \rightarrow X)$ as in Step 1 above. The curve C' is connected on an open and closed substack. There is a similar $\prod S_{\ell_i}$-torsor $\widetilde{M}_{0,n}([\mathrm{Sym}^d X])$ over this substack of $\overline{M}_{0,n}([\mathrm{Sym}^d X])$ of orderings of the preimages in C' of the marked points of C. This is the remaining piece of diagram (4). The map q sends $(C \leftarrow C' \rightarrow X)$ to the stabilization $\overline{C}' \rightarrow X$, and π' sends it to $(C \leftarrow C' \rightarrow \overline{C}')$.

Remark 3.8. Stabilization $s : C \rightarrow \overline{C}$ of prestable maps $C \rightarrow X$ is functorial in that X-automorphisms $\varphi : C \simeq C$ all lie over unique X-automorphisms $\overline{\varphi} : \overline{C} \simeq \overline{C}$. Automorphisms of $C \rightarrow X$ lying over an automorphism of $\overline{C}$ form a subsheaf:

$$i : \underline{\mathrm{Aut}}_X(C \rightarrow \overline{C}) \subseteq \underline{\mathrm{Aut}}_X(C) \qquad \begin{array}{ccc} C & \xrightarrow{\sim} & C \\ \downarrow & & \downarrow \\ \overline{C} & \xrightarrow{\sim} & \overline{C} \end{array} \mapsto (C \simeq C).$$

To argue i is an isomorphism, assume the base is a geometric point [10, 03PU]. But φ must restrict to an automorphism on the union of unstable components over X, so $C \overset{\varphi}{\simeq} C \rightarrow \overline{C}$ is also the stabilization of $C \rightarrow X$. The map $\overline{\varphi}$ comes from canonicity of stabilization.

Lemma 3.9. *The square* (4) *is cartesian.*

Proof. If $\widetilde{M}_{0,n}([\mathrm{Sym}^d X])$ and $\widetilde{\mathfrak{M}}_{0,n}(BS_d)$ and $\overline{M}_{g,\ell}(X)$ are all replaced by their unstable variants, Diagram (4) is immediately commutative and cartesian. Given

$$C \leftarrow C' \rightarrow \overline{C}' \rightarrow X$$

with $\overline{C}' \rightarrow X$ stable but $(C \leftarrow C' \rightarrow \overline{C}')$ not necessarily, we must show $(C \leftarrow C' \rightarrow X)$ is stable if and only if $(C \leftarrow C' \rightarrow \overline{C}')$ is. This is a consequence of Remark 3.8.

□

Diagram (4) is cartesian and p is proper. It remains only to compare the perfect obstruction theories of π and π'.

Lemma 3.10. *The map*

$$\widetilde{\mathfrak{M}}_{0,n}(BS_d) \rightarrow \mathfrak{M}_{0,n}(BS_d) \times_{(BS_d)^n} *$$

forgetting the partial stabilization is étale.

Proof. Apply the formal criterion as in [11, Lemma 7] or [12, Lemma B (ii) for Υ]. □

Corollary 3.11. *The perfect relative obstruction theories on π' induced from the natural ones on π and from $\overline{M}_{0,n}([\mathrm{Sym}^d X]) \rightarrow \mathfrak{M}_{0,n}(BS_d)$ coincide.*

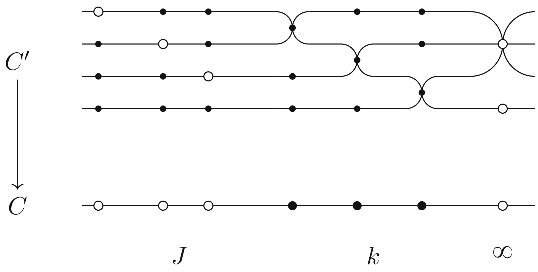

Fig. 2. A cover in Ξ, $g = 3$, $d = 4$. Marked points are colored black if forgotten and white if remembered under the map to $\mathfrak{M}_{g,\ell-A}$

3.2. Costello's example computation

We now compute the pure degree e of $p : \widetilde{\mathfrak{M}}_{0,n}(BS_d) \to \mathfrak{M}_{g,\ell}$ in the setting of Costello [6, Lemma 6.0.1] to check consistency.

