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and the twist number of alternating surface links**

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For D a reduced alternating surface link diagram, we bound the twist number of D in terms of the coefficients of a polynomial invariant. To this end, we introduce a generalization of the homological Kauffman bracket defined by Krushkal. Combined with work of Futer, Kalfagianni and Purcell, this yields a bound for the hyperbolic volume of a class of alternating surface links in terms of these coefficients.

57K10; 57K32

1 Introduction

Since its introduction, the Jones polynomial [8] has been of key interest in the ongoing effort to find explicit relations between quantum invariants of a link and the geometry or topology of the link complement. It is in this spirit that Dasbach and Lin [5] proved a “volume-ish theorem” for alternating, prime, hyperbolic knots, which presents upper and lower bounds for the volume of the knot complement in terms of certain coefficients of the Jones polynomial of the knot. This result was obtained in two steps: first, the authors bound the coefficients in terms of the twist number of the link diagram, and second, work of Lackenby, Agol, and Thurston in [12] bounds the volume in terms of the twist number. Futer, Kalfagianni, and Purcell [7] extended both steps to adequate links.

The latter of these steps was recently extended to alternating links in higher-genus surfaces by Kalfagianni and Purcell in [9]. It is then natural to consider the first step in the same setting. The difficulty, however, is that the classical Jones polynomial does not capture sufficient information about the embedding of links in surfaces. In this paper, we opt for a three-variable generalization of the homological polynomial defined by Krushkal in [11].

Our polynomial, denoted by $\langle D \rangle_\Sigma$, is a Laurent polynomial in $\mathbb{Z}[A^{\pm 1}, Z, W]$ defined by a state-sum. Here, A is the usual Kauffman polynomial variable, while Z and W record homological information about the embedding of D on the surface. We will be

interested in coefficients of terms having certain fixed degrees in the A variable, with minimal degrees in the Z and W variables. The coefficients for the second largest and second smallest degree terms in A are denoted by $\alpha'_{(1)}$ and $\beta'_{(1)}$, while those for the third largest and third smallest are denoted by $\alpha''_{(0)}$ and $\beta''_{(0)}$, respectively. The main purpose of the paper is to establish the following bound for these coefficients in terms of the twist number, by generalizing the method in [7]. The precise definitions of the polynomial and coefficients are given in Sections 2 and 3.

Theorem 1 *Let Σ be a closed, orientable surface and let D be a reduced alternating link diagram on Σ such that every twist region of D has at least three crossings. Let $\star = \alpha'_{(1)} + \beta'_{(1)} - \alpha''_{(0)} - \beta''_{(0)} + 2$. Then*

$$(1) \quad \frac{1}{3} \operatorname{tw}(D) + 1 - g(\Sigma) \leq \star \leq 2 \operatorname{tw}(D).$$

Combined with the result in [9], this yields the following volume bound.

Corollary 2 *Let Σ be a closed orientable surface of genus at least one, and let L be a link that admits a twist-reduced weakly generalized cellularly embedded alternating projection D onto $\Sigma \times \{0\}$ in $Y = \Sigma \times [-1, 1]$. Then the interior of $Y \setminus L$ admits a hyperbolic structure. If Σ is a torus, then we have*

$$(2) \quad \frac{1}{4} v_{\text{oct}} \star \leq \operatorname{vol}(Y \setminus L) < 30 v_{\text{tet}} \star,$$

where v_{tet} is the volume of a regular ideal tetrahedron, and v_{oct} is the volume of a regular ideal octahedron.

If Σ has genus at least two, then we have

$$(3) \quad \frac{1}{4} v_{\text{oct}} \cdot (\star - 6\chi(\Sigma)) \leq \operatorname{vol}(Y \setminus L) < 18 v_{\text{oct}} \cdot (\star + g(\Sigma) - 1).$$

We should note that while this paper was in preparation, related results were independently obtained for different choices of polynomials. Champanerkar and Kofman [3] use the homological Kauffman bracket (see Definition 3), while Bavier and Kalfagianni [1] define a polynomial $\langle D \rangle_0$ which agrees with the classical Kauffman bracket, but is defined over states consisting only of contractible loops. In contrast to Theorem 1, both of these results contain strict equalities between coefficients and the twist number, rather than inequalities. There is, however, an associated cost. In [3] the authors must use a homological version of the twist number, and in [1] the notion of “reduced” is stronger which eliminates many of the terms making up $\star$ in (25). Although we achieve only inequalities, our approach has the advantage of recovering the classical twist number without requiring these further conditions.

1.1 Organization of the paper

In Section 2 we give some basic definitions and construct the more general homological polynomial referred to in the introduction. In Section 3 we define the state graphs $\mathbb{G}_A$ and $\mathbb{G}_B$ and use them to compute certain coefficients of the aforementioned polynomial. Lastly, Section 4 contains proofs of Theorem 1 and Corollary 2 obtained by finding bounds on these coefficients in terms of the twist number of a diagram.

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2 Definitions

Throughout this paper, we will let Σ be a closed, orientable surface. We will implicitly view Σ as $\Sigma \times \{0\} \subset \Sigma \times [-1, 1]$, but since this paper deals primarily with link diagrams, these results will hold for Σ in any compact, orientable, irreducible 3-manifold. For a link $L \subset \Sigma \times [-1, 1]$, a link diagram D for L on Σ is the image of L under the projection, which we may view as a 4-valent graph Γ on Σ . The vertices of Γ correspond to double points of the projection, and these are then equipped with crossing information. There are two possible ways to resolve each crossing. Resolving a crossing means replacing D with a new diagram having one fewer crossing which differs from D only in a neighborhood of that crossing. We call these the A -resolution and the B -resolution, referring to Figure 1. As we resolve crossings, we may draw an arc connecting the strands of the new diagram. These are commonly called “surgery arcs”, as performing surgery along such an arc provides a convenient way to switch between the A and B resolutions of a crossing.

If all crossings are resolved, then all that remains of D is a collection S of disjoint simple closed curves on Σ . We call S a *state*, and denote the set of all states by $\mathcal{S}$. Note that $|\mathcal{S}| = 2^{c(D)}$, where $c(D)$ is the number of crossings of D .

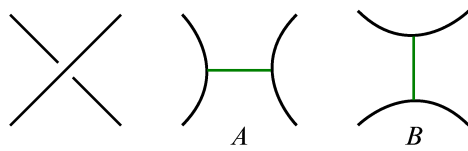


Figure 1: A crossing, an A -resolution, and a B -resolution.

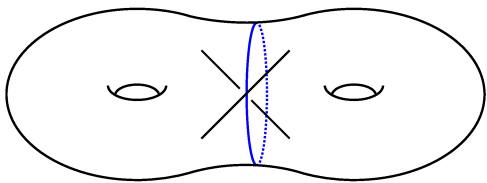


Figure 2: A nugatory crossing.

We will be considering only link diagrams which are alternating and reduced. A link diagram on a surface is *reduced* if it is cellularly embedded and if it contains no nugatory crossings. By cellularly embedded, we mean that the complementary regions of the graph Γ are disks. And a crossing is *nugatory* if there exists a separating curve in Σ that intersects D only at that crossing, as in Figure 2.

The polynomial we will be studying is a generalization of the homological Kauffman bracket, defined by Krushkal in [11]. Let $i : S \rightarrow \Sigma$ be the inclusion map. “Homological” refers to the use of the induced map $i_* : H_1(S) \rightarrow H_1(\Sigma)$, which provides additional information about the embedding of each state. Throughout this paper, we use $\mathbb{Z}$ coefficients unless otherwise indicated.

