# The AD$^+$ Conjecture and the Continuum Hypothesis [y,z]

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July 19, 2022

Abstract

We show that Woodin's AD$^+$ Conjecture follows from various hypotheses extending the Continuum Hypothesis (CH). These results complement Woodin's original result that the AD$^+$ Conjecture follows from MM(c).

## 1 Introduction

This paper concerns Woodin's AD$^+$ Conjecture, [Woo10, Definition 10.7.6]. In [Woo10], Woodin isolates the AD$^+$ Conjecture and shows that it is a consequence of MM(c), the Martin's Maximum for partial orders of size at most the continuum. The conjecture's original motivation was based on speculations from the Inner Model Program and has many important consequences, e.g. the definability of $\Omega$-logic. See [Woo10] for a more detailed discussion. The main results of the paper, Theorems 1.2 and 1.3, complement Woodin's result and show that the AD$^+$ Conjecture is a consequence of theories extending the Continuum Hypothesis (CH).[1]

We identify elements of the Baire space $\omega^\omega$ with reals. Throughout the paper, by a "set of reals $A$", we mean $A \subseteq \omega^\omega$. Given a cardinal $\kappa$, we say $T \subseteq \bigcup_{n<\omega} \omega^n \times \kappa^n$ is a tree on $\omega \times \kappa$ (or just on $\kappa$) if $T$ is closed under initial segments. Given a tree $T$ on $\omega \times \kappa$, we let $[T]$ be the set of its branches, i.e., $b \in [T]$ if $b \in \omega^\omega \times \kappa^\omega$ and

2000 Mathematics Subject Classifications: 03E15, 03E45, 03E60.

[y]Keywords: Divergent models, inner model theory, descriptive set theory.

[z]The author's research was partially supported by the NSF Career Award DMS-1945592.

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[1]It is well-known that MM(c) implies the failure of CH.

letting $b = (b_0, b_1)$, for each $n \in \omega$, $(b_0 \restriction n, b_1 \restriction n) \in T$. We then let $p[T] = \{x \in \omega^\omega : \exists f((x, f) \in [T])\}$. A set $A$ is $\kappa$-Suslin if $A = p[T]$ for some tree on $\omega \times \kappa$. A set $A$ is Suslin if it is $\kappa$-Suslin for some $\kappa$; $A$ is co-Suslin if its complement $\mathbb{R} \setminus A$ is Suslin. A set $A$ is Suslin, co-Suslin if both $A$ and its complement are Suslin. A cardinal $\kappa$ is a Suslin cardinal if there is a set of reals $A$ such that $A$ is $\kappa$-Suslin but $A$ is not $\lambda$-Suslin for any $\lambda < \kappa$. Suslin cardinals play an important role in the study of models of determinacy (see for example, various articles from the Cabal Volumes: [KMM83], [KMS88], [KLS08], [KLS12], [KLS16], [KMM81], [KM78]).

A set of reals $A$ is $\kappa$-universally Baire if there are trees $T, U$ on $\omega \times \lambda$ for some $\lambda$ such that $A = p[T] = \mathbb{R} \setminus p[U]$ and whenever $g$ is $\kappa$-generic (i.e. $g$ is $V$-generic for some forcing $\mathbb{P} \in V$ such that $|\mathbb{P}| \leq \kappa$), in $V[g]$, $p[T] = \mathbb{R} \setminus p[U]$. We write $A^g$ for $p[T]^{V[g]}$; this is the canonical interpretation of $A$ in $V[g]$.[2] We define $<\kappa$-universally Baire in an obvious way.

**Definition 1.1** ($\mathsf{AD}^+$ Conjecture, [Woo21]) Suppose $A_0, A_1 \in \wp(\mathbb{R})$ are such that $L(A_i, \mathbb{R}) \vDash \mathsf{AD}^+$ for $i \in \{0, 1\}$. Let $\Gamma_i$ be the Suslin co-Suslin sets of $L(A_i, \mathbb{R})$. Suppose that for each $B \in \Gamma_0 \cup \Gamma_1$, $B$ is $\omega_1$-universally Baire. Then

$$L(\Gamma_0, \Gamma_1, \mathbb{R}) \vDash \mathsf{AD}^+.$$

It is consistent (relative to large cardinals) that there are divergent models of $\mathsf{AD}^+$, i.e. there are $A_0, A_1 \in \wp(\mathbb{R})$ such that $L(A_i, \mathbb{R}) \vDash \mathsf{AD}^+$ for $i \in \{0, 1\}$ but $A_0 \notin L(A_1, \mathbb{R})$ and $A_1 \notin L(A_0, \mathbb{R})$. This is a theorem of Woodin (cf. [Far]). That the hypothesis of the $\mathsf{AD}^+$ Conjecture is necessary follows from very deep analysis of divergent models of $\mathsf{AD}^+$ and $\mathbb{P}_{max}$-extensions of models of strong $\mathsf{AD}^+$-theories (like "$\mathsf{AD}_\mathbb{R} + \Theta$ is regular").[3] It is beyond the scope of this paper to discuss this any further, but see [Woo21] for related results and discussions.