Remark 3.5 lets one fix discrete data (g, d, n, ℓ_i, Ξ) illustrated in Fig. 2. Let $d = g + 1$ and k be an integer to be specified later. Require $n \leqslant \ell \leqslant dn$, as $\ell = \sum \ell_i$ counts marked points of the source C' and n of the target C.

Consider decompositions $\langle n\rangle = \langle k\rangle \sqcup J \sqcup \{\infty\}$ and $\langle \ell\rangle = \langle k \cdot g\rangle \sqcup J \times \langle g+1\rangle \sqcup I$ and a function $d : I \to \mathbb{Z}_{\geqslant 1}$, with $m(\infty) := \operatorname{lcm}(d(i))$. Define

$$\Xi = \left\{ \begin{array}{c} g(C') = g,\ g(C) = 0, \\ \langle \ell\rangle \to \langle n\rangle \text{ is } \langle kg\rangle = \langle k\rangle \times \langle g\rangle \overset{pr_1}{\mapsto} \langle k\rangle,\ \ J \times \langle g+1\rangle \overset{pr_1}{\mapsto} J,\ \ I \mapsto \infty, \\ \forall j \in J,\ \ r_j = 1, \forall i \in \langle k\rangle\ r_i = 2, r_\infty = m(\infty) \\ \bigsqcup_J B\mu_1 = * \to BS_d,\ \bigsqcup_{i\in\langle k\rangle} B\mu_2 \xrightarrow{\psi} BS_d \end{array} \right\}$$

Here ψ corresponds to the action $\mu_2 \circlearrowright \langle d\rangle$ with one 2-cycle and the other points fixed. We decline to specify the ramification profile $B\mu_{m(\infty)} \to BS_d$ over the point $\infty \in C$. These discrete data define a possibly disconnected substack $\widetilde{M}_\Xi([\mathrm{Sym}^d X]) \subseteq \widetilde{M}_{0,n}([\mathrm{Sym}^d X])$.

Choose $A \subseteq \langle k \cdot g\rangle \sqcup J \times \langle g+1\rangle \subseteq \langle \ell\rangle$ such that $\langle k \cdot g\rangle \subseteq A$ and $J \times \langle g+1\rangle \setminus (A \cap J \times \langle g+1\rangle) \to J$ is a bijection. Write $\ell - A$ abusively for the set $\langle \ell\rangle \setminus A = I \sqcup J$. Define a map $\pi : \widetilde{M}_\Xi([\mathrm{Sym}^d X]) \to \overline{M}_{g,\ell-A}(X)$ as the composite of p from diagram (4) and the map forgetting the marked points A and stabilizing. Define a stack $\widetilde{\mathfrak{M}}_\Xi(BS_d)$ as above to keep track of triples $(C \leftarrow C' \to D)$ with $C' \to D$ a partial stabilization *after forgetting* the points labelled by A, $C' \to C$ a d-sheeted cover, and an ordering on the preimages of C's marked points. This fits in a cartesian square

$$\begin{array}{ccc} \widetilde{M}_{\Xi}([\mathrm{Sym}^d X]) & \longrightarrow & \overline{M}_{g,\ell-A}(X) \\ \downarrow & \ulcorner & \downarrow \\ \widetilde{\mathfrak{M}}_{\Xi}(BS_d) & \xrightarrow{p'} & \mathfrak{M}_{g,\ell-A}. \end{array}$$

One shows this square is cartesian as in Lemma 3.9 and that p' is proper as in Corollary 3.7. We compute its pure degree after [6, Lemma 6.0.1].

Theorem 3.12. *With the above discrete data* Ξ *and* $k = \#I + 3g - 1$, $\dim \widetilde{\mathfrak{M}}_{\Xi}(BS_d) = \dim \mathfrak{M}_{g,\ell-A}$ *and the map* p' *between them is of pure degree*

$$e = \frac{k!(g!)^{\#J}(g!)^k}{2^k \cdot m(\infty)}.$$

Proof. The map from $\widetilde{\mathfrak{M}}_{\Xi}(BS_d)$ that forgets D is étale, so we can ignore D to calculate its dimension. Our moduli spaces are the closure of strata considered by Costello, so we again have

$$\dim \widetilde{\mathfrak{M}}_{\Xi}(BS_d) = k + \#J - 2, \qquad \dim \mathfrak{M}_{g,\ell-A} = 3g - 3 + \#I + \#J.$$

These are equal by definition of k. Because the dimensions are equal, the preimage of the generic point must either be the generic point of the source or empty. The generic point of the source has smooth C', C by design, hence $C' \xrightarrow{\sim} D$ is an isomorphism. We've reduced to the case considered by Costello.