Definition 3 [11] Let $\alpha(S)$ and $\beta(S)$ respectively denote the number of A –resolutions and B –resolutions used to obtain the state S . The *homological Kauffman bracket* is a Laurent polynomial $\langle D \rangle_\Sigma^H \in \mathbb{Z}[A^{\pm 1}, Z]$ defined by the state-sum

(4)
$$\langle D \rangle_\Sigma^H = \sum_{S \in \mathcal{S}} A^{\alpha(S)-\beta(S)} (-A^2 - A^{-2})^{k(S)} Z^{r(S)},$$

where

$$k(S) = \dim(\ker\{i_* : H_1(S) \rightarrow H_1(\Sigma)\}),$$
$$r(S) = \dim(\operatorname{im}\{i_* : H_1(S) \rightarrow H_1(\Sigma)\}).$$

Here $k(S)$ is called the *kernel* of the state, while $r(S)$ is called the *homological rank* of the state. We write

(5)
$$\langle D \mid S \rangle_\Sigma^H = A^{\alpha(S)-\beta(S)} (-A^2 - A^{-2})^{k(S)} Z^{r(S)}$$

for the contribution of the state S to $\langle D \rangle_\Sigma^H$.

In [2, Lemmas 2.1–2.3], Boden and Karimi give a proof for the invariance of $\langle D \rangle_\Sigma^H$ under the second and third Reidemeister moves. Note that the authors use $\mathbb{Z}_2$ coefficients in their definition of $\langle D \rangle_\Sigma^H$. Since states consist of disjointly embedded circles, and

Σ is orientable, the difference in coefficients is inconsequential. To obtain invariance under the first Reidemeister move, a homological version of the Jones polynomial may be obtained by setting

$$(6) \quad J_{\Sigma}^H(t, Z) = (-A)^{-3w(D)} \langle D \rangle_{\Sigma}^H \Big|_{A=t^{-\frac{1}{4}}},$$

where $w(D)$ is the writhe of D . Thus, we obtain a link invariant.

In [11], a homological version of the Tutte polynomial is defined for surface graphs, and then the homological Kauffman bracket is obtained as a specialization of the graph polynomial. More recently, in [6], Fendley and Krushkal define a more general version of this graph polynomial for graphs on the torus. More specifically, for a graph with homological rank equal to one, a third variable W is introduced, which counts the number of components of the graph which are homologically essential. Inspired by this generalization, we present a version of this polynomial for link diagrams and extend it to higher-genus surfaces.

Definition 4 Define $\bar{c}(S)$ to be zero if $r(S) \neq 1$. If $r(S) = 1$, then define $\bar{c}(S)$ to be one-half the number of homologically essential loops in S . The more general homological Kauffman bracket which will be used throughout this paper is given by

$$(7) \quad \langle D \rangle_{\Sigma} = \sum_{S \in \mathcal{S}} A^{\alpha(S) - \beta(S)} (-A^2 - A^{-2})^{k(S)} Z^{r(S)} W^{\bar{c}(S)},$$

and we similarly write

$$(8) \quad \langle D \mid S \rangle_{\Sigma} = A^{\alpha(S) - \beta(S)} (-A^2 - A^{-2})^{k(S)} Z^{r(S)} W^{\bar{c}(S)}$$

for the contribution of a state.

Example 5 As an example, consider the two-component link diagram D in the torus depicted in Figure 3. For the all- A and all- B states, we can see that $k(S_A) = k(S_B) = 1$ and $r(S_A) = r(S_B) = 0$. For the two remaining intermediate states, which differ by diagonal reflection, we have that $k(S) = 1$ and $r(S) = 1$ for both. Furthermore, each intermediate state has exactly two homologically essential loops, so $\bar{c}(S) = 1$. This yields $\langle D \rangle_{\Sigma} = -A^4 - 2A^2ZW - 2 - 2A^{-2}ZW - A^{-4}$. Note that in this example, the third largest and third smallest degree terms in A coincide.

The value of a trivial loop is unaffected by the addition of the W variable, so one can verify that Lemmas 2.1–2.3 in [2] hold for this more general polynomial as well. From

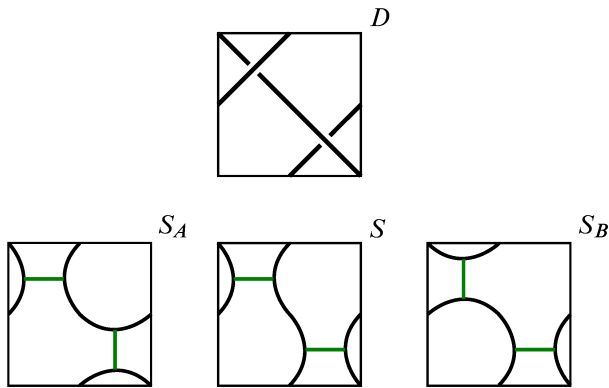


Figure 3: Top: a link diagram in the torus. Bottom: the all- A state S_A , one of two intermediate states, and the all- B state S_B .

here, one can define a link invariant by making the same substitution and renormalization as in (6). This does not affect the coefficients, however, so $\langle D \rangle_\Sigma$ will suffice for our purposes. In [2, Proposition 1.7], the authors prove that any checkerboard colorable link L has $[L] = 0$ in $H_1(\Sigma \times [-1, 1]; \mathbb{Z}_2)$. Under the projection, L is $\mathbb{Z}_2$ -homologous to the sum of all loops in any fixed state S . Thus, $[S] = 0$ in $H_1(\Sigma; \mathbb{Z}_2)$ as well. As a result, if $r(S) = 1$, then there must be an even number of essential loops in S , so we conclude that $\langle D \rangle_\Sigma \in \mathbb{Z}[A^{\pm 1}, Z, W]$ for checkerboard colorable diagrams.

In particular, since alternating, cellularly embedded link diagrams are checkerboard colorable by [2], we will also have $\langle D \rangle_\Sigma \in \mathbb{Z}[A^{\pm 1}, Z, W]$ for the class of links considered by this paper. After having expressed certain coefficients of $\langle D \rangle_\Sigma$ in terms of the state graphs, we will form a connection between the coefficients and the twist number of D .

Definition 6 Let D be a reduced link diagram on a surface Σ . By definition, the complementary regions of D are n -gons with $n \geq 2$. A *twist region* is a connected sequence of bigons arranged crossing-to-crossing of maximal length, as in Figure 4. A single crossing which is adjacent to no bigon regions is also considered to be a twist region. The *twist number* $\text{tw}(D)$ denotes the number of twist regions of D .

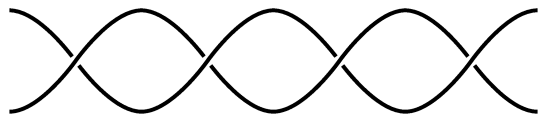


Figure 4: A twist region.

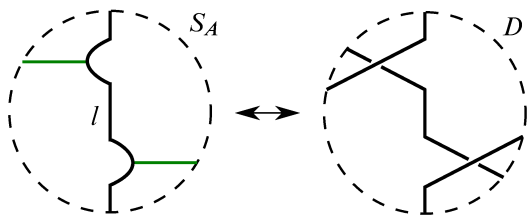


Figure 5: Arcs on opposite sides of l contradict the fact that D is alternating.

3 Computation of coefficients

We first discuss a few properties of reduced alternating diagrams which result in some nice structure on Σ . Let S_A denote the all- A state, and let S_B denote the all- B state, each obtained by selecting only that type of resolution. If D is alternating, and l is any loop in either S_A or S_B , then all surgery arcs attached to l must lie on the same side of l as seen in Figure 5. For $S \in \mathcal{S}$, let $\mathcal{A}(S)$ denote the collection of surgery arcs for S . Consider $U = S \cup \mathcal{A}(S)$. Observe that we can recover the underlying 4-valent graph Γ by contracting each arc of U to a point. This contraction induces a homeomorphism between the complementary regions of U and the complementary regions of Γ . Therefore, if D is cellularly embedded, the complementary regions of U must be disks. As a result, the side of l containing no edges must be one of these disk regions. In other words, every loop of S_A and S_B is contractible. Therefore, we can use the states S_A and S_B to define the following graphs on Σ .

