WEAKLY MINIMAL GROUPS WITH A NEW PREDICATE

GABRIEL CONANT AND MICHAEL C. LASKOWSKI

Abstract. Fix a weakly minimal (i.e., superstable U-rank 1) structure $\mathcal{M}$. Let $\mathcal{M}^*$ be an expansion by constants for an elementary substructure, and let A be an arbitrary subset of the universe M. We show that all formulas in the expansion $(\mathcal{M}^*, A)$ are equivalent to bounded formulas, and so $(\mathcal{M}, A)$ is stable (or NIP) if and only if the $\mathcal{M}$-induced structure $A_{\mathcal{M}}$ on A is stable (or NIP). We then restrict to the case that $\mathcal{M}$ is a pure abelian group with a weakly minimal theory, and $A_{\mathcal{M}}$ is mutually algebraic (equivalently, weakly minimal with trivial forking). This setting encompasses most of the recent research on stable expansions of $(\mathbb{Z}, +)$. Using various characterizations of mutual algebraicity, we give new examples of stable structures of the form $(\mathcal{M}, A)$. Most notably, we show that if $(G, +)$ is a weakly minimal additive subgroup of the algebraic numbers, $A \subseteq G$ is enumerated by a homogeneous linear recurrence relation with algebraic coefficients, and no repeated root of the characteristic polynomial of A is a root of unity, then $(G, +, B)$ is superstable for any $B \subseteq A$.

1. Introduction

Given a structure $\mathcal{M}$, and a set $A \subseteq M$, a common line of investigation concerns model-theoretic properties of $\mathcal{M}$ that are preserved in the expansion $(\mathcal{M}, A)$ of $\mathcal{M}$ by a unary predicate naming A. In this situation, the *$\mathcal{M}$-induced structure on A*, denoted $A_{\mathcal{M}}$ (see Definition 2.3), is interpretable in $(\mathcal{M}, A)$, and so model theoretic complexity in $A_{\mathcal{M}}$ will persist in $(\mathcal{M}, A)$. Altogether, a fundamental question is when some model theoretic property, satisfied by both $\mathcal{M}$ and $A_{\mathcal{M}}$, will be satisfied by $(\mathcal{M}, A)$. In [7], Casanovas and Ziegler define the notion of a set $A \subseteq M$ that is *bounded in* $\mathcal{M}$ (see Definition 2.1), which is a certain "quantifier organization" property of formulas in the expansion $(\mathcal{M}, A)$, and they show that if A is bounded in $\mathcal{M}$ then $(\mathcal{M}, A)$ is stable if and only if $\mathcal{M}$ and $A_{\mathcal{M}}$ are stable. The analogous result for NIP was shown by Chernikov and Simon [9].

A notable instance of the situation above concerns expansions of the complex field $(\mathbb{C}, +, \cdot)$ by a finite rank subgroup Γ of a semi-abelian variety. In this setting, *Lang's conjecture* (now a theorem of Faltings and Vojta) is equivalent to the statement that $(\mathbb{C}, +, \cdot, \Gamma)$ is stable and $\Gamma_{(\mathbb{C},+,\cdot)}$ is 1-based. This equivalence is explained by Pillay in [29], and also describes the model-theoretic ingredients of Hrushovski's [18] proof of Mordell-Lang for function fields. A consequence of Pillay's work is that if $\mathcal{M}$ is strongly minimal, then any $A \subseteq M$ is bounded in $\mathcal{M}$ (see [7, Corollary 5.4]).

Drawing from results of Poizat [30] on "beautiful pairs" of models of a stable theory, Casanovas and Ziegler [7] also isolate the more semantic notion of a *small* set in $\mathcal{M}$ (see Definition 3.2), and show that if $\mathcal{M}$ is stable *and* has nfcp, then small sets are bounded. Altogether, this yields a strategy for proving stability (or NIP) of an expansion $(\mathcal{M}, A)$ of an nfcp structure $\mathcal{M}$: one first shows that A is small in $\mathcal{M}$ and then that $A_{\mathcal{M}}$ is stable (or NIP). This strategy was used by Palacín and Sklinos [27] to give the first examples of stable expansions of the group of integers $(\mathbb{Z}, +)$

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(see also Poizat [31]), and again in subsequent generalizations of these examples by the first author in [10] and [11], and by Lambotte and Point in [21].

The first main result of this paper is that if $\mathcal{M}$ is *weakly minimal* (i.e., superstable of U-rank 1), then *any* set $A \subseteq M$ is bounded in the expansion of $\mathcal{M}$ by constants for some $\mathcal{M}_0 \preceq \mathcal{M}$ (see Theorem 2.9). This generalizes Pillay's result above on strongly minimal structures (modulo the extra constants for $\mathcal{M}_0$, which are necessary; see Remark 2.14), and yields following conclusion about expansions of weakly minimal structures by unary predicates.

Theorem 2.10. Suppose $\mathcal{M}$ is weakly minimal and $A \subseteq M$.

(a) If $A_{\mathcal{M}}$ is stable of U-rank α then $(\mathcal{M}, A)$ is stable of U-rank at most $\alpha \cdot \omega$.
(b) If $A_{\mathcal{M}}$ is NIP then $(\mathcal{M}, A)$ is NIP.

Returning to the work from [10], [21], and [27] on stable expansions of $(\mathbb{Z}, +)$ by unary predicates, we see that the initial step in the above strategy of proving smallness of the predicate is unnecessary. Motivated by this situation, we then focus our attention on abelian groups whose pure theory is weakly minimal (see Proposition 5.1 for an algebraic characterization of such groups). In Proposition 5.4, we observe that if $\mathcal{G} = (G, +)$ is a weakly minimal abelian group, and $A \subseteq G$, then the induced structure $A_{\mathcal{G}}$ consists of the *quantifier-free* induced structure, denoted $A^{\mathrm{qf}}_{\mathcal{G}}$, together with unary predicates for $A \cap nG$ for all $n \geq 1$. Thus the task of analyzing $A_{\mathcal{G}}$ decomposes into understanding solutions in A to linear equations, and the behavior of A modulo any fixed $n \geq 1$.

The focus of [21] and [27] is on expansions of $\mathcal{Z} = (\mathbb{Z}, +)$ by sets $A \subseteq \mathbb{Z}$ that are eventually periodic modulo any fixed $n \geq 1$, which provides a concrete description of the unary predicates needed to complete $A^{\mathrm{qf}}_{\mathcal{Z}}$ to $A_{\mathcal{Z}}$. However, as observed by the first author in [10, 11], the specific sets $A \subseteq \mathbb{Z}$ considered in [10], [11], [21], and [27] have the property that *any* expansion of $A^{\mathrm{qf}}_{\mathcal{Z}}$ by unary predicates is stable, and so this extra assumption of periodicity is unnecessary. In the present paper, we isolate a model-theoretic setting for this phenomenon. Specifically, we consider *mutually algebraic structures*, which were defined by the second author in [22], and shown to satisfy the property that any expansion by unary predicates is stable and has nfcp. For each example of a stable structure $(\mathbb{Z}, +, A)$, considered in [10], [11], [21], and [31], the specific set A has the property that $A_{\mathcal{Z}}$ is mutually algebraic. In Section 5, we show that if $\mathcal{G} = (G, +)$ is a weakly minimal torsion-free abelian group, and $A \subseteq G$ is such that $A_{\mathcal{G}}$ is stable with trivial forking (e.g., mutually algebraic), then A is automatically small in $\mathcal{G}$. In particular, we show that if $A_{\mathcal{G}}$ is not small then it interprets the group $\mathcal{G}$; see Corollary 5.9. While smallness of A is irrelevant for stability of $(\mathcal{G}, A)$ by the above, it does allow one to transfer nfcp from $\mathcal{G}$ and $A_{\mathcal{G}}$ to $(\mathcal{G}, A)$ (by results from [7]). Using this, we prove the following theorem.

Theorem 5.10. Let $\mathcal{G} = (G, +)$ be a weakly minimal abelian group. Fix $A \subseteq G$, and suppose $A^{\mathrm{qf}}_{\mathcal{G}}$ is mutually algebraic. Then, for any finite $F \subset G$ and any $B \subseteq A + F$, $(\mathcal{G}, B)$ is superstable of U-rank at most ω. Moreover, if $\mathcal{G}$ is torsion-free then $(\mathcal{G}, B)$ has nfcp; and if $\mathcal{G} = (\mathbb{Z}, +)$ and B is infinite then $(\mathcal{G}, B)$ has U-rank ω.

Finally, in Section 6, we use this result to find several new examples of stable expansions of weakly minimal abelian groups. In particular, we show that if $\mathcal{G} = (G, +)$ is a weakly minimal abelian group, A is a subset of G, and one of the following situations holds, then $A^{\mathrm{qf}}_{\mathcal{G}}$ is mutually algebraic and so Theorem 5.10 applies.

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* **Section 6.1.** $\mathcal{G}$ is a subgroup of $(\mathbb{C},+)$ and $A=\{a_n\}_{n=0}^{\infty}$, where $\lim_{n\to\infty}\frac{a_{n+1}}{a_n}$ either diverges or converges to some transcendental $\tau\in\mathbb{C}$ with $|\tau|>1$.
* **Section 6.2.** $\mathcal{G}$ is a subgroup of the additive group $(\mathbb{K},+)$ of an algebraically closed field $\mathbb{K}$ of characteristic 0, and A is contained in a finite rank multiplicative subgroup of $\mathbb{K}^*$.
* **Section 6.3.** For any $k\geq 1$, there are $n_k\in\mathbb{N}$ and finite $U_k,V_k\subset G$ such that if $r\notin U_k$, then there are at most n_k tuples $\bar{a}\in(\pm A)^k$ satisfying $a_1+\ldots+a_k=r$ and $\sum_{i\in I}a_i\notin V_k$ for any nonempty $I\subsetneq[k]$.
* **Section 6.4.** $\mathcal{G}$ is a subgroup of the additive group of algebraic numbers, A is enumerated by a linear homogeneous recurrence relation with constant (algebraic) coefficients, and no repeated root of the characteristic polynomial of the recurrence is a root of unity.

The examples in Section 6.1 generalize certain families of "sparse sets" considered in [10], [21], and [27]. In this case, we use methods similar to Lambotte and Point [21] to show that $A^{\mathrm{qf}}_{\mathcal{G}}$ is interdefinable with A in the language of equality.

The examples in Section 6.2 generalize work of the first author from [11], and complement many existing results about expansions of the *field* $(\mathbb{C},+,\cdot)$ by finite rank multiplicative subgroups (e.g., Belegradek & Zilber [2], and Van den Dries & Günaydın [12]). In this case, we use a number-theoretic result of Everste, Schlickewei, and Schmidt [14] to give an extremely quick proof that $A^{\mathrm{qf}}_{\mathcal{G}}$ is mutually algebraic. This proof also highlights a pleasing parallel between the original definition of mutual algebraicity from [22] and the behavior of solutions of linear equations which lie in a finite rank multiplicative group.

The purpose of Section 6.3 is to give a combinatorial generalization of the behavior studied in Section 6.2. One reason for this is to showcase a connection to recent work of the second author and Terry [23] on a new characterization of mutual algebraicity. More importantly, the main technical result of this section (Proposition 6.8) is the key tool needed for Section 6.4. Specifically, fix $\mathcal{G}$ and $A\subseteq G$ enumerated by a recurrence sequence as described above. To prove $A^{\mathrm{qf}}_{\mathcal{G}}$ is mutually algebraic, we first use the work in Section 6.3 to prove mutual algebraicity of an auxiliary structure $\mathbb{N}^{\Phi}_{\mathcal{K}}$, formulated using a number field over which A is defined. We then show that $A^{\mathrm{qf}}_{\mathcal{G}}$ is suitably interpreted in $\mathbb{N}^{\Phi}_{\mathcal{K}}$. To show that $\mathbb{N}^{\Phi}_{\mathcal{K}}$ fits into the combinatorial framework of Section 6.3, we use a quantitative version of work of M. Laurent [24, 25], due to Schlickwei and Schmidt [33], on the number of solutions to polynomial-exponential equations over number fields.

Section 6.4 provides a significant generalization of the examples from [10] and [21] of stable structures of the form $(\mathbb{Z},+,A)$, where A is enumerated by a homogeneous linear recurrence relation. These previous examples imposed fairly restrictive assumptions including irreducibility of the characteristic polynomial $p_A(x)$ of the recurrence. In particular, the question of stability of $(\mathbb{Z},+,A)$ even in the case that $p_A(x)$ is separable was open. In Theorem 6.21, we give a more direct proof of the separable case, which works with any algebraically closed field of characteristic 0 in place of $\mathbb{Q}^{\mathrm{alg}}$. Beyond this, the division between a separable and nonseparable characteristic polynomial is number-theoretically significant, as there are many questions about solutions of linear equations from recurrences sequences, in which the separable case is manageable but the general case is much more difficult (see, e.g., [?], [13, Section 2.5]). So results about $A^{\mathrm{qf}}_{\mathcal{G}}$, with A and $\mathcal{G}$ as in Section 6.4, are interesting in their own right.

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2. Bounded sets in weakly minimal theories

Throughout this section, let T be a complete theory with infinite models in some language $\mathcal{L}$. Given, $\mathcal{M} \models T$, when we say that a set $X \subseteq M^n$ is *$\mathcal{M}$-definable*, we mean definable with parameters from M.

Let $\mathcal{L}(P) = \mathcal{L} \cup \{P\}$ where P is a unary relation symbol not in $\mathcal{L}$. Given $\mathcal{M} \models T$ and $A \subseteq M$, let $(\mathcal{M}, A)$ be the $\mathcal{L}(P)$-structure expanding $\mathcal{M}$ in which P is interpreted as A.

Definition 2.1.

(1) An $\mathcal{L}(P)$-formula $\phi(x_1, \ldots, x_n)$ is **bounded** if it is of the form
$$Q_1 y_1 \in P \ldots Q_m y_m \in P\, \psi(x_1, \ldots, x_n, y_1, \ldots, y_m)$$
for some quantifiers $Q_1, \ldots, Q_m$ and some $\mathcal{L}$-formula $\psi(\bar{x}, \bar{y})$.

(2) Given $\mathcal{M} \models T$, a set $A \subseteq M$ is **bounded in** $\mathcal{M}$ if every $\mathcal{L}(P)$-formula is equivalent, modulo $\mathrm{Th}(\mathcal{M}, A)$, to a bounded $\mathcal{L}(P)$-formula.

Remark 2.2. Suppose $\mathcal{M} \models T$ and $A \subseteq M$ is $\mathcal{M}$-definable over $\emptyset$. Then A is bounded in $\mathcal{M}$.

Definition 2.3. Given $\mathcal{M} \models T$ and a sort S from $\mathcal{L}$, let $\mathcal{L}_S^{\mathcal{M}}$ denote a relational language containing an n-ary relation R_X of sort S^n, for any $n \geq 1$ and any $\mathcal{M}$-definable $X \subseteq (M^S)^n$. Given $A \subseteq M^S$, let $A_{\mathcal{M}}$ denote the $\mathcal{L}_S^{\mathcal{M}}$-structure, with universe A, in which each symbol R_X is interpreted as $A^n \cap X$. We call $A_{\mathcal{M}}$ the **$\mathcal{M}$-induced structure on** A.

The following is Proposition 3.1 of [7].

Proposition 2.4 (Casanovas & Ziegler)**.** *Fix $\mathcal{M} \models T$ and $A \subseteq M$. If A is bounded in $\mathcal{M}$, then $(\mathcal{M}, A)$ is stable if and only if $\mathcal{M}$ and $A_{\mathcal{M}}$ are stable.*

We will use the following characterization of bounded sets in stable theories, which is part of Proposition 5.3 of [7].

Proposition 2.5 (Casanovas & Ziegler)**.** *If T is stable then the following are equivalent for any $\mathcal{M} \models T$ and $A \subseteq M$.*

(*i*) *A is bounded in $\mathcal{M}$.*

(*ii*) *If $(\mathcal{N}, B) \equiv_{\mathcal{L}(P)} (\mathcal{M}, A)$ is $|T|^+$-saturated, f is an $\mathcal{L}$-elementary map in $\mathcal{N}$, which is a finite extension of a permutation of B, and $a \in N$, then there is $b \in N$ such that $f \cup \{(a, b)\}$ is $\mathcal{L}$-elementary.*

For the rest of the paper, we will focus on expansions of weakly minimal theories.

Definition 2.6. T is **weakly minimal** if it is stable and, for any $\mathcal{M} \models T$, $B \subseteq M$, and $p \in S_1(B)$, any forking extension of p is algebraic.

In other words, T is weakly minimal if and only if it is stable of U-rank 1. In this case, we also call models of T **weakly minimal**.

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Recall that any stable theory has a U-rank in $\mathrm{Ord}\cup\{\infty\}$, which is an ordinal if and only if the theory is superstable. Multiplication of ordinals (denoted $\cdot$) extends to $\mathrm{Ord}\cup\{\infty\}$ in the obvious way. The following result is [10, Theorem 2.11], and is proved using Proposition 2.4 and techniques similar to the work of Palacín and Sklinos [27] on the expansion of $(\mathbb{Z},+)$ by $\{2^n : n\in\mathbb{N}\}$.

