COUNTABLE LENGTH EVERYWHERE CLUB UNIFORMIZATION

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ABSTRACT. Assume $\mathsf{ZF} + \mathsf{AD}$ and all sets of reals are Suslin. Let Γ be a boldface pointclass closed under $\wedge$, $\vee$, and $\forall^{\mathbb{R}}$ with the scale property. Let $\kappa = \delta(\Gamma)$ be the supremum of the length of prewellorderings on $\mathbb{R}$ which belong to $\Delta = \Gamma \cap \check{\Gamma}$. Let club denote collection of club subsets of κ. The countable length everywhere club uniformization holds for κ: For every relation $R \subseteq {}^{<\omega_1}\kappa \times \mathsf{club}$ with the property that for all $\ell \in {}^{<\omega_1}\kappa$ and clubs $C \subseteq D \subseteq \kappa$, $R(\ell, D)$ implies $R(\ell, C)$, there is a uniformization function $\Lambda : \mathrm{dom}(R) \to \mathsf{club}$ with the property that for all $\ell \in \mathrm{dom}(R)$, $R(\ell, \Lambda(\ell))$.

In particular, under these assumptions, for all $n \in \omega$, $\boldsymbol{\delta}^1_{2n+1}$ has the countable length everywhere club uniformization.

1. Introduction

Intuitively, club uniformization is a selection principle for club subsets of certain cardinals. These uniformization principles are useful in the study of combinatorics of partition measures under determinacy axioms.

If $X \subseteq \kappa$ and $\epsilon \leq \kappa$, then $[X]^\epsilon_*$ denote the set of increasing functions $f : \epsilon \to X$ which have the correct type (everywhere discontinuous and has uniform cofinality ω). The (correct type) partition relation $\kappa \to_* (\kappa)^\epsilon_2$ asserts that for all $P : [\kappa]^\epsilon_* \to 2$, there exists an $i \in 2$ and a club subset $C \subseteq \kappa$ so that for all $f \in [C]^\epsilon_*$, $P(f) = i$. (The correct type partition relation is essentially equivalent to the ordinary partition relation. Club homogeneous sets are often easier to handle due to their closure properties. In practice, partition relations under determinacy are established through the correct type version.)

Martin showed under AD that the partition relation $\omega_1 \to_* (\omega_1)^{\omega_1}_2$ holds. This implies that for each $\epsilon \leq \omega_1$, the filter μ_ϵ defined on $[\omega_1]^\epsilon_*$ by $X \in \mu_\epsilon$ if and only if there is a club $C \subseteq \omega_1$ so that $[C]^\epsilon_* \subseteq X$ is a countably complete ultrafilter. (See [2] for a survey of partition relations on ω_1.) To study the combinatorics of the partition measures μ_ϵ, it is frequently necessary to select clubs witnessing μ_ϵ-largeness or are homogeneous for partitions.

The most challenging partition measure on ω_1 is the strong partition measure μ_{ω_1}. There are several interesting combinatorial questions surrounding the strong partition measures. For instance, one can ask if every function $\Phi : [\omega_1]^{\omega_1}_* \to \omega_1$ is continuous μ_{ω_1}-almost everywhere. Another class of questions involve the stable theory of the strong partition measure. Since for each $\epsilon \leq \omega_1$, μ_ϵ is an ultrafilter, for any sentence φ in the language $\{\dot{\in}, \dot{E}\}$ (where $\dot{\in}$ is a binary relation symbol and $\dot{E}$ is a unary relation symbol), either μ_ϵ-almost all f satisfies $L[f] \models \varphi$ or μ_ϵ-almost all f satisfies $L[f] \models \neg\varphi$. The ϵ-stable theory, denoted $\mathfrak{T}_\epsilon$, is the collection of sentences φ so that μ_ϵ-almost all f satisfies $L[f] \models \varphi$. One can naturally ask whether important statements of set theory, such as GCH, belong to the stable theory of the strong partition measure $\mathfrak{T}_{\omega_1}$.

To answer these types of questions concerning the strong partition measure, [3] Theorem 3.10 proved under AD the almost everywhere short length club uniformization at ω_1: Let club denote the collection of club subsets of ω_1. For every relation $R \subseteq [\omega_1]^{<\omega_1}_* \times \mathsf{club}$ which is $\subseteq$-downward closed in the club-coordinate (which means for all $\sigma \in [\omega_1]^{<\omega_1}_*$ and clubs $D \subseteq E$, $R(\sigma, E)$ implies $R(\sigma, D)$), then there is a club $C \subseteq \omega_1$ and a function $\Lambda : ([C]^{<\omega_1}_* \cap \mathrm{dom}(R)) \to \mathsf{club}$ so that for all $\sigma \in [C]^{<\omega_1}_* \cap \mathrm{dom}(R)$, $R(\sigma, \Lambda(\sigma))$.