Definition 7 The all- A state graph $\mathbb{G}_A$ is a graph embedded on Σ whose vertices correspond to the loops of S_A and whose edges correspond to the surgery arcs seen in Figure 1. The all- B state graph $\mathbb{G}_B$ is defined similarly for S_B .

Let v_A and v_B denote the number of vertices in $\mathbb{G}_A$ and $\mathbb{G}_B$ (alternatively, the number of loops in S_A and S_B), respectively. Since all the loops in S_A and S_B are contractible, they are homologically trivial. This shows that the states S_A and S_B contribute

(9)
$$\langle D \mid S_A \rangle_\Sigma = A^{c(D)}(-A^2 - A^{-2})^{v_A},$$

(10)
$$\langle D \mid S_B \rangle_\Sigma = A^{-c(D)}(-A^2 - A^{-2})^{v_B},$$

to $\langle D \rangle_\Sigma$. For a polynomial $p \in \mathbb{Z}[A^{\pm 1}, Z, W]$, we write $\deg^A_{\max}(p)$ and $\deg^A_{\min}(p)$ for the maximal and minimal degrees of p in the variable A , respectively. Note from the

above, that

$$(11) \quad \deg_{\max}^A \langle D \mid S_A \rangle_{\Sigma} = c(D) + 2v_A,$$

$$(12) \quad \deg_{\min}^A \langle D \mid S_B \rangle_{\Sigma} = -c(D) - 2v_B.$$

In fact, these degrees are precisely the maximal and minimal degrees in A of the polynomial $\langle D \rangle_{\Sigma}$. The authors of [2] show this for $\langle D \rangle_{\Sigma}^H$ in their Proposition 2.8, by proving that reduced alternating diagrams on surfaces are homologically adequate.

Definition 8 A diagram D is *homologically A -adequate* if, for any state S with exactly one B -resolution, we have $k(S) \leq k(S_A)$. A diagram D is *homologically B -adequate* if, for any state S with exactly one A -resolution, we have $k(S) \leq k(S_B)$. A diagram D is called *homologically adequate* if it is both homologically A -adequate and homologically B -adequate.

In [2, Lemma 2.6] it is shown that for homologically adequate diagrams, S_A and S_B are the unique states which contribute maximal and minimal degree terms. This can be restated in terms of coefficients as the following.

Proposition 9 [2, Lemma 2.6] *Let D be a reduced alternating projection of a link L onto a closed, orientable surface Σ . Then $\langle D \rangle_{\Sigma}^H$ has unique terms of maximal and minimal degree in the variable A of the form*

$$(13) \quad (-1)^{v_A} A^{c(D)+2v_A},$$

$$(14) \quad (-1)^{v_B} A^{-c(D)-2v_B}.$$

Clearly, the addition of the variable W does not affect the degrees in A or Z , so the same holds for $\langle D \rangle_{\Sigma}$. This proposition directly corresponds to the classical results of Kauffman [10], Thistlethwaite [15] and Murasugi [13], which led to a proof of the Tait conjectures using the Jones polynomial.

In the classical planar case, Dasbach and Lin in [4] as well as Stoimenow in [14] compute more coefficients. These coefficients have more complicated expressions in terms of data coming from the graphs $\mathbb{G}_A$ and $\mathbb{G}_B$. We will extend their work to links in surfaces using the more general homological Kauffman bracket. First, we set some terminology. Typically, the term “loop” is used for an edge in a graph which is incident to only one vertex. Since we will be using the word “loop” to refer to the simple closed curves in a state, we will instead call such an edge a *self-edge*. An edge which is incident to two distinct vertices is called a *simple edge*.

Definition 10 Let e be a self-edge in either $\mathbb{G}_A$ or $\mathbb{G}_B$. We may view e as a well-defined element $[e]$ of $H_1(\Sigma)$ up to a choice of orientation. We define an equivalence relation $\sim^*$ on self-edges, where $e_1 \sim^* e_2$ if $[e_1] = \pm[e_2]$ and if they are adjacent, meaning incident to the same vertex.

Similarly, let e_1 and e_2 be simple edges in either $\mathbb{G}_A$ or $\mathbb{G}_B$ which are incident to the same pair of vertices. The edges e_1 and e_2 form a cycle in the graph, which may also be viewed as an element of $H_1(\Sigma)$ up to orientation. We define an equivalence relation $\sim$ on simple edges, where $e_1 \sim e_2$ if they form a nullhomologous cycle.

Let e_A^* and e_B^* denote the number of distinct equivalence classes of self-edges in $\mathbb{G}_A$ and $\mathbb{G}_B$, and let $\tilde{e}_A$ and $\tilde{e}_B$ be the number of distinct equivalence classes of simple edges in $\mathbb{G}_A$ and $\mathbb{G}_B$, respectively.

Proposition 11 Let D be a reduced alternating projection of a link L onto a closed, orientable surface Σ . Then the terms of $\langle D \rangle_\Sigma$ having the second-highest degree in the A variable are of the form

(15)
$$(-1)^{v_A} (\alpha'_{(1)} W + \alpha'_{(2)} W^2 + \cdots + \alpha'_{(N)} W^N) Z A^{c(D)+2v_A-2},$$

where

(16)
$$\alpha'_{(1)} = e_A^*,$$

and the terms having the second-lowest degree are of the form

(17)
$$(-1)^{v_B} (\beta'_{(1)} W + \beta'_{(2)} W^2 + \cdots + \beta'_{(M)} W^M) Z A^{-c(D)-2v_B+2},$$

where

(18)
$$\beta'_{(1)} = e_B^*$$

for some $N, M \geq 1$.

Proof We look first at states which contribute to the second-highest degree. Let $S \in \mathcal{S}$ be any state with at least one B -resolution. S contributes

(19)
$$\langle D \mid S \rangle_\Sigma = A^{c(D)-2\beta(S)} (-A^2 - A^{-2})^{k(S)} Z^{r(S)} W^{\bar{c}(S)},$$

with

(20)
$$\deg_{\max}^A(\langle D \mid S \rangle_\Sigma) = c(D) - 2\beta(S) + 2k(S),$$

so we need only consider states such that $k(S) - \beta(S)$ is maximal.

Suppose $\beta(S) = n \geq 1$ and let $e_1, \dots, e_n$ be the edges of $\mathbb{G}_A$ that correspond to B -resolved crossings in S , ordered arbitrarily. Surgering along the edges one at a time produces a sequence of states $S_A = S_0, S_1, \dots, S_n = S$. By homological adequacy, $k(S_1) \leq k(S_0)$, but for any other surgery, $k(S_{i+1}) \leq k(S_i) + 1$. In order for $k(S)$ to be maximal, these must all be equalities, in which case $k(S) = v_A + n - 1$ and the corresponding degree in the A variable will be $c(D) + 2v_A - 2$. We will see that such states have homological rank one, thus proving (15).

We should note here, that in the binomial expansion of $(-A^2 - A^{-2})^{v_A}$ in $\langle D \mid S_A \rangle_\Sigma$ all degrees of the variable A will be of the form $c(D) + 2v_A - 4k$ for some k , so S_A will not contribute to these coefficients. We now turn our attention to $\alpha'_{(1)}$, so assume further that $\bar{c}(S) = 1$. Our goal is to understand to what types of surgeries the edges $e_1, \dots, e_n$ correspond. A surgery from S_i to S_{i+1} is one of the following three types:

- (a) A *merge*, where the total number of loops is reduced: in this case either $k(S_{i+1}) = k(S_i) - 1$ and $r(S_{i+1}) = r(S_i)$, or $k(S_{i+1}) = k(S_i)$ and $r(S_{i+1}) = r(S_i) - 1$.
- (b) A *split*, where the total number of loops is increased: in this case either $k(S_{i+1}) = k(S_i) + 1$ and $r(S_{i+1}) = r(S_i)$, or $k(S_{i+1}) = k(S_i)$ and $r(S_{i+1}) = r(S_i) + 1$.
- (c) A *single cycle smoothing* (defined in [2, Section 2]), where the total number of loops remains the same: in this case $k(S_{i+1}) = k(S_i)$ and $r(S_{i+1}) = r(S_i)$.