Theorem 2.7 (Conant)**.** *Assume T is weakly minimal and fix $\mathcal{M}\models T$. Suppose $A\subseteq M$ is bounded in $\mathcal{M}$ and is such that $A_{\mathcal{M}}$ is stable of U-rank α. Then $(\mathcal{M},A)$ is stable of U-rank at most $\alpha\cdot\omega$.*

Definition 2.8.

(1) Given $\mathcal{M}\models T$, let $\mathcal{L}_M$ be the expansion of $\mathcal{L}$ by adding a constant symbol for each element of M, and let $T_{\mathcal{M}}$ be the elementary diagram of $\mathcal{M}$ in the expanded language $\mathcal{L}_M$.

(2) Fix $\mathcal{M}\models T$ and $\mathcal{M}_0\preceq\mathcal{M}$. A set $A\subseteq M$ is **bounded in $\mathcal{M}$ with respect to $\mathcal{L}_{M_0}$** if it is bounded in the canonical $\mathcal{L}_{M_0}$-expansion of $\mathcal{M}$, i.e., for all $\mathcal{L}(P)$-formulas $\phi(\bar{x};\bar{y})$ and all $\bar{b}\in M_0^{\bar{y}}$, there is a bounded $\mathcal{L}(P)$-formula $\psi(\bar{x};\bar{z})$ and $\bar{c}\in M_0^{\bar{z}}$ such that $(\mathcal{M},A)\models\forall\bar{x}(\phi(\bar{x};\bar{b})\leftrightarrow\psi(\bar{x};\bar{c}))$.

We now state our first main result.

Theorem 2.9. *If T is weakly minimal, $\mathcal{M}\models T$, and $\mathcal{M}_0\preceq\mathcal{M}$, then every $A\subseteq M$ is bounded in $\mathcal{M}$ with respect to $\mathcal{L}_{M_0}$.*

Before continuing to the proof, we use Theorem 2.9 to establish the second main result of this section.

Theorem 2.10. *Assume T is weakly minimal. Fix $\mathcal{M}\models T$ and $A\subseteq M$.*

(a) If $A_{\mathcal{M}}$ is stable of U-rank α, then $(\mathcal{M},A)$ is stable of U-rank at most $\alpha\cdot\omega$.
(b) If $A_{\mathcal{M}}$ is NIP then $(\mathcal{M},A)$ is NIP.

Proof. Fix $A\subseteq M$. By definition of $A_{\mathcal{M}}$, we may assume without loss of generality that $\mathcal{L}=\mathcal{L}_M$ and $T=T_{\mathcal{M}}$. By Theorem 2.9, A is bounded in $\mathcal{M}$. So part (a) follows from Theorem 2.7, and part (b) follows from [9, Corollary 2.5]. $\square$

The proof of Theorem 2.9 breaks into several pieces. We first note various facts about weakly minimal theories. First off, note that if T is weakly minimal and $\mathcal{M}\models T$, then $T_{\mathcal{M}}$ is weakly minimal.

Lemma 2.11. *Suppose T is weakly minimal, $\mathcal{M}_0\models T$, $\mathcal{M}_0\preceq\mathcal{M}$, and $M_0\subseteq A\subseteq M$. Then $\mathrm{acl}(A)\models T$ and $\mathcal{M}_0\preceq\mathrm{acl}(A)\preceq\mathcal{M}$. Moreover, if $\mathcal{M}_0$ is $|T|^+$-saturated, then $\mathrm{acl}(A)$ is $|T|^+$-saturated as well.*

Proof. Without loss of generality, assume $A=\mathrm{acl}(A)$. To show $A\preceq\mathcal{M}$, we choose an $\mathcal{L}$-formula $\phi(x;\bar{a})$, with $\bar{a}\subset A$, that has a solution $b\in M$, and we show that $\phi(x;\bar{a})$ has a solution in A. If $b\in A$ we are done, so assume otherwise. As $A=\mathrm{acl}(A)$ and $M_0\subset A$, we have $A=\mathrm{acl}_{\mathcal{L}_{M_0}}(A)$ as well. So, as $T_{\mathcal{M}_0}$ is weakly minimal, $b\notin A$ implies $b\mathop{\smile\!\!\!\!\!\!\mid}_{M_0}\bar{a}$ with respect to $T_{\mathcal{M}_0}$. Hence also, $b\mathop{\smile\!\!\!\!\!\!\mid}_{M_0}\bar{a}$ with respect to T. Thus, by finitely satisfiability, there is $b^*\in M_0$ such that $\mathcal{M}\models\varphi(b^*;\bar{a})$, as desired. Next, by the same argument applied to $T_{\mathcal{M}_0}$, we have $A=\mathrm{acl}_{\mathcal{L}_{M_0}}(A)\models T_{\mathcal{M}_0}$, which clearly implies $\mathcal{M}_0\preceq A$.

Now assume $\mathcal{M}_0$ is $|T|^+$-saturated. We argue that any model $\mathcal{N}\succeq\mathcal{M}_0$ must also be $|T|^+$-saturated, which suffices. The proof is essentially the same as [17,

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Proposition 3.5] (in fact, the following argument can be adapted to any non-multidimensional theory by replacing the use of weak minimality with an appropriate version of the "three-model lemma"). Let $\mathcal{N}^*$ be the $|T|^+$-prime model over N. If $\mathcal{N} = \mathcal{N}^*$ we finish, so assume otherwise. Choose $b \in N^* \backslash N$. Then $\operatorname{tp}(b/N)$ is a non-algebraic extension of $\operatorname{tp}(b/M_0)$ and so $b \mathop{\smile\!\!\!\!\!\!\mid}_{M_0} N$ by weak minimality. Since N^* is dominated by N over M_0 (adapt [28, Lemma 1.4.3.4(*iii*)] to the category of $|T|^+$-saturated models), we have $b \mathop{\smile\!\!\!\!\!\!\mid}_{M_0} N^*$, which is a contradiction. □

Suppose now that T is weakly minimal. Then a type over a model of T is regular if and only if it is non-algebraic. Suppose $\mathcal{M} \preceq \mathcal{N}$ are $|T|^+$-saturated models of T. Then, by weak minimality and exchange for algebraic independence, we have that for any regular $p, q \in S_1(M)$, if p and q are non-orthogonal, and $I \subseteq p(N)$ and $J \subseteq q(N)$ are maximal M-independent sets, then $|I| = |J|$ (note that by $|T|^+$ saturation, orthogonality and weak orthogonality coincide for regular types over M; see [28, Lemma 1.4.3.1]). So, for any regular type p over some other model of T, we have a well-defined dimension $\dim_p(N/M)$, namely, the cardinality of a maximal M-independent set of realizations in N of any regular $q \in S_1(M)$ non-orthogonal to p. In fact, $\dim_p(N/M)$ coincides with the cardinality of a maximal M-independent $I \subseteq N$ such that, for any $a \in I$, $\operatorname{tp}(a/M)$ is regular and non-orthogonal to p. The following properties of $\dim_p$ are standard exercises (see [28], [35]).

Fact 2.12. *Assume T is weakly minimal and $\mathcal{M} \preceq \mathcal{N} \models T$ are $|T|^+$-saturated. Suppose p and q are regular types over models.*

(*a*) *If p and q are non-orthogonal then* $\dim_p(N/M) = \dim_q(N/M)$.
(*b*) $\dim_p(N/M) \leq \dim_{\mathrm{acl}}(N/M)$.
(*c*) *If $\mathcal{N}^* \succeq \mathcal{N}$ then* $\dim_p(N^*/M)$ *is finite if and only if* $\dim_p(N^*/N)$ *and* $\dim_p(N/M)$ *are finite, and in this case* $\dim_p(N^*/M) = \dim_p(N^*/N) + \dim_p(N/M)$.

We now prove a proposition that carries additional hypotheses, which we subsequently remove in the proof of Theorem 2.9.

Proposition 2.13. *Suppose T is weakly minimal, $\mathcal{M}_0 \models T$ is $|T|^+$-saturated, $\mathcal{M}_0 \preceq \mathcal{M}$, and $A \subseteq M$ is $\mathcal{L}_{M_0}$-algebraically closed (so, in particular, $M_0 \subseteq A \subseteq M$). Then A is bounded in $\mathcal{M}$ with respect to $\mathcal{L}_{M_0}$.*

Proof. We will apply Proposition 2.5 with respect to the $\mathcal{L}_{M_0}$-theory $T_{\mathcal{M}_0}$. Given A as in the statement, choose any sufficiently saturated $(\mathcal{M}^*, A^*) \succeq_{\mathcal{L}_{M_0}(P)} (\mathcal{M}, A)$. Choose any finite $\bar{b}, \bar{c} \subset M^*$ and any $\mathcal{L}_{M_0}$-elementary bijection $f \colon A^*\bar{b} \to A^*\bar{c}$ extending a permutation of A^*. Choose any $d \in M^* \backslash A^* b$. It suffices to find $d' \in M^*$ such that $f \cup \{(d, d')\}$ is $\mathcal{L}_{M_0}$-elementary. By Lemma 2.11, the structures A^*, $\mathcal{M}_1 := \operatorname{acl}_{\mathcal{L}_{M_0}}(A^*\bar{b})$, and $\mathcal{M}_2 := \operatorname{acl}_{\mathcal{L}_{M_0}}(A^*\bar{c})$ are all $|T|^+$-saturated models of $T_{\mathcal{M}_0}$. Choose an $\mathcal{L}_{M_0}$-elementary bijection $f^* \colon \mathcal{M}_1 \to \mathcal{M}_2$ extending f. Let $p := \operatorname{tp}(d/M_1)$ and $p' := f^*(p) \in S_1(M_2)$. We want to show that p' is realized by some $d' \in M^*$.

Now, if $d \in M_1$ then we are done, so assume otherwise. Then p and p' are regular, and have the same restriction to M_0 since f^* is $\mathcal{L}_{M_0}$-elementary. In particular, p and p' are non-orthogonal. To show p' is realized in $\mathcal{M}^*$, it suffices to show $\dim_{p'}(M^*/M_2) > 0$. By Fact 2.12(a), it suffices to show $\dim_p(M^*/M_2) > 0$.

By Fact 2.12(b), $\dim_p(M_1/A^*)$ and $\dim_p(M_2/A^*)$ are both finite. Moreover, these dimensions are equal since (for the inequality in one direction), if $q \in S_1(A^*)$

is regular and non-orthogonal to p, and $I \subseteq q(M_1)$ is A^*-independent, then $f^*(q) \in S_1(A^*)$ is regular and non-orthogonal to p, and $f^*(I) \subseteq f^*(q)(M_2)$ is A^*-independent.

Suppose first that $\dim_p(M^*/A^*)$ is infinite, witnessed by $q \in S_1(A^*)$ and $I \subseteq q(M^*)$, where q is regular and non-orthogonal to p, and I is infinite and A^*-independent. Since $\dim_p(M_2/A^*)$ is finite, there is an infinite M_2-independent set of realizations of $(q|M_2)(M^*)$ contained in I. Thus $\dim_p(M^*/M_2)$ is infinite since $q|M_2$ is non-orthogonal to p.

Finally, suppose that $\dim_p(M^*/A^*)$ is finite. By Fact 2.12(c), we have

$$\dim_p(M^*/M_1) + \dim_p(M_1/A^*) = \dim_p(M^*/A^*) = \dim_p(M^*/M_2) + \dim_p(M_2/A^*),$$

and all dimensions involved are finite. Since $\dim_p(M_1/A^*) = \dim_p(M_2/A^*)$ and p is realized in M^*, we have $\dim_p(M^*/M_2) = \dim_p(M^*/M_1) > 0$, as desired. □

We can now prove Theorem 2.9.

Proof of Theorem 2.9. Assume T is weakly minimal, $\mathcal{M} \models T$, and $\mathcal{M}_0 \preceq \mathcal{M}$. Choose $A \subseteq M$ arbitrarily. We want to show A is bounded in $\mathcal{M}$ with respect to $\mathcal{L}_{M_0}$. Consider the $\mathcal{L}(P,Q)$-structure $(\mathcal{M}, A, M_0)$. Choose a $|T|^+$-saturated $\mathcal{L}(P,Q)$-elementary extension $(\mathcal{M}^*, A^*, M_0^*)$, and note that M_0^* is the universe of a $|T|^+$-saturated $\mathcal{L}$-elementary extension $\mathcal{M}_0^*$ of $\mathcal{M}_0$.

We now work with the theory $T_{\mathcal{M}_0^*}$ in the language $\mathcal{L}^* := \mathcal{L}_{M_0^*}$. Let $(\mathcal{N}^*, B) \equiv_{\mathcal{L}^*(P)} (\mathcal{M}^*, A^*)$ be $|T_{\mathcal{M}_0^*}|^+$-saturated. Let $B^* = \operatorname{acl}_{\mathcal{L}^*}(B)$. We have that T is weakly minimal, $\mathcal{M}_0^* \models T$ is $|T|^+$-saturated, $\mathcal{M}_0^* \preceq \mathcal{N}^*$, and $B^* \subseteq N^*$ is $\mathcal{L}^*$-algebraically closed. So we may apply Proposition 2.13 to conclude that B^* is bounded in $\mathcal{N}^*$ with respect to $\mathcal{L}^*$. Now, suppose $\bar{c}, \bar{d} \subset N^*$ are finite and $h\colon B\bar{c} \to B\bar{d}$ is an $\mathcal{L}^*$-elementary bijection in $\mathcal{N}^*$ extending a permutation of B. Then h extends to an $\mathcal{L}^*$-elementary bijection $h^*\colon B^*\bar{c} \to B^*\bar{d}$. Since B^* is bounded in $\mathcal{N}^*$ with respect to $\mathcal{L}^*$, Proposition 2.5 implies that for every $a \in N^*$ there is an $a' \in N^*$ such that $h^* \cup \{(a, a')\}$ is $\mathcal{L}^*$-elementary in $\mathcal{N}^*$. Applying Proposition 2.5 again, we conclude that B is bounded in $\mathcal{N}^*$ with respect to $\mathcal{L}^*$. By elementarity, A^* is bounded in $\mathcal{M}^*$ with respect to $\mathcal{L}^*$.

Now, fix any $\mathcal{L}(P)$-formula $\phi(\bar{x};\bar{y})$ and let $\Gamma(\bar{y})$ be the $\mathcal{L}(P,Q)$-type

$$\{\bar{y} \in Q\} \cup \{\forall \bar{z} \in Q\, \neg\forall \bar{x}(\phi(\bar{x};\bar{y}) \leftrightarrow \psi(\bar{x};\bar{z})) : \psi(\bar{x};\bar{z}) \text{ is a bounded } \mathcal{L}(P)\text{-formula}\}.$$

Since A^* is bounded in $\mathcal{M}^*$ with respect to $\mathcal{L}^*$, $\Gamma(\bar{y})$ is not realized by $\mathcal{N} = (\mathcal{M}^*, A^*, M_0^*)$. By saturation of $\mathcal{N}$, $\Gamma(\bar{y})$ is inconsistent with $\operatorname{Th}(\mathcal{N})$. By compactness, there are finitely many bounded $\mathcal{L}(P)$-formulas $\psi_1(\bar{x};\bar{z}_1), \ldots, \psi_\ell(\bar{x};\bar{z}_\ell)$ such that

$$\mathcal{N} \models \forall \bar{y} \in Q \bigvee_{i=1}^{\ell} \exists \bar{z}_i \in Q\, \forall \bar{x}(\phi(\bar{x};\bar{y}) \leftrightarrow \psi_i(\bar{x};\bar{z}_i)).$$

So $(\mathcal{M}, A, M_0)$ realizes this sentence, and so we see that for every $\bar{a} \in M_0^{\bar{y}}$ there is $1 \leq i \leq \ell$ and $\bar{c} \in M_0^{\bar{z}_i}$ such that $(\mathcal{M}, A) \models \forall \bar{x}(\phi(\bar{x};\bar{b}) \leftrightarrow \psi_i(\bar{x};\bar{c}))$.

As the $\mathcal{L}(P)$-formula $\phi(\bar{x};\bar{y})$ above was arbitrary, we conclude that A is bounded in $\mathcal{M}$ with respect to $\mathcal{L}_{M_0}$. □

Remark 2.14. We make some comments on the assumptions in Theorem 2.9

(1) Theorem 2.9 cannot be generalized to arbitrary stable theories. For example, Poizat [30] constructed an ω-stable theory T and models $\mathcal{N} \prec \mathcal{M} \models T$ such that the pair $(\mathcal{M}, N)$ is unstable. By stability of T, the induced structure $N_{\mathcal{M}}$

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is the same as $\mathcal{N}$ and so, by Proposition 2.4, N is not bounded in $\mathcal{M}$ (or in any expansion of $\mathcal{M}$ by constants). In [6], Bouscaren shows that if T is superstable, then every theory of pairs of models of T is stable if and only if T does not have the dimensional order property.

(2) The additional constants naming a substructure $\mathcal{M}_0$ are necessary in order to prove Theorem 2.9. For example, let T be the theory of an equivalence relation E with two infinite classes. Fix $\mathcal{M} \models T$ and distinct $a_1, a_2, b \in M$ such that $E(a_1, a_2)$ and $\neg E(a_1, b)$. Then $A = M\backslash\{a_1, a_2, b\}$ is not bounded in $\mathcal{M}$. To see this, note that a_1 and b clearly have different $\mathcal{L}(P)$-types while, on the other hand, there is an $\mathcal{L}$-elementary map from Aa_1 to Ab, extending a permutation of A, and so a_1 and b satisfy the same bounded $\mathcal{L}(P)$-formulas.