Fix general points $q_1, \ldots, q_s \in D$ and write $B = \sum d(i)[q_i]$ for the induced divisor of degree $g + 1$.

Claim: There are no special effective subdivisors $0 \leqslant B' < B$ of degree g.

We outsource the proof to Lemma 3.13. We conclude as in Costello's original argument. Any effective $B' < B$ of degree g is not special, so $h^1(\mathcal{O}(B')) = h^1(\mathcal{O}(B)) = 0$. Riemann–Roch gives $h^0(\mathcal{O}(B')) = 1$ and $h^0(\mathcal{O}(B)) = 2$. This means there is at most one map $f : D \to \mathbb{P}^1$ with $f^*\infty \leqslant B$ up to isomorphism and no such maps with $f^*\infty \leqslant B'$; i.e., $f^*\infty = B$.

The dimension of the moduli space of covers of $\mathbb{P}^1$ is determined by the number of marked and branch points [13, 1.G]. A more-ramified cover has fewer branch points, so general maps $f : D \to \mathbb{P}^1$ as above are *simply* ramified at distinct points away from B. We do not control the ramification profile over $\infty \in C$.

It remains to promote the source and target of $f : D \to \mathbb{P}^1$ to $\langle \ell \rangle$- and $\langle n \rangle$-marked curves. Endowing $D, \mathbb{P}^1$ with stack structure to make f étale, we must then trivialize the $\mu_2^k \times \mu_{m(\infty)}$ marked gerbes of $\mathbb{P}^1$ as per our conventions in Remark 3.4. On moduli, this is a gerbe of pure degree $\dfrac{1}{2^k m(\infty)}$. Ordering the k images of the simple ramification points, each of their fibers, and the fibers of the marked points labelled by J constitutes an $S_k \ltimes (S_g)^k \times (S_g)^{\#J}$-torsor (the black points in Fig. 2). This gives the multiplicity e.

□

Noam Elkies' response to [3] led to this lemma.

Lemma 3.13. *Fix multiplicities* $d : I \to \mathbb{Z}_{\geqslant 1}$ *with* $\sum d(i) = g+1$. *General points* $q_1, \ldots, q_s \in D$ *on a general smooth curve engender a divisor* $B = \sum d(i)[q_i]$. *There are no special effective subdivisors* $0 \leqslant B' < B$ *of degree* g.

Proof. Fix numbers $b_1, \ldots, b_s \in \mathbb{N}$ adding up to $\sum b_i = g$ to obtain a map

$$D^s \to \mathrm{Div}^g; \qquad (p_1, \ldots, p_s) \mapsto \sum b_i[p_i].$$

The locus of special divisors is closed in Div^g, as can be seen by applying upper semicontinuity theorem [14, Theorem III.12.8] to the universal sheaf $\mathcal{O}(\mathscr{B})$ for the fibers of the projection $\pi : D \times \mathrm{Div}^g \to \mathrm{Div}^g$. We argue the pullback $P_{\{b_i\}} \subseteq D^s$ of the locus in Div^g of special divisors is a *proper* closed subscheme. If $P_{\{b_i\}}$ contained the diagonal Δ_D, a general point $p \in D$ would be a Weierstrass point. There are finitely many Weierstrass points on a curve over $\mathbb{C}$, so $P_{\{b_i\}}$ is a proper closed subscheme.

For any $1 \leqslant j \leqslant s$, we obtain a sequence

$$b_i := \begin{cases} d(i) & \text{if } i \neq j \\ d(i) - 1 & \text{if } i = j. \end{cases}.$$

Our general points $q_1, \ldots, q_s \in D$ are not in any $P_{\{b_i\}}$, so our divisor $B := \sum d(i)[q_i]$ contains no special effective divisors of degree g. □

Remark 3.14. Taking C alone to be general in Costello's original proof [6, Lemma 6.0.1] does not suffice – one must assume $I = \mathrm{Supp} D \subseteq C$ general as well to guarantee $\dim \Gamma(\mathcal{O}(D')) = 1$ for *any* divisor $0 \leqslant D' < D$. Otherwise, take a generic genus two curve with g^1_2 mapping $f : C \to \mathbb{P}^1$ and let $D = 2p + q$ with p a Weierstrass point. If $D' = 2p$, $\dim \Gamma(\mathcal{O}(D')) = 2$.