The key difference between the surface case and the planar case in [4] and [14] is the possibility that the kernel is preserved between states, as well as the potential for single cycle smoothings. However, as the authors of [2] point out, single cycle smoothings cannot occur for checkerboard colorable diagrams. Also, since every loop in $S_A = S_0$ is trivial, any merge would reduce the kernel. The only remaining possibility for the first surgery is a split with $k(S_1) = k(S_0)$ and $r(S_1) = r(S_0) + 1 = 1$. This means that e_1 is a self-edge for a loop l and that surgering l along e_1 produces parallel loops l_1 and l_2 in S_1 , which are both homologous to e_1 . Thus, $\bar{c}(S_1) = 1$. Note that if there is no such sequence, ie if $r(S) = 0$ for all states S adjacent to S_0 , then there are no self-edges. Hence, $\alpha'_{(1)} = e_A^* = 0$ and we are done. Otherwise, each remaining surgery in such a sequence must be a split, where $k(S_{i+1}) = k(S_i) + 1$, $r(S_{i+1}) = r(S_i)$ and $\bar{c}(S_{i+1}) = \bar{c}(S_i)$.

Next, we will show by induction that the remaining edges are in the same equivalence class as e_1 . Suppose that for all $j \leq i$, e_j is adjacent to e_1 and $i_*(H_1(S_j)) = i_*(H_1(S_1)) = \langle e_1 \rangle \subset H_1(\Sigma)$. Let l' be the loop in S_i corresponding to the vertex incident to e_{i+1} . There are two possibilities.

Case 1 Suppose l' is nullhomologous. Since $r(S_{i+1}) = r(S_i) = 1$ and $\bar{c}(S_{i+1}) = \bar{c}(S_i)$, surgering l' along e_{i+1} must produce two homologically trivial loops, as opposed to two parallel essential loops. Note that this would violate homological adequacy if l' had been a loop originating in S_0 , since changing the order of edge surgeries so that e_{i+1} is done first would increase the kernel. Thus, l' must have been created by a previous surgery, so e_{i+1} must have been adjacent to e_j for some $j \leq i$. Therefore, $i_*(H_1(S_{i+1})) = \langle e_1 \rangle \subset H_1(\Sigma)$ and the inductive step holds.

Case 2 Suppose $[l'] = [e_1]$. Since $\bar{c}(S_i) = \bar{c}(S_1) = 1$, S_i has two essential loops, one of which is l' . Thus, e_{i+1} must have been adjacent to e_j for some $j \leq i$. Furthermore, the other essential loop homologous to e_1 is unaffected by the surgery. Since $r(S_{i+1}) = r(S_i) = 1$, surgering l' along e_{i+1} must produce one nullhomologous loop and one loop homologous to e_1 . This gives $i_*(H_1(S_{i+1})) = i_*(H_1(S_i)) = \langle e_1 \rangle$ and thus, the inductive step holds.

Therefore, $i_*(H_1(S)) = \langle e_1 \rangle \subset H_1(\Sigma)$ and all the edges are adjacent to e_1 . Changing the order of edge surgeries so that e_i is done first will still yield the same state S , implying $[e_i] = [e_1]$ for all i . This means precisely that $e_i \sim^* e_1$ for all i .

Let S be a state whose B -resolutions occur within a single equivalence class of self-edges. Let e_i^A be the i^{th} of these classes, and let k_i be the number of self-edges in that class. The total contribution of these states is

$$\begin{aligned}
 & \sum_{i=1}^{e_A^*} \sum_{j=1}^{k_i} \binom{k_i}{j} A^{c(D)-2j} (-A^2 - A^{-2})^{v_A+j-1} ZW \\
 &= \sum_{i=1}^{e_A^*} \sum_{j=1}^{k_i} \binom{k_i}{j} A^{c(D)-2j} (-1)^{v_A+j-1} (A^2 + A^{-2})^{v_A+j-1} ZW \\
 &= \sum_{i=1}^{e_A^*} \sum_{j=1}^{k_i} \binom{k_i}{j} A^{c(D)-2j} (-1)^{v_A+j-1} (A^{2v_A+2j-2} + \text{lower-order terms}) ZW \\
 &= \sum_{i=1}^{e_A^*} \sum_{j=1}^{k_i} \binom{k_i}{j} (-1)^{v_A+j-1} A^{c(D)+2v_A-2} ZW + \text{lower-order terms} \\
 &= \sum_{i=1}^{e_A^*} (-1)^{v_A} A^{c(D)+2v_A-2} ZW + \text{lower-order terms} \\
 &= (-1)^{v_A} e_A^* A^{c(D)+2v_A-2} ZW + \text{lower-order terms},
 \end{aligned}$$

proving (16). The second-to-last equality follows from the fact that $\sum_{j=0}^k (-1)^j \binom{k}{j} = 0$.

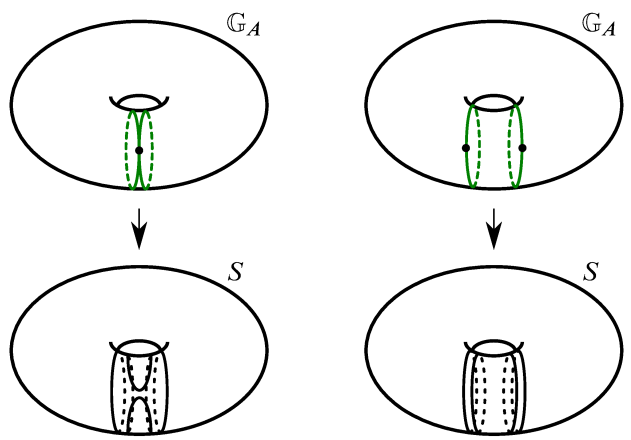


Figure 6: Left: surgering along self-edges in the same equivalence class. Right: surgering along merely homologous self-edges.

An analogous argument holds for states whose A -resolutions lie in the same equivalence class of self edges and the coefficient $\beta'_{(1)}$.

In the calculation above, note that in order for $\bar{c}(S)$ to equal one, the self-edges in S must be adjacent. In general, $\bar{c}(S)$ counts the number of vertices which house edges that are homologous, but not equivalent under $\sim^*$. This is illustrated in Figure 6. The numbers N and M in the statement of the proposition are the maximum numbers of these vertices, taken over the set of all homology classes of self-edges. $\square$

Since it is possible for a diagram to have no self-edges, we cannot establish a meaningful lower bound for any combination of $\alpha'_{(1)}$ and $\beta'_{(1)}$. It is then necessary to look at more coefficients.

Definition 12 Let e_1 and e_2 be adjacent self-edges in either $\mathbb{G}_A$ or $\mathbb{G}_B$ whose homology classes have intersection number $\text{int}([e_1], [e_2]) = \pm 1$. We call the unordered double (e_1, e_2) a *transverse pair* of self-edges.

If (e_1, e_2) is a transverse pair, it is possible that there exists a third self-edge e_3 such that $[e_3] = \pm[e_1] \pm [e_2]$. In this case, we call the unordered triple (e_1, e_2, e_3) a *self-triangle*.

The equivalence relation $\sim^*$ on self-edges also induces equivalence relations on transverse pairs and self-triangles. Let $\mathfrak{h}_A^*$ and $\mathfrak{h}_B^*$ denote the number of equivalence classes of transverse pairs of self-edges, and let τ_A^* and τ_B^* denote the number of equivalence classes of self-triangles in $\mathbb{G}_A$ and $\mathbb{G}_B$, respectively. See Figure 7 for an example of a transverse pair and a self-triangle.

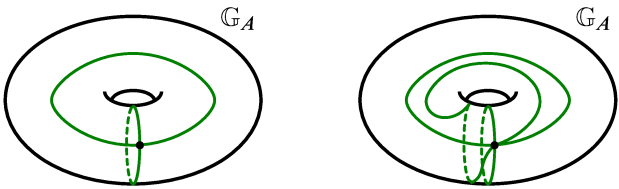


Figure 7: A transverse pair, left, and a self-triangle, right.

