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COUNTABLE MODELS OF THE THEORIES OF BALDWIN–SHI HYPERGRAPHS AND THEIR REGULAR TYPES

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Abstract. We continue the study of the theories of Baldwin–Shi hypergraphs from [5]. Restricting our attention to when the rank δ is rational valued, we show that each countable model of the theory of a given Baldwin–Shi hypergraph is isomorphic to a generic structure built from some suitable subclass of the original class used in the construction. We introduce a notion of dimension for a model and show that there is a an elementary chain $\{\mathfrak{M}_\beta : \beta \leq \omega\}$ of countable models of the theory of a fixed Baldwin–Shi hypergraph with $\mathfrak{M}_\beta \preccurlyeq \mathfrak{M}_\gamma$ if and only if the dimension of $\mathfrak{M}_\beta$ is at most the dimension of $\mathfrak{M}_\gamma$ and that each countable model is isomorphic to some $\mathfrak{M}_\beta$. We also study the regular types that appear in these theories and show that the dimension of a model is determined by a particular regular type. Further, drawing on a large body of work, we use these structures to give an example of a pseudofinite, ω-stable theory with a nonlocally modular regular type, answering a question of Pillay in [11].

§1. Introduction. Following Hrushovski's discovery [7] of a new strongly minimal set via a generalized Fraisse construction, many authors looked at variants of his method. In [3], Baldwin and Shi, taking the weight α as a parameter, consider the theory of a Fraisse limit $\mathfrak{M}_\alpha$ obtained from classes K_α of structures defined via a dimension function $\delta(\mathfrak{A}) = |A| - \alpha|E^{\mathfrak{A}}|$ and an associated notion of strong substructure. Here, we consider these theories in the slightly more general context of a finite, relational language, but under the assumption that

each of the weights $\overline{\alpha}(E)$ *is rational and* $0 < \overline{\alpha}(E) < 1$.

Baldwin and Shi [3] prove that these assumptions imply the theory is ω-stable. In [5], the author proves that the theory is $\forall\exists$-axiomatizable and describes the family of definable sets, bringing together and extending results from [3], [9], and [8]. Here, we continue the study of Baldwin–Shi hypergraphs from [5] using the tools developed therein.

We begin in Section 3 by studying the countable models of $S_{\overline{\alpha}}$ where $S_{\overline{\alpha}}$ denotes the $\forall\exists$ axiomatization of $Th(\mathfrak{M}_{\overline{\alpha}})$. A key result is Theorem 3.5, where we prove that all countable models of $S_{\overline{\alpha}}$ can be obtained as a generic structures. We then use this result, along with a notion of dimension for models, to prove Theorem 3.7, which establishes that the countable spectrum is $\aleph_0$. In Theorem 3.8 we sharpen this result and show that the countable models of $S_{\overline{\alpha}}$ form an elementary chain $\{\mathfrak{M}_\beta : \beta \leq \omega\}$ with $\mathfrak{M}_\beta \preccurlyeq \mathfrak{M}_\gamma$ for $\beta \leq \gamma$ with each model of $S_{\overline{\alpha}}$ isomorphic to some $\mathfrak{M}_\beta$.

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In Section 4 we study the regular types of $S_{\overline{\alpha}}$. A key result is Theorem 4.10 which identifies certain types as being regular. In Theorem 4.11 we establish that a certain class of types are nonorthogonal. We also show that there is a regular type whose realizations determine the dimension of a model that was introduced in Section 3. We show in Theorem 4.12, that these types are in fact not locally modular. We end the section with Theorem 4.13, which establishes that a large class of types are not regular.

In Section 5 drawing on a large body of work, we observe that certain of these generic structures have pseudofinite theories. Thus we obtain pseudofinite ω-stable theories with nonlocally regular modular types. This answers a question of Pillay in [11] on whether all regular types in a pseudofinite stable theory are locally modular.

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§**2. Preliminaries.** Fix a finite relational language L where each relation symbol has arity at least 2. Let K_L denote the class of finite structures where each relation symbols is interpreted reflexively and symmetrically. Fix a *rational valued* function $\overline{\alpha} : L \to (0, 1)$ and define a *rank function* $\delta : K_L \to \mathbb{Q}$ by $\delta(\mathfrak{A}) = |A| - \sum_{E \in L} \overline{\alpha}(E)|E^{\mathfrak{A}}|$ where $|E^{\mathfrak{A}}|$ is the number of subsets of A on which E holds. We include $\emptyset$, the empty structure in K_L for technical reasons and note $\delta(\emptyset) = 0$. Given $\mathfrak{A}, \mathfrak{B} \in K_L$, we say that $\mathfrak{A}$ *is strong in* $\mathfrak{B}$, denoted by $\mathfrak{A} \leq \mathfrak{B}$ if and only if $\mathfrak{A} \subseteq \mathfrak{B}$ and $\delta(\mathfrak{A}) \leq \delta(\mathfrak{A}')$ for all $\mathfrak{A} \subseteq \mathfrak{A}' \subseteq \mathfrak{B}$. Let $K_{\overline{\alpha}} = \{\mathfrak{A} \in K_L : \delta(\mathfrak{A}') \geq 0 \text{ for all } \mathfrak{A}' \subseteq \mathfrak{A}\}$.

Remark 2.1. The relation $\leq$ on $K_L \times K_L$ is reflexive, transitive and has the property that given $\mathfrak{A}, \mathfrak{B}, \mathfrak{C} \in K_L$, if $\mathfrak{A} \leq \mathfrak{C}$, $\mathfrak{B} \subseteq \mathfrak{C}$ then $\mathfrak{A} \cap \mathfrak{B} \leq \mathfrak{B}$ (use (1) of Fact 2.4). The same statement holds true if we replace K_L by $K_{\overline{\alpha}}$ in the above equation. Further we note that $\emptyset \in K_{\overline{\alpha}}$ and for any given $\mathfrak{A} \in K_{\overline{\alpha}}$, $\emptyset \leq \mathfrak{A}$.

Typically the notion of $\leq$ is defined on $K_{\overline{\alpha}} \times K_{\overline{\alpha}}$. However several proofs require verifying that certain finite hypergraphs constructed over some $\mathfrak{A} \in K_{\overline{\alpha}}$ are in $K_{\overline{\alpha}}$. We find that defining the notion of $\leq$ on $K_L \times K_L$ allows us to make the exposition significantly simpler via the following remark.

Remark 2.2. Let $\mathfrak{A}, \mathfrak{B} \in K_L$. Further assume that $\mathfrak{A} \subseteq \mathfrak{B}$ with $\mathfrak{A} \in K_{\overline{\alpha}}$. If $\mathfrak{A} \leq \mathfrak{B}$, then $\mathfrak{B} \in K_{\overline{\alpha}}$. (use (1) of Fact 2.4).

Let n be a positive integer. A set $\{\mathfrak{B}_i : i < n\}$ of elements of K_L is *disjoint over* $\mathfrak{A}$ if $\mathfrak{A} \subseteq \mathfrak{B}_i$ for each $i < n$ and $B_i \cap B_j = A$ for $i < j < n$. If $\{\mathfrak{B}_i : i < n\}$ is disjoint over $\mathfrak{A}$, then $\mathfrak{D}$ is the *free join* of $\{\mathfrak{B}_i : i < n\}$, denoted by $\mathfrak{D} = \oplus_{\mathfrak{A}}^{i<n} \mathfrak{B}_i$, if the universe $D = \bigcup\{B_i : i < n\}$, $\mathfrak{B}_i \subseteq \mathfrak{D}$ for all i and, there are no additional relations in $\mathfrak{D}$, i.e., $E^{\mathfrak{D}} = \bigcup\{E^{\mathfrak{B}_i} : i < n\}$ for all $E \in L$. In the case $n = 2$ we will use the notation $\mathfrak{B}_0 \oplus_{\mathfrak{A}} \mathfrak{B}_1$ for $\oplus_{\mathfrak{A}}^{i<2} \mathfrak{B}_i$.

The following fact, in the terminology of Baldwin and Shi in [3], shows that $(K_{\overline{\alpha}}, \leq)$ has *full amalgamation*. It is the key to showing that $(K_{\overline{\alpha}}, \leq)$ is a *Fraïssé class*, i.e., $(K_{\overline{\alpha}}, \leq)$ has the *Joint Embedding Property* and the *Amalgamation Property*. The *generic* for $(K_{\overline{\alpha}}, \leq)$, i.e., the Fraïssé limit of $(K_{\overline{\alpha}}, \leq)$ will be called the *Baldwin–Shi hypergraph for* $\overline{\alpha}$.