In [29], Pillay proves that if T is *strongly minimal*, $\mathcal{M} \models T$, and $A \subseteq M$, then A is bounded in $\mathcal{M}$, without the use of any extra constants (see also [7, Corollary 5.4]). Although it will not be necessary for our later results, it is interesting to see that the same holds for weakly minimal *expansions of groups*.

Theorem 2.15. *Suppose T is the theory of a weakly minimal expansion of a group, and $\mathcal{M} \models T$. Then every $A \subseteq M$ is bounded in $\mathcal{M}$.*

Proof. Fix $A \subseteq M$ and let $(\mathcal{G}, B) \succeq_{\mathcal{L}(P)} (\mathcal{M}, A)$ be $|T|^+$-saturated. Fix finite $\bar{c}, \bar{d} \subset G$ and suppose $f \colon B\bar{c} \to B\bar{d}$ is a partial $\mathcal{L}$-elementary map extending a permutation of B. Fix $a \in G$. We want to find $b \in G$ such that $f \cup \{(a, b)\}$ is $\mathcal{L}$-elementary. Given this, we will have that A is bounded in $\mathcal{M}$ by Proposition 2.5.

For the rest of the proof, we work in T. Given a strong type p over $\emptyset$, and sets $B \subseteq C \subseteq G$, let $\dim_p(C/B)$ be the cardinality of a maximal B-independent subset of $p(\operatorname{acl}(C))$ (which is well-defined by weak minimality). We will use properties of $\dim_p$ analogous to parts (b) and (c) of Fact 2.12, along with the following key claim.

Claim: Suppose p, q are non-algebraic strong types over $\emptyset$, and $C \subseteq G$ is such that p and q are both realized in $\operatorname{acl}(C)$. Then $\dim_p(G/C) = \dim_q(G/C)$.

Proof: It suffices to show $\dim_p(G/C) \leq \dim_q(G/C)$. Fix $b_0, c_0 \in \operatorname{acl}(C)$ realizing p and q, respectively. Given any C-independent set $I \subseteq p(G)$, let $J = \{ab_0^{-1}c_0 : a \in I\}$. Then we clearly have that J is C-independent, and that $|J| = |I|$. Moreover, for any $a \in I$, we have $\operatorname{stp}(a) = \operatorname{stp}(b_0)$, and so $b_0a^{-1} \in G^0 = \operatorname{Stab}(q)$, which implies $ab_0^{-1}c_0 \models q$. So $J \subseteq q(G)$ and, altogether, $\dim_p(G/C) \leq \dim_q(G/C)$. $\dashv_{\text{claim}}$

Now let $C_1 = \operatorname{acl}(B\bar{c})$ and $C_2 = \operatorname{acl}(B\bar{d})$. Without loss of generality, we may extend f so that it is a map from C_1 to C_2. Let $p = \operatorname{stp}(a)$, and let $\mathcal{G}^*$ be a sufficiently saturated elementary extension of $\mathcal{G}$. Choose $b_* \in G^*$ such that $f \cup \{(a, b_*)\}$ is elementary, and let $q = \operatorname{stp}(b_*)$. If $b^* \in G$ then we are done, so assume otherwise. In particular, $b_* \notin C_2$, which implies $a \notin C_1$ and $b_* \mathop{\smile\!\!\!\!\!\mid}_{\emptyset} C_2$. To find our desired b, it suffices by stationarity of q to find $b \in G\backslash C_2$ realizing q. In other words, we want to show $\dim_q(G/C_2) > 0$.

Suppose first that p is not realized in C_1. Since $\mathcal{G}$ is $|T|^+$-saturated, there is a realization b of q in G. Toward a contradiction, suppose $b \in C_2$. Then $\operatorname{stp}(b_*) = \operatorname{stp}(b)$, and so $b_*^{-1}b \in (G^*)^0$. Then $a^{-1}f^{-1}(b) \in G^0$, and so $\operatorname{stp}(a) = \operatorname{stp}(f^{-1}(b))$, which contradicts that p is not realized in C_1.

Next, let $r \in S_1(\operatorname{acl}^{\mathrm{eq}}(\emptyset))$ be the principal generic. Suppose r is not realized in C_1. Since r is $\emptyset$-invariant, it is also not realized in C_2. Note that if $b_1, b_2 \models q$, with

$b_1 \mathop{\smile\!\!\!\!\!\!\!\;\mid}_{\emptyset} b_2$, then $b_1^{-1}b_2 \models r$. So we have $\dim_q(C_2/\emptyset) \leq 1$. Since $\dim_q(G/\emptyset)$ is infinite (by $|T|^+$-saturation of $\mathcal{G}$), it follows that $\dim_q(G/C_2)$ is infinite.

Finally, suppose p and r are both realized in C_1. As above, r is realized in C_2. Also q is realized in C_2 since $f(p(C_1)) \subseteq q(C_2)$. By the claim,

$$\dim_p(G/C_1) = \dim_r(G/C_1) \quad \text{and} \quad \dim_q(G/C_2) = \dim_r(G/C_2).$$

In particular, we may assume $\dim_r(G/C_2)$ is finite. Note also that $\dim_r(C_2/B)$ is finite since it is bounded above by $\dim_{\mathrm{acl}}(C_2/B)$. By additivity,

$$\dim_r(G/C_1) + \dim_r(C_1/B) = \dim_r(G/B) = \dim_r(G/C_2) + \dim_r(C_2/B).$$

Since $f\colon C_1 \to C_2$ extends a permutation of B, and r is $\emptyset$-invariant, we also have $\dim_r(C_1/B) = \dim_r(C_2/B)$, and so $\dim_r(G/C_1) = \dim_r(G/C_2)$. Altogether, this yields $\dim_p(G/C_1) = \dim_q(G/C_2)$. Since $\dim_p(G/C_1) > 0$ (witnessed by a), we have $\dim_q(G/C_2) > 0$. $\square$

3. Small sets and nfcp

We again let T denote a complete $\mathcal{L}$-theory. Recall that T has **nfcp** (no finite cover property) if for any formula $\phi(\bar{x};\bar{y})$ there is some $k \geq 1$ such that, for any $\mathcal{M} \models T$ and $B \subseteq M^{\bar{y}}$, the partial type $\{\phi(\bar{x};\bar{b}) : \bar{b} \in B\}$ is consistent if and only if it is k-consistent.

Fact 3.1.

(a) *T has nfcp if and only if it is stable and eliminates $\exists^\infty$ in all imaginary sorts.*
(b) *If T is weakly minimal then it has nfcp.*

Proof. Part (a) is one of the equivalences of Shelah's *fcp theorem* [35, Theorem II.4.4]. For part (b), it follows from Section 2 of [15] that any weakly minimal theory eliminates $\exists^\infty$ in all imaginary sorts. $\square$

Given $\mathcal{M} \models T$ and $A \subseteq M$, Casanovas and Ziegler [7] also provide a test for transferring nfcp from $\mathcal{M}$ and $A_{\mathcal{M}}$ to $(\mathcal{M}, A)$.

Definition 3.2. Suppose $\mathcal{M} \models T$ and $A \subseteq M$. Then $A \subseteq M$ is **small in** $\mathcal{M}$ if there is $(\mathcal{N}, B) \equiv_{\mathcal{L}(P)} (\mathcal{M}, A)$ such that, for any finite tuple $\bar{b}$ from N, any type $p \in S_1^{\mathcal{L}}(B\bar{b})$ is realized in $\mathcal{N}$.

Remark 3.3. If $\mathcal{M} \models T$ and $A \subseteq M$ is $\mathcal{M}$-definable, then A is small in $\mathcal{M}$ if and only if it is finite.

Proposition 3.4 (Casanovas & Ziegler). *Fix $\mathcal{M} \models T$ and $A \subseteq M$.*

(a) *If $\mathcal{M}$ has nfcp and A is small in $\mathcal{M}$, then A is bounded in $\mathcal{M}$.*
(b) *If A is small in $\mathcal{M}$, then $(\mathcal{M}, A)$ has nfcp if and only if $\mathcal{M}$ and $A_{\mathcal{M}}$ have nfcp.*

Proof. These are Propositions 2.1 and 5.7 of [7], respectively. $\square$

Next we will give a characterization of small sets in weakly minimal structures, and then refine this characterization for the unidimensional case.

Lemma 3.5. *Suppose T is weakly minimal and $\mathcal{M} \models T$. Given $A \subseteq M$, the following are equivalent.*

(i) *A is not small in $\mathcal{M}$.*
(ii) *A is not small in the $\mathcal{L}_M$-expansion of $\mathcal{M}$ by constants.*

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(iii) *There is an $\mathcal{L}_M$-formula $\phi(x;\bar{y})$ such that such that $\{\phi(x;\bar{a}) : \bar{a} \in A^{\bar{y}}\}$ is finitely satisfiable in $\mathcal{M}$ but not realized in $\mathcal{M}$.*
(iv) *There are $\mathcal{L}_M$-formulas $\psi(x)$ and $\phi(x;\bar{y})$ such that $\psi(x)$ is non-algebraic, $\phi(x;\bar{a})$ is algebraic for all $\bar{a} \in M^{\bar{y}}$, and $\psi(M) \subseteq \bigcup_{\bar{a}\in A^{\bar{y}}} \phi(M;\bar{a})$.*
(v) *There is a non-algebraic $\mathcal{L}_M$-formula $\psi(x)$ such that if $(\mathcal{N}, B) \equiv_{\mathcal{L}_M(P)} (\mathcal{M}, A)$ then $\psi(N) \subseteq \operatorname{acl}_{\mathcal{L}_M}(B)$.*

Proof. We first show (i), (ii), and (iv) are equivalent. $(i) \Rightarrow (ii)$ is clear.

$(ii) \Rightarrow (iv)$. Assume (ii). By elimination of $\exists^\infty$ for $T_{\mathcal{M}}$ it suffices to find $\mathcal{L}_M$-formulas $\psi(x)$ and $\phi(x;\bar{y})$ satisfying the desired conditions for some $(\mathcal{N}, B) \equiv_{\mathcal{L}_M(P)} (\mathcal{M}, A)$. So fix an $|M|^+$-saturated extension $(\mathcal{N}, B) \succ_{\mathcal{L}_M(P)} (\mathcal{M}, A)$. Since A is not small in the expansion of $\mathcal{M}$ by constants, there is a tuple $\bar{c} \in N^{\bar{z}}$ and a type $p \in S_1^{\mathcal{L}_M}(B\bar{c})$ such that p is not realized in $\mathcal{N}$. In particular, p is not algebraic. Let $p_0 \in S_1^{\mathcal{L}_M}(\emptyset)$ be the restriction of p. We claim that $p_0(N) \subseteq \operatorname{acl}_{\mathcal{L}_M}(B\bar{c})$. Suppose otherwise that there is $a \in p_0(N)\setminus \operatorname{acl}_{\mathcal{L}_M}(B\bar{c})$ and let $q = \operatorname{tp}_{\mathcal{L}_M}(a/B\bar{c})$. Then p and q have the same restriction to $\emptyset$, and so $p = q$ by stationarity, which contradicts that p is not realized in $\mathcal{N}$.

Since $p_0(N) \subseteq \operatorname{acl}_{\mathcal{L}_M}(B\bar{c})$, the following $\mathcal{L}_M(P)$-type is omitted in $(\mathcal{N}, B)$:

$$p_0(x) \cup \left\{\forall \bar{y} \in P \left(\exists^{<\infty} v\, \phi(v;\bar{y};\bar{c}) \to \neg\phi(x;\bar{y};\bar{c})\right) : \phi(x;\bar{y};\bar{z}) \text{ an } \mathcal{L}_M\text{-formula}\right\}.$$

By saturation of $(\mathcal{N}, B)$, there are $\mathcal{L}_M$-formulas $\psi(x)$, $\phi_1(x;\bar{y}_1;\bar{z}), \ldots, \phi_n(x;\bar{y}_n;\bar{z})$ such that $\psi(x) \in p_0$ and

$$(\mathcal{N}, B) \models \theta(\bar{c}) := \forall x \left(\psi(x) \to \bigvee_{t=1}^{n} \exists \bar{y}_t \in P \left(\phi_t(x;\bar{y}_t;\bar{c}) \wedge \exists^{<\infty} v\, \phi_t(v;\bar{y}_t;\bar{c})\right)\right).$$

By elementarity there is $\bar{d} \in M^{\bar{y}}$ such that $(\mathcal{N}, B) \models \theta(\bar{d})$. Let $\bar{y} = (\bar{y}_1, \ldots, \bar{y}_n)$ and set $\phi(x;\bar{y}) = \bigvee_{t=1}^n (\phi_t(x;\bar{y}_t;\bar{d}) \wedge \exists^{<\infty} v \phi_t(v;\bar{y}_t;\bar{d}))$. Then we have (iv).

$(iv) \Rightarrow (i)$. If (iv) holds then there are $\mathcal{L}$-formulas $\psi(x;\bar{u})$ and $\phi(x;\bar{y};\bar{z})$ such that the following sentence holds in $(\mathcal{M}, A)$:

$$\exists \bar{u} \exists \bar{z} \left(\exists^\infty x\, \psi(x;\bar{u}) \wedge \forall \bar{y} \in P\, \exists^{<\infty} x\, \phi(x;\bar{y};\bar{z}) \wedge \forall x (\psi(x;\bar{u}) \to \exists \bar{y} \in P\, \phi(x;\bar{y};\bar{z}))\right).$$

Fix $(\mathcal{N}, B) \equiv_{\mathcal{L}(P)} (\mathcal{M}, A)$. Then the sentence above holds in $(\mathcal{N}, B)$, witnessed by some $(\bar{c}, \bar{d}) \in N^{\bar{z}} \times N^{\bar{u}}$. Since $\psi(x;\bar{d})$ is non-algebraic and $\phi(x;\bar{b};\bar{c})$ is algebraic for all $\bar{b} \in N^{\bar{y}}$, there is some $p \in S_1^{\mathcal{L}}(B\bar{c}\bar{d})$ extending $\{\psi(x;\bar{d})\} \cup \{\neg\phi(x;\bar{b};\bar{c}) : \bar{b} \in B^{\bar{y}}\}$. By construction, p is not realized in $\mathcal{N}$, and so A is not small in $\mathcal{M}$.

To finish the proof, we show $(iv) \Rightarrow (iii) \Rightarrow (ii)$ and $(iv) \Rightarrow (v) \Rightarrow (ii)$.

$(iv) \Rightarrow (iii)$. Let $\psi(x)$ and $\phi(x;\bar{y})$ be as in (iv), and consider the formula $\theta(x;\bar{y}) := \psi(x) \wedge \neg\phi(x;\bar{y})$. Then $\{\theta(x;\bar{a}) : \bar{a} \in A^{\bar{y}}\}$ is finitely satisfiable in $\mathcal{M}$, but not realized in $\mathcal{M}$.

$(iii) \Rightarrow (ii)$. Assume (iii) and suppose $(\mathcal{N}, B) \equiv_{\mathcal{L}_M(P)} (\mathcal{M}, A)$. By (iii), and nfcp for $T_{\mathcal{M}}$, it follows that the partial type $\pi(x) := \{\phi(x;\bar{b}) : \bar{b} \in B^{\bar{y}}\}$ is consistent with $T_{\mathcal{M}}$. But $\pi(x)$ is not realized in $\mathcal{N}$ by (iii) and $\mathcal{L}_M(P)$-elementarity.

$(iv) \Rightarrow (v)$. By elimination of $\exists^\infty$, (iv) is an $\mathcal{L}_M$-elementary property for any given $\psi(x)$ and $\phi(x;\bar{y})$. So this is implication is clear.

$(v) \Rightarrow (ii)$. Let $\psi(x)$ be an $\mathcal{L}_M$-formula witnessing (v). Fix $(\mathcal{N}, B) \equiv_{\mathcal{L}_M(P)} (\mathcal{M}, A)$. Let $p \in S_1^{\mathcal{L}_M}(B)$ be a non-algebraic type containing $\psi(x)$. Then p is not realized in $\mathcal{N}$ since $\psi(N) \subseteq \operatorname{acl}_{\mathcal{L}_M}(B)$. $\square$

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Recall that T is **unidimensional** if any two non-algebraic stationary types are non-orthogonal. This setting is of interest to us due to the following standard fact.