Proposition 13 *Let D be a reduced alternating projection of a link L onto a closed, orientable surface Σ . Then the terms of $\langle D \rangle_\Sigma$ having the third-highest degree in the variable A are of the form*

(21)
$$(-1)^{v_A}(\alpha''_{(0)} + \alpha''_{(2)}Z^2)A^{c(D)+2v_A-4},$$

where

(22)
$$\alpha''_{(0)} = v_A - \tilde{e}_A + \mathfrak{h}_A^* - \tau_A^* \quad \text{and} \quad \alpha''_{(2)} = \binom{e_A^*}{2} - \mathfrak{h}_A^*,$$

and the terms having the third-lowest degree in the variable A are of the form

(23)
$$(-1)^{v_B}(\beta''_{(0)} + \beta''_{(2)}Z^2)A^{-c(D)-2v_B+4},$$

where

(24)
$$\beta''_{(0)} = v_B - \tilde{e}_B + \mathfrak{h}_B^* - \tau_B^* \quad \text{and} \quad \beta''_{(2)} = \binom{e_B^*}{2} - \mathfrak{h}_B^*.$$

Note that these coefficients contain elements of both second and third coefficients from the planar case in [4; 14]. The reason for this is the possibility for the kernel to be preserved at the first surgery, whereas in the planar case the number of loops can only decrease at the first surgery. For the first surgery, only self-edges preserve the kernel, which is why the terms for self-triangles and pairs of self-edges appear here, while corresponding terms for simple edges do not.

Proof By (20), the third-highest possible degree of $\langle D \rangle_\Sigma$ in the variable A is $c(D) + 2v_A - 4$. Notice that this time, S_A will make a contribution. We get

$$\begin{aligned} \langle D \mid S_A \rangle_\Sigma &= A^{c(D)}(-A^2 - A^{-2})^{v_A} \\ &= (-1)^{v_A} A^{c(D)}(\text{higher-order terms} + v_A A^{2v_A-4} + \text{lower-order terms}) \\ &= \text{higher-order terms} + (-1)^{v_A} v_A A^{c(D)+2v_A-4} + \text{lower-order terms}. \end{aligned}$$

On the other hand, states of the form described in the previous proposition will not contribute anything (the degrees of A in the binomial expansion are of the wrong form: $c(D) + 2v_A - 2 - 4k$ for some k).

Now suppose that S is a state with $\deg_{\max}(\langle D \mid S \rangle_{\Sigma}) = c(D) + 2v_A - 4$. As in the previous proof, let $e_1, \dots, e_n$ be a sequence of edges of $\mathbb{G}_A$, and $S_A = S_0, S_1, \dots, S_n = S$ the sequence of states obtained by surgering along each edge in order. As before, we seek to determine which types of surgeries could have yielded S . By homological adequacy, $k(S_1) \leq k(S_0)$. There are two cases, depending on whether the first surgery reduces the kernel or preserves the kernel.

Case 1 If $k(S_1) = k(S_0) - 1$, then e_1 is a simple edge, whose surgery merges two trivial loops l_1 and l_2 in S_0 into a single trivial loop l' . Then $r(S_1) = 0$. In order for the remaining surgeries to yield the desired degree of A , they must all be splits which increase the kernel. In particular, e_2 must split a trivial loop in S_1 into two nullhomologous loops. By homological adequacy, this initial loop must be l' and furthermore, in $\mathbb{G}_A$, e_2 must have been a simple edge connecting the same two vertices as e_1 . It follows that e_1 and e_2 must have formed a nullhomologous curve. This means precisely that $e_1 \sim e_2$. By reordering the sequence of edges we see that $e_i \sim e_1$ for all $2 \leq i \leq n$. The total contribution of states whose B -resolutions occur within a single equivalence class of simple edges is

$$\sum_{i=1}^{\tilde{e}_A} \sum_{j=1}^{k_i} \binom{k_i}{j} A^{c(D)-2j} (-A^2 - A^{-2})^{v_A+j-2} = (-1)^{v_A-1} \tilde{e}_A A^{c(D)+2v_A-4} + \text{lower-order terms},$$

where the equality follows from the same steps used in the proof of [Proposition 11](#).

Case 2 If $k(S_1) = k(S_0)$ then e_1 was a self-edge as in the previous proposition, and $r(S_1) = 1$. In order for the state S to contribute to the desired degree of A , there must be a j with $1 < j \leq n$ such that $k(S_j) = k(S_{j-1})$ and for all $i \neq 1, j$ we have $k(S_i) = k(S_{i-1}) + 1$, while the homological rank is preserved. Note that the j^{th} surgery could either be a merge resulting in $r(S_n) = r(S_j) = 0$ or a split resulting in $r(S_n) = r(S_j) = 2$. First, let us deal with the scenario that the j^{th} surgery is a merge.

Case 2a We first claim that in such a scenario, all of the edges had to be adjacent self-edges. Indeed, the proof of the previous proposition implies that e_i for $1 \leq i < j$ are all homologous and that $H_1(S_i) = \langle e_1 \rangle$ (but we cannot yet conclude they are adjacent). However, since $r(S_j)$ must be zero, surgery along e_j must be a merge which reduces the homological rank. This is only possible if S_{j-1} contains exactly two essential loops l_1 and l_2 , making $\bar{c}(S_{j-1}) = 1$. We saw that this can only happen if the e_i are adjacent,

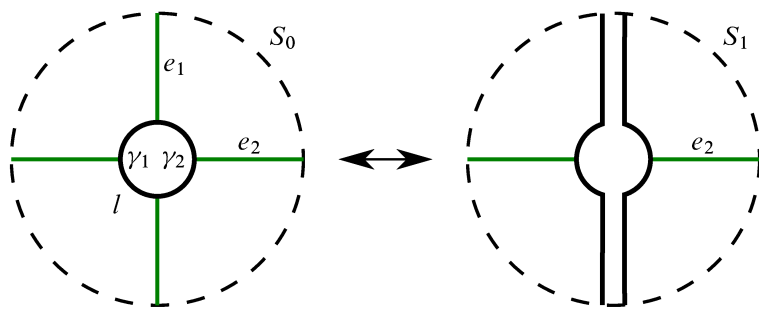


Figure 8: A state in case 2a.

and therefore belong to the same equivalence class for $1 \leq i < j$. Surgery along e_j merges l_1 and l_2 into a single nullhomologous loop l' , so e_j must have been adjacent to e_i for some $i < j$. At this point, all loops in S_j are nullhomologous, so the remaining surgeries must split off more nullhomologous loops. By a similar homological adequacy argument as in the proof of the previous proposition, e_i for $j < i \leq n$ also must be adjacent to the previous edges.

Now that we know all edges in such a state must be incident to a single vertex v in $\mathbb{G}$, it will suffice to look at a local neighborhood of the loop l in S_A corresponding to v . Without loss of generality, assume that $j = 2$. As seen in Figure 8, e_1 splits l into two arcs γ_1 and γ_2 , which correspond to the loops l_1 and l_2 . Since surgery along e_2 merges l_1 and l_2 , it must connect γ_1 and γ_2 outside of the disk region bounded by l . (This can only happen if $g(\Sigma) > 0$.) In this way, we see that e_1 and e_2 are transverse.

Now, let us address the edges e_i , $2 < i \leq n$, whose surgeries increase the kernel. As seen in Figure 9, e_1 and e_2 divide l into four arcs γ_k , $k = \text{I, II, III, IV}$. The loop $l' \in S_2$ is a boundary component of a regular neighborhood of $l \cup e_1 \cup e_2$, which may be viewed as the union of corresponding arcs γ'_k and two parallel copies of each of the edges e_1 and e_2 . Consider e_3 . By homological adequacy, the endpoints of e_3 must lie in different arcs. This leaves us with two possibilities.