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FACT 2.3. *If* $\mathfrak{B}, \mathfrak{C} \in K_{\overline{\alpha}}$, $\mathfrak{A} = \mathfrak{B} \cap \mathfrak{C}$, *and* $\mathfrak{A} \leq \mathfrak{B}$, *then* $\mathfrak{B} \oplus_{\mathfrak{A}} \mathfrak{C} \in K_{\overline{\alpha}}$ *and* $\mathfrak{C} \leq \mathfrak{B} \oplus_{\mathfrak{A}} \mathfrak{C}$.

Before proceeding further, we record three easy computations for future use. Given an L-structure $\mathfrak{Z}$ whose finite structures lie in K_L and finite $A, B \subseteq Z$, the *relative rank of B over A* is obtained by taking $\delta(B/A) := \delta(BA) - \delta(A)$ where BA is is the finite structure with universe $B \cup A$ in $\mathfrak{Z}$.

FACT 2.4. *Let $\mathfrak{Z}$ be an L-structure whose finite structures lie in K_L and let $A, B, B_i, C \subseteq Z$ be finite.*

1. *Let $A' = A \cap B$. Now $\delta(B/A') \geq \delta(B/A) = \delta(AB/A)$. Further if B, C are disjoint and freely joined over A, then $\delta(B/AC) = \delta(B/A)$.*
2. *If $\{B_i : i < n\}$ is disjoint over A and $Z = \oplus_{i<n} B_i$ is their free join over A, then $\delta(Z/A) = \sum_{i<n} \delta(B_i/A)$. In particular, if $A \leq B_i$ for each $i < n$, then $A \leq \oplus_{i<n} B_i$.*
3. $\delta(B_1 B_2, \dots, B_k/A) = \delta(B_1/A) + \sum_{i=2}^{k} \delta(B_i/AB_1, \dots, B_{i-1})$.

A useful notion in the study of these structures is the extension of the notion of $\leq$ to arbitrary L-hypergraphs. Fix $\mathfrak{Z}$ whose finite structures all lie in K_L. Given a finite $\mathfrak{A} \subseteq \mathfrak{Z}$ we say that $\mathfrak{A}$ is *strong in* $\mathfrak{X}$ if $\mathfrak{A} \leq \mathfrak{B}$ for all finite $\mathfrak{B} \subseteq \mathfrak{Z}$ with $\mathfrak{A} \subseteq \mathfrak{B}$. Given $\mathfrak{A}, \mathfrak{B} \in K_L$ with $\mathfrak{A} \subseteq \mathfrak{B}$, we say that $(\mathfrak{A}, \mathfrak{B})$ is a *minimal pair* if and only if $\mathfrak{A} \subseteq \mathfrak{B}$, $\mathfrak{A} \leq \mathfrak{C}$ for all $\mathfrak{A} \subseteq \mathfrak{C} \subset \mathfrak{B}$ but $\mathfrak{A} \not\leq \mathfrak{B}$. Given $X \subseteq Z$. We say X *is closed in* $\mathfrak{Z}$ if and only if for all finite $A \subseteq X$, if (A, B) is a minimal pair with $B \subseteq Z$, then $B \subseteq X$. It is easily established that given $\mathfrak{Z} \in K_L$ and $\mathfrak{A} \subseteq \mathfrak{Z}$ is finite, then $\mathfrak{A} \leq \mathfrak{Z}$ if and only if $\mathfrak{A}$ is closed in $\mathfrak{Z}$ and thus the notion of a closed set generalizes the notion of $\leq$ on finite structures.

It is immediate that any such $\mathfrak{Z}$, Z is closed in $\mathfrak{Z}$ and that the intersection of a family of closed sets of $\mathfrak{Z}$ is again closed. Thus we define the *intrinsic closure* of X in Z, denoted by $\mathrm{icl}_{\mathfrak{Z}}(X)$ is the smallest set X' such that $X \subseteq X' \subseteq Z$ and X' is closed in Z. As $\overline{\alpha}(E)$ is rational for all $E \in L$, it follows that for any finite $A \subseteq Z$, $\mathrm{icl}_{\mathfrak{Z}}(A)$ is finite. The important definition of the *induced dimension* function is thus simplified in our context.

Fix a monster model $\mathbb{M}$ of the theory of the generic and let $A, B \subseteq \mathbb{M}$ be finite. Then the *induced dimension* is given by $d(A) = \min\{\delta(A') | A \subseteq A' \subseteq N, A' \text{ is finite.}\} = \mathrm{icl}_{\mathbb{M}}(A)$. Further $d(B/A) = d(AB) - d(B)$. If $X \subseteq \mathbb{M}$ is infinite, then $d(A/X) = \min\{d(A/X_0) | X_0 \subseteq X \text{ is finite}\}$. For finite A, B, C, it is easily observed that $d(A/C) \geq 0$, $d(AB/C) = d(A/BC) + d(B/C)$.

The theory $S_{\overline{\alpha}}$ is the smallest set of sentences insuring that if $\mathfrak{M} \models S_{\overline{\alpha}}$, then every finite substructure of $\mathfrak{M}$ is in $K_{\overline{\alpha}}$ and for all $\mathfrak{A} \leq \mathfrak{B}$ from $K_{\overline{\alpha}}$, every (isomorphic) embedding $f : \mathfrak{A} \to \mathfrak{M}$ extends to an embedding $g : \mathfrak{B} \to \mathfrak{M}$. Clearly $S_{\overline{\alpha}}$ is a collection of $\forall\exists$ sentences. Further given $\mathfrak{A} \in K_L$ with a fixed enumeration $\overline{a}$ of A, we write $\Delta_{\mathfrak{A}}(\overline{x})$ for the atomic diagram of $\mathfrak{A}$. Also for $\mathfrak{A}, \mathfrak{B} \in K_L$ with $\mathfrak{A} \subseteq \mathfrak{B}$ and fixed enumerations $\overline{a}, \overline{b}$ respectively with $\overline{a}$ an initial segment of $\overline{b}$; we let $\Delta_{\mathfrak{A},\mathfrak{B}}(\overline{x}, \overline{y})$ the atomic diagram of $\mathfrak{B}$ with the universe of $\mathfrak{A}$ enumerated first according to the enumeration $\overline{a}$. Let $\mathfrak{A}, \mathfrak{B} \in K$ and assume $\mathfrak{A} \subseteq \mathfrak{B}$. Let $\Psi_{\mathfrak{A},\mathfrak{B}}(\overline{x}) = \Delta_{\mathfrak{A}}(\overline{x}) \wedge \exists \overline{y} \Delta_{(\mathfrak{A},\mathfrak{B})}(\overline{x}, \overline{y})$. Such formulas are collectively called *extension formulas* (over $\mathfrak{A}$). A *chain minimal extension formula* is an extension formula $\Psi_{\mathfrak{A},\mathfrak{B}}$ where $\mathfrak{B}$ us the union of a minimal chain over $\mathfrak{A}$, i.e., there is some sequence $\langle \mathfrak{A}_i \rangle_{i \leq n}$ with $n \in \omega$, $\mathfrak{A}_0 = \mathfrak{A}$, $\mathfrak{A}_n = \mathfrak{B}$ and $(\mathfrak{A}_i, \mathfrak{A}_{i+1})$ is a minimal pair.

We finish this section by collecting some key results about $S_{\overline{\alpha}}$ from various sources.