Remark 3.15. The pure degree e is different from that computed by Costello:

$$e' = \frac{k!(g!)^{\#J}((g-1)!)^k}{2^k m(\infty)}.$$

Consider the substack $\widetilde{\mathfrak{M}}'_\Xi \subseteq \widetilde{\mathfrak{M}}_\Xi(BS_d)$ ordering points of equal ramification separately by fixing τ as in Remark 3.5. The restriction of p' to $\widetilde{\mathfrak{M}}'_\Xi$ is of pure degree e' because the simple ramification points must have the greatest label in their fiber for each of the k-marked points. Ordering the other unramified points is a S_{g-1}-torsor. The other terms count degree of the gerbes and ordering of the other points identically to Theorem 3.12.

The main computation of virtual fundamental classes in [6, Lemma 8.0.2] thus applies to our above modifications:

$$q_*[\widetilde{M}_\Xi([\mathrm{Sym}^d X])]^{vir} = e \cdot [\overline{M}_{g,\ell-A}(X)]^{vir}.$$

Remark 3.16. We employed the technology of Abramovich and Vistoli [8] for convenience and brevity, but the same results may be achieved with Costello's original technology of weighted graphs. The data of a partial stabilization can be encoded on the level of graphs, and the stabilization of a map $C \to X$ can be reconstructed from the weighting of components by their curve classes in X.

4. Applications of the pushforward formula

This section addresses myriad articles which use Costello's Formula. The papers [15–18] reference but don't use Costello's Formula. The paper [19] uses other results from Costello's paper and not his formula, while [20] uses it only for motivation.

The use of Costello's Formula in [12] will be addressed alongside other simplifications in forthcoming work by Sam Molcho, Rahul Pandharipande, and the authors. Similar techniques also apply to Cavalieri et al. [21] and Marcus and Wise [22], although both are subsumed by the suitably proper diagram in [23, 5.5].

4.1. An algebraic proof of the hyperplane property of the genus-one GW-invariants of quintics

The application of a "cosection-localized version" of Costello's Formula proposed in Eq. (1.4) [24] is spelled out at the end of Sect. 2. There is a cartesian diagram

$$\begin{array}{ccccccccc}
D(\widetilde{\sigma}) & \longrightarrow & \widetilde{Y} & \xrightarrow{\widetilde{f}} & \widetilde{X} & \longrightarrow & \widetilde{\mathcal{D}} & \longrightarrow & \widetilde{\mathcal{M}}^w \\
\downarrow & \ulcorner & \downarrow{\scriptstyle p} & \ulcorner & \downarrow{\scriptstyle q} & \ulcorner & \downarrow & \ulcorner & \downarrow \\
D(\sigma) & \longrightarrow & Y & \xrightarrow{f} & X & \longrightarrow & \mathcal{D} & \longrightarrow & \mathcal{M}^w,
\end{array}$$

where the map $\widetilde{\mathcal{M}}^w \to \mathcal{M}^w$ is a blowup and the obstruction theories of Y, X relative to $\mathcal{D}$ pull back to those of $\widetilde{Y}, \widetilde{X}$ relative to $\widetilde{\mathcal{D}}$. Properness and birationality of the blowup lets one apply Costello's pushforward formula to show $q_*[\widetilde{X}]^{vir} = [X]^{vir}$.

One of two proofs they offer of Proposition 2.3 claims that $\widetilde{f}_*[\widetilde{Y}]^{vir}_{loc} = [\widetilde{X}]^{vir}$ and $f_*[Y]^{vir}_{loc} = [X]^{vir}$. From this claim and the valid application of Costello's pushforward formula, we see the Proposition is correct:

$$\deg[\widetilde{Y}]^{vir}_{loc} = \deg[Y]^{vir}_{loc}.$$

4.2. Virtual pull-backs

The final result [1, Proposition 5.29] in the latest version assumes the morphism is projective.

4.3. *Log Gromov–Witten theory with expansions*

The paper only uses Costello's Formula to address pushforwards along logarithmic modifications that are pulled back from the target of the perfect obstruction theory [25, Proposition 3.6.1]. The definition of logarithmic modification includes a properness assumption [25, 3.2].