Case 2a(i) Suppose that e_3 connects two adjacent arcs. Since $k(S_3) = k(S_2) + 1$, surgery along e_3 splits off a nullhomologous loop from l' . In other words, e_3 along with an arc in l' forms a nullhomologous loop, independent of the choice of arc. As seen in Figure 9, this arc may be taken to be one of the edges e_i , $i = 1, 2$. Thus, $e_3 \sim^* e_i$. If all remaining e_i , $3 \leq i \leq n$, behave this way, then S is a state whose B -resolutions appear within a single equivalence class of the transverse pair (e_1, e_2) , which we denote by

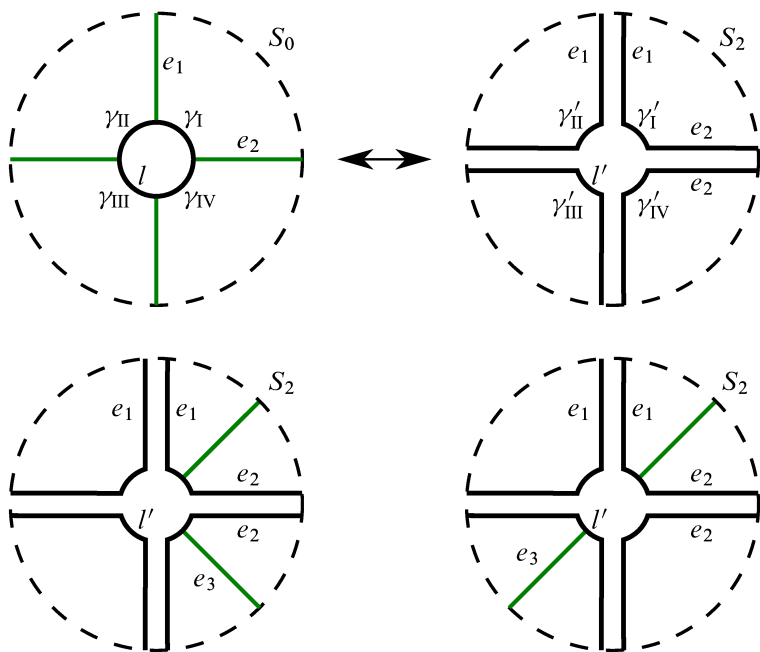


Figure 9: Bottom left, a state in case 2a(i). Bottom right, a state in case 2a(ii).

$(e_1, e_2)^*$. The total contribution of such states is

$$\sum_{(e_1, e_2)^*} \sum_{i=1}^{k_{e_1}} \sum_{j=1}^{k_{e_2}} \binom{k_{e_1}}{i} \binom{k_{e_2}}{j} A^{c(D)-2(i+j)} (-A^2 - A^{-2})^{v_A+i+j-2} \\ = (-1)^{v_A} \cap_A^* A^{c(D)+2v_A-4} + \text{lower-order terms},$$

where k_{e_1} and k_{e_2} denote the number of edges in the equivalence classes of e_1 and e_2 , respectively.

Case 2a(ii) Suppose that e_3 connects two opposite arcs. This time, the arc in l' is a combination of e_1 and e_2 . This means precisely $[e_3] = \pm[e_1] \pm [e_2] \in H_1(\Sigma)$. Thus (e_1, e_2, e_3) is a self-triangle. Note that if such an e_3 exists, no edge can connect the other pair of opposite γ_k in l . Otherwise, if there did exist such an edge e' , consider the intersection number $\text{int}(e', e_3) = \text{int}(e', e_1) + \text{int}(e', e_2)$. Since $\mathbb{G}_A$ is embedded in Σ , we would have $\text{int}(e', e_1) = \text{int}(e', e_2) = \text{int}(e', e_1) + \text{int}(e', e_2) = \pm 1$, which is impossible. Therefore, for $3 < i \leq n$, e_i falls under case 2a(i) or 2a(ii) and as such belongs to the equivalence class of either e_1 , e_2 or e_3 . Then S is a state whose B -resolutions appear within a single equivalence class of the self-triangle (e_1, e_2, e_3) ,

which we denote by $(e_1, e_2, e_3)^*$. The total contribution of such states is

$$\begin{aligned} \sum_{(e_1, e_2, e_3)^*} \sum_{i=1}^{k_{e_1}} \sum_{j=1}^{k_{e_2}} \sum_{k=1}^{k_{e_3}} \binom{k_{e_1}}{i} \binom{k_{e_2}}{j} \binom{k_{e_3}}{k} A^{c(D)-2(i+j+k)} (-A^2 - A^{-2})^{v_A+i+j+k-2} \\ = (-1)^{v_A-1} \tau_A^* A^{c(D)+2v_A-4} + \text{lower-order terms.} \end{aligned}$$

Case 2b Lastly, we address the other scenario described at the beginning of case 2, where the j^{th} surgery is a split which increases the homological rank rather than the kernel. The only possibility left is that e_j is a self-edge which is neither transverse nor homologous to e_1 . Once again, all other edges must belong to the equivalence class of either e_1 or e_j . Letting e_1^* and e_2^* denote the two classes, the total contribution of these states is

$$\begin{aligned} \sum_{\substack{e_1^* \neq e_2^* \\ \text{not transverse}}} \sum_{i=1}^{k_{e_1}} \sum_{j=1}^{k_{e_2}} \binom{k_{e_1}}{i} \binom{k_{e_2}}{j} A^{c(D)-2(i+j)} (-A^2 - A^{-2})^{v_A+i+j-2} Z^2 \\ = (-1)^{v_A} \left(\binom{e_A^*}{2} - \mathfrak{h}_A^* \right) A^{c(D)+2v_A-4} Z^2 + \text{lower-order terms.} \end{aligned}$$

Combining the contributions from S_A , cases 1 and 2a, we get $\alpha''_{(0)} = v_A - \tilde{e}_A + \mathfrak{h}_A^* - \tau_A^*$, as claimed. Likewise, case 2b gives $\alpha''_{(2)} = \binom{e_A^*}{2} - \mathfrak{h}_A^*$. An analogous argument holds for $\beta''_{(0)}$ and $\beta''_{(2)}$. □

4 Bounds on the twist number

The goal of this section is to prove [Theorem 1](#). Let $e_A = e_A^* + \tilde{e}_A$ and $e_B = e_B^* + \tilde{e}_B$. We consider the quantity

$$\begin{aligned} (25) \quad \star &= \alpha'_{(1)} + \beta'_{(1)} - \alpha''_{(0)} - \beta''_{(0)} + 2 \\ &= e_A + e_B - v_A - v_B + 2 + \tau_A^* + \tau_B^* - \mathfrak{h}_A^* - \mathfrak{h}_B^* . \end{aligned}$$

We now seek to bound $\star$ in terms of the twist number of the diagram. We follow the method used for the planar case in [\[7\]](#). As transverse edge pairs (and by extension self-triangles) do not arise in the planar case, we deal with these terms first.

Proposition 14 Suppose Σ has genus g . Then

$$(26) \quad -2g \leq \tau_A^* + \tau_B^* - \mathfrak{h}_A^* - \mathfrak{h}_B^* \leq 0.$$

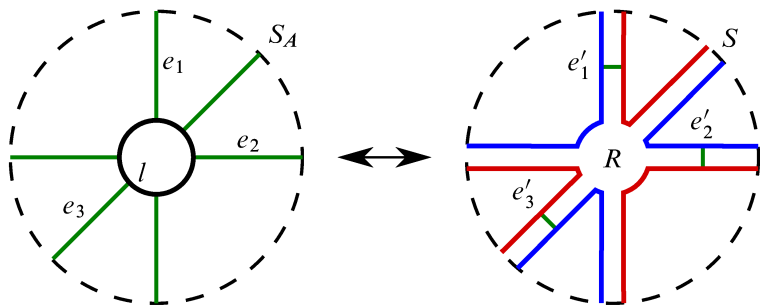


Figure 10: The result of surgering along a self-triangle. The red and blue loops bound separate punctured surfaces.