Theorem 2.5. 1. *The theory $S_{\overline{\alpha}}$ is complete and is the theory of the generic for $(K_{\overline{\alpha}}, \leq)$. (see* [8] *or* [5]*).*

2. *Every L-formula is $S_{\overline{\alpha}}$-equivalent to a boolean combination of chain-minimal extension formulas. (see* [5], [1]*).*
3. *$S_{\overline{\alpha}}$ is ω-stable. (see* [3], [13]*).*
4. *Given any $\mathfrak{M} \models S_{\overline{\alpha}}$ and $X \subseteq M$, X is algebraically closed in M if and only if X is intrinsically closed in M. (see* [3], [13] *or* [5]*).*
5. *The theory $S_{\overline{\alpha}}$ has weak elimination of imaginaries, i.e., every complete type over an algebraically closed set in the home sort is stationary. (see* [3], [5] *or* [12]*).*
6. *Let $\mathfrak{M} \models S_{\overline{\alpha}}$ be $\aleph_0$-saturated and let A be a finite closed set of $\mathfrak{M}$. Suppose that π is a consistent partial type over A such that for any $\overline{b}, \overline{c} \models \pi$, $qftp(\overline{b}/A) = qftp(\overline{c}/A)$. If any realization $\overline{b}$ of π in $\mathfrak{M}$ has the property that $\overline{b}A$ is closed in $\mathfrak{M}$, then π has a unique completion to a complete type p over A. (see* [5]*).*
7. *$S_{\overline{\alpha}}$ has finite closures, i.e., given any $\mathfrak{N} \models S_{\overline{\alpha}}$ for any finite $\mathfrak{A} \subseteq \mathfrak{N}$, there exists $\mathfrak{C} \in K_{\overline{\alpha}}$ such that $\mathfrak{A} \subseteq \mathfrak{C} \leq \mathfrak{N}$.*
8. *Let $\mathbb{M}$ be a monster model of $S_{\overline{\alpha}}$. For algebraically closed X, Y, Z with $Z = X \cap Y$, $X \mathop{\smile\!\!\!\!\!\!\mid}_Z Y$ if and only if XY is algebraically closed and X, Y are freely joined over Z if and only if for any finite $X_0 \subseteq X$, $Y_0 \subseteq Y$, $d(X_0/Z) = d(X_0/ZY_0)$ and $acl(X_0Z) \cap acl(Y_0Z) = acl(Z)$ (or equivalently $icl(X_0Z) \cap icl(Y_0Z) = icl(Z)$: see* [3], [12] *or* [13]*).*
9. *$S_{\overline{\alpha}}$ does not interpret infinite groups. (see Section 7 of* [13]*).*

2.1. Essential minimal pairs. We now recall *essential minimal pairs*, which were introduced in [5].

Definition 2.6. Let $\mathfrak{B} \in K_{\overline{\alpha}}$ with $\delta(\mathfrak{B}) > 0$. We call $\mathfrak{D} \in K_{\overline{\alpha}}$ with $\mathfrak{B} \subseteq \mathfrak{D}$ an *essential minimal pair* if $(\mathfrak{B}, \mathfrak{D})$ is a minimal pair and for any $\mathfrak{D}' \subsetneq \mathfrak{D}$, $\delta(\mathfrak{D}'/\mathfrak{D}' \cap \mathfrak{B}) \geq 0$.

The following, in more general form, appears in [5] as Theorem 3.33. It will form the backbone of many of the results to follow. Let c be the least common multiple of the denominators of $\overline{\alpha}(E)$ (in reduced form), i.e., $c = \mathrm{lcm}(q_E)_{E \in L}$, with $\overline{\alpha}(E) = p_E/q_E$ with p_E, q_E relatively prime.

Theorem 2.7. *Let $\mathfrak{A} \in K_{\overline{\alpha}}$ with $\delta(\mathfrak{A}) = k/c > 0$. There are $\mathfrak{D} \in K_{\overline{\alpha}}$ such that $(\mathfrak{A}, \mathfrak{D})$ is an essential minimal pair and satisfies $\delta(\mathfrak{D}/\mathfrak{A}) = -1/c$.*

We immediately obtain the following useful lemma.

Lemma 2.8. *Let $k \in \omega$. Given any $\mathfrak{B} \in K_{\overline{\alpha}}$, there is some $\mathfrak{D} \in K_{\overline{\alpha}}$ such that $\mathfrak{D} \supseteq \mathfrak{B}$, $\delta(\mathfrak{D}) = k/c$ and for any $\mathfrak{A} \leq \mathfrak{B}$ with $\delta(\mathfrak{A}) \leq k/c$, $\mathfrak{A} \leq \mathfrak{D}$.*

Proof. Given $\mathfrak{B}$ take $\mathfrak{D}_0$ to be the free join of $\mathfrak{B}$ with a structure with $k+1$ many points with no relations among them over $\emptyset$. Note that $\mathfrak{B} \leq \mathfrak{D}_0$. Let $l = c\delta(\mathfrak{D}_0) - k$. Consider a sequence $\mathfrak{D}_0 \subseteq \cdots \subseteq \mathfrak{D}_l$ where each $(\mathfrak{D}_i, \mathfrak{D}_{i+1})$ is an essential minimal pair with $\delta(\mathfrak{D}_{i+1}/\mathfrak{D}_1) = -1/c$. We claim that $\mathfrak{D} = \mathfrak{D}_l$ is as required. Fix any $\mathfrak{A} \leq \mathfrak{B}$ with $\delta(\mathfrak{A}) \leq k/c$. We show by induction on $i < l$ that if $\mathfrak{A} \leq \mathfrak{D}_i$, then $\mathfrak{A} \leq \mathfrak{D}_{i+1}$. Clearly $\mathfrak{A} \leq \mathfrak{D}_0$ as $\mathfrak{A} \leq \mathfrak{B} \leq \mathfrak{D}_0$. Fix $i < l$ and consider any $\mathfrak{F}$ such that $\mathfrak{A} \subseteq \mathfrak{F} \subseteq \mathfrak{D}_{i+1}$. If $\mathfrak{F} = \mathfrak{D}_{i+1}$ then $\delta(\mathfrak{F}) \geq k/c \geq \delta(\mathfrak{A})$ and so $\delta(\mathfrak{F}/\mathfrak{A}) \geq 0$. On the other hand, if $\mathfrak{F} \neq \mathfrak{D}_{i+1}$, then, $\delta(\mathfrak{F}/\mathfrak{D}_{i+1} \cap \mathfrak{F})$ since $(\mathfrak{D}_i, \mathfrak{D}_{i+1})$ is an essential

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minimal pair and $\delta(\mathfrak{D}_i \cap \mathfrak{F}/\mathfrak{A}) \geq 0$ as $\mathfrak{A} \leq \mathfrak{D}_i$. Thus $\delta(\mathfrak{F}/\mathfrak{A}) = \delta(\mathfrak{F}/\mathfrak{D}_i \cap \mathfrak{F}) + \delta(\mathfrak{F} \cap \mathfrak{D}_i/\mathfrak{A}) \geq 0$ as required. ⊣

§3. **Countable models of $S_{\overline{\alpha}}$.** Our goal in this section is to study the countable models of $S_{\overline{\alpha}}$. We begin by defining a notion of dimension for (countable) models. We then show that this notion of dimension is able to categorize countable models up to both isomorphism and elementary embeddability. Recall that c is the least common multiple of the denominators of the $\overline{\alpha}_E$ (in reduced form).

Definition 3.1. Let $\mathfrak{M} \models S_{\overline{\alpha}}$. Let $\mathfrak{A} \leq \mathfrak{M}$. We let $\dim(\mathfrak{M}/\mathfrak{A}) = \max\{\delta(\mathfrak{B}/\mathfrak{A}) : \mathfrak{A} \leq \mathfrak{B} \leq \mathfrak{M}\}$. If there is no maximum, i.e., given any $z > 0$, there will be some $\mathfrak{B} \leq \mathfrak{M}$ with $\delta(\mathfrak{B}/\mathfrak{A}) > z$, we let $\dim(\mathfrak{M}/\mathfrak{A}) = \infty$. We write $\dim(\mathfrak{M})$ for $\dim(\mathfrak{M}/\emptyset)$.