Woodin, in [Woo21], has shown that $\mathsf{MM}(\mathfrak{c})$, the Martin's Maximum for partial orders of size at most the continuum, implies the $\mathsf{AD}^+$ Conjecture. This is the strongest known result regarding the conjecture in the context where the Continuum Hypothesis (CH) fails.[4] The following two theorems show that the $\mathsf{AD}^+$ Conjecture can also hold with CH.

Recall, for an infinite cardinal $\kappa$, the principle $\square_\kappa$ asserts the existence of a sequence $\langle C_\alpha \mid \alpha < \kappa^+ \rangle$ such that for each $\alpha < \kappa^+$,

[2] One can show $A^g$ does not depend on the choice of $T, U$.

[3] $\Theta$ is the supremum of ordinals $\alpha$ such that there is a surjection from $\mathbb{R}$ onto $\alpha$. "$\mathsf{AD}_\mathbb{R} + \Theta$ is regular" is consistent relative to large cardinals. This is an important result of G. Sargsyan, cf. [Sar14].

[4] $\mathsf{MM}(\mathfrak{c})$ implies $\mathfrak{c} = \omega_2$.

C is club in ;

for each limit point of C, C = C \ ;

the order type of C is at most .

The principle () asserts the existence of a sequence hC j < i such that 1. for each < ,

each C is club in ;

for each limit point of C, C = C \ ; and

2. there is no thread through the sequence, i.e., there is no club E such that C = E \ for each limit point of E.

We remark that the hypothesis of Theorem 1.2 is consistent relative to large cardinals.

Theorem 1.2 Suppose CH holds and :(!$_2$) + :(!$_3$) holds. Then the AD$^+$ Conjecture holds.

We say that an ideal I on !$_1$ is !$_1$-dense if the associated poset P$_I$ = }(!$_1$)=I has a dense set of size !$_1$.[5] The hypothesis of Theorem 1.3 is consistent relative to \AD$_R$+ is regular" (see [STW21] for a proof).

Theorem 1.3 Suppose CH holds and there is an !$_1$-dense ideal on !$_1$. Then the AD$^+$ Conjecture holds.

As mentioned, the AD$^+$ Conjecture was motivated by inner model theoretic considerations. One may attempt to prove the full conjecture (i.e. without any extra hypothesis) by extending the HOD analysis for all AD$^+$ models. This is an active area of research in descriptive inner model theory and has many other applications (cf. [Sar14, ST] for a treatment of the HOD analysis for all AD$^+$ models below the minimal model of AD$_R$+\ is regular", The Largest Suslin Axiom, [Ste22] for a different, more general treatment of the HOD analysis, and [Tra16, ST19, ST] for recent applications of the HOD analysis in computing consistency strength.)

Acknowledgment. The author is grateful for H.W. Woodin for his many insightful conversations concerning the topic and for his inspiring work in this direction. The author would also like to thank the NSF for its generous support through the CAREER grant DMS-1945592.

[5] ~~The ideals considered in this paper~~ are proper, normal, ne, and countably complete. Being !$_1$-dense is a very strong property; it implies for example that I is saturated.

# 2 Preliminaries

## 2.1 AD and AD$^+$ Facts

We review basic facts about the Axiom of Determinacy (AD). Suppose $A \subseteq \omega^\omega$. Let $G_A$ be the following game. Players I and II alternatively play natural numbers; player I starts round 0 by playing $x_0$, player II responds by playing $x_1$ and so on for infinitely many rounds. Let $x_k$ be the natural number played at the kth move.

| Round | 0 | | 1 | | $\dots$ | $n$ | | $\dots$ |
|---|---|---|---|---|---|---|---|---|
| Player I | $x_0$ | | $x_2$ | | | $x_{2n}$ | | |
| Player II | | $x_1$ | | $x_3$ | | | $x_{2n+1}$ | |

Let

$$x = (x_k)_{k<\omega}$$

$$\text{Then I wins iff } x \in A.$$

$G_A$ is determined if one of the players has a winning strategy. Often times, we will say $A$ is determined instead.

**Definition 2.1** Axiom of Determinacy, AD: For every $A \subseteq \omega^\omega$, $G_A$ is determined.

One particular game that is relevant to this paper is the Wadge game. We review it here. Let $A, B \subseteq \mathbb{R}$, the Wadge game $G_{A,B}$ for $A, B$ is defined as follows. Players I and II take turns to play integers $(n_i : i < \omega)$ and $(m_i : i < \omega)$ respectively. After $\omega$ many rounds (i.e. when the play is finished), letting $x = (n_i : i < \omega)$ and $y = (m_i : i < \omega)$, player II wins the play if and only if

$$x \in A \Leftrightarrow y \in B.$$

AD implies that $G_{A,B}$ is determined and therefore $A, B$ are Wadge comparable. More precisely, if player II has a winning strategy $\tau$, then $\tau$ induces a continuous (in fact, Lipschitz) function $f_\tau : \mathbb{R} \to \mathbb{R}$ such that $f_\tau^{-1}[B] = A$; otherwise, there is a continuous function $g : \mathbb{R} \to \mathbb{R}$ such that $g^{-1}[\mathbb{R} \setminus A] = B$. In the first case, we say that $A$ is Wadge reducible to $B$ and we denote this by $A \leq_w B$; in the second case, we say $B$ is Wadge reducible to $\mathbb{R} \setminus A$.