Proof The second inequality is immediate from the fact that every self-triangle consists of three self-edges which are all pairwise transverse. Furthermore, no pair can be part of any other self-triangle since the third self-edge is already determined by the pair.

For the first inequality, we will show that $\tau_A^* + \tau_B^* \leq g$. Suppose without loss of generality that $\mathbb{G}_A$ contains a self-triangle class $(e_1, e_2, e_3)^*$ at a vertex corresponding to the loop l . Figure 10 shows the result of surgering l along e_1, e_2 and e_3 to obtain the state S , where e'_1, e'_2 and e'_3 are the surgery arcs dual to these edges. In place of the self-triangle, there is a region R which is a punctured torus. Note that $\Sigma \setminus R$ then consists of two punctured surfaces with total genus $g - 1$. Iterating this argument, we immediately see that $\mathbb{G}_A$ can have at most g self-triangle classes.

More is true, however. Notice that e'_1, e'_2 and e'_3 are the only surgery arcs joining these punctured surfaces. Thus, when we complete the remaining surgeries to obtain S_B , the arcs e'_1, e'_2 and e'_3 become identified with simple edges of $\mathbb{G}_B$. By reordering, this will also be the case for the other edges making up $(e_1, e_2, e_3)^*$. Therefore, the remaining surgeries induce an isotopy on R taking it to a punctured torus in Σ which contains no self-edges of $\mathbb{G}_B$. We have shown that the presence of a self-triangle class in either $\mathbb{G}_A$ or $\mathbb{G}_B$ obstructs the existence of a potential set of transverse self-edge classes (and by extension a self-triangle) in the other. Iterating this argument, we have $\tau_A^* + \tau_B^* \leq g$, as claimed.

Recall that across both state graphs, there are a total of $3(\tau_A^* + \tau_B^*)$ classes of transverse pairs coming from self-triangles. By an analogous argument, in $\mathbb{G}_A$ there are at most $g - (\tau_A^* + \tau_B^*)$ pairs which are not part of any self-triangle. By contrast, these do not obstruct a transverse pair in $\mathbb{G}_B$: for an example, see the left-hand graph depicted in Figure 7 and note that it is isotopic to its dual.

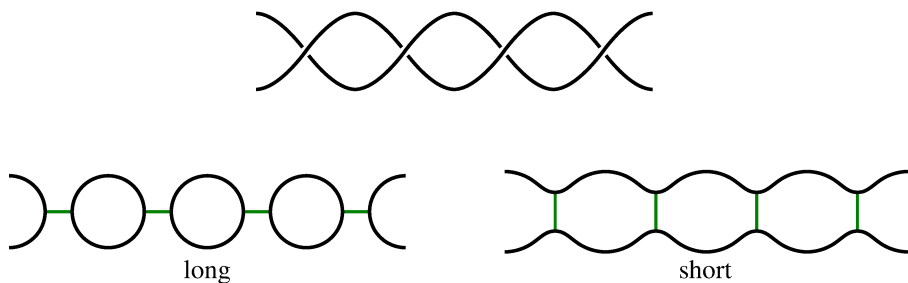


Figure 11: Left, the long resolution of a twist region. Right, the short resolution of a twist region.

Therefore, the same bound holds for $\mathbb{G}_B$. By adding across both state graphs, we have

(27)
$$\mathfrak{h}_A^* + \mathfrak{h}_B^* \leq 3(\tau_A^* + \tau_B^*) + 2g - 2(\tau_A^* + \tau_B^*) = 2g + \tau_A^* + \tau_B^*.$$

This implies

(28)
$$\tau_A^* + \tau_B^* - \mathfrak{h}_A^* - \mathfrak{h}_B^* \geq -2g,$$

as desired. □

In [7], the authors bound the quantity analogous to $\star$ from above by recategorizing the vertices and edges based on how twist regions in D reveal themselves in two different ways in each of the state graphs. Referring to Figure 11, we call these “long” and “short” resolutions. Edges are also called “long” and “short” if they come from that type of resolution. Notice that short edges coming from the same twist region are all in the same equivalence class. This is the exact same property exhibited by the “reduced edges” in [7], which leads to Proposition 4.6, the upper bound in the planar case. As there is no difference in the definition of a twist region between the planar and higher-genus cases, the proof goes through without changes as long as one replaces “reduced edges” with “equivalence classes of edges”.

Proposition 15 [7, Proposition 4.6] *Let D be a reduced alternating diagram on a closed, orientable surface Σ . Then*

(29)
$$e_A + e_B - v_A - v_B + 2 \leq 2 \operatorname{tw}(D).$$

For the lower bound, only a slight modification of the methods used in the planar case is required. Let us first summarize the argument, which appears in Section 4.3 of [7].

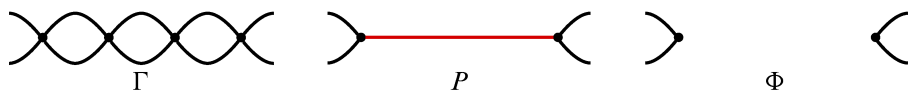


Figure 12: The graphs Γ , P and Φ .

The authors first assume that there are no single-crossing twist regions, so that each region contains at least one bigon. Recall that D may be viewed as a 4-valent graph Γ . They obtain a new 3-valent graph P by collapsing strings of bigons in Γ to red edges. A 2-valent graph, Φ may be obtained by deleting the red edges. Figure 12 depicts these graphs.

The authors call the regions of P “provinces” and the regions of Φ “countries”. In [7], the graphs are all constructed on the Turaev surface of D , but since we only deal with alternating links, the graphs P and Φ embed naturally on Σ . Note that P inherits the cellular embedding of D , so all provinces are disks, while countries may be nontrivial regions. For each country, its provinces correspond to vertices in either $\mathbb{G}_A$ or $\mathbb{G}_B$ which are incident to more than two edges (the authors call these “ n -gon” vertices). The red edges which divide countries into provinces are dual to sets of parallel short edges which connect the two provinces. Recall that if short edges are dual to the same red edge, then they are in the same equivalence class. The converse is not true, however; there may exist short edges dual to distinct red edges which will become identified. The key to the authors’ argument is to realize that these different short edges must lie in the same country. So, to find a lower bound the authors investigate the number of short edges in each country that will survive when we pass to equivalence classes. We now prove a version of their result [7, Lemma 4.8].

Proposition 16 *Let D be a reduced alternating diagram on a closed, orientable surface Σ such that every twist region of D has at least two crossings. Let N be a country of D , and let $e_{\text{short}}(N)$ denote the number of equivalence classes of short edges contained in N and $e_{\partial}^*(N)$ the number of self-edge equivalence classes in N which are homologous to a boundary component of N . Let $\text{tw}(N)$ denote the number of twist regions in N and let $|\partial N|$ denote the number of loops in Φ which bound N . Then*

(30)
$$e_{\text{short}}(N) \geq \text{tw}(N) + 1 - |\partial N| + e_{\partial}^*(N).$$

Proof Consider the dual graph of N , defined to be a connected ribbon graph G whose vertices are the provinces of N , edges are dual to the red edges in N , and boundary components are the loops in Φ which bound N . In [7], the authors obtain a lower bound

by finding a spanning tree of G , or alternatively by cutting N along red edges until a disk remains. In our case, it suffices to do the same with a spanning quasi-tree of G .

A *quasi-tree* is a ribbon graph with exactly one boundary component. We can obtain a spanning quasi-tree by removing edges as long as they reduce the number of boundary components of the graph, which corresponds to cutting N along dual red edges. Thus, it suffices to cut N along $|\partial N| - 1$ red edges.

Note that if G is a quasi-tree, then N cannot contain any short edges which are in the same equivalence class, yet are dual to different red edges. Otherwise, these short edges form a nullhomologous curve separating N , forcing there to be at least two boundary components. Also note that there can be no self-edges which are homologous to a boundary component of N , for the exact same reason. Recall that a cellularly embedded diagram must be connected, so that all self-edges are short.