Definition 3.2. Fix an integer $k \geq 0$ and let $K_{k/c} = \{\mathfrak{A} : \mathfrak{A} \in K_{\overline{\alpha}} \text{ and } \delta(\mathfrak{A}) = k/c\}$. Let $(K_{k/c}, \leq)$ be such that $\leq$ is *inherited by* $K_{\overline{\alpha}}$ i.e., $\mathfrak{A} \leq \mathfrak{B}$ for $\mathfrak{A}, \mathfrak{B} \in K_{k/c}$ if and only if for all $\mathfrak{A} \subseteq \mathfrak{B}' \subseteq \mathfrak{B}$ with $\mathfrak{B}' \in K_{\overline{\alpha}}$, $\mathfrak{A} \leq \mathfrak{B}'$

We begin with the following technical lemma:

Lemma 3.3. *Let $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D} \in K_{\overline{\alpha}}$ with $\mathfrak{A} \leq \mathfrak{B}, \mathfrak{C}$; $\delta(\mathfrak{C}/\mathfrak{A}) \geq \delta(\mathfrak{B}/\mathfrak{A})$ and $\mathfrak{D} = \mathfrak{B} \oplus \mathfrak{C}$ the free join of $\mathfrak{B}, \mathfrak{C}$ over $\mathfrak{A}$. We can construct $\mathfrak{H} \in K_{\overline{\alpha}}$ such that $\mathfrak{A}, \mathfrak{B}, \mathfrak{C} \leq \mathfrak{H}$, $\mathfrak{D} \subseteq \mathfrak{H}$ and $\delta(\mathfrak{H}/\mathfrak{C}) = 0$. Further if $\delta(\mathfrak{B}/\mathfrak{A}) = \delta(\mathfrak{C}/\mathfrak{A})$, the $\mathfrak{H}$ that was constructed has the property $\delta(\mathfrak{H}/\mathfrak{B}) = 0$.*

Proof. This follows from an easy application of Lemma 2.8 on $\mathfrak{D}$. ⊣

We now work toward showing that certain countable models of $S_{\overline{\alpha}}$ can be built as Fraïssé limits $(K_{k/c}, \leq)$. In Theorem 3.7 we show that these are in fact, all of the countable models up to isomorphism.

Lemma 3.4. *For any fixed integer $k \geq 0$, $(K_{k/c}, \leq)$, where $\leq$ is inherited from $K_{\overline{\alpha}}$ is a Fraïssé class.*

Proof. Fix an integer $k \geq 0$ and consider $K_{k/c}$. Let $\mathfrak{A}, \mathfrak{B}, \mathfrak{C} \in K_{k/c}$. Note that for the purposes of proving amalgamation, we may as well assume $\mathfrak{B}, \mathfrak{C}$ are freely joined over $\mathfrak{A}$ and that $\mathfrak{A} \leq \mathfrak{B}, \mathfrak{C}$. Note that $\delta(\mathfrak{B}/\mathfrak{A}) = \delta(\mathfrak{C}/\mathfrak{A}) = 0$. The required statement follows by a simple application of Lemma 3.3 on $\mathfrak{B} \oplus_{\mathfrak{A}} \mathfrak{C}$. For joint embedding consider $\emptyset \leq \mathfrak{B}, \mathfrak{C}$. Note that $\delta(\mathfrak{B}/\emptyset) = \delta(\mathfrak{C}/\emptyset) = k/c$. Apply Lemma 3.3 on $\mathfrak{B} \oplus_{\emptyset} \mathfrak{C}$, the free join of $\mathfrak{B}, \mathfrak{C}$ *over* $\emptyset$. ⊣

We now prove the following theorem that the theory of the generic for the Fraïssé class $(K_{k/c}, \leq)$ also models $S_{\overline{\alpha}}$ and has dimension k/c.

Theorem 3.5. *Let k be a fixed integer with $k \geq 0$. Let $\mathfrak{M}_{k/c}$ be the generic for the Fraïssé class $(K_{k/c}, \leq)$ where $\leq$ is inherited from $K_{\overline{\alpha}}$. Now $\mathfrak{M}_{k/c} \models S_{\overline{\alpha}}$ and* $\dim(\mathfrak{M}_{k/c}) = k/c$.

Proof. Fix an integer $k \geq 0$. From Lemma 3.4, it follows that $(K_{k/c}, \leq)$ where $\leq$ is inherited from $K_{\overline{\alpha}}$ is a Fraïssé class. Let $\mathfrak{M}_{k/c}$ be the $(K_{k/c}, \leq)$ generic. Note that given $\mathfrak{B} \in K_{\overline{\alpha}}$, there is some $\mathfrak{D} \in K_{k/c}$ such that $\mathfrak{D} \supseteq \mathfrak{B}$ by Lemma 2.8. Thus it suffices to show that $\mathfrak{M}_{k/c}$ satisfies the extension formulas in $S_{\overline{\alpha}}$.

Let $\mathfrak{A}, \mathfrak{B} \in K_{\overline{\alpha}}$ with $\mathfrak{A} \leq \mathfrak{B}$ and assume that $\mathfrak{A} \subseteq \mathfrak{M}_{k/c}$. As $\mathfrak{M}_{k/c}$ is the $(K_{k/c}, \leq)$ generic, there is some $\mathfrak{C} \leq \mathfrak{M}_{k/c}$ with $\mathfrak{A} \subseteq \mathfrak{C}$ and $\delta(\mathfrak{C}) = k/c$. By Fact 2.3, we have that $\mathfrak{D} = \mathfrak{B} \oplus_{\mathfrak{A}} \mathfrak{C}$ is in $K_{\overline{\alpha}}$ and that $\mathfrak{C} \leq \mathfrak{D}$. Now using Lemma 2.8, we

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can find $\mathfrak{G} \in K_{k/c}$ such that $\mathfrak{D} \subseteq \mathfrak{G}$ and $\mathfrak{C} \leq \mathfrak{G}$. But as $\mathfrak{M}_{k/c}$ is the $(K_{k/c}, \leq)$ generic we can find a strong embedding of $\mathfrak{G}$ into $\mathfrak{M}_{k/c}$ over $\mathfrak{C}$. Thus it follows that $\mathfrak{M}_{k/c} \models \forall \overline{x} \exists \overline{y} (\Delta_A(\overline{x}) \implies \Delta_{A,B}(\overline{x}, \overline{y}))$. Hence it follows that $\mathfrak{M}_{k/c} \models S_{\overline{\alpha}}$. Further as noted above, given any finite $\mathfrak{A} \subseteq \mathfrak{M}_{k/c}$, there is some $\mathfrak{C} \leq \mathfrak{M}_{k/c}$ with $\mathfrak{A} \subseteq \mathfrak{C}$ and $\delta(\mathfrak{C}) = k/c$. Hence $\dim(\mathfrak{M}_{k/c}) = k/c$. ⊣

We now work toward classifying the countable models of $S_{\overline{\alpha}}$ up to isomorphism using our notion of dimension.

Lemma 3.6. *Let $\mathfrak{M} \models S_{\overline{\alpha}}$ and $\mathfrak{A} \leq \mathfrak{M}$ be finite. Let $\mathfrak{D} \in K_{\overline{\alpha}}$ be such that $\mathfrak{A} \leq \mathfrak{D}$. Then $\dim(\mathfrak{M}/\mathfrak{A}) \geq \delta(\mathfrak{D}/\mathfrak{A})$ if and only if there is some g that strongly embeds $\mathfrak{D}$ into $\mathfrak{M}$ over $\mathfrak{A}$ (i.e., $g(\mathfrak{D}) \leq \mathfrak{M}$).*

Proof. The statement that if there is some g such that g strongly embeds $\mathfrak{D}$ into $\mathfrak{M}$ over $\mathfrak{A}$, then $\dim(\mathfrak{M}/\mathfrak{A}) \geq \delta(\mathfrak{D}/\mathfrak{A})$ is immediate from the definition. Thus we prove the converse. Let $\mathfrak{A} \leq \mathfrak{M}$ be finite. Let $\mathfrak{D} \in K_{\overline{\alpha}}$ be such that $\mathfrak{A} \leq \mathfrak{D}$.

First assume that $\delta(\mathfrak{D}/\mathfrak{A}) = 0$. Now as $S_{\overline{\alpha}} \models \forall \overline{x} \exists \overline{y} (\Delta_{\mathfrak{A}}(\overline{x}) \implies \Delta_{\mathfrak{A},\mathfrak{D}}(\overline{x}, \overline{y}))$. Thus there is some $\mathfrak{A} \subseteq \mathfrak{D}' \subseteq \mathfrak{M}$ such that $\mathfrak{D} \cong_{\mathfrak{A}} \mathfrak{D}'$. Further as $\delta(\mathfrak{D}'/\mathfrak{A}) = 0$, from (2) of Lemma 2.5, $\mathfrak{D}' \leq \mathfrak{M}$. Thus regardless of the value of $\dim(\mathfrak{M}/\mathfrak{A})$, if $\delta(\mathfrak{D}/\mathfrak{A}) = 0$ then there is some g such that g strongly embeds $\mathfrak{D}$ into $\mathfrak{M}$ over $\mathfrak{A}$.