We continue with the definition of Woodin's theory of AD$^+$. As mentioned above, we use $\Theta$ to denote the sup of ordinals $\alpha$ such that there is a surjection $\pi : \mathbb{R} \to \alpha$.

Under AC, is just the successor cardinal of the continuum. In the context of AD, is shown to be the supremum of w(A)[6] for A R (cf. [Sol78]). The denition of relativizes to any determined pointclass (with sucient closure properties). We denote for the supremum of ordinals such that there is a surjection from R onto coded by a set of reals in .

Denition 2.2 $AD^+$ is the theory ZF + AD+$DC_R$ and

1. for every set of reals A, there are a set of ordinals S and a formula ' such that x 2 A , L[S; x] '[S; x]. (S; ') is called an 1-Borel code for A;

2. for every < , for every continuous : ! ! !!, for every A R, the set $^1$[A] is determined.

$AD^+$ is equivalent to \AD + the set of Suslin cardinals is closed". Another, perhaps more useful, characterization of $AD^+$ is \AD+$_1$ statements reect into the Suslin co-Suslin sets" (see [ST10] for the precise statement).

## 2.2 Term capturing under $AD^+$

The denition of mice and iteration strategies used in this paper are standard, see [Ste10]. In the following, suppose M is a premouse[7] and is an iteration strategy for M, then we say that (M; ) is a mouse pair. A more general denition of mouse pairs can be found in [Ste22].

Denition 2.3 Let A R, (M; ) is a (countable) mouse pair, and a cardinal in M.

1. (M; ) term captures A at if there is a term 2 $M^{Col(!;)}$ such that whenever i : M ! N is according to , and g Col(!; i()) is N-generic, then A \ N[g] = $i()_g$.

2. (M; ) Suslin captures A at if there is a pair of trees (T; U) 2 M such that whenever i : M ! N is according to , and g Col(!; i()) is N-generic, then A \ N[g] = $p[i(T)]^{N[g]}$ = $R^{N[g]}$np[i(U)].

In the above, is a $!_1$ + 1-iteration strategy of M.[8] M need not be ne structural.

[6] w(A) is the Wadge rank of A. Under AD the Wadge reducibility relation is a prewellorder on }(R).

[7] A premouse can be either ne-structural or coarse.

[8] Iteration trees according to are normal trees in the sense of [Ste10].

Clearly, Suslin capturing implies term capturing. The relationship between de-terminacy and term capturing is best expressed by the following theorem.

Lemma 2.4 (Neeman, [Nee95, Nee10]) Suppose is a Woodin cardinal in a countable mouse $M$ and $A$ $\mathbb{R}$ is term captured by $(M; )$ at . Then $A$ is de-termined.

We will call the triple $(M; ; )$ in Lemma 2.4 a Woodin mouse pair or coarse Woodin mouse pair if $M$ is not ne structural. Under $AD^+$, the following theorem, due to Woodin, gives the existence of coarse Woodin mouse pairs capturing Suslin co-Suslin sets of reals. See [Ste08, Section 10] for a more detailed version and its proof. In the following, we say that has condensation if whenever $T$ is an iteration tree according to and $U$ is a hull of $T$ then $U$ is according to . As usual, $T$ is equipped with a tree order $<_T$. $<_T$ implies $<$ and the interval $[; ]_T$ is the set of such that $_T$ $_T$. See [SS, Ste08] for more details. What we mean by an iteration strategy being Suslin co-Suslin is: the set of reals Code[] is Suslin co-Suslin, where Code is a xed, simple coding function of elements of $HC$ by reals.

Theorem 2.5 (Woodin, [Ste08]) Suppose $AD^+$ holds and $A$ is a Suslin co-Suslin set. Then there is a (coarse) Woodin mouse pair $(M; ; )$ that Suslin captures $A$, has condensation (and hence has the Dodd-Jensen property), and is Suslin co-Suslin.

## 2.3 Axiom of Strong Condensation

In this section, we briey discuss the Axiom of Strong Condensation, isolated by Woodin. This axiom roughly abstracts essential condensation properties typically seen in canonical inner models (like $L$). For more details, see [Woo10].

Denition 2.6 (Axiom of Strong Condensation) For each cardinal $> !$, there is a bijection $F : !$ $H$ such that for all countable $X$ $(H; F)$, letting $F_X$ be the image of $F \cap X$ under the transitive collapse map,

$$F_X \quad F.$$

We say that $F$ witnesses Strong Condensation at .

Remark 2.7 By absoluteness, the $X$ above can be taken to be in any outer model of $V$. Furthermore, [Woo21, Theorem 4.3] shows that the Axiom of Strong Condensa-tion implies many consequences that typically hold in $L$, like GCH and there are no measurable cardinals. [Woo21, Theorem 4.3] also shows that there is a "global" $F$ witnessing Strong Condensation, i.e. there is an $F : ON \to V$ that is $\Sigma_2$-definable from $F|\omega_1$ and for every cardinal $\kappa > \omega$, $F|\kappa : \kappa \to H_\kappa$ witnesses Strong Condensa-tion at $\kappa$.