Thus, any classes we are left with correspond to a unique one of the remaining $\text{tw}(N) + 1 - |\partial N|$ red edges left in the country. Adding back in the self-edge classes which were removed, we obtain the desired inequality. $\square$

The final step in [7] is to find a bound on the number of countries. A small but important modification is needed here. In the planar case, the authors show that every component of Φ contains at least three vertices. As we will see, this does not hold in the surface case, and this issue must be resolved in order to avoid canceling the term for the twist number in the lower bound.

Proposition 17 *Let D be a reduced alternating diagram on a closed, orientable surface Σ such that every twist region of D has at least two crossings. Let e_{short} denote the total number of equivalence classes of short edges in $\mathbb{G}_A$ and $\mathbb{G}_B$. Then*

$$(31) \quad e_{\text{short}} \geq \frac{1}{3} \text{tw}(D) + 1 - g(\Sigma).$$

Proof Let ϕ be a component of Φ , which may be thought of as a simple closed curve in Σ . Since D is reduced, ϕ must contain at least two vertices, or else it would bound a monogon region corresponding to a nugatory crossing. We call ϕ “bad” if it has exactly two vertices and “good” if it has at least three. Let $|\Phi|_{\text{bad}}$ denote the number of bad curves and $|\Phi|_{\text{good}}$ the number of good curves, and $|\Phi|$ the total number of curves.

Suppose ϕ is bad. Then because P is cellularly embedded, the two red edges incident to the vertices of ϕ must lie on opposite sides of ϕ . Otherwise, ϕ would bound a bigon

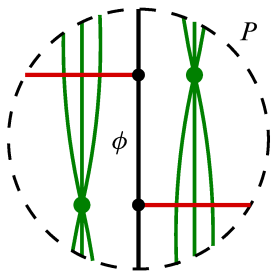


Figure 13: A bad loop. The boundary-homologous self-edges are drawn in green.

region, and all bigons were collapsed in the construction of P . Also note that Φ inherits the checkerboard coloring of D , so we see that these are indeed distinct red edges, which lie in separate countries. By following along ϕ , we see that on each side of ϕ the red edge forms a border between a province and itself. Thus, the red edge is dual to a self-edge which represents the same homology class as ϕ in $H_1(\Sigma)$. This is illustrated in Figure 13. By homological adequacy, we have that this self-edge along with ϕ are homologically essential. In a country N , there can be at most two essential boundary components which are homologous. To see this, recall that these two homologous curves separate Σ , so a third one would separate N . Therefore, for each bad curve there are two boundary-homologous classes of self-edges in the neighboring countries, and there is at worst a 2:1 correspondence between bad curves and a given self-edge class. Therefore, $e_\partial^* \geq |\Phi|_{\text{bad}}$, where $e_\partial^* = \sum_N e_\partial^*(N)$.

The total number of vertices in P is at least three times the number of good edges. Since every red edge has two vertices, this gives us that $\text{tw}(D) \geq \frac{3}{2}|\Phi|_{\text{good}}$. Let $n(D)$ denote the number of countries in Φ . By summing (30) over all countries, we get

$$\begin{aligned} e_{\text{short}} &\geq \text{tw}(D) + n(D) - 2|\Phi| + e_\partial^* \\ &\geq \text{tw}(D) + n(D) - 2|\Phi| + |\Phi|_{\text{bad}} \\ &= \text{tw}(D) + n(D) - |\Phi| - |\Phi|_{\text{good}} \\ &\geq \frac{1}{3} \text{tw}(D) + n(D) - |\Phi| \\ &\geq \frac{1}{3} \text{tw}(D) + 1 - g(\Sigma). \end{aligned}$$

The last inequality follows from the fact that if we cut Σ along the simple closed curves in Φ , at most g of these cuts are nonseparating. □

We combine these results into a proposition mirroring [7, Theorem 4.10].

Proposition 18 *Let D be a reduced alternating diagram on a closed, orientable surface Σ such that every twist region of D has at least three crossings. Then*

(32)
$$\star \geq \frac{1}{3} \operatorname{tw}(D) + 1 - g(\Sigma).$$

Proof Like the authors in [7], we define v_{bigon} as the total number of bigon vertices in $\mathbb{G}_A$ and $\mathbb{G}_B$. Bigon vertices are incident to exactly two edges and correspond to the vertices in the long resolution of a twist region. We define $v_{n\text{-gon}}$ to be the total number of n -gon vertices. We have already seen that e_{short} counts the total number of equivalence classes of short-edges, which are edges coming from a short resolution. Likewise, we define e_{long} to be the total number of equivalence classes of edges coming from a long resolution. We remark that an edge is long if and only if it is incident to at least one bigon vertex. (When every twist region has at least three crossings, no two long edges can be in the same equivalence class.) Every vertex and edge across both state graphs falls into exactly one of these categories, so we can regroup $e_A + e_B - v_A - v_B = e_{\text{short}} + e_{\text{long}} - v_{\text{bigon}} - v_{n\text{-gon}}$.

The long resolution of a twist region with c crossings has exactly $c - 1$ bigon vertices and c long edges. By summing over every twist region, we see that $v_{\text{bigon}} = c(D) - t(D)$ and $e_{\text{long}} = c(D)$. Recall that the n -gon vertices become provinces in P , and the twist regions become red edges. Therefore, by summing over countries, we can compute $\chi(\Sigma) = v_{n\text{-gon}} - \operatorname{tw}(D)$. Putting this all together,

$$\begin{aligned} \star &= e_A + e_B - v_A - v_B + 2 + \tau_A^* + \tau_B^* - \hbar_A^* - \hbar_B^* \\ &= e_{\text{short}} + e_{\text{long}} - v_{\text{bigon}} - v_{n\text{-gon}} + 2 + \tau_A^* + \tau_B^* - \hbar_A^* - \hbar_B^* \\ &= e_{\text{short}} + c(D) - (c(D) - \operatorname{tw}(D)) - v_{n\text{-gon}} + 2 + \tau_A^* + \tau_B^* - \hbar_A^* - \hbar_B^* \\ &= e_{\text{short}} - \chi(\Sigma) + 2 + \tau_A^* + \tau_B^* - \hbar_A^* - \hbar_B^* \\ &\geq e_{\text{short}} - \chi(\Sigma) + 2 - 2g(\Sigma) = e_{\text{short}} \\ &\geq \frac{1}{3} \operatorname{tw}(D) + 1 - g(\Sigma). \end{aligned}$$

□

Combining (29) and (32), we obtain a proof of Theorem 1.

In [9], the authors bound the hyperbolic volume of the complement of a “weakly generalized” alternating link in terms of the twist number. A weakly generalized alternating link is a link in a compact, irreducible, orientable 3-manifold with a projection onto a surface Σ , with some additional properties that guarantee the complement admits a complete hyperbolic structure. They also require the diagram to be twist-reduced. See [9] for the specific definitions of these terms.

Theorem 19 [9, Theorem 1.4] *Let Σ be a closed orientable surface of genus at least one, and let L be a link that admits a twist-reduced weakly generalized cellularly embedded alternating projection D onto $\Sigma \times \{0\}$ in $Y = \Sigma \times [-1, 1]$. Then the interior of $Y \setminus L$ admits a hyperbolic structure. If Σ is a torus, then we have*

$$(33) \quad \frac{1}{2} v_{\text{oct}} \cdot \text{tw}(D) \leq \text{vol}(Y \setminus L) < 10 v_{\text{tet}} \cdot \text{tw}(D),$$

where v_{tet} is the volume of a regular ideal tetrahedron, and v_{oct} is the volume of a regular ideal octahedron.

If Σ has genus at least two,

$$(34) \quad \frac{1}{2} v_{\text{oct}} \cdot (\text{tw}(D) - 3\chi(\Sigma)) \leq \text{vol}(Y \setminus L) < 6 v_{\text{oct}} \cdot \text{tw}(D).$$

Direct substitution of (1) into these volume bounds yields [Corollary 2](#).

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