4.4. *The cohomological crepant resolution conjecture for the Hilbert-Chow morphisms*

This paper uses Costello's Formula for a cartesian square

$$\begin{CD}
\overline{\mathfrak{M}}(\widehat{V}_1 \times_T \widehat{V}_2) @>>> \overline{\mathfrak{M}}(\widehat{V}_1) \times_T \overline{\mathfrak{M}}(\widehat{V}_2) \\
@VVV @VVV \\
T \times \mathcal{D}(d_1, d_2) @>>> T \times \mathcal{M}_{0,3}(d_1) \times \mathcal{M}_{0,3}(d_2)
\end{CD}$$

in the proof of [26, Lemma 5.5]. Immediately before, Li and Qin [26, Lemma 5.4] shows that the lower horizontal arrow without T, $\mathcal{D}(d_1, d_2) \to \mathcal{M}_{0,3}(d_1) \times \mathcal{M}_{0,3}(d_2)$, is proper and birational.

4.5. *Gromov-Witten theory of étale gerbes, I: root gerbes*

Costello's result is used in [27, Theorem 4.3]. The morphism $Y^g_{0,n,\beta} \to \mathfrak{M}_{0,n,\beta}$ is an example of the Matsuki–Olsson construction, which is finite [28, Theorem 4.1].

4.6. *The degeneration formula for logarithmic expanded degenerations*

The map $\mathfrak{T}_0^{u,spl} \to \mathfrak{T}_0^u$ is observed to be a normalization in [29, 7.2], subject to Remark A.7. This proper map is the base of a diagram

$$\begin{CD}
K_{\mathfrak{Q}} @>>> \mathfrak{Q} @>>> \mathfrak{Q}^{ext} @>>> \mathfrak{T}_0^{u,spl} \\
@VVV @VVV @VVV @VVV \\
K @>>> \mathfrak{T}_0^{etw} @>>> \mathfrak{T}^{tw} @>>> \mathfrak{T}_0^u
\end{CD}$$

to which Chen applies Costello's Formula.

4.7. *Virtual classes of Artin stacks*

The result [30, Theorem 5.2] includes a properness assumption.

4.8. Virtual normalization and virtual fundamental classes

Theorem 1 applies Costello's pushforward formula to a pullback along the map

$$\widehat{Log} \subseteq Log^1 \to Log.$$

This pullback entails saturation of log structures, which is finite [31, Proposition III.2.1.5 (2)].

4.9. Orbifold techniques in degeneration formulas

Costello's formula is used several times in [32].

Theorem 4.7. *the maps* $\mathfrak{T}_0^{\mathfrak{r}'} \to \mathfrak{T}_0^{\mathfrak{r}}$, $\mathscr{T}^{\mathfrak{r}'} \to \mathscr{T}^{\mathfrak{r}}$ *along the bottom of the two squares written as one in Proposition* 4.4.2 (2) *are proper by Proposition* 2.12 [31].

Lemma 4.16. *the proof applies Costello's formula to the diagram*

$$\begin{array}{ccc} K_{\Xi} & \longrightarrow & \prod K_{\Gamma_v} \\ \downarrow & \ulcorner & \downarrow \\ \mathscr{T}' & \longrightarrow & (\mathscr{T})^h. \end{array}$$

We need to argue $\mathscr{T}' \to (\mathscr{T}^{tw})^h$ *is proper. We don't know how to define the contraction maps unless the rooting order is the same at each node, but this suffices.*

Recall the description of $\mathscr{T}$ given in [23, 5.2]. The strict-étale topology on fs log schemes supports a sheaf of groups:

$$\mathbb{G}_m^{trop}(S) := \Gamma(S, \overline{M}_S^{gp}).$$

An (oriented) *tropical line* (*bundle*) is a torsor in the strict-étale topology for $\mathbb{G}_m^{trop}$. A map $S \to \mathscr{T}$ is a tropical line P together with a subsheaf of sets $Q \subseteq P$ for which there locally exists a nonempty chain $\{\gamma_1 \leqslant \cdots \leqslant \gamma_n\} \subseteq \Gamma(S, \overline{M}_S^{gp})$ such that Q is the subsheaf of sections of P locally comparable to all the γ_i.