## 2.4 $\omega_1$-dense ideals on $\omega_1$

Suppose $I$ is an $\omega_1$-dense ideal on $\omega_1$. The following are standard facts; see [Woo10, Definition 6.19] and the discussion after it.

**Fact 2.8** 1. $\mathbb{P}_I$ is a homogeneous forcing.[9]

2. There is a boolean isomorphism $\pi : \mathbb{P}_I \to RO(Coll(\omega,\omega_1))$[10]. In particular, $\mathbb{P}_I$ is forcing equivalent to $Coll(\omega,\omega_1)$.

3. For any $V$-generic filter $G \subseteq Coll(\omega,\omega_1)$, $\pi$ induces a $V$-generic filter $H \subseteq \mathbb{P}_I$, and letting $j : V \to M =_{def} Ult(V,H) \subseteq V[H]$ be the associated generic ultrapower map, we have:

   (a) $j(f)(\omega_1^V) = G$ for some $f : \omega_1 \to H_{\omega_1}$; in particular, $V[H] = V[G]$.

   (b) $j(\omega_1^V) = \omega_2^V$.

   (c) $M$ is well-founded and $M^\omega \subseteq M$ in $V[H]$.

# 3 Proof of Theorem 1.2

We first start with a definition.

**Definition 3.1** Given a set of reals $A$, $\Sigma^1_\omega(A)$ is the collection of $B \subseteq \mathbb{R}$ that is defin-able in the structure $(V_{\omega+1}, \in, A)$ without parameters; this is the collection of (lightfaced) projective sets in $A$. We say that $A \subseteq \mathbb{R}$ is projective-like if every $B \in \Sigma^1_\omega(A)$ has a scale which is $\Sigma^1_\omega(A)$. We can relativize the above definitions to $\Sigma^1_\omega(A,x)$ for any real $x$ in an obvious way.

It is a basic AD fact, essentially following from the Moschovakis Periodicity The-orem, that every Suslin co-Suslin set is Wadge reducible to a set which is projective-like. Every projective-like set is clearly Suslin co-Suslin.

[9] A forcing $\mathbb{P}$ is homogeneous if whenever $p,q \in \mathbb{P}$, there is an automorphism $\pi : \mathbb{P} \to \mathbb{P}$ such that $\pi(p)$ is compatible with $q$.

[10] $RO(Coll(\omega,\omega_1))$ is the regular open algebra of $Coll(\omega,\omega_1)$.

Now we start the proof of Theorem 1.2. Suppose $A_0; A_1; _0; _1$ are as in the hypothesis of the $AD^+$ Conjecture. Let $B_i$ 2 $_i$ be projective-like. It is enough to show that the Wadge game $G_{B_0;B_1}$ is determined for all projective-like $B_0$ 2 $_0$ and $B_1$ 2 $_1$. Since projective-like sets are Wadge-conal in $_0; _1$, we easily get that either $_0$ $_1$ or $_1$ $_0$ and the conclusion $L(_0; _1; R)$ AD $^+$ easily follows.

By Theorem 2.5, there are coarse Woodin pairs $(M_i; _i; _i)$ that Suslin capture $B_i$ for each i 2 f0; 1g. Let $(T_i; U_i)$ witness $_i$ is $!_1$-universally Baire. In the following, $M_1^{(_0;_1);]}$ is the minimal active mouse with a Woodin cardinal that is closed under $_0$ and $_1$. This is a kind of strategy mice and its general theory has been fully developed in for example [ST16]. Recall we x a canonical coding Code : $H_{!_1}$ ! R. This coding is simply denable and generically absolute (for example, take the one dened in [Woo10, Chapter 2]).

Lemma 3.2 $M_1^{(_0;_1);]}$ exists.

Proof. For each i, since $_i$ is $!_1$-universally Baire, $_i$ can be uniquely extended to a strategy (which we will also call $_i$) on $H_{!_2}$. For a tree T 2 $H_{!_2}$, according to $_i$,

$$_i(T) = b \; i \; _{Coll(!;!_1)} (Code(T); Code(b)) \; 2 \; p[T_i].$$

Fix i and let = $_i$. We let $^g$ = $p[T_i] \setminus V[g]$ for any generic g Coll(!; $!_1$). It is easy to see that the denition of $^g(T)$ = b is independent of generics and therefore b 2 V. To see this, suppose there is an ordinal and conditions p; q such that p $_{Coll(!;!_1)}$ 2 $^g(T)$ i q $_{Coll(!;!_1)}$ 2 $^g(T)$. But then, by the homogeneity of Coll(!; $!_1$), we can nd generics $g_0; g_1$ such that

$V[g_0] = V[g_1]$.

p 2 $g_0$ and q 2 $g_1$.