Now assume that $m/c = \delta(\mathfrak{D}/\mathfrak{A}) \leq \dim(\mathfrak{M}/\mathfrak{A})$ with $m \geq 1$ and further assume that $\dim(\mathfrak{M}/\mathfrak{A}) \geq k/c$ with $k \geq m$. Let $\mathfrak{A} \leq \mathfrak{F} \leq \mathfrak{M}$ be such that $\delta(\mathfrak{F}/\mathfrak{A}) = k/c$. Let $\mathfrak{G} = \mathfrak{D} \oplus_{\mathfrak{A}} \mathfrak{F}$. By Lemma 3.3, there exists $\mathfrak{H} \in K_{\overline{\alpha}}$ with $\mathfrak{G} \subseteq \mathfrak{H}$ and $\mathfrak{A}, \mathfrak{D}, \mathfrak{F} \leq \mathfrak{H}$ and $\delta(\mathfrak{H}/\mathfrak{F}) = 0$. Since $\mathfrak{F} \leq \mathfrak{M}$ and $\delta(\mathfrak{H}/\mathfrak{F}) = 0$ we are in the setting above. So take a strong embedding g of $\mathfrak{H}$ into $\mathfrak{M}$ over $\mathfrak{F}$. Clearly g fixes $\mathfrak{A}$ and $\mathfrak{D}$ has the property that $g(\mathfrak{D}) \leq \mathfrak{F} \leq \mathfrak{M}$ and thus $g(\mathfrak{D}) \leq \mathfrak{M}$. ⊣

Abusing notation and letting $K_{k/c} = K_{\overline{\alpha}}$ in the case that $k = \infty$, we now obtain:

Theorem 3.7. *Let $\mathfrak{M}, \mathfrak{N} \models S_{\overline{\alpha}}$ be countable. Now $\mathfrak{M} \cong \mathfrak{N}$ if and only if $\dim(\mathfrak{M}) = \dim(\mathfrak{N})$ and $\dim(\mathfrak{M}) = \infty$ if and only if $\mathfrak{M}$ is the generic for $K_{\overline{\alpha}}$. Thus there are precisely $\aleph_0$ many nonisomorphic models of $S_{\overline{\alpha}}$ of size $\aleph_0$. Further each countable model of $S_{\overline{\alpha}}$ is isomorphic to some $(K_{k/c}, \leq)$ generic.*

Proof. Since δ is invariant under isomorphism, it immediately follows that if $\mathfrak{M} \cong \mathfrak{N}$, then $\dim(\mathfrak{M}) = \dim(\mathfrak{N})$. Now from Theorem 3.5, it follows that the number of nonisomorphic countable models is at least $\aleph_0$.

Case 1. $\dim(\mathfrak{M}) = \dim(\mathfrak{N}) = k/c$ for some $k \in \omega$. Fix enumerations for M, N. Let $\mathfrak{A} \leq \mathfrak{M}$ with $\dim(\mathfrak{M}/\mathfrak{A}) = 0$. Thus $\delta(\mathfrak{A}) = \dim(\mathfrak{M}) = \dim(\mathfrak{N})$. Assume that we have constructed a *strong embedding* $g : \mathfrak{A} \to \mathfrak{N}$. Pick $b \in \mathfrak{N} - g(\mathfrak{A})$, where b in the enumeration corresponds to the element of N with least index not in $g(A)$. Consider $icl_{\mathfrak{N}}(\{b\} \cup g(\mathfrak{A})) = \mathfrak{B} \leq \mathfrak{N}$. Now $\mathfrak{B}$ is finite. Since $g(\mathfrak{A}) \leq \mathfrak{N}$ and $g(\mathfrak{A}) = \dim(\mathfrak{N})$, it follows that $\delta(\mathfrak{B}/g(\mathfrak{A})) = 0$ and $g(\mathfrak{A}) \leq \mathfrak{B}$. Now as $\mathfrak{A} \cong g(\mathfrak{A})$ by Lemma 3.6, there exists a *strong embedding* $g' : \mathfrak{B} \to \mathfrak{M}$ and $g'|_{g(\mathfrak{A})} = g^{-1}$. Clearly this allows us to form a back and forth system between $\mathfrak{M}, \mathfrak{N}$.

Thus all that remains to be shown is that we can find a strong embedding of $\mathfrak{A} \leq \mathfrak{M}$ where $\delta(\mathfrak{A}) = \dim(\mathfrak{M})$. To see this first note that $\emptyset \leq \mathfrak{N}$. Further $\dim(\mathfrak{N}/\emptyset) = \delta(\mathfrak{A}/\emptyset)$. Thus there exists some strong embedding of $\mathfrak{A}$ over $\emptyset$ into $\mathfrak{N}$ by an application of Lemma 3.6 as required.

CASE 2. $\mathfrak{M} \models S_{\overline{\alpha}}$ and $\dim(\mathfrak{M}) = \infty$. We claim that in this case $\mathfrak{M}$ is isomorphic to the generic. Clearly $\mathfrak{M}$ has finite closures and hence condition (1) of the generic is satisfied. Note that if we show that $\dim(\mathfrak{M}) = \infty$ implies that for any $\mathfrak{A} \leq \mathfrak{M}$, $\dim(\mathfrak{M}/\mathfrak{A}) = \infty$, then condition (2) follows immediately from Lemma 3.6. We claim that this is indeed the case. By way of contradiction, assume that there is some $\mathfrak{A} \leq \mathfrak{M}$ such that $\dim(\mathfrak{M}/\mathfrak{A})$ is finite. Now there is some $\mathfrak{A} \leq \mathfrak{D} \leq \mathfrak{M}$ such that $\dim(\mathfrak{M}/\mathfrak{A}) = \delta(\mathfrak{D}/\mathfrak{A})$. It is immediate from the definition that $\dim(\mathfrak{M}/\mathfrak{D}) = 0$. As $\dim(\mathfrak{M}) = \infty$, fix a $\mathfrak{B} \leq \mathfrak{M}$ with $\delta(\mathfrak{B}) > \delta(\mathfrak{D})$. Consider G, the closure of BD in M. Now G is finite and since $B, D \leq M$, $B, D \leq G$. Further $\delta(G/D) = 0$ as $\dim(M/D) = 0$. So $\delta(G) = \delta(D)$. But $B \leq M$, so $\delta(G/B) \geq 0$ and hence $\delta(G) \geq \delta(B)$. Thus $\delta(B) \leq \delta(D)$, a contradiction to our choice of B that establishes the claim. Hence it follows that the number of nonisomorphic countable models of $S_{\overline{\alpha}}$ is $\aleph_0$.

From Theorem 3.5, it follows that we can construct a countable model of a fixed dimension (the $\dim(\mathfrak{M}) = \infty$ case being the generic as seen above) as the generic of a subclass of $(K_{\overline{\alpha}}, \leq)$. But as the dimension determines the countable model up to isomorphism, we obtain the result. ⊣

We now use our notion of dimension to characterize elementary embeddability.

THEOREM 3.8. *Let $\mathfrak{M}, \mathfrak{N}$ be countable models of $S_{\overline{\alpha}}$. If* $\dim(\mathfrak{M}) \leq \dim \mathfrak{N}$, *then there is some elementary embedding $f : \mathfrak{M} \to \mathfrak{N}$. Thus there is an elementary chain $\mathfrak{M}_0 \preccurlyeq \cdots \preccurlyeq \mathfrak{M}_n \cdots \preccurlyeq \mathfrak{M}_\omega$ of countable models of $S_{\overline{\alpha}}$ with each countable model isomorphic to some element of the chain.*

PROOF. Let $\mathfrak{M}, \mathfrak{N}$ be countable models of $S_{\overline{\alpha}}$ with $\dim(\mathfrak{M}) \leq \dim(\mathfrak{N})$. Note that if $\dim(\mathfrak{M}) = \dim(\mathfrak{N})$, then by Theorem 3.7, $\mathfrak{M} \cong \mathfrak{N}$. So assume that $\dim(\mathfrak{M}) < \dim(\mathfrak{N})$ and fix an enumeration $\{m_i : i \in \omega\}$. Now we have that $\dim(\mathfrak{M}) < \infty$. Let $\mathfrak{A} \leq \mathfrak{M}$ be such that $\delta(\mathfrak{A}) = \dim(\mathfrak{M})$. Now by Lemma 3.6, there exists a strong embedding f_1 of $\mathfrak{A}$ into $\mathfrak{N}$. Let $\mathfrak{B} \leq \mathfrak{M}$ be such that $Am_i \subseteq B$ where i is the least index such that $m_i \notin A$. Note that as $\delta(\mathfrak{A}) = \dim(\mathfrak{M})$, $\delta(\mathfrak{B}) = \delta(\mathfrak{A})$. Again using Lemma 3.6, we can extend f_1 to f_2 so that f_2 is a strong embedding of $\mathfrak{B}$ into $\mathfrak{N}$ *over* $\mathfrak{A}$.