Fact 3.6. *If T is the theory of a weakly minimal expansion of a group, then T is unidimensional.*

Proof. This is essentially contained in the proof of [28, Remark 4.5.11]. It suffices to show any two non-algebraic 1-types, over a sufficiently saturated $G \models T$, are non-orthogonal. So fix such types p and q. Then p and q are generic by weak minimality, and so $q = gp$ for some $g \in G$. Therefore p and q are non-orthogonal. $\square$

Corollary 3.7. *Suppose T is weakly minimal and unidimensional. Given $A \subseteq M$, the following are equivalent.*

(i) *A is not small in $\mathcal{M}$.*

(ii) *There is an $\mathcal{L}_M$-formula $\phi(x;\bar{y})$ such that $\phi(x;\bar{a})$ is algebraic for all $\bar{a} \in M^{\bar{y}}$ and $M = \bigcup_{\bar{a}\in A^{\bar{y}}} \phi(M;\bar{a})$.*

(iii) *If $(\mathcal{N}, B) \equiv_{\mathcal{L}_M(P)} (\mathcal{M}, A)$ then $N = \operatorname{acl}_{\mathcal{L}_M}(B)$.*

Proof. $(i) \Rightarrow (ii)$. Assuming (i), let $\psi(x)$ be as in Lemma 3.5(v). Suppose $(\mathcal{N}, B) \succeq_{\mathcal{L}_M(P)} (\mathcal{M}, A)$ is $|T_{\mathcal{M}}|^+$-saturated, and let $(\mathcal{N}^*, B^*) \succeq_{\mathcal{L}_M(P)} (\mathcal{N}, B)$ be $|N|^+$-saturated. Then we have $\psi(N^*) \subseteq \operatorname{acl}_{\mathcal{L}_M}(B^*)$. Let $p \in S_1^{\mathcal{L}}(N)$ be a non-algebraic type containing $\psi(x)$, and fix $c \in N^*\backslash N$. Then $\operatorname{tp}(c/N)$ is non-algebraic, and thus non-orthogonal to p by unidimensionality. By saturation of $\mathcal{N}$, $\operatorname{tp}(c/N)$ is not weakly orthogonal to p. So there is $a \models p$ such that $a \in \operatorname{acl}_{\mathcal{L}}(cN) \subseteq N^*$. By choice of p, we have $a \in \psi(N^*) \subseteq \operatorname{acl}_{\mathcal{L}_M}(B^*)$. So $c \in \operatorname{acl}_{\mathcal{L}}(aN) \subseteq \operatorname{acl}_{\mathcal{L}}(B^*N)$. Altogether, $N^* = \operatorname{acl}_{\mathcal{L}}(B^*N)$ and so $(\mathcal{N}^*, B)$ omits the type

$$\Gamma(x) := \left\{\forall \bar{y} \in P\left(\exists^{<\infty} u\, \phi(u;\bar{y}) \rightarrow \neg\phi(x;\bar{y})\right) : \phi(x;\bar{y}) \text{ is an } \mathcal{L}_N\text{-formula}\right\}.$$

By saturation of $(\mathcal{N}^*, B^*)$, we may choose an $\mathcal{L}$-formula $\phi(x;\bar{y};\bar{z})$, and some $\bar{c} \in N^{\bar{z}}$ such that $\phi(x;\bar{b};\bar{c})$ is algebraic for all $\bar{b} \in B^*$, and $N^* = \bigcup_{\bar{b}\in(B^*)^{\bar{y}}} \phi(N^*;\bar{b};\bar{c})$. Now (ii) follows using $\mathcal{L}(P)$-elementarity.

$(ii) \Rightarrow (iii)$ is trivial (given elimination of $\exists^\infty$ for $T_{\mathcal{M}}$).

$(iii) \Rightarrow (i)$ is immediate from Lemma 3.5[$(v) \Rightarrow (i)$]. $\square$

Remark 3.8. In Corollary 3.7, the assumption that T is unidimensional cannot be removed. For example, let T be the theory of an equivalence relation E with two infinite classes. Fix $\mathcal{M} \models T$ and let $A \subseteq M$ be one E-class. Then T is weakly minimal (but not unidimensional), A is not small in $\mathcal{M}$ by Remark 3.3, and if $(\mathcal{N}, B) \equiv_{\mathcal{L}_M(P)} (\mathcal{M}, A)$ is $\aleph_1$-saturated then $N \neq \operatorname{acl}_{\mathcal{L}_M}(B)$.

Remark 3.9. Corollary 3.7 yields an alternate proof that if T is weakly minimal and unidimensional, $\mathcal{M} \models T$, and $\mathcal{L} = \mathcal{L}_M$, then any $A \subseteq M$ is bounded in $\mathcal{M}$ (a special case of Theorem 2.9). The argument splits into two cases. If A is small in $\mathcal{M}$ then it is bounded in $\mathcal{M}$ by Proposition 3.4(a) and Fact 3.1. If A is not small in $\mathcal{M}$ then condition (iii) of Corollary 3.7 holds, and one easily sees that Proposition 2.5 applies to conclude A is small in $\mathcal{M}$.

4. Mutually algebraic structures

The notion of a *mutually algebraic structure* was introduced in [22] by the second author, and we now recall the definition. Throughout this section, let $\mathcal{M}$ be an $\mathcal{L}$-structure, with universe M and complete theory T. Let $T_{\mathcal{M}} = \operatorname{Th}_{\mathcal{L}_M}(\mathcal{M})$.

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Definition 4.1.

(1) A set $X \subseteq M^n$ is **mutually algebraic** if there is an integer $N \geq 1$ such that, for any $1 \leq i \leq n$ and any $b \in M$, the fiber
$$\{(a_1, \dots, a_{n-1}) \in M^{n-1} : (a_1, \dots, a_{i-1}, b, a_i, \dots, a_{n-1}) \in X\}$$
has size at most N.

(2) An $\mathcal{L}_M$-formula $\phi(x_1, \dots, x_n)$ is **mutually algebraic** if $\phi(M^n)$ is a mutually algebraic subset of M^n.

(3) $\mathcal{M}$ is **mutually algebraic** if every $\mathcal{L}_M$-formula is equivalent, modulo $T_{\mathcal{M}}$, to a Boolean combination of mutually algebraic $\mathcal{L}_M$-formulas.

This property has many interesting consequences; here are two.

Theorem 4.2 (Laskowski)**.** *Suppose $\mathcal{M}$ is mutually algebraic.*

(a) *Any reduct of $\mathcal{M}$ is mutually algebraic.*

(b) *Any expansion of $\mathcal{M}$ by unary predicates is mutually algebraic.*

Proof. This follows from [22, Theorem 3.3] (and the fact that mutual algebraicity is preserved by elementary equivalence, see [22, Lemma 2.10]). □

We now recall several useful characterizations of mutual algebraicity, which will be used in later results. These require the following definitions.

Definition 4.3 (T stable)**.** $\mathcal{M}$ has **trivial forking** if, for any $\mathcal{N} \models T$ and $A \subseteq N$, if $\bar{a}, \bar{b}, \bar{c} \subset N$ are pairwise forking independent over A, then $\bar{a} \mathop{\smile\!\!\!\!\!\!\mid}_A \bar{b}\bar{c}$.

Definition 4.4. Fix an $\mathcal{L}$-formula $R(\bar{z})$, and let $\mathcal{L}_R$ be the language containing just the relation $R(\bar{z})$.

(1) Given a nonempty tuple $\bar{x} \subseteq \bar{z}$ and a finite set $B \subseteq M$, let $S^R_{\bar{x}}(B)$ be the set of complete quantifier-free $\mathcal{L}_R$-types realized in $\mathcal{M}$, which are in the variables $\bar{x}$, and over parameters from B.

(2) Fix $m \geq 1$, $\bar{x} \subseteq \bar{z}$ nonempty, and $B \subseteq M$ finite. A type $p \in S^R_{\bar{x}}(B)$ **supports an m-array** if there are realizations $\bar{a}_1, \dots, \bar{a}_m$ of p in $\mathcal{M}$ such that $\bar{a}_i \cap \bar{a}_j = \emptyset$ for all distinct $i, j \leq m$.

(3) R has **uniformly bounded arrays in $\mathcal{M}$** if there are $m, N \in \mathbb{N}$ such that, for any nonempty tuple $\bar{x} \subseteq \bar{z}$ and any finite $B \subseteq M$, at most N types in $S^R_{\bar{x}}(B)$ support an m-array.

Theorem 4.5. *The following are equivalent.*

(i) *$\mathcal{M}$ is mutually algebraic.*

(ii) (Laskowski) *Every atomic $\mathcal{L}$-formula is equivalent, modulo $T_{\mathcal{M}}$, to a Boolean combination of mutually algebraic $\mathcal{L}_M$-formulas.*

(iii) (Laskowski) *$\mathcal{M}$ is weakly minimal with trivial forking.*

(iv) (Laskowski & Terry) *Every atomic $\mathcal{L}$-formula has uniformly bounded arrays in $\mathcal{M}$.*

Proof. See [22, Proposition 2.7], [22, Theorem 3.3], and [23, Theorem 7.3] for the equivalence of (i) with (ii), (iii), and (iv), respectively. □

Given an $\mathcal{L}$-formula $\phi(\bar{x})$ (possibly over parameters A from some model of T), recall the that *U-rank of $\phi(\bar{x})$ in T* is supremum of the U-ranks of all types (over A) containing $\phi(\bar{x})$.

Corollary 4.6. *Suppose $\mathcal{M}$ is mutually algebraic, and $\mathcal{N}$ is a first-order structure interpretable in $\mathcal{M}$. Assume that the universe of $\mathcal{N}$ has U-rank 1 as a definable set in $\mathcal{M}^{\mathrm{eq}}$. Then $\mathcal{N}$ is mutually algebraic.*

Proof. Let N be the universe of $\mathcal{N}$, which we view as a definable set in $\mathcal{M}^{\mathrm{eq}}$. Then $\mathcal{N}$ is a reduct of the $\mathcal{M}$-induced structure on N and so, by Theorem 4.2(b), we may assume $\mathcal{N} = N_{\mathcal{M}}$. Since N is definable, it is bounded in $\mathcal{M}^{\mathrm{eq}}$. Since $\mathcal{M}^{\mathrm{eq}}$ is stable and N has U-rank 1 as a definable set, it follows that $\mathcal{N}$ is weakly minimal (see, e.g., [10, Theorem 2.10]). Since $\mathcal{M}$ has trivial forking, so does $\mathcal{M}^{\mathrm{eq}}$ by [16, Lemma 1]. From this one can show that $\mathcal{N}$ has trivial forking (see, e.g., [10, Proposition 2.7]). So $\mathcal{N}$ is mutually algebraic by the characterization in Theorem 4.5(iii). □

Remark 4.7. In the previous corollary, the restriction on the U-rank of the universe of $\mathcal{N}$ is necessary. For example, let $\mathcal{M}$ be an infinite set in the language of equality, and let $\mathcal{N}$ be the $\mathcal{M}$-induced structure on M^2. Then $\mathcal{M}$ is mutually algebraic, but $\mathcal{N}$ has U-rank 2 and so is not mutually algebraic.

Combining previous results, we obtain the following theorem about expansions of weakly minimal structures by sets with mutually algebraic induced structure.

Theorem 4.8. *Suppose $\mathcal{M}$ is weakly minimal and $A \subseteq M$ is such that $A_{\mathcal{M}}$ is mutually algebraic. Then, for any $B \subseteq A$, $(\mathcal{M}, B)$ is superstable of U-rank at most ω. Moreover, if B is small in $\mathcal{M}$ then $(\mathcal{M}, B)$ has nfcp.*

Proof. Fix $B \subseteq A$. We may asume B is infinite. By Theorem 4.2(b), the expansion $\mathcal{A} = (A_{\mathcal{M}}, B)$ is mutually algebraic. Therefore B has U-rank 1 as an $\mathcal{A}$-definable set. Since $B_{\mathcal{M}}$ is interpretable in $\mathcal{A}$ as a structure with universe B, we conclude from Corollary 4.6 that $B_{\mathcal{M}}$ is mutually algebraic (and, in particular, weakly minimal). By Theorem 2.10, $(\mathcal{M}, B)$ is superstable of U-rank at most ω. If B is small in $\mathcal{M}$ then $(\mathcal{M}, B)$ has nfcp by Fact 3.1 and Proposition 3.4(b). □

5. Weakly minimal abelian groups

The goal of this section is strengthen Theorem 4.8 in the case of pure abelian groups. By a *pure* group, we mean a group as a structure in the group language. Recall that if $(G, +)$ is an abelian group, then the pure theory of $(G, +)$ is stable, and has quantifier elimination in the expansion by binary relations for equivalence modulo n, for all $n \geq 1$ (see, e.g., [20]). By a **weakly minimal abelian group**, we mean an infinite abelian group $(G, +)$ whose pure theory is weakly minimal. It is not difficult to give an algebraic characterization of all such groups. Given an abelian group $(G, +)$ and $n \geq 1$, let $nG = \{nx : x \in G\}$ and $t_n(G) = \{x \in G : nx = 0\}$. Note that nG and $t_n(G)$ are $(G, +)$-definable subgroups of G.

Proposition 5.1. *An infinite abelian group $(G, +)$ is weakly minimal if and only if, for all $n \geq 1$, nG and $t_n(G)$ are each either finite or of finite index.*

Proof. It is a standard fact that a weakly minimal expansion of a group has no infinite definable subgroups of infinite index (see, e.g., [4, Corollary 8.2]). Conversely, suppose that for all $n \geq 1$, nG and $t_n(G)$ are each either finite or of finite index. Let $\mathcal{L}$ be the expansion of the group language by constants for G, and let $\mathcal{M} \models \mathrm{Th}_{\mathcal{L}}(G)$. By quantifier elimination, any definable subset of M is a finite Boolean combination of sets of one of the following two forms:

(i) $X_n(r) := \{x \in M : nx = r\}$ for some $n \geq 1$ and $r \in M$,

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(*ii*) $Y_{m,n}(r) := \{x \in M : mx \equiv_n r\}$ for some $m, n \geq 1$ and $r \in M$.

We claim that any such set is either finite or $\mathcal{L}$-definable over $\emptyset$, which implies $(G, +)$ is weakly minimal (e.g., by [1, Theorem 21]). To see this, fix some $n \geq 1$. If $t_n(G)$ is finite then $X_n(r)$ is finite for any $r \in M$. If $t_n(G)$ has finite index then $nM \subseteq G$, which implies $X_n(r) = \emptyset$ for any $r \in M \backslash G$, and so $X_n(r)$ is $\emptyset$-definable for any $r \in M$. Next, if nG has finite index then any element of M is equivalent modulo n to some element of G, and so $Y_{m,n}(r)$ is $\emptyset$-definable for any $m \geq 1$ and $r \in M$. Suppose nG is finite. Then $nM = nG$ and so, for any $r \in M$ and $m \geq 1$, we have $Y_{m,n}(r) = \bigcup_{s \in nG} X_m(r + s)$. By the above, $Y_{m,n}(r)$ is either finite or $\emptyset$-definable for any $m \geq 1$ and $r \in M$. □

Remark 5.2. In [19, Proposition 2.1], Hrushovski and Loveys show that if $\mathcal{M}$ is an expansion of an abelian group by any number of predicates naming subgroups, then $\mathcal{M}$ is weakly minimal if and only if any infinite definable subgroup has finite index. It is also worth mentioning the result of Cherlin and Shelah that any weakly minimal group is definably abelian-by-finite (see [8, Theorem 62]).

The next goal is to give a more explicit description of the induced structure on subsets of weakly minimal abelian groups.

Definition 5.3. Given an $\mathcal{L}$-structure $\mathcal{M}$, and a set $A \subseteq M$, let $A^{\text{qf}}_{\mathcal{M}}$ denote the reduct of $A_{\mathcal{M}}$ to relations of the form $A^n \cap X$, for any $n \geq 1$ and $X \subseteq M^n$ definable by a quantifier-free $\mathcal{L}_M$-formula.

We say that two structures $\mathcal{M}_1$ and $\mathcal{M}_2$, with the same universe M (but possibly different languages), are **interdefinable** if, for any $n \geq 1$ and $X \subseteq M^n$, X is $\mathcal{M}_1$-definable if and only if it is $\mathcal{M}_2$-definable.

Proposition 5.4. *Suppose $\mathcal{G} = (G, +)$ is a weakly minimal abelian group. Then, for any $A \subseteq G$, $A_{\mathcal{G}}$ is interdefinable with the expansion of $A^{\text{qf}}_{\mathcal{G}}$ by unary predicates for $A \cap nG$, for all $n \geq 1$.*

Proof. Fix $A \subseteq G$. By quantifier elimination, $A_{\mathcal{G}}$ is interdefinable with its reduct to relations of the following two forms:

(*i*) $\{\bar{a} \in A^k : c_1 a_1 + \ldots + c_k a_k = r\}$ where $k \geq 1$, $\bar{c} \in \{1, \text{-}1\}^k$, and $r \in G$,

(*ii*) $\{\bar{a} \in A^k : c_1 a_1 + \ldots + c_k a_k \equiv_n r\}$ where $k, n \geq 1$, $\bar{c} \in \{1, \text{-}1\}^k$, and $r \in G$.

Note that $A^{\text{qf}}_{\mathcal{G}}$ is interdefinable with the reduct of $A_{\mathcal{G}}$ to type (*i*) relations. Moreover, any type (*ii*) relation, where $n \geq 1$ is such that nG finite, is a finite union of type (*i*) relations. So, by Proposition 5.1 (really, [4, Corollary 8.2]), it suffices to show that type (*ii*) relations, where $n \geq 1$ is such that G/nG finite, are definable using unary predicates for $A \cap nG$. This is straightforward, and exactly as in the case of $(\mathbb{Z}, +)$ (see [10, Proposition 5.2] and [11, Proposition 2.11]). □

The final step needed before the main result of this section (Theorem 5.10 below) is a finer analysis of small sets in torsion-free weakly minimal abelian groups. In light of Theorem 2.10, smallness is no longer relevant in proving stability for expansions of weakly minimal structures by new predicates. On the other hand, in light of Proposition 3.4(*b*), smallness is still relevant for proving nfcp.

Definition 5.5. Let $(G, +)$ be an abelian group, and fix $A \subseteq G$.