Since $p[T_i] \setminus V[g_0] = p[T_i] \setminus V[g_1]$, $^{g_0}(T) = {}^{g_1}(T)$. This contradicts what p; q force. We have shown that can be extended to a (necessarily) unique strategy, also called , acting on trees in $H_{!_2}$. Now we extend to $H_{!_3}$. Suppose T is a normal tree of length $!_2$ in $H_{!_3}$. If cof(lh(T)) = $!_2$, using $:(!_2)$, we can easily nd a conal branch b through T.[11] This branch is necessarily unique and well founded; this is simply because cof(lh(T)) > ! and the branch b is a conal, closed subset

[11] This is a standard argument. The set $\tilde{C}$ = f[0; ]$_T$ : < lh(T)g is a coherent sequence on lh(T). Fix a continuous, increasing function f : $!_2$ ! lh(T), we can use f to pull back $\tilde{C}$ into a coherent sequence D in $!_2$. Now apply $:(!_2)$ to get a thread E for D. Then f[E] is a thread through $\tilde{C}$ and gives a conal branch through lh(T).

of lh(T ), so it must be unique. Well-foundedness follows from the fact that any countable sequence $(x_i : i < \omega)$ of elements in the last model of T have preimages in some model $M^T$ for $< lh(T)$. We dene (T ) = b.

Suppose cof(lh(T )) $\omega_1$. We dene X to be good if

$$X \quad (H_{\omega_3}, \in), |X| = \omega_1, \text{ and } X^{\omega} \quad X.$$

Note that such good hulls exist by CH. For a good X such that $T \in X$, we let $_X : M_X \to X$ be the uncollapse map, $T_X = {}_X^{-1}(T)$ and $b_X = (T_X)$. We claim.

**Claim 3.3** There is a good X such that for any good X Y, letting $_{X,Y} = {}^{-1}_{Y} {}_{X}$, $c_{X,Y} = {}_{X,Y}[b_X]$, then $c_{X,Y} \quad b_Y$.

Proof. Since cof(lh(T )) $\omega_1$, for any good X, $X \cap T$ is conal in T. Suppose cof(lh(T )) = $\omega_1$, then it is easy to see that the claim holds for any good X such that $T \in X$. To see this, x any good X such that $T \in X$ and let Y be good such that X Y. Note that $X \cap T, Y \cap T$ are conal in lh(T ). Therefore, $c_{X,Y}$ is a conal branch of $T_Y$. Both $c_{X,Y}, b_Y$ are club subsets of $lh(T_Y)$ and since $cof(lh(T_Y)) = \omega_1$, $b_Y = c_{X,Y}$ as desired.

Now suppose cof(lh(T )) = $\omega$, then note that for any good X, since $X^{\omega} \quad X$, $b_X \in M_X$. The argument is as in [Ste05]. Suppose there is no such X as in the claim, we can form an elementary chain $(X : \in \omega_2)$ such that:

1. If is a limit ordinal then $X = \bigcup_{<} X$.
2. If is a successor ordinal, then X is good.
3. For each successor or for each limit such that cof() > $\omega$, $c_{,+1} =_{def} c_{X,X_{+1}} = b_{+1} =_{def} b_{X_{+1}}$.

We also write $_{,}$ for $_{X,X}$ etc. An easy argument (using that for each $< \omega_2$ with cof() > $\omega$, $b \in M_X$) gives a stationary S $\omega_2$ such that

$\in S \Rightarrow$ cof() > $\omega$, and

$, \in S \Rightarrow {}_{,}(b) = b$.[12]

[12] Suppose the set S as dened is not stationary. So there is a club C such that $C \cap S = \emptyset$. For any $< \in C$ with uncountable conality, $_{,}(b) = b$. Let $\in lim(C)$ be of uncountable conality and is a ~~limit of points in C of uncountable cona~~lity. Since C is club below and cof(lh(T)) = $\omega$, we can easily nd $\in C \cap$ such that $rng(_{,}) \cap b$ is conal in b and $b \in rng(_{,})$, but then $_{,}[b] = b$ by condensation of , i.e. $_{,}(b) = b$. Contradiction.

For < 2 S, $_{+1;}$ witnesses $(T_{+1})^a{}_{;+1}(b)$ is a hull of $T^a b$, and so $_{;+1}(b)$ is according to , i.e. $c_{;+1} = b_{+1}$. Contradiction.

Using the claim, we can dene (T ) to be the downward closure of $_X[b_X]$, where X is as in the claim.

Now we show $M_1^{(_0;_1);]}$ exists and is $!_3$-iterable. The rst step is to show $H^V$ is $_{!_3}$ closed under $(_0;_1)^]$. More precisely, for any A 2 $H_!$ , $A^{(_0;_1);]}_3$ exists.[13] If not, then by covering, letting $M = L^{(_0;_1)}[A]$,[14] $_{!_3}$ where A is some subset of $!_2$, letting $= (!^V)^{+;M}$, $_2$

$$\mathrm{cof}() \ !_2.$$

The inequality above follows from weak covering of M; this is a straightforward generalization of the Jensen's weak covering theorem for L. The proof of the weak covering theorem for L generalizes straightforwardly to M, using the fact that $_0; _1$ have sucient condensation and therefore M has the ne structure needed to run the covering lemma proof.

But $:(!_3)$ ) < $!_3$ . This is because if = $!_3$ , letting C be the canonical $_{!^V}$ -sequence dened over M, then $:(!_3)$ implies there is a thread D. The thread D, as usual, gives a collapsing structure for , i.e. some sound model N such that 2 N and $_!(N) = !_2$ (this means N projects to $!_2$). This is a contradiction as was assumed to be a cardinal in V. So < $!_3$ . Then $:(!_2)$ ) $_V$cof() < $!_2$ by a similar argument. Contradiction.