Lemma 4.1. *The map* $\mathscr{T}' \to (\mathscr{T}^{tw})^h$ *is a log blowup, hence proper.*

Proof. For any map $S \to (\mathscr{T})^h$, take the fs pullback

$$\begin{array}{ccc} R & \longrightarrow & S \\ \downarrow & \ulcorner \ell & \downarrow \\ \mathscr{T}' & \longrightarrow & (\mathscr{T})^h. \end{array}$$

By strict-étale localization, assume each map $S \to \mathscr{T}$ corresponds to a $\mathbb{G}_m^{trop}$-torsor P which is subdivided by sections $\gamma_1^j \leqslant \cdots \leqslant \gamma_{n_j}^j$. Use γ_1^j to trivialize this torsor:

$$P \simeq \mathbb{G}_m^{trop} \qquad p \mapsto p - \gamma_1^j.$$

Write

$$s_i^j = \gamma_i^j - \gamma_1^j \in \overline{M}_S$$

for the image of γ_i^j under this isomorphism.

In the present language, the map $\mathscr{T}' \to (\mathscr{T})^h$ corresponds to a subdivided tropical line $\gamma_1' \leqslant \cdots \leqslant \gamma_n'$ which induces all the others by forgetting some elements γ_i'. The first γ_1' is never forgotten, since it corresponds to the unexpanded target X in the expansion of (X, D). Thus the element γ_1' maps to 0 under our trivializations above. Write $s_i' := \gamma_i' - \gamma_1'$ similarly.

Take the fs product B of all log blowups of S at ideals given by pairs $(s_i^j, s_{i'}^{j'})$ for $1 \leqslant j \leqslant h$. Each of these blowups may be fs pulled back from the ideal of universal elements of $\overline{M}_{\mathscr{A}^2}$ on $\mathscr{A}^2$. A map $T \to S$ factors through B (and uniquely) if and only if the set $(s_i^j|_T) \subseteq \overline{M}_T$ is totally ordered.

Since the γ_i^j's all arise by forgetting parts of the subdivision $\gamma_1' \leqslant \cdots \leqslant \gamma_n'$, they are totally ordered on R. This means $R \to S$ factors through B. Observe also that the fs product $R \times_S B \to B$ is an isomorphism – if the s_i^j's are totally ordered, their sums with γ_1 yield a unique subdivision. Thus $R \to B$ is an isomorphism. □

Lemma 5.11. *Costello's formula is applied to the cartesian diagram*

$$\begin{array}{ccc} K_{\mathfrak{Q}} & \longrightarrow & K \\ \downarrow & \ulcorner & \downarrow \\ \mathfrak{Q} & \longrightarrow & \mathfrak{T}_0^{tw} \\ \downarrow & \ulcorner & \downarrow \\ \mathfrak{T}_0^{u,spl} & \longrightarrow & \mathfrak{T}_0^{u} \end{array}$$

beginning Sect. 5.4 [31]. They observe that the bottom horizontal arrow is a normalization of locally finite type stacks, hence subject to Remark A.7.

Lemma 5.12. *The bottom map in the diagram*

$$\begin{array}{ccc} K_r^{spl} & \longrightarrow & K_{\mathfrak{Q}_r} \\ \downarrow & \ulcorner & \downarrow \\ \mathfrak{T}_0^{r,spl} & \longrightarrow & \mathfrak{Q}_r \end{array}$$

is the reduced induced closed substack, hence a proper map.

Lemma 5.15. *Costello's formula is applied to a gerbe banded by μ_r, which is proper.*

4.10. Birational invariance in log Gromov–Witten theory

The paper [33] uses Costello's Pushforward Formula on the cartesian square (1) in [33, 1.6]:

$$\begin{array}{ccc} \overline{M}(Y) & \xrightarrow{\quad\quad} & \overline{M}(X) \\ \big\downarrow & \ulcorner & \big\downarrow \\ \mathfrak{M}'(\mathcal{Y} \to \mathcal{X}) & \xrightarrow{\mathfrak{M}(h)} & \mathfrak{M}(\mathcal{X}). \end{array}$$