(1) Let $\pm A = \{x \in G : x \in A \text{ or } \text{-}x \in A\}$.
(2) Given $n \geq 1$, let $nA = \{nx : x \in A\}$.

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(3) Given $n \geq 1$, let $\Sigma_n(A) = \{a_1 + \ldots + a_k : 1 \leq k \leq n \text{ and } a_1, \ldots, a_k \in A\}$.
(4) A is **generic** if there is a finite set $F \subset G$ such that $G = A + F$.
(5) A is **sufficiently sparse** if $\Sigma_n(\pm A)$ is not generic for any $n \geq 1$.

Proposition 5.6. *Suppose $\mathcal{G} = (G, +)$ is a torsion-free weakly minimal abelian group. Then $A \subseteq G$ is small in $\mathcal{G}$ if and only if it is sufficiently sparse.*

Proof. Let $\mathcal{L}$ denote the language of groups. Suppose first that there is $F \subset G$ finite and some $n \geq 1$ such that $G = \Sigma_n(\pm A) + F$. Then, for any $(\mathcal{N}, B) \equiv_{\mathcal{L}_G(P)} (\mathcal{G}, A)$ we have $N = \Sigma_n(\pm B) + F \subseteq \operatorname{acl}_{\mathcal{L}_G}(B)$. So A is not small in $\mathcal{G}$ by Corollary 3.7[$(iii) \Rightarrow (i)$].

Now suppose $A \subseteq G$ is not small in $\mathcal{G}$. Let $(\mathcal{N}, B) \succ_{\mathcal{L}_G(P)} (\mathcal{G}, A)$ be saturated. By Corollary 3.7[$(i) \Rightarrow (ii)$], there is a finite set $F \subset G$ such that $N = \operatorname{acl}_{\mathcal{L}}(B \cup F)$. Given $k, n \geq 1$, set

$$X_{k,n} = \{x \in N : mx \in \Sigma_n(\pm(B \cup F)) \text{ for some } 1 \leq m \leq k\}.$$

Note that each $X_{k,n}$ is $(\mathcal{N}, B)$-definable. Since $\mathcal{G}$ is torsion-free and weakly minimal, we have $nG \cong G/t_n(G) \cong G$ for any $n \geq 1$. So $\mathcal{G}$ is torsion-free and nG is infinite for all $n \geq 1$, which implies $\operatorname{acl}_{\mathcal{L}}(B \cup F) = \bigcup_{k,n} X_{k,n}$. By saturation of $(\mathcal{N}, B)$, and since $N = \operatorname{acl}_{\mathcal{L}}(B \cup F)$, there are $k, n \geq 1$ such that $N = X_{k,n}$. By $\mathcal{L}_G(P)$-elementarity, it follows that for any $x \in G$ there is $m \leq k$ such that $mx \in \Sigma_n(\pm(A \cup F))$.

Given $m \leq k$, let $C_m = \{x \in G : mx \in \Sigma_n(\pm(A \cup F))\}$. Then $G = C_1 \cup \ldots \cup C_k$, so we may fix some $m \leq k$ such that C_m is *piecewise syndetic*, i.e., there is a finite set $E \subset G$ such that, if $D := E + C_m$ then, for any finite $U \subset G$, there is $g \in G$ such that $g + U \subseteq D$ (see, e.g., [3, Theorem 3.5]). In particular, for any $u \in G$, there is $g \in G$ such that $\{g, g + u\} \subseteq D$, and so $u \in D - D$. So we have

$$mG \subseteq m(D - D) \subseteq mE - mE + \Sigma_{2n}(\pm(A \cup F)) = \Sigma_{2n}(\pm A) + \Sigma_{2n}(\pm F) + mE - mE.$$

So $\Sigma_{2n}(\pm A)$ is generic since mG is generic and $\Sigma_{2n}(\pm F) + mE - mE$ is finite. $\square$

Remark 5.7. Using results from [26], one can show that $A \subseteq \mathbb{Z}$ is sufficiently sparse if and only if $m\mathbb{Z} \not\subseteq \Sigma_n(\pm A)$ for all $m, n \geq 1$ (see [10, Section 4]). Using Proposition 5.6, it follows that for any $A \subseteq \mathbb{Z}$, either A is small in $(\mathbb{Z}, +)$ or $N = \operatorname{acl}_{\mathcal{L}}(B)$ for any $(\mathcal{N}, B) \equiv_{\mathcal{L}(P)} (\mathbb{Z}, +, A)$ (where $\mathcal{L}$ is the group language). So the same argument as outlined in Remark 3.9 yields an alternate proof that all subsets of $\mathbb{Z}$ are bounded in $(\mathbb{Z}, +)$, which is a special case of Theorem 2.15.

Proposition 5.8. *Suppose $\mathcal{G} = (G, +)$ is a torsion-free weakly minimal abelian group, and $A \subseteq G$ is not small in $\mathcal{G}$. Then $A_{\mathcal{G}}$ interprets $\mathcal{G}$.*

Proof. Suppose $A \subseteq G$ is not small in $\mathcal{G}$. By Proposition 5.6, we may fix a finite set $F \subset G$ and some $n \geq 1$ such that $G = \Sigma_n(\pm A) + F$. We work in the structure $\mathcal{M} := A^{\mathrm{eq}}_{\mathcal{G}}$, and so *definable* means $\mathcal{M}$-definable with parameters. Fix $F_* \subseteq A$, with $|F| = |F_*|$, and let $\sigma \colon F_* \to F$ be a bijection. Let $\mathfrak{l}_1, \ldots, \mathfrak{l}_n, \mathfrak{o}, \mathfrak{n}, \mathfrak{p}$, be $n+3$ pairwise distinct elements of $A \backslash F^*$. Set

$$X = F^* \times \bigcup_{k=1}^{n} (A^k \times \{\mathfrak{o}\}^{n-k} \times \{\mathfrak{n}, \mathfrak{p}\}^k \times \{\mathfrak{o}\}^{n-k} \times \{\mathfrak{l}_k\}),$$

and note that X is a definable subset of A^{2n+2}. Given $f \in F$, $1 \leq k \leq n$, $\bar{a} \in A^k$, and $\bar{s} \in \{\mathfrak{n}, \mathfrak{p}\}^k$, let $\langle f, \bar{a}, \bar{s}, k \rangle$ denote the element $(\sigma(f), \bar{a}, \mathfrak{o}, \overset{n-k}{\ldots}, \mathfrak{o}, \bar{s}, \mathfrak{o}, \overset{n-k}{\ldots}, \mathfrak{o}, \mathfrak{l}_k)$

of X. Given $1 \leq k \leq n$, $\bar{s} \in \{\mathfrak{n},\mathfrak{p}\}^k$, and $\bar{z} \in \mathbb{Z}^k$, we let $\Sigma_{\bar{s}}\bar{z}$ denote the integer $\dot{s_1}z_1 + \ldots + \dot{s_k}z_k$, where $\dot{\mathfrak{n}} = $ -1 and $\dot{\mathfrak{p}} = 1$.

Let $\sim$ be the equivalence relation on Y such that $\langle f, \bar{a}, \bar{s}, j\rangle \sim \langle g, \bar{b}, \bar{t}, k\rangle$ if and only if $f + \Sigma_{\bar{s}}\bar{a} = g + \Sigma_{\bar{t}}\bar{b}$. Then $\sim$ is definable (as a subset of Y^2) using induced relations of the form $A^{j+k} \cap \{(\bar{x}, \bar{y}) \in \mathbb{Z}^j \times \mathbb{Z}^k : f + \Sigma_{\bar{s}}\bar{x} = g + \Sigma_{\bar{t}}\bar{y}\}$, for some fixed $f, g \in F$, $1 \leq j, k \leq n$, $\bar{s} \in \{\mathfrak{n},\mathfrak{p}\}^j$, and $\bar{t} \in \{\mathfrak{n},\mathfrak{p}\}^k$. Let $Z = Y/{\sim}$, which is definable. Given $f \in F$, $1 \leq k \leq n$, $\bar{a} \in A^k$, and $\bar{s} \in \{\mathfrak{n},\mathfrak{p}\}^k$, let $[f, \bar{a}, \bar{s}, k]$ denote the $\sim$-class of $\langle f, \bar{a}, \bar{s}, k\rangle$.

For any $z \in G$, we may choose $f(z) \in F$, $1 \leq k(z) \leq n$, $\bar{a}(z) \in A^{k(z)}$, and $\bar{s}(z) \in \{\mathfrak{n},\mathfrak{p}\}^{k(z)}$ such that $z = f(z) + \Sigma_{\bar{s}(z)}\bar{a}(z)$. Let $[\![z]\!] = [f(z), \bar{a}(z), \bar{s}(z), k(z)]$. By definition of Y, $[\![z]\!] \in Y$ for all $z \in G$. Note that, for any $\langle f, \bar{a}, \bar{s}, k\rangle \in Y$, we have some $z \in G$ such that $z = f + \Sigma_{\bar{s}}\bar{a}$, and so $[f, \bar{a}, \bar{s}, k] = [\![z]\!]$ by definition of $\sim$. Altogether, we have a surjective function $f : G \to Z$ such that $f(z) = [\![z]\!]$. It is easy to check that f is injective.

Given $x, y \in G$, let $[\![x]\!] \oplus [\![y]\!] = [\![x + y]\!]$. Since f is a bijection, $\oplus$ is a well-defined binary operation on Z, and $(Z, \oplus)$ is isomorphic to $(G, +)$ as structures in the language of groups. Therefore, to finish the proof, it suffices to show $\oplus$ is definable in $\mathcal{M}$. By arguments similar to the above, if $W \subseteq Y^3$ is the set of triples $(\langle f, \bar{a}, \bar{s}, i\rangle, \langle g, \bar{b}, \bar{t}, j\rangle, \langle h, \bar{c}, \bar{u}, k\rangle)$ such that $f + \Sigma_{\bar{s}}\bar{a} + g + \Sigma_{\bar{t}}\bar{b} = h + \Sigma_{\bar{u}}\bar{c}$, then W is definable and the graph of $\oplus$ is defined by $W/{\sim}$. □

Since a stable structure with trivial forking cannot interpret an infinite group, we obtain the following corollary.

Corollary 5.9. *Suppose $\mathcal{G} = (G, +)$ is a torsion-free weakly minimal abelian group. If $A \subseteq G$ is such that $A_{\mathcal{G}}$ is stable with trivial forking (e.g., $A_{\mathcal{G}}$ is mutually algebraic), then A is small in $\mathcal{G}$.*

We now state and prove the main result of this section.

Theorem 5.10. *Let $\mathcal{G} = (G, +)$ be a weakly minimal abelian group. Fix $A \subseteq G$, and suppose $A^{\mathrm{qf}}_{\mathcal{G}}$ is mutually algebraic. Then, for any finite $F \subset G$ and any $B \subseteq A + F$, $(\mathcal{G}, B)$ is superstable of U-rank at most ω. Moreover, if $\mathcal{G}$ is torsion-free then $(\mathcal{G}, B)$ has nfcp; and if $\mathcal{G} = (\mathbb{Z}, +)$ and B is infinite then $(\mathcal{G}, B)$ has U-rank ω.*

Proof. We may assume A is infinite. Fix a finite set $F \subset G$. Then $(A + F)_{\mathcal{G}}$ is interpretable in $A_{\mathcal{G}}$ as a structure on $(A \times F)/E$, where E is the $A_{\mathcal{G}}$-definable equivalence relation $\{((a_1, f_1), (a_2, f_2)) \in (A \times F)^2 : a_1 + f_1 = a_2 + f_2\}$. Since F is finite and $A_{\mathcal{G}}$ is mutually algebraic, $(A \times F)/E$ has U-rank 1 as an interpretable set in $A_{\mathcal{G}}$. So $(A + F)_{\mathcal{G}}$ is mutually algebraic by Corollary 4.6. So, for any $B \subseteq A + F$, $(\mathcal{G}, B)$ is superstable of U-rank at most ω by Theorem 4.8.

Fix $B \subseteq A + F$. If G is torsion-free then B is small in $\mathcal{G}$ by Corollary 5.9. So $(\mathcal{G}, B)$ has nfcp by Fact 3.1 and Proposition 3.4(b). Note also that if B is infinite then it is not $\mathcal{G}$-definable by Remark 3.3. So if $\mathcal{G} = (\mathbb{Z}, +)$ and B is infinite then $(\mathcal{G}, B)$ does not have finite U-rank by [27, Theorem 1]. □

Remark 5.11. Suppose $\mathcal{K} = (K, +)$ is an abelian group and $A \subseteq K$ is such that $A^{\mathrm{qf}}_{\mathcal{K}}$ is mutually algebraic. Let $\mathcal{G} = (G, +)$ be a subgroup of $\mathcal{K}$, such that $A \subseteq G$. Then $A^{\mathrm{qf}}_{\mathcal{G}}$ is a reduct of $A^{\mathrm{qf}}_{\mathcal{K}}$ and so, if $\mathcal{G}$ is weakly minimal, then the conclusion of Theorem 5.10 holds.

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All examples of stable expansions of $(\mathbb{Z},+)$, considered in [10], [11], [21], and [27], fall under the umbrella of Theorem 5.10. In particular, given $d \geq 1$, let $\mathcal{N}^d_{\mathfrak{s}}$ denote the structure $(\mathbb{N}^d, \mathfrak{s}_1, \ldots, \mathfrak{s}_d)$, where $\mathfrak{s}_i$ is the successor function on the i^{th} coordinate and the identity on all other coordinates. Then $\mathcal{N}^d_{\mathfrak{s}}$ is mutually algebraic for any $d \geq 1$ (in fact, it follows from [22] that any structure containing only unary injective functions is mutually algebraic). In each example of a stable expansion of the form $(\mathbb{Z},+,A)$ considered in the sources above, it is shown that $A_{(\mathbb{Z},+)}$ is interpretable in an expansion of $\mathcal{N}^d_{\mathfrak{s}}$ by unary predicates, for some $d \geq 1$ (in fact, $d = 1$ suffices for all examples considered in [10], [21], and [27]).

It is worth emphasizing that in the sources cited above, a considerable amount of work is still required to show that $A_{(\mathbb{Z},+)}$ is interpretable in an expansion of some $\mathcal{N}^d_{\mathfrak{s}}$ by unary predicates. On the other hand, as we will see later, there are some cases where it is significantly easier to just show $A_{(\mathbb{Z},+)}$ is mutually algebraic. Theorem 6.5 is a notable example. Moreover, once it is shown that $A_{(\mathbb{Z},+)}$ is mutually algebraic, it then follows rather quickly that $B_{(\mathbb{Z},+)}$ is mutually algebraic for any $B \subseteq A + F$, with $F \subset \mathbb{Z}$ finite. This also eliminates a nontrivial amount of technical and tedious work in some examples considered in the sources above (e.g., [11, Lemma 4.17]).

6. Stable expansions of weakly minimal abelian groups

In this section, we give several new families of stable expansions of weakly minimal abelian groups. The main results are Theorems 6.3, 6.5, 6.6, and 6.14. Each one of these theorems is formulated for a weakly minimal abelian group $\mathcal{G} = (G,+)$ satisfying certain further properties, which always hold for $(\mathbb{Z},+)$. The conclusion of each of these theorems is that some expansion of the form $(\mathcal{G}, B)$ superstable of U-rank *at most* ω. For each result, we obtain this by showing that the induced structure $B_{\mathcal{G}}$ is mutually algebraic. Therefore, if $\mathcal{G} = (\mathbb{Z},+)$ and B is infinite, then $(\mathcal{G}, B)$ has U-rank *exactly* ω by Theorem 5.10.

Given an integer $n \geq 1$, we let $[n] = \{1, \ldots, n\}$.

6.1. Strongly lacunary sets in $\mathbb{C}$.

A strictly increasing sequence $(a_n)_{n=0}^{\infty}$ of (positive) real numbers is often called *lacunary* if $\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} > 1$. This motivates the following definition.

Definition 6.1. A countable set $A \subseteq \mathbb{C}$ is **strongly lacunary** if there is an enumeration $A\backslash\{0\} = \{a_n\}_{n=0}^{\infty}$ such that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ either diverges, or converges to some $\kappa \in \mathbb{C}$ with $|\kappa| > 1$.

Suppose $A \subseteq \mathbb{C}$ is strongly lacunary, witnessed by an enumeration $\{a_n\}_{n=0}^{\infty}$. Then there is some $N \geq 0$ such that $|a_{n+1}| > |a_n|$ for all $n \geq N$. It follows from this that if $\{c_n\}_{n=0}^{\infty}$ is another enumeration witnessing that A is strongly lacunary, then $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ and $\lim_{n\to\infty} \frac{c_{n+1}}{c_n}$ either both diverge or are equal. In the former case we call A **divergent**, and in the latter case we call A **convergent** and call this unique limit the **Kepler limit of** A (this terminology is often used in the context of Fibonacci sequence, whose Kepler limit is the golden ratio).