Proceeding iteratively we can find a $\leq$ chain $\{\mathfrak{A}_i : i \in \omega\}$ such that $\mathfrak{M} = \bigcup_{i<\omega} \mathfrak{A}_i$ and $f : \mathfrak{M} \to \mathfrak{N}$ such that $f(\mathfrak{A}_i) \leq \mathfrak{N}$ for each $i \in \omega$. It is easily seen that f is an isomorphic embedding. We claim that f is actually an elementary embedding of $\mathfrak{M}$ into $\mathfrak{N}$. Note that given $\mathfrak{C} \leq \mathfrak{M}$ with C finite, there is some $\mathfrak{A}_{\mathrm{i}}$ with $\mathfrak{C} \leq \mathfrak{A}_i \leq \mathfrak{M}$. Using the transitivity of $\leq$, it easily follows that $f(\mathfrak{C}) \leq \mathfrak{N}$. In particular $f(\mathfrak{M})$ is (algebraically) closed in $\mathfrak{N}$. For notational convenience we will assume that $\mathfrak{M} \subseteq \mathfrak{N}$.

Let $\psi(\overline{x}, \overline{y})$ be an L formula. Let $\overline{a} \in M^{lg(\overline{x})}$. Assume that $\mathfrak{N} \models \exists \overline{y} \psi(\overline{a}, \overline{y})$. But $\psi(\overline{x}, \overline{y})$ is equivalent to the boolean combination of chain minimal formulas, say $S_{\overline{\alpha}} \vdash \forall(\overline{x})(\exists \psi(\overline{x}, \overline{y}) \leftrightarrow \bigwedge_{i<n} \varphi_i(\overline{x}, \overline{y}))$ where each $\varphi(\overline{x}, \overline{y})$ is either a chain minimal formula or the negation of a chain minimal formula. Suppose that $\overline{b} \in N^{lg(\overline{y})}$ is such that $\mathfrak{N} \models \psi(\overline{a}, \overline{b})$. If φ_i is a chain minimal formula then it follows that $\overline{b} \in M^{lg(\overline{y})}$ as M is a closed set. So assume that each φ_i is the negation of a chain minimal formula. Note that we may split $\overline{b} = \overline{b}_1 \overline{b}_2$ where $\overline{b}_1$ is formed via a minimal chain and $A\overline{b}_1 \leq N$. As above, it follows that $\overline{b}_1 \subseteq\in M^{lg(\overline{y}) - lg(\overline{b}_1)}$. But as $\mathfrak{M} \models S_{\overline{\alpha}}$, it follows that there exists a $\overline{b'}_2 \in M^{lg(\overline{y}) - lg(\overline{b}_1)}$ that is isomorphic to $\overline{b}_2$ over $A\overline{b}_1$. It is

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now easily seen that the $\overline{b}_1\overline{b'}_2 \in M^{lg(\overline{y})}$ and $\mathfrak{N} \models \varphi_i(\overline{a}, \overline{b}_1\overline{b'}_2)$ for each i. Thus $\mathfrak{N}$ is an elementary extension of $\mathfrak{M}$.

Note that given an elementary chain $\mathfrak{M}_1 \preccurlyeq \cdots \preccurlyeq \mathfrak{M}_n$ of models of $S_{\overline{\alpha}}$ we may construct $\mathfrak{M}_{n+1}$ such that $\mathfrak{M}_1 \preccurlyeq \cdots \preccurlyeq \mathfrak{M}_n \preccurlyeq \mathfrak{M}_{n+1}$. Note that we may also insist that $\dim(\mathfrak{M}_k) = k/c$. Now given an elementary chain $\mathfrak{M}_0 \preccurlyeq \cdots \preccurlyeq \mathfrak{M}_n \preccurlyeq \cdots \preccurlyeq$ set $\mathfrak{M}_\omega = \bigcup_{n<\omega} \mathfrak{M}_n$. As elementary embeddings preserve closed sets it is easily seen that $\dim(\mathfrak{M}_\omega) = \infty$. The rest of the claim now follows from Theorem 3.7. ⊣

§4. Regular types.

In Section 4 we turn our attention toward the study of regular types. We fix a monster model $\mathbb{M}$ of $S_{\overline{\alpha}}$. Recall the notions of $d(A)$ and $d(B/X)$ for some finite $A \subseteq \mathbb{M}$ and $X \subseteq \mathbb{M}$ from Section 2. We begin by extending this notion to a type as follows (see also [2]).

Definition 4.1. Let $\mathbb{M}$ be a monster model of $S_{\overline{\alpha}}$ and let X be a small subset of $\mathbb{M}$. Let $p \in S(X)$. We let $d(p/X) = d(\overline{b}/X)$ for some (equivalently any) realization $\overline{b}$ of p.

Now, due to ω-stability and weak elimination of imaginaries (see (3) and (5) of Theorem 2.5), it suffices to restrict our attention to nonalgebraic types over finite algebraically closed sets in the home sort for the study of regular types. So fix some finite $A \leq \mathbb{M}$ (recall that algebraically closed sets are precisely the intrinsically closed ones). In what follows we freely use regular types, orthogonality, modular types etc. and facts about them. The relevant definitions and results can be found in [10].

Remark 4.2. Let $A \leq \mathbb{M}$ be finite and $\overline{b}$ be finite such that $\overline{b} \cap A = \emptyset$. Now let $A \subseteq C$ also be finite. Note that $\overline{b} \downarrow_A C$ if and only if $\mathrm{acl}(\overline{b}A) \downarrow_{\mathrm{acl}(A)} \mathrm{acl}(C)$. Since $S_{\overline{\alpha}}$ has finite closures it follows that $\mathrm{acl}(bA), \mathrm{acl}(C)$ are both finite. Thus in order to understand nonforking, it suffices to look at types $p \in S(A)$ such that $x \neq a \in p$ for all $a \in A$ such that for any $\overline{b} \models p$, $\overline{b}A \leq \mathbb{M}$. Note that this information, along with the atomic diagram of some (of any) realization of p is sufficient to determine p uniquely as noted in (1) of Lemma 2.5. Also such a type p is nonalgebraic and stationary as A is algebraically closed.

In light of our comments at the beginning of Section 4 and Remark 4.2 it suffices to study *basic types over finite sets* in order to understand regular types (i.e., we can choose a basic type to represent the required parallelism class).

Definition 4.3. Let $A \leq \mathbb{M}$ be finite and $p \in S(A)$, we say that p is a *basic type* if $x \neq a \in p$ for all $a \in A$ and for some (equivalently any) $\overline{b} \models p$, $\overline{b}A \leq \mathbb{M}$.

Recall that c is the least common multiple of the denominators of the $\overline{\alpha}_E$ (in reduced form).

Lemma 4.4. *Let $\mathfrak{A} \in K_{\overline{\alpha}}$. Then there exists $\mathfrak{B} \in K_{\overline{\alpha}}$ such that $\mathfrak{A} \leq \mathfrak{B}$ and $\delta(\mathfrak{B}/\mathfrak{A}) = 1/c$.*

Proof. Consider the structure given by $\mathfrak{A}^* = \mathfrak{A} \oplus_\emptyset \mathfrak{A}_0$ where $\mathfrak{A}_0 \in K_{\overline{\alpha}}$ consists of a single point. Now an application of Lemma 2.8 to $\mathfrak{A}^*$ yields the required result. ⊣

We begin by studying basic types such that $d(p/A) = 0, 1/c$ where $A \leq \mathbb{M}$ is finite. The choice to restrict our attention to such types will be justified by Theorem 4.13, where we show any type p with $d(p/A) \geq 2/c$ *cannot* be regular. We begin our analysis of types that can be regular types by defining nuggets and nugget-like types.

Definition 4.5. Let $\mathfrak{A}, \mathfrak{D} \in K_{\overline{\alpha}}$ with $\mathfrak{A} \subsetneq \mathfrak{D}$ with $D = AB$. Let $k \in \omega$. We say that B is a *k/c-nugget over* $\mathfrak{A}$ if $A \cap B = \emptyset$, $\delta(B/A) = k/c$ and $\delta(B'/A) > k/c$ for all $A \subsetneq AB' \subsetneq AB$.