Lemma 4.2. *The map $\mathfrak{M}(h)$ is proper.*

Proof. The map sends a square

$$\begin{array}{ccc} C & \longrightarrow & \mathcal{Y} \\ \big\downarrow & & \big\downarrow \\ \overline{C} & \longrightarrow & \mathcal{X} \end{array}$$

to the bottom horizontal arrow. Write $\overline{\mathscr{C}}$ for the universal curve on $\mathfrak{M}(\mathcal{X})$ and

$$P := \overline{\mathscr{C}} \times_{\mathcal{X}} \mathcal{Y}$$

for the pullback. Then $\mathfrak{M}(h)$ factors through the inclusion of components of $\overline{M}(P/\mathfrak{M}(\mathcal{X}))$ on which the universal map $C \to P \to \overline{\mathscr{C}}$ is a partial stabilization and Lemma 3.6 concludes. □

4.11. Relative and Orbifold Gromov–Witten Invariants

In [34, Diagram 2.3.1], we see another application of Costello's pushforward formula. This square is a special case of a more general class of diagrams investigated in Sect. 7.3 [33]:

$$\begin{array}{ccc} \overline{M}^{tr}_{\Gamma}(\mathscr{X}^{rel}/\mathscr{T}) & \xrightarrow{\phi_{\mathscr{X}}} & \overline{M}_{\Gamma}(\mathscr{X}) \\ \big\downarrow & \ulcorner & \big\downarrow \\ \mathfrak{M}^{rel}_{0,n}(\mathscr{A}, B\mathbb{G}_m)' & \xrightarrow{\phi_{\mathscr{A}}} & \mathfrak{M}_{0,n}(\mathscr{A})'. \end{array}$$

The stack $\mathfrak{M}_{0,n}(\mathscr{A})'$ is an open substack of $\mathfrak{M}_{0,n}(\mathscr{A})$.

Lemma 4.3. *The map*

$$\mathfrak{M}^{rel}_{0,n}(\mathscr{A}, B\mathbb{G}_m)' \to \mathfrak{M}_{0,n}(\mathscr{A})'$$

is proper.

Proof. This map is pulled back from the map

$$\mathfrak{M}^{rel}_{0,n}(\mathscr{A}, B\mathbb{G}_m)^* \to \mathfrak{M}_{0,n}(\mathscr{A}).$$

This map sends a square

$$\begin{array}{ccc} C & \longrightarrow & \widetilde{\mathscr{A}_r} \\ \downarrow & & \downarrow \\ \overline{C} & \longrightarrow & \mathscr{A}_r \times S \end{array}$$

to the lower horizontal arrow. We again employ Lemma 3.6 by describing this map as the locus among relative moduli of stable curves where a particular morphism is a partial stabilization. ◻

The same techniques handle the square [34, 7.1.2]:

$$\begin{array}{ccc} \overline{M}^{rel}_{g=0}(X_r, D_r) & \longrightarrow & \overline{M}^{orb}_{g=0}(X_r) \\ \downarrow & \ulcorner & \downarrow \\ \mathfrak{M}^{rel}_{g=0}(\mathscr{A}_r, \mathscr{D}_r) & \longrightarrow & \mathfrak{M}^{orb}_{g=0}(\mathscr{A}_r). \end{array}$$

We still must address the map $\phi_{\mathscr{A}}$ in

$$\begin{array}{ccc} \overline{M}^{rel}(X_r, D_r) & \xrightarrow{\phi_{\mathscr{X}}} & \overline{M}^{rel}(X, D) \\ \downarrow & \ulcorner & \downarrow \\ \mathfrak{M}^{rel}_{0,n}(\mathscr{A}_r, \mathscr{D}_r) & \xrightarrow{\phi_{\mathscr{A}}} & \mathfrak{M}^{rel}_{0,n}(\mathscr{A}, \mathscr{D}). \end{array}$$

Remark 4.4. No stabilization occurs in $\phi_{\mathscr{X}}$.

Lemma 4.5. *The map*

$$\phi_{\mathscr{A}} : \mathfrak{M}^{rel}_{0,n}(\mathscr{A}_r, \mathscr{D}_r) \to \mathfrak{M}^{rel}_{0,n}(\mathscr{A}, \mathscr{D})$$

is of pure degree 1.