In [10, Theorem 7.16(a)] the first author showed that *any* divergent strongly lacunary set $A \subseteq \mathbb{Z}^+$ admits a stable expansion $(\mathbb{Z},+,A)$ (this was shown independently by Lambotte and Point [21] under the extra assumption that the set is eventually periodic modulo any $n \geq 1$). We will reprove this below in a more general setting. On the other hand, there are strongly lacunary sets $A \subseteq \mathbb{Z}$ such that $(\mathbb{Z},+,A)$ is unstable (the existence of such sets was questioned in [10] and

[21]). For example, given $q \geq 2$, if $A_q = \{q^n + n : n \in \mathbb{N}\}$, then $(\mathbb{Z}, +, A_q)$ is interdefinable with $(\mathbb{Z}, +, <, x \mapsto q^x)$ (see [11, Theorem 4.8]). The proof generalizes to $\{F_n + n : n \in \mathbb{N}\}$, where F_n is the n^{th} Fibonacci number, and so we also have a strongly lacunary set $A \subseteq \mathbb{Z}^+$, with an irrational Kepler limit, such that $(\mathbb{Z}, +, A)$ is unstable. In this section, we show that this cannot happen for a strongly lacunary set with a transcendental Kepler limit.

Lemma 6.2. *Suppose $A \subseteq \mathbb{C}$ is strongly lacunary, and either divergent or convergent with transcendental Kepler limit. Then $A^{\text{qf}}_{(\mathbb{C},+)}$ is interdefinable with A in the language of equality.*

Proof. The proof uses techniques similar to those of Palacín and Sklinos [27] and Lambotte and Point [21] (see also Remark 6.4). Let $A = \{a_n\}_{n=0}^{\infty}$ be an enumeration of A such that either $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ diverges or converges to a transcendental $\tau \in \mathbb{C}$, with $|\tau| > 1$. Without loss of generality, we may assume $|a_{n+1}| > |a_n|$ for all $n \in \mathbb{N}$.

Let $\mathcal{N}$ be the structure on $\mathbb{N}$ induced from $A^{\text{qf}}_{(\mathbb{C},+)}$ via the map $a_n \mapsto n$. It suffices to show $\mathcal{N}$ is interdefinable with the structure $\mathbb{N}$ in the language of equality, which we denote by $\dot{\mathbb{N}}$. Given $k \geq 1$, $\bar{d} \in \mathbb{Z}^k$, and $r \in \mathbb{C}$, define

$$X_{\bar{d};r} = \left\{ \bar{n} \in \mathbb{N}^k : n_i \neq n_j \text{ for all distinct } i, j \in [k] \text{ and } \sum_{i=1}^{k} d_i a_{n_i} = r \right\}.$$

Note that any $X_{\bar{d};r}$ is $\emptyset$-definable in $\mathcal{N}$. Let $\mathcal{N}_0$ be the reduct of $\mathcal{N}$ to symbols for $X_{\bar{d};r}$, where $\bar{d} \in (\mathbb{Z}^*)^k$ and $r \in \mathbb{C}$. It is easy to see that $\mathcal{N}$ is interdefinable with $\mathcal{N}_0$, and so it suffices to show that $\mathcal{N}_0$ is interdefinable with $\dot{\mathbb{N}}$. Fix $k \geq 1$, $\bar{d} \in (\mathbb{Z}^*)^k$, and $r \in \mathbb{C}$. Toward a contradiction, suppose $X_{\bar{d};r}$ is infinite.

By pigeonhole, there are infinitely many tuples in $X_{\bar{d};r}$ of the same order type. After permuting the coordinates, we may fix an infinite sequence $(\bar{n}(t))_{t=0}^{\infty}$ from $X_{\bar{d};r}$, such that $n(t)_1 < \ldots < n(t)_k$ for all $t \in \mathbb{N}$. Since $(\bar{n}(t))_{t=0}^{\infty}$ is infinite, we may pass to a subsequence and assume that $(n(t)_k)_{t=0}^{\infty}$ diverges. For $t \in \mathbb{N}$ and $i \in [k]$, let $u(t)_i = n(t)_k - n(t)_i$. Then $u(t)_1 > \ldots > u(t)_k$ for all $t \in \mathbb{N}$. Let $u_k = 0$, and note that $u(t)_k = u_k$ for all $t \in \mathbb{N}$. If the sequence $(u(t)_{k-1})_{t=0}^{\infty}$ does not diverge then, by pigeonhole, it contains a constant subsequence. So, after passing to a subsequence, we may assume that either $(u(t)_{k-1})_{t=0}^{\infty}$ diverges, or $u(t)_{k-1} = u_{k-1}$ for all $t \in \mathbb{N}$ and some $u_{k-1} \in \mathbb{N}$. Repeating this process, we may assume that for some $\ell \in [k]$ and $u_k, u_{k-1}, \ldots, u_\ell \in \mathbb{N}$, we have $u(t)_i = u_i$ for all $t \in \mathbb{N}$ and $\ell \leq i \leq k$, and $\lim_{t\to\infty} u(t)_i = \infty$ for all $1 \leq i < \ell$ (note that $\ell = 1$ is possible, making the second condition vacuous).

For any $1 \leq i < \ell$, since $(u(t)_i)_{t=0}^{\infty}$ diverges, we have that, for any $u \in \mathbb{N}$,

$$0 \leq \lim_{t\to\infty} \frac{|a_{n(t)_i}|}{|a_{n(t)_k}|} = \lim_{t\to\infty} \frac{|a_{n(t)_k - u(t)_i}|}{|a_{n(t)_k}|} \leq \lim_{t\to\infty} \frac{|a_{n(t)_k - u}|}{|a_{n(t)_k}|}.$$

So we have $\lim_{t\to\infty} \frac{a_{n(t)_i}}{a_{n(t)_k}} = 0$ for all $1 \leq i < \ell$ (if A is divergent this is clear, and if A is convergent then this follows from $|\tau| > 1$). Recall that $\bar{n}(t) \in X_{\bar{d};r}$ for all $t \in \mathbb{N}$, and that $(n(t)_k)_{k=0}^{\infty}$ diverges. Altogether,

$$(\dagger) \qquad 0 = \lim_{t\to\infty} \frac{r}{a_{n(t)_k}} = \lim_{t\to\infty} \sum_{i=1}^{k} d_i \frac{a_{n(t)_i}}{a_{n(t)_k}} = \lim_{t\to\infty} \sum_{i=\ell}^{k} d_i \frac{a_{n(t)_k - u_i}}{a_{n(t)_k}}.$$

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Recall that $u_\ell > \ldots > u_k = 0$. Therefore, if A is divergent then the rightmost limit in (†) is d_k, and if A is convergent then the rightmost limit in (†) is $\sum_{i=\ell}^{k} d_i \tau^{-u_i}$. In either case, this contradicts $d_i \neq 0$ for all $\ell \leq i \leq k$. □

Theorem 6.3. *Suppose $\mathcal{G} = (G,+)$ is a weakly minimal subgroup of $(\mathbb{C},+)$, and $A \subseteq G$ is strongly lacunary and either divergent or convergent with transcendental Kepler limit. Then, for any finite $F \subseteq G$ and infinite $B \subseteq A+F$, $(\mathcal{G}, B)$ has nfcp and is superstable of U-rank at most ω.*

Proof. Apply Lemma 6.2 and Theorem 5.10 (via Remark 5.11). □

Remark 6.4. In [21], Lambotte and Point prove stability of $(\mathbb{Z},+,A)$ for certain strongly lacunary sets $A \subseteq \mathbb{Z}^+$ with transcendental Kepler limit, namely, if A is eventually periodic modulo n, for every $n \geq 1$, and is enumerated by a strictly increasing sequence $(a_n)_{n=0}^{\infty}$ such that $\lim_{n\to\infty} \frac{a_n}{\tau^n} \in \mathbb{R}_{>1}$ for some fixed transcendental $\tau > 1$. Note, however, that this condition does not hold for sets such as $A = \{\lfloor n\tau^n \rfloor : n \in \mathbb{N}\}$ (where $\tau > 1$ is transcendental), which is a strongly lacunary set with Kepler limit τ.

6.2. **Finite rank multiplicative groups.** Throughout this section, we fix an algebraically closed field $\mathbb{K}$ of characteristic 0 and a subgroup $\mathcal{G} = (G,+)$ of the additive group $(\mathbb{K},+)$. Let $\mathbb{K}^*$ denote the multiplicative subgroup of nonzero elements of $\mathbb{K}$. Recall that the *rank* of an abelian group is the cardinality of a maximal $\mathbb{Z}$-linearly independent set. We will give a short proof of the following theorem.

Theorem 6.5. *Suppose $\mathcal{G}$ is weakly minimal, and $A \subseteq G$ is contained in a finite rank subgroup of $\mathbb{K}^*$. Then, for any finite $F \subset G$ and any $B \subseteq A+F$, $(\mathcal{G}, B)$ has nfcp and is superstable of U-rank at most ω.*

For the case $\mathcal{G} = (\mathbb{Z},+)$, this was proved by the first author in [11, Theorem 3.1] (although explicitly only for $A \subseteq \mathbb{Z}^+$ and $F = \{0\}$). The proof relies on results concerning the structure of solutions to linear equations from finite rank multiplicative groups. This goes back to work of Mann, and is connected to number-theoretic results around Lang's Conjecture (proved by Faltings and Vojta) and Schmidt's Subspace Theorem. See [29] for a model-theoretic account of this relationship.

In [2], Belegradek and Zilber use these type of results to prove stability for the expansion of the *field* $(\mathbb{C},+,\cdot)$ by a finite rank multiplicative subgroup of the unit circle. Similar results for arbitrary finite rank subgroups of $\mathbb{C}^*$ were proved by Van den Dries and Günaydın [12]. Note however that the full conclusion of Theorem 6.5 does not hold for expansions of fields. For instance, if $\Gamma = \{2^n : n \in \mathbb{Z}\}$ and $\Pi = \{2^n : n \in \mathbb{N}\}$, then $(\mathbb{C},+,\cdot,\Gamma)$ is stable while $(\mathbb{C},+,\cdot,\Pi)$ defines the ordering on Π. Note also that $\Gamma_{(\mathbb{C},+,\cdot)}$ is interdefinable with $(\mathbb{Z},+)$, and thus is weakly minimal but does not have trivial forking.

The work in [11] uses the following result, which is [14, Theorem 1.1].

Theorem 6.6 (Evertse, Schlickewei, Schmidt). *Suppose Γ is a subgroup of $(\mathbb{K}^*)^k$ of rank at most ρ, for some $k, \rho \in \mathbb{N}$. Then there is an integer $N = N(k,\rho)$ such that, for any $c_1, \ldots, c_k \in \mathbb{K}$ and any $r \in \mathbb{K}^*$, there are at most N tuples $(x_1, \ldots, x_k) \in \Gamma$ such that $c_1x_1 + \ldots + c_kx_k = r$ and $\sum_{i\in I} c_ix_i \neq 0$ for all nonempty $I \subseteq [k]$.*

We will use this result to directly show that, for $A \subseteq G$ as in Theorem 6.5, $A^{\mathrm{qf}}_{\mathcal{G}}$ is mutually algebraic.

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Proof of Theorem 6.5. Let $A \subseteq G$ be as in the statement. We may assume A is infinite. By Theorem 5.10, it suffices to show $A^{\mathrm{qf}}_{\mathcal{G}}$ is mutually algebraic. Given $k \geq 1$, $\bar{c} \in \{1,\text{-}1\}^k$, and $r \in G$, define $A(\bar{c};r) := \{\bar{a} \in A^k : c_1a_1 + \ldots + c_ka_k = r\}$ and define $A_0(\bar{c};r)$ to be the set of $\bar{a} \in A(\bar{c};r)$ such that $\sum_{i\in I} c_ia_i \neq 0$ for all nonempty $I \subsetneq [k]$. Note that any $A(\bar{c};r)$ is a finite Boolean combination of sets of the form $A_0(\bar{c}';r')$ for some k'-tuple $\bar{c}'$ and $r' \in G$. So it suffices to show that, for any $k \geq 1$, $\bar{c} \in \{1,\text{-}1\}^k$, and $r \in G$, $A_0(\bar{c};r)$ is a mutually algebraic subset of A^k.

Fix $k \geq 1$, $\bar{c} \in \{1,\text{-}1\}^k$, and $r \in G$. Suppose $A \subseteq \Gamma$, where Γ is a subgroup of $\mathbb{K}^*$ of rank $\rho \in \mathbb{N}$. Note that Γ^k is a subgroup of $(\mathbb{K}^*)^k$ of rank $k\rho$. Let $\Gamma_0(\bar{c};r)$ be the set of $\bar{x} \in \Gamma^k$ such that $c_1x_1 + \ldots + c_kx_k = r$ and $\sum_{i\in I} c_ix_i \neq 0$ for all nonempty $I \subsetneq [k]$. We have $A_0(\bar{c};r) \subseteq \Gamma_0(\bar{c};r)$, and so if $r \in \mathbb{K}^*$ then $A_0(\bar{c};r)$ is finite by Theorem 6.6. So we may assume $r = 0$. Given $i \in [k]$, set $\Gamma_{0,i} = \Gamma_0(c_1,\ldots,c_{i-1},c_{i+1},\ldots,c_k;\text{-}c_i)$. By Theorem 6.6, there is some $N \geq 0$ such that $|\Gamma_{0,i}| \leq N$ for all $i \in [n]$. Fix $i \in [n]$ and $b \in A$ and set

$$X = \left\{(a_1,\ldots,a_{k-1}) \in A^{k-1} : (a_1,\ldots,a_{i-1},b,a_i,\ldots,a_{k-1}) \in A_0(\bar{c};0)\right\}.$$

Then $b^{\text{-}1}X \subseteq \Gamma_{0,i}$, and so $|X| \leq N$, as desired. $\square$

6.3. **The ESS Property.** In this section, we generalize the behavior found in Theorem 6.6 to define a certain combinatorial property of subsets A of weakly minimal abelian groups $(G,+)$, which implies $A_{(G,+)}$ is mutually algebraic. In contrast to Section 6.2 however, we will need to use the characterization of mutual algebraicity involving uniformly bounded arrays (see Theorem 4.5(iv)).

Throughout this section, we fix an infinite set A, an abelian group $\mathcal{G} = (G,+)$, and a set Φ of functions from A to G. (For now, we do not assume $A \subseteq G$.)

Definition 6.7.

(1) Given $k \geq 1$, $\bar{\varphi} \in \Phi^k$, $r \in G$, and $V \subseteq G$, define
$$A(\bar{\varphi};r) := \left\{\bar{a} \in A^k : \varphi_1(a_1) + \ldots + \varphi_k(a_k) = r\right\}, \text{ and}$$
$$A_V(\bar{\varphi};r) := \left\{\bar{a} \in A(\bar{\varphi};r) : \sum_{i\in I} \varphi_i(a_i) \notin V \text{ for all } \emptyset \neq I \subsetneq [k]\right\}.$$

(2) We say A has the **ESS property with respect to Φ and $\mathcal{G}$** if, for any $k \geq 1$, there are $n_k \in \mathbb{N}$ and finite sets $U_k, V_k \subseteq G$ such that $|A_{V_k}(\bar{\varphi};r)| \leq n_k$ for any $\bar{\varphi} \in \Phi^k$ and $r \notin U_k$.

(3) Let $A^{\Phi}_{\mathcal{G}}$ be the relational structure with universe A and a k-ary relation $R_{\bar{\varphi};r}$ interpreted as $A(\bar{\varphi};r)$, for any $k \geq 1$, $\bar{\varphi} \in \Phi^k$, and $r \in G$.

Proposition 6.8. *If A has the ESS property with respect to Φ and $\mathcal{G}$ then $A^{\Phi}_{\mathcal{G}}$ is mutually algebraic.*

Proof. For $k \geq 1$, let $n_k \in \mathbb{N}$ and $U_k, V_k \subseteq G$ be as Definition 6.7(2). Given $k \geq 1$, $\bar{\varphi} \in \Phi^k$, $r \in G$, $\bar{x} \subseteq \bar{z} = (z_1,\ldots,z_k)$, and finite $B \subseteq A$, set
$$S^{\bar{\varphi};r}_{\bar{x}}(B) = S^{R_{\bar{\varphi};r}}_{\bar{x}}(B)$$
(working in $A^{\Phi}_{\mathcal{G}}$). We show, by induction on $k \geq 1$, that there are $m_k, N_k \in \mathbb{N}$ such that, for any $\bar{\varphi} \in \Phi^k$, $r \in G$, finite $B \subseteq A$, and any nonempty $\bar{x} \subseteq \bar{z} = (z_1,\ldots,z_k)$, there are at most N_k types in $S^{\bar{\varphi};r}_{\bar{x}}(B)$ supporting an m_k-array.

For the base case $k = 1$, note that any unary relation $R(z)$ has uniformly bounded arrays. Indeed, given finite $B \subseteq A$, there are at most two types in $S^R_z(B)$ which contain $z \neq b$ for all $b \in B$. So fix $k > 1$ and suppose we have defined m_{k-1} and N_{k-1} satisfying the desired properties. Let $\bar{z} = (z_1, \ldots, z_k)$. Given $\bar{\varphi} \in \Phi^k$, $r \in G$, finite $B \subseteq A$, $\bar{x} \subseteq \bar{z}$, and an equivalence relation E on $\bar{x}$, let $S^{\bar{\varphi};r}_{\bar{x},E}(B)$ be the set of $p \in S^{\bar{\varphi};r}_{\bar{x}}(B)$ such that:

(*i*) $z_i \neq b \in p$ for all $z_i \in \bar{x}$ and $b \in B$, and
(*ii*) given $z_i, z_j \in \bar{x}$, $z_i = z_j \in p$ if and only if $E(z_i, z_j)$.