Similarly, we can then show $M_1^{(_0;_1);]}$ exists. Otherwise, the core model $K = K^{(_0;_1)}$ exists.[15] Let $= (!^V)^{+;K}_2$. By covering, cf. [JS13], cof() $!_2$, but as before $:(!_3) + :(!_2)$ ) cof() < $!_2$. Contradiction.

Let $H = M_1^{(_0;_1);]}$, $^H$ be the Woodin cardinal of H, be H's canonical strategy, and let $_i$ 2 H be $\mathrm{Coll}(!;^H)$-terms for $B_i$. In particular, for any g $\mathrm{Coll}(!;^H)$ in

[13] This is the theory of the indiscernibles for the model $L^{(_0;_1)}_{!_3}[A]$. Here the language is the language of set theory augmented by the following predicate symbols: a unary predicate symbol A and two binary predicate symbols $_0; _1$.

[14] Again, $L^{(_0;_1)}_{!_3}[A]$ is the minimal model over A of height $!_3$ and is closed under strategies $_0$ and $_1$. [ST16] gives a detailed treatment of how to feed strategy information of $_0$ and $_1$ into the model. This model is L-like in that it satises all condensation properties L satises. The key here, of course, is because $_0$ and $_1$ have condensation properties.

[15] Here we construct the Jensen-Steel core model as in [JS13] up to $!_3^V$ . Again, the core model and the corresponding $K^c$-construction are hybrid, relativized to $(_0;_1)$. $K^c$-constructions relativized to $(_0;_1)$ will construct $(_0;_1)$-mice because $(_0;_1)$ relativize well and condense well. Comparisons of $(_0;_1)$-mice is not an issue; these are simply extender comparisons because all iterates are $(_0;_1)$-mice, so no strategy disagreements will occur. The theory of strategy mice is developed more fully in [ST16, SS].

$V$, $(_i)_h$ = $B_i \setminus H[g]$.[16]. First, we note that the Wadge game $G_{B_0;B_1}$ is determined in $H[g]$ for any $H$-generic $g$ Coll(!; ) in $V^H$(via a strategy in $H$). This follows from the fact that the corresponding Neeman's game $G_{B_0;B_1}$ ^is determined in $H[g]$ (see [Nee95]).

Now we let $(M_i; _i)_{i<!}$ enumerate the coarse Woodin pairs that Suslin capture all sets in $^1_!(B_0)$ [ $^1_!(B_1)$. By a similar proof, $H = M_1^{(_i : i<!);]}$ exists and $G_{B_0;B_1}$ is determined in $H^{Coll(!;)}$ via a strategy in $H$, where is the Woodin cardinal of $H$. We note that $H$ has Coll(!; )-terms that capture $B_0; B_1$ as well as scales on $B_0$, $B_1$. Therefore,

$$(H[g] \setminus V_{!+1}; H[g] \setminus B) \quad (V_{!+1}; B); \qquad (1)$$

for any $H$-generic $g$ Coll(!; ) such that $g$ 2 $V$, where $B = B_0$ or $B = B_1$. By AD and the fact that $B$ is projective-like, each $^1_!(B)$-relation can be uniformized by a function whose graph is in $^1_!(B)$. Therefore, each non-empty $^1_!(B; x)$-set, for each real $x$ 2 $H[g]$, has a member $u$ such that $fug$ is $^1_!(B; x)$. (1) follows from this fact.

Finally, (1) implies that $G_{B_0;B_1}$ is determined in $V$. Suppose 2 $H$ witnesses $G_{B_0;B_1}$ is determined in $H[g]$ for any $H$-generic $g$ Coll(!; ). Without loss of generality, we assume $_1[B_1] = B_0$ in $H[g]$; we show that this holds in $V$. Suppose not and let $x$ 2 $R$ be such that

$$x \; 2 \; B_0 \; , \quad (x) \; 2\!\!\!/ \; B_1:$$

Let $T$ be an $x$-genericity iteration tree of $H$ using ; in particular, we have:

$T$ is countable, according to with last model $K$.

The iteration embedding $i : H \; ! \; K$ exists.

There is a $K$-generic $g$ Coll(!; $i(^H)$) such that $x$ 2 $K[g]$.

We note that (1) holds for $K$ in place of $H$. By elementarity, $^1[B_1] = B_0$ in $K[g]$ and since $x$ 2 $K[g]$, in $V$ and in $K[g]$,

$$x \; 2 \; B_0 \; , \quad (x) \; 2 \; B_1:$$

This is clearly a contradiction and completes the proof of the theorem.

[16] In fact, we can take $_i$ to be some tree $T_i$ 2 ! for some 2 $H$.

# 4 Proof of Theorem 1.3

Let $I$ be an $\omega_1$-dense ideal on $\omega_1$. Let $A_0, A_1, \Gamma_0, \Gamma_1$ be as in the hypothesis of the AD$^+$ Conjecture. As before, let $B_i \in \Gamma_i$ for $i \in \{0, 1\}$ be projective-like. We note that by our hypothesis, for any $C \in \Sigma_\omega(B_0) \cup \Sigma_\omega(B_1)$, $C$ is $\omega_1$-UB, and hence for any generic $g$ for a poset $P \in H_{\omega_2}$,

$$(V_{\omega+1}, C) \prec (V[g]_{\omega+1}, C_g); \tag{2}$$

where $C_g$ is the canonical interpretation of $C$ in $V[g]$.