Definition 4.6. Let $A \leq \mathbb{M}$ be finite. We say that a basic type $p \in S(A)$ is *nugget-like* over A, if given B where B realizes the quantifier free type of p over $\mathfrak{A}$, then B is a k/c-nugget over A for some $k \in \omega$.

Lemma 4.7. *Let $A \leq \mathbb{M}$ be finite and let $p \in S(A)$ be nugget-like. Let $A \subseteq X$ with X closed. For any $\overline{b} \models p$, either $\overline{b} \cap X = \emptyset$ or $\overline{b} \subseteq X$.*

Proof. Assume that $\overline{b} \cap X \neq \emptyset$. Let $\overline{b}' = \overline{b} \cap X$ assume that $\overline{b}' \neq \overline{b}$. Then as $\delta(\overline{b}'/A) > \delta(\overline{b}/A)$, it follows that there is some minimal pair $(A\overline{b}', D)$ with $D \subseteq A\overline{b}$ but $D \nsubseteq X$. But this contradicts that X is closed. Hence $\overline{b} \subseteq X$. ⊣

We now explore how the behavior of the d function interacts with nugget-like types. The following results are well known (see e.g., Theorem 3.28 of [3] or Lemma 3.13 of [12] and Lemma 2.6 of [2]).

Lemma 4.8. 1. *Suppose B is finite and $X \subseteq Y$. Then $d(B/X) \geq d(B/Y)$.*
2. *Let $A \leq \mathbb{M}$ be finite and let $p \in S(A)$. Suppose that for some $k \in \omega$, $d(p/A) = k/c$. Let $A \subseteq X \leq \mathbb{M}$. Suppose that $q \in S(X)$ extends p. If $d(q/X) < d(p/A)$, then q is a forking extension of p.*

We now obtain the following fact about the forking of nugget-like types:

Lemma 4.9. *Let $A \leq \mathbb{M}$ be finite and let $p \in S(A)$ is nugget-like. Let $A \subseteq Y \subseteq \mathbb{M}$ with Y closed. Let q be an extension of p to Y. Now q is a forking extension of p if and only if $d(q/Y) < d(p/A)$ or given $\overline{b} \models q$, $\overline{b} \subseteq Y$.*

Proof. If $d(q/Y) < d(p/A)$, then Lemma 4.8 tells us that q is a forking extension of p. Further Y is algebraically closed. So if for any $\overline{b} \models q$, $\overline{b} \subseteq Y$, it follows that b is an algebraic type over Y. Since p is not an algebraic type over A, it follows that q is a forking extension of p.

For the converse assume that q is a forking extension of p and that $d(q/Y) = d(p/A)$. As q is a forking extension of p, it follows from (8) of Theorem 2.5 that $\mathrm{icl}(\overline{b}A) \cap \mathrm{icl}(Y) \supsetneq \mathrm{icl}(A)$. But $\mathrm{icl}(A) = A$, $\mathrm{icl}(Y) = Y$ and as $\overline{b}$ realizes p over A, $\mathrm{icl}(\overline{b}A) = \overline{b}A$. Thus $\overline{b} \cap Y \neq \emptyset$. Now by Lemma 4.7, $\overline{b} \subseteq Y$. ⊣

The following theorem allows us to identify certain regular types. Further it establishes that 0-nuggets are, in some sense, orthogonal to almost all other types.

Theorem 4.10. *Let $A \leq \mathbb{M}$ be finite and let $p \in S(A)$ be nugget-like. Now if $d(p/A) = 0$ or $d(p/A) = 1/c$, then p is regular. Further if $d(p/A) = 0$, then p is orthogonal to any other nugget-like type over A.*

Proof. Under the given conditions p is clearly nonalgebraic and stationary. We directly establish that it will be orthogonal to any forking extension of itself. Let

$A \subseteq X \subseteq \mathbb{M}$ with X closed. Since $S_{\overline{\alpha}}$ is ω-stable and has finite closures we may as well assume that X is finite, i.e., if $q \in S(X)$ with $q \supseteq p$ a forking extension, there is some finite closed $X_0 \subseteq X$ such that $q \restriction_{X_0}$ is a forking extension. Let $\overline{b} \models p$. We have that $\overline{b} \perp_A X$. As $A\overline{b}$, X are closed and $A\overline{b} \cap X = A$, from an application of (8) of Theorem 2.5, we obtain that $X\overline{b}$ is closed.

First assume that $d(p/A) = 0$. Let p' be a forking extension of p to X and let $\overline{f} \models p'$. It follows easily from Lemma 4.8, that $d(\overline{f}/A) \geq d(\overline{f}/X)$. As $d(\overline{f}/A) = 0$ and $d(\overline{f}/X) \geq 0$, it now follows that $d(\overline{f}/X) = 0$. Thus by Lemma 4.9, we have that $\overline{f} \subseteq X$ and hence $\overline{b} \perp_X \overline{f}$ as $\overline{b} \perp_A X$.

So assume that $d(p/A) = 1/c$. Let $p', \overline{f}$ be as above. By Lemma 4.9, $d(p'/X) = 0$ or $\overline{f} \subseteq X$. As above $\overline{f} \subseteq X$ yields that $\overline{b} \perp_X \overline{f}$. So assume that $\overline{f} \not\subseteq X$ and note that by Lemma 4.7 we have that $\overline{f} \cap X = \emptyset$. Now by (8) of Theorem 2.5 it suffice to show that $X\overline{b} \cap \operatorname{acl}(X\overline{f}) = X$ to establish that $\overline{b} \perp_X \overline{f}$. Consider $d(\operatorname{acl}(X\overline{f})\overline{b}/X)$. On the one hand, as $X\overline{b} \subseteq \operatorname{acl}(X\overline{f})\overline{b}$, $d(\operatorname{acl}(X\overline{f})\overline{b}) \geq d(X\overline{b})$ and thus we have that $d(\operatorname{acl}(X\overline{f})\overline{b}/X) \geq d(\overline{b}/X) = 1/c$. On the other hand $d(\operatorname{acl}(X\overline{f})\overline{b}/X) = d(\overline{b}/acl(X\overline{f})) + d(\operatorname{acl}(X\overline{f})/X)$. As $d(\operatorname{acl}(X\overline{f})/X) = d(\overline{f}/X) = 0$, we obtain that $d(\overline{b}/\operatorname{acl}(X\overline{f})) \geq 1/c$. In particular $\overline{b} \not\subseteq \operatorname{acl}(X\overline{f})$. But then by Lemma 4.7, $\overline{b} \cap \operatorname{acl}(X\overline{f}) = \emptyset$ and thus $X\overline{b} \cap \operatorname{acl}(X\overline{f}) = \emptyset$ as required.

For the second half of the claim, assume that $d(p/A) = 0$. Let $q \in S(A)$ be nugget-like and distinct from p. Now $d(p/A) = d(p|_X/X)$ and $d(q/A) = d(q|_X/X)$. Let $\overline{f} \models q|_X$. Note that $\overline{f} \perp_A X$ implies that $X\overline{f}$ is closed. Now using Lemma 4.7, we can easily show that $\overline{b}X \cap \overline{f}X \neq X$, then $\overline{b} = \overline{f}$. But this contradicts $p \neq q$. Thus it follows that $\overline{b}X \cap \overline{f}X = X$. Further $0 = d(\overline{b}/X) \geq d(\overline{b}/X\overline{f}) \geq 0$. Again by (8) of Theorem 2.5, we obtain that $\overline{b} \perp_X \overline{f}$ and thus p, q are orthogonal. ⊣

The following theorem shows that while there are many regular types with $d(p/A) = 1/c$, all such types are nonorthogonal. Thus up to nonorthogonality, there is only one regular type with $d(p/A) = 1/c$. This is in contrast to distinct 0-nuggets, any two of which are orthogonal to each other. We also show that the number of independent realizations of a $1/c$ nugget determines the dimension of a model.