We claim that it suffices to find m^*_k and N^*_k such that, for any $\bar{\varphi} \in \Phi^k$, $r \in G$, finite $B \subseteq A$, nonempty $\bar{x} \subseteq \bar{z}$, and any equivalence relation E on $\bar{x}$, at most N^*_k types in $S^{\bar{\varphi};r}_{\bar{x},E}(B)$ support an m^*_k-array. Indeed, there are only finitely many choices for $\bar{x}$ and E; and if $p \in S^{\bar{\varphi};r}_{\bar{x}}(B)$ is such that $z_i = b \in p$ for some $z_i \in \bar{x}$ and $b \in B$, then p cannot support a 2-array. Therefore, setting $m_k = \max\{m^*_k, 2\}$ and $N_k = hN^*_k$, where h is the number of pairs $(\bar{x}, E)$ as above, it follows that m_k and N_k satisfy the desired properties. Define

$$\begin{aligned} N_{k,1} &= 1 + \max\{N_{k-1}(2^\ell - 2)|V_\ell| : 1 \leq \ell \leq k\}, \\ N_{k,2} &= \max\{|U_\ell| : 1 \leq \ell \leq k\}, \\ N^*_k &= N_{k,1} + N_{k,2}, \text{ and} \\ m^*_k &= 1 + \max\{n_\ell + (m_{k-1} - 1)(2^\ell - 2)|V_\ell| : 1 \leq \ell \leq k\}. \end{aligned}$$

Fix $\bar{\varphi} \in \Phi^k$, $r \in G$, finite $B \subseteq A$, $\bar{x} \subseteq \bar{z}$ nonempty, and an equivalence relation E on $\bar{x}$. Let S^* be the set of types in $S^{\bar{\varphi};r}_{\bar{x},E}(B)$ that support an m^*_k-array. We want to show $|S^*| \leq N^*_k$.

For $\bar{u} \subseteq \bar{z}$ and $\bar{a} \in A^{\bar{u}}$, let $\Sigma_{\bar{u}}\bar{a}$ denote $\sum_{z_i \in \bar{u}} \varphi_i(a_i)$. Let $\bar{y} = \bar{z} \backslash \bar{x}$. Given $t \in G$, let $S^*(t)$ be the set of types $p \in S^*$ such that $p \models R_{\bar{\varphi};r}(\bar{x}; \bar{b})$ for some $\bar{b} \in B^{\bar{y}}$ satisfying $\Sigma_{\bar{y}}\bar{b} = t$. We claim that $|S^*(t)| \leq 1$ for any $t \in G$. Indeed, suppose we have $p, q \in S^*(t)$ for some $t \in G$. By construction, p and q agree on atomic formulas in the language of equality. So we just need to show that they agree on instances of $R_{\bar{\varphi};r}(\bar{x}; \bar{y})$. Let $\bar{a}^1, \bar{a}^2 \in A^{\bar{x}}$ realize p and q, respectively. Since $p, q \in S^*(t)$, we have $r - \Sigma_{\bar{x}}\bar{a}^1 = t = r - \Sigma_{\bar{x}}\bar{a}^2$. Given $\bar{d} \in B^{\bar{y}}$, we have

$$p \models R_{\bar{\varphi};r}(\bar{x}; \bar{d}) \Leftrightarrow \Sigma_{\bar{y}}\bar{d} = r - \Sigma_{\bar{x}}\bar{a}^1 \Leftrightarrow \Sigma_{\bar{y}}\bar{d} = r - \Sigma_{\bar{x}}\bar{a}^2 \Leftrightarrow q \models R_{\bar{\varphi};r}(\bar{x}; \bar{d}).$$

Altogether, we have $p = q$.

Let $X = \{t \in G : S^*(t) \neq \emptyset\}$ and, for $t \in X$, let q_t be the unique type in $S^*(t)$. Note that there is at most one type in $S^{\bar{\varphi};r}_{\bar{x},E}(B)$ which contains $\neg R_{\bar{\varphi};r}(\bar{x}; \bar{b})$ for all $\bar{b} \in B^{\bar{y}}$. Altogether, $|S^*| \leq |X| + 1$.

Let $\ell = |\bar{x}|$. Partition $X = X_1 \cup X_2$ where $X_1 = \{t \in X : r - t \notin U_\ell\}$ and $X_2 = X \backslash X_1$. Then $|X_2| \leq |U_\ell| \leq N_{k,2}$. So, to finish the proof, it suffices to show $|X_1| \leq N_{k,1} - 1$. Suppose, for a contradiction, that we have pairwise distinct $t_1, \ldots, t_{N_{k,1}} \in X_1$. For $1 \leq i \leq N_{k,1}$, let $p_i = q_{t_i}$.

Fix $i \in [N_{k,1}]$. Since $p_i \in S^*$, we may fix pairwise disjoint realizations $\bar{a}^1, \ldots, \bar{a}^{m^*_k}$ of p_i in $A^{\bar{x}}$. Moreover, there is $\bar{b}^i \in B^{\bar{y}}$ such that $\Sigma_{\bar{y}}\bar{b}^i = t_i$ and $p_i \models R_{\bar{\varphi};r}(\bar{x}; \bar{b}^i)$. So we have $\Sigma_{\bar{x}}\bar{a}^j = s_i := r - t_i$ for all $j \in [m^*_k]$. In particular, $\bar{a}^1, \ldots, \bar{a}^{m^*_k} \in A((\varphi_j)_{z_j \in \bar{x}}; s_i)$. Since $s_i \notin U_\ell$, we have $|A_{V_\ell}((\varphi_j)_{z_j \in \bar{x}}; s_i)| \leq n_\ell$ and so, after renaming the tuples, we may assume $\bar{a}^1, \ldots, \bar{a}^m \notin A_{V_\ell}((\varphi_j)_{z_j \in \bar{x}}; s_i)$, where $m := m^*_k - n_\ell \geq 1 + (m_{k-1} - 1)(2^\ell - 2)|V_\ell|$. Let Ω be the set of nonempty proper subtuples of $\bar{x}$, and note that $|\Omega| = 2^\ell - 2$. For each $j \in [m]$, there are $\bar{x}^{i,j} \in \Omega$ and $v_{i,j} \in V_\ell$ such that $\Sigma_{\bar{x}^{i,j}}(a^j_l)_{z_l \in \bar{x}^{i,j}} = v_{i,j}$. Since $m \geq 1 + (m_{k-1} - 1)(2^\ell - 2)|V_\ell|$, there are

$\bar{x}' \in \Omega$, $v_i \in V_\ell$, and $I \subseteq [m]$ such that $|I| = m_{k-1}$ and, for all $j \in I$, $\bar{x}^{i,j} = \bar{x}'$ and $v_{i,j} = v_i$. After renaming tuples, we may assume $I = [m_{k-1}]$. Set $r_i = r - v_i$ and $\bar{x}^i = \bar{x} \backslash \bar{x}' \in \Omega$. For $j \in [m_{k-1}]$, let $\bar{a}^j_* = (a^j_l)_{z_l \in \bar{x}^i}$. Then $\bar{a}^1_*, \ldots, \bar{a}^{m_{k-1}}_*$ are pairwise disjoint tuples, which all realize the same type $p^*_i \in S^{\bar{\varphi}^i;r_i}_{\bar{x}^i}(B)$, where $\bar{\varphi}^i = (\varphi_j)_{z_j \in \bar{x}^i}$. So p^*_i supports an m_{k-1}-array. Note also that $p^*_i \models R_{\bar{\varphi}^i;r_i}(\bar{x}^i, \bar{b}^i)$.

Since $N_{k,1} \geq 1 + N_{k-1}(2^\ell - 2)|V_\ell|$, there are $\bar{x}^* \in \Omega$, $v \in V_\ell$, and $I \subseteq [N_{k,1}]$ such that $|I| = N := N_{k-1} + 1$ and, for all $i \in I$, we have $\bar{x}^i = \bar{x}^*$ and $v_i = v$. After renaming the types, we may assume $I = [N]$. Let $\bar{\varphi}^* = (\varphi_j)_{z_j \in \bar{x}^*}$. Let $r^* = r - v$. Then $p^*_1, \ldots, p^*_N$ are types in $S^{\bar{\varphi}^*;r^*}_{\bar{x}^*}(B)$. For each $i \in [N]$, we have $p^*_i \models R_{\bar{\varphi}^*;r^*}(\bar{x}^*, \bar{b}^i)$ and $\Sigma_{\bar{y}} \bar{b}^i = t_i$. So, if $i, j \in [N]$ are distinct, then $p^*_i \neq p^*_j$ since $t_i \neq t_j$. So we have N types in $S^{\bar{\varphi}^*;r^*}_{\bar{x}^*}(B)$, each of which supports an m_{k-1}-array. This is a contradiction, since $N = N_{k-1} + 1$ and $|\bar{x}^*| \leq k - 1$. $\square$

Remark 6.9. In the previous proof, we showed that for any atomic relation R in $A^\Phi_{\mathcal{G}}$, the parameters m and N from the definition of uniformly bounded arrays depend only on the arity of R. Thus the ESS property does not characterize mutual algebraicity of $A^\Phi_{\mathcal{G}}$ (ad hoc counterexamples can be constructed).

The canonical example of the above situation is when A is a subset of G and Φ consists of the maps $x \mapsto x$ and $x \mapsto$ -x, in which case $A^\Phi_{\mathcal{G}}$ is precisely $A^{\text{qf}}_{\mathcal{G}}$. So we introduce specific terminology for this case.

Definition 6.10. A set $A \subseteq G$ has the **ESS property in** $\mathcal{G}$ if, for any $k \geq 1$, there are $n_k \in \mathbb{N}$ and finite sets $U_k, V_k \subseteq G$ such that if $r \notin U_k$, then there are at most n_k tuples $\bar{a} \in (\pm A)^k$ satisfying $a_1 + \ldots + a_k = r$ and $\sum_{i \in I} a_i \notin V_k$ for any nonempty $I \subsetneq [k]$.

Proposition 6.8 and Theorem 5.10 together imply the following result.

Theorem 6.11. *Assume $\mathcal{G}$ is weakly minimal, and $A \subseteq G$ has the ESS property in $\mathcal{G}$. Then, for any finite $F \subset G$ and any $B \subseteq A + F$, $(\mathcal{G}, B)$ is superstable of U-rank at most ω.*

Example 6.12.

1. (1) If $\mathcal{G}$ is a subgroup of the additive group $(\mathbb{K}, +)$ of an algebraically closed field $\mathbb{K}$ of characteristic 0, and $A \subseteq G$ is contained in a finite rank subgroup of $\mathbb{K}^*$, then A has the ESS property in $\mathcal{G}$, with $U_k = V_k = \{0\}$ for all $k \geq 1$. This is immediate from Theorem 6.6.
2. (2) Suppose $A \subseteq \mathbb{C}$ is strongly lacunary and divergent. Then A has the ESS property in $(\mathbb{C}, +)$, with $U_2 = \{0\}$, $U_k = \emptyset$ for all $k \neq 2$, and $V_k = \{0\}$ for all $k \geq 1$. We leave this as an exercise.

Remark 6.13. It follows from Propositions 5.6 and 6.8 that if $\mathcal{G}$ is torsion-free and $A \subseteq G$ has the ESS property in $\mathcal{G}$, then it is sufficiently sparse in $\mathcal{G}$. However, one can further show that, for all $n \geq 1$, $\Sigma_n(\pm A)$ does not contain arbitrarily large finite arithmetic progressions. This requires a straightforward modification of [11, Lemma 3.3]. It is also easy to show that if $A \subseteq G$ has the ESS property in $\mathcal{G}$ and $F \subset G$ is finite, then any $B \subseteq A + F$ has the ESS property in $\mathcal{G}$.

6.4. **Linear recurrence relations.** In this section, we consider sets of algebraic numbers, which are enumerated by linear homogeneous recurrence relations, with constant coefficients.

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Let $\mathbb{Q}^{\mathrm{alg}}$ denote the field of algebraic numbers. We say that a set $A \subseteq \mathbb{Q}^{\mathrm{alg}}$ is **enumerated by a linear recurrence relation** if A is enumerated by a sequence $(a_n)_{n=0}^{\infty}$ such that, for some $d \geq 1$ and $\beta_1, \ldots, \beta_d \in \mathbb{Q}^{\mathrm{alg}}$, we have

$$a_{n+d} = \beta_1 a_{n+d-1} + \ldots + \beta_d a_n$$

for any $n \in \mathbb{N}$.

Suppose $A \subseteq \mathbb{Q}^{\mathrm{alg}}$ is enumerated by a linear recurrence relation, witnessed by $d \geq 1$ and $\beta_1, \ldots, \beta_d \in \mathbb{Q}^{\mathrm{alg}}$. The **characteristic polynomial** of A is $p_A(x) := x^d - \beta_1 x^{d-1} - \ldots - \beta_{d-1}x - \beta_d$. We assume that d is minimal, and so $p_A(x)$ is uniquely determined. In particular, $\beta_d \neq 0$, and so 0 is not a root of $p_A(x)$. Let $\mu_1, \ldots, \mu_{d_*} \in \mathbb{Q}^{\mathrm{alg}}$ be the distinct roots of $p_A(x)$, for some $d_* \leq d$. By the general theory, there are nonzero polynomials $\alpha_1(x), \ldots, \alpha_{d_*}(x) \in \mathbb{Q}^{\mathrm{alg}}[x]$ such that $\alpha_i(x)$ has degree strictly less than the multiplicity of μ_i as a root of $p_A(x)$, and, for any $n \in \mathbb{N}$,

$$a_n = \alpha_1(n)\mu_1^n + \ldots + \alpha_{d_*}(n)\mu_{d_*}^n.$$

As the set A is completely determined by $\beta_1, \ldots, \beta_d$, and $a_0, \ldots, a_{d-1}$, we sometimes identify A with the notation $\mathrm{LRR}(\beta_1, \ldots, \beta_d; a_0, \ldots, a_{d-1})$.

We are interested in stable expansions of weakly minimal subgroups of $\mathbb{Q}^{\mathrm{alg}}$ by sets enumerated by a linear recurrence relation. For expansions of $(\mathbb{Z}, +)$, the previous literature on this question is as follows. In [27], Palacín and Sklinos proved stability for the expansion of $(\mathbb{Z}, +)$ by $\Pi(q) = \mathrm{LRR}(q; 1)$. In [10], the first author proved stability of $(\mathbb{Z}, +, A)$, for any $A \subseteq \mathbb{Z}$, enumerated by linear recurrence relation, such $p_A(x)$ is irreducible over $\mathbb{Q}$ (so $d_* = d$) and there is some $1 \leq t \leq d$ such that $\mu_t \in \mathbb{R}_{>1}$ and $|\mu_i| \leq 1$ for all $i \neq t$ (e.g., the Fibonacci sequence $\mathrm{LRR}(1, 1, 0, 1)$). In [21], Lambotte and Point proved stability for a more general class of expansions of $(\mathbb{Z}, +)$, namely when $p_A(x)$ is irreducible over $\mathbb{Q}$ and there is some $1 \leq t \leq d$ such that $\mu_t \in \mathbb{R}_{>1}$ and $|\mu_i| < |\mu_t|$ for all $i \neq t$. There are also easy examples of unstable expansions of $(\mathbb{Z}, +)$ by linear recurrences. For instance, given $k \geq 1$, the set $P_k := \{n^k : n \in \mathbb{N}\}$ is enumerated by a linear recurrence with characteristic polynomial $(x-1)^{k+1}$. Recall that $(\mathbb{Z}, +, P_k)$ defines the ordering by the Hilbert-Waring Theorem, and even defines multiplication when $k \geq 2$ (see [5, Proposition 6]). Another unstable example is the expansion of $(\mathbb{Z}, +)$ by $\{q^n + n : n \in \mathbb{N}\}$, for any fixed integer $q \geq 2$, which is enumerated by a recurrence relation with characteristic polynomial $(x-q)(x-1)^2$ (see [11, Theorem 4.8]). In this section, we separate the stable examples from the unstable ones using the observation that, in each unstable example, 1 is a repeated root of $p_A(x)$.

Theorem 6.14. *Suppose $\mathcal{G} = (G, +)$ is a weakly minimal subgroup of $(\mathbb{Q}^{\mathrm{alg}}, +)$, and $A \subseteq G$ is enumerated by a linear recurrence relation such that no repeated root of the characteristic polynomial is a root of unity. Then, for any finite $F \subseteq G$ and any $B \subseteq A + F$, $(\mathcal{G}, B)$ has nfcp and is superstable of U-rank at most ω.*

Note that this is a significant generalization of the previous results described above, since if $p_A(x)$ is irreducible over $\mathbb{Q}$ then it is separable (i.e., has no repeated roots). On the other hand, the absence of roots of unity as repeated roots of $p_A(x)$ does not characterize stability of $(G, +, A)$ (see Remark 6.20).

To prove Theorem 6.14, we will use the material in Section 6.3 together with a number-theoretic tool of a similar flavor as Theorem 6.6. To state this result, we need some further notation. For the rest of this section, let $A \subseteq \mathbb{Q}^{\mathrm{alg}}$ be enumerated by linear recurrence relation. We may assume A is infinite. Fix a number field

$K \subseteq \mathbb{Q}^{\text{alg}}$ containing $\mu_1, \ldots, \mu_{d_*}$ and the coefficients of $\alpha_1(x), \ldots, \alpha_{d_*}(x)$. Given an integer $k \geq 1$ and a tuple $\bar{\lambda} = (\lambda_1, \ldots, \lambda_k) \in (K^*)^k$, define the function $\bar{\lambda}^{\bar{x}} \colon \mathbb{Z}^k \to K$ such that $\bar{\lambda}^{(n_1,\ldots,n_k)} = \lambda_1^{n_1} \cdot \ldots \cdot \lambda_k^{n_k}$.