Let $\mathcal{P}$ be the term relation for $\Sigma_\omega(B_0) \cup \Sigma_\omega(B_1)$. More precisely, $\mathcal{P}$ consists of tuples $(i, \varphi, P, \tau, q)$ such that

- $i \in \{0, 1\}$.
- $P \in H_{\omega_2}$ is a poset.
- $\tau \in V^P \cap H_{\omega_2}$ is a term for a real.
- For a closed unbounded set of countable $X \prec H_{\omega_2}$, for a comeager set of $X$-generic filter $g \subset X \cap P$: if $i = 0$ and $q \in g$ then $(V_{\omega+1}, B_0) \models \varphi[\tau_g]$; and if $i = 1$ and $q \in g$ then $(V_{\omega+1}, B_1) \models \varphi[\tau_g]$.[17]

Furthermore, by the proof of [Woo21, Theorem 5.13] and (2), for all generic $g$ for a poset in $H_{\omega_2}$, for all bounded $Z \subset \omega_2^V$ in $V[g]$,

$$L_{\omega_2^V}[Z, \mathcal{P}]^{\mathrm{Coll}(\omega, \sup(Z))} \models \text{ZFC} + \text{Axiom of Strong Condensation}.$$

Let $G \subseteq \mathrm{Coll}(\omega, \omega_1)$ be $V$-generic. Note that $\mathrm{Coll}(\omega, \omega_1) \in H_{\omega_2^V}$ and $G$ induces a generic $g \subseteq \wp(\omega_1)/I$ and a generic elementary embedding $j : V \to M \subseteq V[g]$. Similarly, over $M$, we let $k : M \to N$ be the generic embedding induced by an $M$-generic $h \subseteq \mathrm{Coll}(\omega, \omega_1^M)$. We note that by Section 2.4, $M^\omega \subseteq M$ in $V[G]$; in particular, $\mathbb{R}^M = \mathbb{R}^{V[G]}$.

We then have by strong condensation of $\mathcal{P}$ and the fact that $M^\omega \subseteq M$ in $V[G]$:

$$j(\mathcal{P}) = \mathcal{P}_G \cap M.\text{[18]} \tag{3}$$

[17] The existence of a club of such $X$ for each $P \in H_{\omega_2}$ follows from the fact that $B_0, B_1$ are $\omega_1$-UB and [Ste09, Lemma 4.1].

[18] Here $\mathcal{P}_G$ is $\mathcal{P}$ as interpreted in $V[G]$. Note here that $j(A_i) = (A_i)_G$, the canonical interpretation of $A_i$ in $V[G]$.

Claim 4.1 For all bounded $Z \subseteq \omega_2^V$, there is a closed and unbounded set $C \subseteq \omega_2$ of indiscernibles for the structure $(N_Z =_{def} L_{\omega_2^V}[Z; P]; P \cap N_Z)$.

Proof. First let $Z$ be a bounded subset of $\omega_1$. We note the following:

1. $N_Z = L_{\omega_2^V}[Z; j(P)]$.

2. $\omega_1^V$ and $\omega_2^V = j(\omega_1) = \omega_1^M$ are strongly inaccessible in $N_Z$.

Item (1) is proved in [Woo21] using the fact that $N_Z$ satisfies the Axiom of Strong Condensation. We give a quick proof. Let $F : ON^{N_Z} \to N_Z$ be a function witnessing Strong Condensation of $N_Z$. For all uncountable cardinals $\alpha < \omega_2^V$ of $N_Z$, $j(F)|\alpha$ witnesses Strong Condensation at $\alpha$. By Remark 2.7, $j(F)|\alpha = F|\alpha$ because

$$(j[H_\alpha^{N_Z}]; j[F|\alpha]) \prec (j(H_\alpha^{N_Z}); j(F|\alpha)) \cong (H_{j(\alpha)}^{j(N_Z)}; j(F)|j(\alpha)).$$

Since the transitive collapse of $j[F|\alpha]$ is precisely $F|\alpha$ and by Strong Condensation of $j(F)$, $F|\alpha \subseteq j(F)$, so $j(F)|\alpha = F|\alpha$. Using this and the fact that $j(Z) = Z$, we get (1).