Theorem 4.11. *Let A be closed and finite and let $p, q \in S(A)$ be distinct basic types and satisfy $d(p/A) = d(q/A) = 1/c$. Then p, q are nonorthogonal. Hence any two regular types over $p', q' \in S(X)$ where X is closed and $d(p'/X) = d(q'/X) = 1/c$ are nonorthogonal. Further if we take $A = \emptyset$ and let $\mathfrak{M} \preccurlyeq \mathbb{M}$. The dimension of $\mathfrak{M}$ is determined by the number of independent realizations of p in $\mathfrak{M}$. Thus a single regular type determines the dimension of $\mathfrak{M}$.*

Proof. Let A be as given. Consider A as a finite structure that lives in $K_{\overline{\alpha}}$. Now consider the finite structures AB, AC where B, C realize the quantifier free types of p, q respectively. Consider $D = AB \oplus_A AC$. Apply Lemma 2.8 to obtain a finite G with $\delta(G/D) = -1/c$ and $A, AB, AC \leq G$. Let f be a strong embedding of G into $\mathbb{M}$ where f is the identity on A. From (6) of Theorem 2.5 and the transitivity of $\leq$ it follows that $f(B) \models p$ and $f(C) \models q$. Now from (8) of Theorem 2.5, it follows that $f(B) \not\perp_A f(C)$ and thus $p \not\perp q$. Now given $p', q' \in S(X)$, there exists a finite closed set, which by an abuse of notation we call A, such that p', q' are based and stationary over A. Since regularity is parallelism invariant both $p|_A$ and $q|_A$ are regular. Arguing as above we see that $p'|_A \not\perp q'|_A$ and thus they are nonorthogonal.

Let $\mathfrak{M} \preccurlyeq \mathbb{M}$ and assume that $A = \emptyset$. Given $n \in \omega$, consider the finite structure C_n that is the free join of n-copies of the quantifier free type of p over $\emptyset$. If $\dim(\mathfrak{M}) \geq n/c$, by Lemma 3.6, there is a strong embedding of C_n into $\mathfrak{M}$. It is easily checked that the strong embedding witnesses n-independent realizations of p. The rest follows easily. ⊣

The following shows that $1/c$ nugget-like types are not locally modular.

Theorem 4.12. *Let $A \leq \mathbb{M}$ be finite and let $p \in S(A)$ be a nugget-like with $d(p/A) = 1/c$. Then p is not locally modular, in particular it is nontrivial.*

Proof. Recall that given a regular type p, the realizations of p form a pregeometry with respect to forking closure. In order to simplify the presentation, we will let $A = \emptyset$.

We begin with a proof that p is nontrivial. Let B_0, B_1, B_2 be three finite structures that has the same quantifier free type as p and are disjoint over $\emptyset$. Consider $C = \oplus_{\emptyset}^{i<3} B_i$. Using Lemma 2.8 we obtain a finite structure $D \in K_{\overline{\alpha}}$ with $\delta(D) = 2/c$, $B_i \leq C$ and $B_i \oplus_{\emptyset} B_j \leq C$ for any $i \neq j$. Note that $C \nleq D$ as $\delta(C) > \delta(D)$. Let g be a strong embedding of C into $\mathbb{M}$. An argument similar to that found in Theorem 4.11 shows that $g(B_0), g(B_1), g(B_2)$ are pairwise independent but dependent realizations of p and thus p is nontrivial.

By well known results of Hrushovski in [6], any stable theory with a nontrivial locally modular regular type interprets a group. As these structures do not interpret groups (see [13] by Wagner for detailed discussion) the result now follows. ⊣

The following result shows that a broad class of types cannot be regular types and justifies the choice to study types $p \in S(A)$ with $d(p/A) = 0, 1/c$ in our study of regular types.

Theorem 4.13. *Let A be finite and closed in $\mathbb{M}$. Let $p \in S(A)$ be a basic type such that $d(p/A) \geq 2/c$. Then p is not regular.*

Proof. Recall that a regular type has weight 1. We establish the above result by showing that p has preweight at least 2 and hence weight at least 2. Our strategy is similar to the one used in Theorem 4.11: we consider A as living inside of $K_{\overline{\alpha}}$. We then construct a finite structure G over the finite structure A that we then embed strongly into $\mathbb{M}$ over A using saturation. Finally we argue that the strong embedding witnesses the fact that the preweight of p is at least 2.

Consider A as a finite structure that lives in $K_{\overline{\alpha}}$. By Lemma 4.4 we may construct $D \in K_{\overline{\alpha}}$ such that the $D = AC$, $A \cap C = \emptyset$ (as sets), and $A \leq D$ with $\delta(D/A) = \delta(C/A) = 1/c$. Let AB be such that B realizes the quantifier free type of p over A. Consider the finite structures F_i, $i = 1, 2$ where each F_i is the free join of AB and an isomorphic copy of D over A and $F_1 \cap F_2 = AB$. We label the isomorphic copies of D as AC_1, AC_2 and thus $F_i = ABC_i$, the free join of AB, AC_i over A. Apply Theorem 2.7 to obtain G_i for $i = 1, 2$ such that (F_i, G_i) is an essential minimal pair and $\delta(G_i/F_i) = -1/c$. It is easily verified that $A, AB, AC_i \leq G_i$. Let $G = G_1 \oplus_{AB} G_2$. Note that $G \in K_L$ and that we may now regard the finite structures A, AB, AC_1 etc. as substructures of G.

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We claim that $G \in K_{\overline{\alpha}}$, $A, AB, AC_1, AC_2, AC_1C_2 \leq G$ but F_1, F_2, is *not* strong in G. Using Remark 2.2 and the transitivity of $\leq$, we obtain that it suffices to show that $AB, AC_1C_2 \leq G$ along with $F_1, F_2 \nleq G$ to obtain the claim.

First, as $AB \leq G_i$ and G is the free join of G_1, G_2 over AB, we obtain $AB \leq G$ by an application of (4) of Fact 2.4. We now show that $AC_1C_2 \leq G$. Let $AC_1C_2 \subseteq G' \subseteq G$ and let $B' = B \cap G'$, $G_i' = G_i - AC_i$. Now $\delta(G'/AC_1C_2) = \delta((G_1' - B')(G_2' - B')/AC_1C_2B') + \delta(B'/AC_1C_2)$ using (5) of Fact 2.4. Further, since AB, AC_1C_2 is freely joined over A $\delta(B'/AC_1C_2) = \delta(B'/A)$ follows from (2) of Fact 2.4. Arguing similarly we obtain that $\delta(G_i' - B'/AC_1C_2B') = \delta(G_i'/AB'C_i)$. Thus it follows that $\delta(G'/AC_1C_2) = \delta(G_1'/AC_1B') + \delta(G_2'/AC_2B') + \delta(B'/A)$. Now as $A \leq AB$, it follows that $\delta(B'/A) \geq 0$. The claim now follows by considering the cases $B' \neq B$ and $B' = B$ using that fact that (ABC_i, G_i) forms an essential minimal pair. Finally, and easy calculation shows that $\delta(G/F_1F_2) = -2/c$. Now $\delta(G/F_i) = \delta(G/F_1F_2) + \delta(F_1F_2/F_i) = -2/c + 1/c = -1/c$.

Arguing as we did in Theorem 4.11, we easily obtain that a strong embedding of G into $\mathbb{M}$ over A witnesses that the preweight of p is at least 2. We leave the details to the reader. ⊣

§5. A pseudofinite ω-stable theory with a nonlocally modular regular type.

In this section we draw on some known results to prove that there are pseudofinite ω-stable theories with nonlocally modular regular types. This answers a question of Pillay's in [11] regarding whether pseudofinite stable theories always have locally modular regular types. We assume that the reader is familiar with basic facts about pseudofinite theories.

THEOREM 5.1. *There is a pseudofinite ω-stable theory with a nonlocally modular regular type.*

PROOF. Consider the case where $L = \{E\}$ contains only one relation symbol (recall E has arity at least 2). We claim that $S_{\overline{\alpha}}$ has the required properties.

Let $\{\alpha_n\}$ be an increasing sequence of irrationals in $(0, 1)$ that converge to $\overline{\alpha}(E)$. By the results of [1], it follows that $Th(\mathfrak{M}_{\alpha_n})$ can be obtained as a almost sure theory with respect to a certain probability measure. Thus, in particular, each theory $Th(\mathfrak{M}_{\alpha_n})$ is pseudofinite. Now by Theorem 4.2 of [4], it follows that $S_{\overline{\alpha}} = Th(\Pi_{\mathcal{U}}\mathfrak{M}_{\alpha_n})$ where $\mathcal{U}$ is a nonprincipal ultrafilter on ω. Since taking the ultraproduct of structures with pseudofinite theories results in a structure with a pseudofinite theory, it follows that $S_{\overline{\alpha}}$ is pseudofinite. Further as we have shown in Theorem 4.12 that $1/c$-nuggets are nonlocally modular, the result follows. ⊣

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