The following result (which holds for any number field) is a quantitative version of work of Laurent [24, 25], due to Schlickewei and Schmidt [33] (see also [34, Theorem 12.1]).

Theorem 6.15 (Schlickewei & Schmidt). *Fix $k, m \geq 1$ and, for each $i \in [m]$, fix $\bar{\lambda}_i \in (K^*)^k$ and $P_i(x_1, \ldots, x_k) \in K[x_1, \ldots, x_k]$ of degree δ_i. Assume:*

(i) no $P_i(\bar{x})$ is identically 0, and

(ii) for any $\bar{n} \in \mathbb{Z}^k$, if $\bar{\lambda}_1^{\bar{n}} = \ldots = \bar{\lambda}_m^{\bar{n}}$ then $\bar{n} = \bar{0}$.

Then there are $O_{K,m,k,\delta_1,\ldots,\delta_m}(1)$ tuples $\bar{n} \in \mathbb{Z}^k$ such that $\sum_{i=1}^m P_i(\bar{n})\bar{\lambda}_i^{\bar{n}} = 0$ and $\sum_{i \in I} P_i(\bar{n})\bar{\lambda}_i^{\bar{n}} \neq 0$ for any nonempty $I \subsetneq [m]$.

Given $i \in [d_*]$, will use the notation $\alpha^*_{\mu_i}(x)$ for $\alpha_i(x)$. Set $\Lambda = \{\mu_1, \ldots, \mu_{d_*}\}$, and partition $\Lambda = \Lambda_0 \cup \Lambda_1$ so that $\mu_i \in \Lambda_1$ if and only if μ_i is a root of unity. Let Φ denote the set of functions from $\mathbb{N}$ to K of the form $x \mapsto c\alpha^*_\lambda(x)\lambda^x$ for some $\lambda \in \Lambda_0$ and $c \in \{1,\text{-}1\}$. Let $\mathcal{K} = (K, +)$ be the additive group in K.

Lemma 6.16. *$\mathbb{N}^\Phi_{\mathcal{K}}$ is mutually algebraic.*

Proof. By Proposition 6.8, it suffices to show that $\mathbb{N}$ has the ESS property with respect to Φ and $\mathcal{K}$. In particular, we show that for any $k \geq 1$, there is some $w_k \in \mathbb{N}$ such that $|\mathbb{N}_0(\bar{\varphi}; r)| \leq w_k$ for any $\bar{\varphi} \in \Phi^k$ and $r \in K^*$. In particular, let δ be the maximum degree of any $\alpha_i(x)$, for $i \in [d_*]$. Given $k \geq 1$, let $w_k \in \mathbb{N}$ be greater than the $O_{K,k+1,k,\delta_1,\ldots,\delta_{k+1}}(1)$ bound from Theorem 6.15, for any $\delta_1, \ldots, \delta_{k+1} \leq \delta$.

Fix $k \geq 1$, $\bar{\varphi} \in \Phi^k$, and $r \in K^*$. For $i \in [k]$, let $c_i \in \{1,\text{-}1\}$ and $\lambda_i \in \Lambda_0$ be such that $\varphi_i(x) = c_i\alpha^*_{\lambda_i}(x)\lambda_i^n$, and let $P_i(\bar{x}) \in K[x_1, \ldots, x_k]$ be the polynomial $c_i\alpha_{\lambda_i}(x_i)$. Let $P_{k+1}(\bar{x}) = \text{-}r$. For $i \in [k]$, let $\bar{\lambda}_i = (1, \overset{i-1}{\ldots}, 1, \lambda_i, 1, \overset{k-i}{\ldots}, 1) \in (K^*)^k$. Let $\bar{\lambda}_{k+1} = (1, \overset{k}{\ldots}, 1)$. Note that for any $\bar{n} \in \mathbb{Z}^k$, $\bar{\lambda}_{k+1}^{\bar{n}} = 1$ and $\bar{\lambda}_i^{\bar{n}} = \lambda_i^{n_i}$ for any $i \in [k]$. In particular, $\mathbb{N}_0(\bar{\varphi}; r)$ is precisely the set of solutions to $\sum_{i=1}^m P_i(\bar{x})\bar{\lambda}_i^{\bar{x}} = 0$ in $\mathbb{N}^k$ such that $\sum_{i \in I} P_i(\bar{x})\bar{\lambda}_i^{\bar{x}} \neq 0$ for all nonempty $I \subsetneq [k]$.

Suppose $\bar{n} \in \mathbb{Z}^k$ is such that $\bar{\lambda}_1^{\bar{n}} = \ldots = \bar{\lambda}_{k+1}^{\bar{n}}$. Then $\lambda_1^{n_1} = \ldots = \lambda_k^{n_k} = 1$, and so $n_i = 0$ for all $i \in [k]$ since λ_i is not a root of unity. Altogether, by Theorem 6.15, we have $|\mathbb{N}_0(\bar{\varphi}; r)| \leq w_k$. $\square$

We now assume that no $\lambda \in \Lambda_1$ is a repeated root of $p_A(x)$, and so $\alpha^*_\lambda(x)$ is a constant $\alpha^*_\lambda \in K^*$. Define

$$B = \left\{ \sum_{\lambda \in \Lambda_0} \alpha^*_\lambda(n)\lambda^n : n \in \mathbb{N} \right\} \quad \text{and} \quad F = \left\{ \sum_{\lambda \in \Lambda_1} \alpha^*_\lambda \lambda^n : n \in \mathbb{N} \right\}.$$

Note that $B, F \subseteq K$ and $A \subseteq B + F$. Moreover, since any $\lambda \in \Lambda_1$ is a root of unity, it follows that $\{\lambda^n : n \in \mathbb{N}\}$ is finite. So F is finite and B is infinite.

Lemma 6.17. *$B^{\text{qf}}_{\mathcal{K}}$ is mutually algebraic.*

Proof. Let $\Lambda_0 = \{\lambda_1, \ldots, \lambda_\ell\}$ for some $\ell \in [d_*]$ Given $k \geq 1$, $\bar{c} \in \{1,\text{-}1\}^k$, and $r \in K$, define

$$D_{\bar{c};r} = \left\{ \bar{n} \in \mathbb{N}^k : \sum_{t=1}^k \sum_{i=1}^\ell c_t \alpha^*_{\lambda_i}(n_t)\lambda_i^{n_t} = r \right\}.$$

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Then $D_{\bar{c};r}$ is $\emptyset$-definable in $\mathbb{N}^{\Phi}_{\mathcal{K}}$ since $\bar{n} \in D_{\bar{c};r}$ if and only if, setting

$$\bar{n}^t = (n_t, \overset{\ell}{\ldots}, n_t) \quad \text{and} \quad \bar{\varphi}_t = (c_t \alpha^*_{\lambda_1}(x)\lambda_1^x, \ldots, c_t \alpha^*_{\lambda_\ell}(x)\lambda_\ell^x)$$

for $t \in [k]$, we have $(\bar{n}^1, \ldots, \bar{n}^k) \in A((\bar{\varphi}_1, \ldots, \bar{\varphi}_k); r)$. Let E be the equivalence relation on $\mathbb{N}$ such that $E(m, n)$ holds if and only if

$$\sum_{i=1}^{\ell} \alpha^*_{\lambda_i}(m)\lambda_i^m = \sum_{i=1}^{\ell} \alpha^*_{\lambda_i}(n)\lambda_i^n.$$

Then E is defined by $D_{(1,\text{-}1);0} \subseteq \mathbb{N}^2$, and thus is $\emptyset$-definable in $\mathbb{N}^{\Phi}_{\mathcal{K}}$. Note also that, for any $\bar{c} \in \{1, \text{-}1\}^k$ and $r \in K$, $D_{\bar{c};r}$ is E-invariant as a subset of $\mathbb{N}^k$.

Now $B^{\mathrm{qf}}_{\mathcal{K}}$ is clearly interdefinable with the structure with universe $\mathbb{N}/E$ and relations $D_{\bar{c};r}/E$ for all $k \geq 1$, $\bar{c} \in \{1, \text{-}1\}^k$, and $r \in K$. So $B^{\mathrm{qf}}_{\mathcal{K}}$ is mutually algebraic by Lemma 6.16 and Corollary 4.6. $\square$

Corollary 6.18. *$A^{\mathrm{qf}}_{\mathcal{K}}$ is mutually algebraic.*

Proof. Let $\mathcal{M} = (B + F)^{\mathrm{qf}}_{\mathcal{K}}$. Then $A^{\mathrm{qf}}_{\mathcal{K}}$ is a reduct of $A_{\mathcal{M}}$, and so, as in the proof of Theorem 5.10, it suffices to show that $\mathcal{M}$ is mutually algebraic. Fix a finite set $F_0 \subseteq B$ with $|F| = |F_0|$, and let $\sigma \colon F_0 \to F$ be a bijection. Let $D = B \times F_0$, and note that $D \subseteq B^2$ is $B^{\mathrm{qf}}_{\mathcal{K}}$-definable of U-rank 1. Given $k \geq 1$, $\bar{c} \in \{1, \text{-}1\}^k$, and $r \in K$, define

$$D_{\bar{c};r} = \left\{ ((b_1, f_1), \ldots, (b_k, f_k)) \in D^k : \sum_{i=1}^{k} c_i(b_i + \sigma(f_i)) = r \right\}.$$

Then, for any $k \geq 1$, $\bar{c} \in \{1, \text{-}1\}^k$, and $r \in K$, we have

$$D_{\bar{c};r} = \bigcup_{\bar{f} \in F_0^k} \left\{ ((b_1, f_1), \ldots, (b_k, f_k)) : \sum_{i=1}^{k} c_i b_i = r - \sum_{i=1}^{k} c_i \sigma(f_i) \right\},$$

and so $D_{\bar{c};r}$ is $B^{\mathrm{qf}}_{\mathcal{K}}$-definable. Moreover, the equivalence relation E on D given by $b_1 + \sigma(f_1) = b_2 + \sigma(f_2)$ is $B^{\mathrm{qf}}_{\mathcal{K}}$-definable by $D_{(1,1,\text{-}1,\text{-}1);0}$, and any $D(\bar{c}; r)$ is E-invariant. Finally, $\mathcal{M}$ is clearly interdefinable with the structure with universe D/E and relations for $D(\bar{c}; r)/E$, for any $k \geq 1$, $\bar{c} \in \{1, \text{-}1\}^k$, and $r \in K$. By Lemma 6.17 and Corollary 4.6, $\mathcal{M}$ is mutually algebraic. $\square$

As before, Corollary 6.18, Theorem 5.10, and Remark 5.11 yield Theorem 6.14.

Remark 6.19. Theorem 6.14 implies that if $\mathcal{G} = (G, +)$ is a weakly minimal abelian group, $A \subseteq G$ is enumerated by a linear recurrence relation, and no repeated root of $p_A(x)$ is a root of unity, then A is sufficiently sparse in $(G, +)$. We expect a direct proof of this could be given using Theorem 6.15. In fact, if one assumes that $p_A(x)$ has no repeated roots at all then, similar to Remark 6.13, one can use Theorem 6.6 to show that for any $n \geq 1$, $\Sigma_n(\pm A)$ does not contain arbitrarily large finite arithmetic progressions (see [11, Remark 3.6]).

Remark 6.20. A root of unity appearing as a repeated root of $p_A(x)$ does not necessarily mean $(G, +, A)$ is unstable. For example, $\mathbb{Z} = \mathrm{LRR}(2, 0, \text{-}1, 0; 0, 0, 1, \text{-}1)$, which has characteristic polynomial $(x-1)^2(x+1)^2$. This situation would likely be clarified by focusing on recurrence relations which are *non-degenerate*, i.e., there do not exist distinct roots μ_i and μ_j of $p_A(x)$ such that μ_i/μ_j is a root of unity.

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In general, any recurrence relation can be effectively partitioned into finitely many non-degenerate pieces (see [13, Theorem 1.2]). Note also that if A is non-degenerate and some root μ of $p_A(x)$ is a root of unity, then μ is the unique such root and $\mu \in \{1,\text{-}1\}$. A tentative conjecture is that if $A \subseteq \mathbb{Z}$ is enumerated by a linear recurrence relation as above, and some repeated root of $p_A(x)$ is a root of unity, then either $(\mathbb{Z}, +, A)$ is unstable or A is degenerate.

Finally, we point out that the only reason we have restricted to sets of *algebraic numbers* enumerated by linear recurrence relations is so that we can work in a number field K and apply Theorem 6.15. Suppose instead that we have a set A, enumerated by a recurrence relation as above, but with $a_0, \dots, a_{d-1}, \beta_1, \dots, \beta_d$ in an arbitrary algebraically closed field $\mathbb{K}$ of characteristic 0. In order to carry out the work in this section, one would need a version of Theorem 6.15, where $O_{K,m,k,\delta_1,\dots,\delta_m}(1)$ is replaced by some bound depending only on k, m, and A. Such a result is known to hold in the case that $p_A(x)$ is separable, due to various "specialization" techniques (see [34]). On the other hand, we can use Theorem 6.6, and arguments similar to the proof of Theorem 6.5, to give a more direct argument.

Theorem 6.21. *Let $\mathbb{K}$ be an algebraically closed field of characteristic 0, and let $\mathcal{G} = (G, +)$ be a weakly minimal subgroup of the additive group of $\mathbb{K}$. Fix $A \subseteq G$ enumerated by a linear homogeneous recurrence relation with constant coefficients in $\mathbb{K}$ and separable characteristic polynomial. Then, for any finite $F \subset G$ and any $B \subseteq A + F$, $(\mathcal{G}, B)$ has nfcp and is superstable of U-rank at most ω.*

Proof. We use the same notation for A as above, but with $\mathbb{Q}^{\mathrm{alg}}$ replaced by $\mathbb{K}$. Since $p_A(x)$ is separable, we have $d_* = d$. Moreover, for all $i \in [d]$, $\alpha_i(x)$ is a constant $\alpha_i \in \mathbb{K}^*$, which we also denote by $\alpha^*_{\mu_i}$. Let $\Lambda = \{\mu_1, \dots, \mu_d\}$, and partition $\Lambda = \Lambda_0 \cup \Lambda_1$ as above. Let Φ denote the set of functions from $\mathbb{N}$ to $\mathbb{K}$ of the form $x \mapsto c\alpha^*_\lambda \lambda^x$ for some $\lambda \in \Lambda_0$ and $c \in \{1,\text{-}1\}$. Let $\mathcal{K}$ denote the additive group of $\mathbb{K}$. If we can show that $\mathbb{N}^\Phi_{\mathcal{K}}$ is mutually algebraic, then the rest of the proof follows as above.

To show that $\mathbb{N}^\Phi_{\mathcal{K}}$ is mutually algebraic, we fix $k \geq 1$, $\bar{\varphi} \in \Phi^k$, and $r \in \mathbb{K}$, and show that $\mathbb{N}_0(\bar{\varphi}; r)$ is a mutually algebraic subset of $\mathbb{N}^k$. Let $\bar{\varphi} = (\varphi_1, \dots, \varphi_k)$ where $\varphi_i \colon x \mapsto c_i \alpha^*_{\lambda_i} \lambda_i^x$ for some $\lambda_i \in \Lambda_0$ and $c_i \in \{1,\text{-}1\}$. Let Γ be the subgroup of $\mathbb{K}^*$ generated by $\lambda_1, \dots, \lambda_k$, and let Δ be the set of $\bar{x} \in \Gamma^k$ such that $\sum_{i=1}^k c_i \alpha^*_{\lambda_i} x_i = r$ and $\sum_{i \in I} c_i \alpha^*_{\lambda_i} x_i \neq 0$ for all nonempty $I \subsetneq [k]$. Then the map $\sigma \colon \bar{n} \to (\lambda_1^{n_1}, \dots, \lambda_k^{n_k})$ is well-defined from $\mathbb{N}_0(\bar{\varphi}; r)$ to Δ, and is also injective since no λ_i is a root of unity. So it suffices to show Δ is a mutually algebraic subset of Γ^k. This follows from Theorem 6.6 exactly as in the proof of Theorem 6.5. $\square$

Remark 6.22. A recurrence sequence $(a_n)_{n=0}^\infty$ as above can be extended to to $(a_n)_{n \in \mathbb{Z}}$ using the same recurrence relation, and the representation of a_n using the roots of $p_A(x)$ still holds. Thus the analogues of Theorems 6.14 and 6.21 hold for a set $A \subseteq G$ enumerated in this fashion as well. In the proofs one only needs to replace $\mathbb{N}^\Phi_{\mathcal{K}}$ by $\mathbb{Z}^\Phi_{\mathcal{K}}$, where the maps in Φ are extended to $\mathbb{Z}$ in the obvious way.

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Department of Mathematics, University of Notre Dame, Notre Dame, IN, 46656, USA
E-mail address: gconant@nd.edu

Department of Mathematics, University of Maryland, College Park, MD 20742, USA
E-mail address: mcl@math.umd.edu