For item (2), first note that $j(N_Z)$ has the form $L_{j(\omega_2^V)}[Z; j(P)] = L_{j(\omega_2^V)}[Z; P_G]$. We note that $\omega_1^V$ is a cardinal in both $N_Z$ and $j(N_Z)$. The first part holds because $\omega_1^V$ is a cardinal in $V$ and $N_Z \subseteq V$. The second part follows from the following facts:

$N_Z = j(N_Z)|\omega_2^V \models$ "$\omega_1^V$ is a cardinal",

$\omega_2^V = j(\omega_1^V)$ is a cardinal of $j(N_Z)$,

the Axiom of Strong Condensation holds in $N_Z$ and hence in $j(N_Z)$, in particular, $\mathcal{P}(\omega_1^V) \cap j(N_Z) \subseteq j(N_Z)|\omega_2^V$.[19]

Suppose there is a $\kappa < \omega_1^V$ such that in $N_Z$, $\mathcal{P}(\kappa) \geq \omega_1^V$, we may assume $\kappa^+ = \omega_1^V$ in $N_Z$. But this means in $j(N_Z)$,

$$\kappa^+ = j(\omega_1^V) = \omega_2^V$$

This contradicts the agreement of $N_Z$ and $j(N_Z)$ in item (1) and the fact that $\omega_1^V; \omega_2^V$ are cardinals of $N_Z; j(N_Z)$. Now use the fact that $\omega_2^V = j(\omega_1^V)$ and apply the above argument to $j(N_Z)$ and k, we conclude that $\omega_2^V$ is strongly inaccessible in $j(N_Z)$. By item (1), $\omega_2^V$ is also strongly inaccessible in $N_Z$.

[19] This fact is proved by the same proof of the condensation property of $L$, using the Axiom of Strong Condensation here.

We then get the conclusion of the claim for this particular choice of $Z$. In fact, one gets that $(N_Z, P)^\sharp$ exists. This is simply an adaptation of Kunen's construction of $0^\sharp$ from the existence of a nontrivial embedding $j : L \to L$. Roughly, we can show that letting be the measure over $N_Z$ derived from $j$, $M_Z = Ult(N_Z, )$, and $i : N_Z \to M_Z$ be the ultrapower map, then there is a canonical factor map $: M_Z \to j(N_Z)$ such that $j \restriction N_Z = i$. We also have $crt() = i(\omega^V_1)$. In fact by condensation of $N_Z, j(N_Z)$, $M_Z = N_Z$. We can then derive a measure over $M_Z$ from . We continue this process, showing that $(N_Z, )$ is iterable. A similar argument has been given in similar contexts, see for example [Tra16, Lemma 3.64] or [SZ14, Theorem 28].

Now assume $Z \subseteq \omega^V_2$ is bounded. Note then that in $M$, $Z$ is a bounded subset of $\omega_1$. We apply the above proof to $L_{\omega^M_2}[Z, j(P)] =_{def} N_Z$ and use $k$. We conclude that in $k(N_Z)$, $\omega^M_2$ is a limit of a club of indiscernibles. By elementarity and the fact that $k(\omega^M_2) = \omega^M_1$, in $N_Z$, $\omega^M_1$ is a limit of a club of indiscernibles. But again by strong condensation, $N_Z \restriction \omega^M_1 = N_Z$. This completes the proof of the claim.

Now let $R = L_{\omega^V_2}(\mathbb{R})[P]$. Let $(M_0, _0, _0)$, $(M_1, _1, _1) \in R$ be coarse Woodin pairs that capture $B_0, B_1$ respectively. As in the proof of Theorem 1.2, it suces to show $M_1^{(_0, _1), \sharp}$ exists. Suppose not. Then for any $H \subseteq Coll(\omega_1, \mathbb{R})$ in $V$, $K = K^{(_0, _1)}$ exists in $R[H]$.[20] We note that:

- $R[H]$ has the form $L[Z, P]$, where $Z$ is a bounded subset of $\omega^V_2$ coding $(\mathbb{R}^V, H)$.
- $K \in R$ by homogeneity of $Coll(\omega_1, \mathbb{R})$.
- $j(K) \in V$ by homogeneity of $Coll(\omega, \omega_1)$ and $j(R)$ being denable in $V[g]$ from parameters in $V$.[21]

From the above, the proof of [SS, Section 2.11] goes through and shows that $\omega^V_1$ must be Shelah in $j(K)$. Contradiction. Hence $M_1^{(_0, _1), \sharp}$ exists. As before, we let $(M_i, _i)_{i<\omega}$ enumerate the coarse Woodin pairs that Suslin capture all sets in $^1_\omega(B_0) \cup ^1_\omega(B_1)$. By a similar proof, $H = M_1^{(_i : i<\omega), \sharp}$ exists and $G_{B_0, B_1}$ is determined in $H^{Coll(\omega, )}$ via a strategy in $H$, where is the Woodin cardinal of $H$. This implies $G_{B_0, B_1}$ is determined in $V$ and completes the proof of the theorem.

[20] That we can nd such a $H$ in $V$ follows from CH. $K$ is the Jensen-Steel core model (constructed relative to $(_0, _1)$), cf [JS13]. Since $R[H]$ contains all reals in $V$ and is closed under $(_0, _1)$, whether $M_1^{(_0, _1), \sharp}$ exists is absolute between $V$ and $R[H]$.

[21] The parameters are the collection of trees $\{(T_C, S_C) : C \in ^1_\omega(A_0) \cup ^1_\omega(A_1)\}$ witnessing each $C \in ^1_\omega(A_0) \cup ^1_\omega(A_1)$ is $\omega_1$-universally Baire. Here, we also use equation (3) to get that $j(R)$ has the form $L[j(Z), P_G]$ to be able to use homogeneity of $Coll(\omega, \omega_1)$.

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