Proof. Write $u : \mathfrak{D} \times_{\widetilde{\mathscr{A}}} \widetilde{\mathscr{A}_r} \to \mathfrak{M}(\mathscr{A}, \mathscr{D})$ for the pullback of the universal curve along the map between universal expansions. The space $\mathfrak{M}^{rel}(\mathscr{A}_r, \mathscr{D}_r)$ lies inside the spaces of relative stable map $\overline{M}(u)$ as the locus with S-points where $C \to \mathfrak{D}|_S$ is an isomorphism. Denote the closure of this locus by $\overline{\mathfrak{M}}$. Then $\overline{\mathfrak{M}} \to \mathfrak{M}(\mathscr{A}, \mathscr{D})$ is proper and birational, so restriction to the dense open $\mathfrak{M}(\mathscr{A}_r, \mathscr{D}_r)$ is pure degree 1. ◻

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Appendix A. Degree of a Generically Finite Morphism

The stacks project offers two definitions of generic finiteness. We assume our stacks are locally noetherian and elaborate on definition (1) of [10, 073A].

Definition A.1. Let $f : X \to Y$ be locally of finite type and $\eta \in Y$ be a maximal point. We say f is *generically finite at* η if the preimage $X \times_Y \eta$ is a finite, *nonempty* set. Equivalently, there's an affine open $V \subseteq Y$ and finitely many $U_1, \ldots, U_n$ such that $U_i \to V$ is finite and $\eta \in V$ and $X \times_Y \eta \subseteq \bigcup_n U_i$ [10, 02NW].
Given that $f : X \to Y$ is generically finite at some maximal η, we say it is *of degree d* at η if [10, 02NY]

$$d = \sum_{\xi \in f^{-1}(\eta)} \dim_{R(\eta)} \mathcal{O}_{X,\xi}.$$

A morphism $f : X \to Y$ locally of finite type is said to be *generically finite* or *of degree d* if it is so at every maximal point $\eta \in Y$.
A representable morphism $X \to Y$ locally of finite type between algebraic stacks is said to be generically finite or of degree d (at a specific maximal point $\eta \in Y$ or for all) if the same is true for pulling back along some smooth cover $V \to Y$ by a scheme (with $\xi \in V$ mapping to η).

Remark A.2. Generically finite and degree d both pull back along flat, quasicompact morphisms $Y' \to Y$ and may be checked after some (equivalently any) flat, quasicompact cover. This is because generalizations lift along flat, quasicompact morphisms, ensuring that maximal points map to each maximal point.

Lemma A.3. *Let $X \to \operatorname{Spec} k$ be a finite morphism from a DM stack to a field. Then X admits a finite étale cover from a scheme.*

Proof. Pick a finite type étale cover $P \to X$. Then $P \to X$ is locally quasifinite [10, 03WS] and hence quasifinite [10, 01TD]. The composite $P \to \operatorname{Spec} k$ is quasifinite, hence finite [10, 02NH]. The map $P \to X$ is then finite. □

Definition A.4. A finite DM-type morphism $X \to \operatorname{Spec} k$ is of pure degree d if, for some (equiv. any) finite étale cover $P \to X$ by a scheme,

$$\frac{\deg(P/\operatorname{Spec} k)}{\deg(P/X)} = d.$$

A DM-type morphism $X \to Y$ of locally noetherian artin stacks is *generically finite* if, for all maximal points $\eta \to Y$, the pullback

$$X \times_Y \eta \to \eta$$

is finite.

Remark A.5. The definition of *degree d* for generically finite morphisms is determined by its properties:

- A composite $X \xrightarrow{f} Y \xrightarrow{g} Z$ for which $\deg f$, $\deg g$, $\deg g \circ f$ are well defined satisfies

$$\deg(g \circ f) = \deg f \cdot \deg g.$$

- Given a pullback square

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow{\scriptstyle f'} & \ulcorner & \downarrow{\scriptstyle f} \\ Y' & \longrightarrow & Y \end{array}$$

  with $Y' \to Y$ flat and quasicompact, f is generically finite (of degree d) if and only if f' is.
- Agreement with the notion for representable morphisms in Definition A.1.

We conclude with two folklore observations that we use in the body of the text.

Remark A.6. ("*Stability is an open condition*") Suppose $f : X \to Y$ is locally finite type and X, Y are algebraic stacks. There is a substack $U \subseteq X$ representing morphisms $T \to X$ such that $f|_T$ is DM type, and this substack is open. A map is DM type when the diagonal is unramified, which is an open condition by The Stacks Project Authors [10, 0475].

This shows that the locus where a family of prestable maps is stable is open in the base.

Remark A.7. If X is an algebraic stack locally of finite type, then its normalization $X^\nu \to X$ is finite. This is because normalizations are integral [10, 035Q] and the map is locally of finite type [10, 01WJ].

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