To illustrate a typical application, [3] Theorem 4.5 showed that under AD, every function $\Phi : [\omega_1]^{\omega_1}_* \to \omega_1$ is continuous μ_{ω_1}-almost everywhere, which means there is a club $C \subseteq \omega_1$ with the property that for all $f \in [C]^{\omega_1}_*$, there is an $\alpha < \omega_1$ so that for all $g \in [C]^{\omega_1}_*$, if $f \restriction \alpha = g \restriction \alpha$, then $\Phi(f) = \Phi(g)$. Define a partition $P : [\omega_1]^{\omega_1}_* \to 2$ by $P(f) = 0$ if and only if there exists an $\alpha < \omega_1$ so that for all clubs $D \subseteq \omega_1$,

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there exists a $g \in [D]^{\omega_1}_*$ with $\sup(f \restriction \alpha) < g(0)$ and $\Phi((f \restriction \alpha)\hat{}g) < g(0)$. By $\omega_1 \to_* (\omega_1)^{\omega_1}_2$, there is a club C homogeneous for P. The most important step is to show that C is homogeneous for P taking value 0. Suppose otherwise. Define a relation $R \subseteq [C]^{<\omega_1}_* \times \mathsf{club}$ by $R(\sigma, D)$ if and only if for all $g \in [D]^{\omega_1}_*$, $\Phi(\sigma\hat{}g) \geq g(0)$. C being homogeneous for P taking value 1 implies $\mathrm{dom}(R) = [C]^{<\omega_1}_*$. Applying the almost everywhere club uniformization to R, there is a club $E \subseteq C$ and a function $\Lambda : [E]^{<\omega_1}_* \to \mathsf{club}$ so that for all $\sigma \in [E]^{<\omega_1}_*$, $R(\sigma, \Lambda(\sigma))$. Using Λ, one can recursively construct a function $h \in [E]^{\omega_1}_*$ so that for all $\alpha < \omega_1$, $R(h \restriction \alpha, \Lambda(h \restriction \alpha))$. By definition of R, this means that for all $\alpha < \omega_1$, $\Phi(h) \geq h(\alpha)$. Since h is an increasing function, this implies $\Phi(h) \geq \omega_1$ which is impossible since Φ takes values in ω_1. Thus C must be homogeneous for P taking value 0 and this will eventually lead to the μ_{ω_1}-almost everywhere continuity of Φ. Following this template, in forthcoming work by the authors, it is shown that many familiar statements of set theory like GCH belong to the stable theory $\mathfrak{T}_\epsilon$ for all $\epsilon \leq \omega_1$. It is also shown that for μ_ϵ-almost all f, there is a sequence of normal measures $\bar{\nu}_f$ with a discontinous sequence of critical points $\bar{\kappa}$ so that f is a generic over $L[\bar{\nu}_f]$ for a generalized Prikry forcing $\bar{\mathbb{P}}_{\bar{\nu}_f}$ consider by Fuchs [4].

The argument in [3] to prove the almost everywhere short length club uniformization at ω_1 (although uses just AD) appears peculiar and inefficient in that it passes first through an everywhere club uniformization principle whose argument requires generic coding, category arguments, and uniformization for certain relations on $\mathbb{R} \times \mathbb{R}$. [3] Theorem 3.7 shows that if $R \subseteq [\omega_1]^{<\omega_1}_* \times \mathsf{club}$ is a $\subseteq$-downward closed relation so that its coded version $\tilde{R} \subseteq \mathbb{R} \times \mathbb{R}$ has a uniformization, then there is a uniformization function $\Lambda : \mathrm{dom}(R) \to \mathsf{club}$ for R. Thus under $\mathsf{ZF} + \mathsf{AD}_{\frac{1}{2}\mathbb{R}}$ (which Kechris [6] showed is equivalent to AD and all relations on $\mathbb{R} \times \mathbb{R}$ can be uniformized), every $\subseteq$-downward closed relation $R \subseteq [\omega_1]^{<\omega_1} \times \mathsf{club}$ can be uniformized everywhere on its domain. Using the Moschovakis coding lemma, a Martin good coding system for ${}^{\omega_1}\omega_1$, and the almost everywhere good code uniformization ([2] Theorem 3.8), it can be shown that there is a club $C \subseteq \omega_1$ so that $R \cap ([C]^{<\omega_1}_* \times \mathsf{club})$ has a coded version $\tilde{R}$ which is projective and hence uniformizable under AD. The prior result ([3] Theorem 3.7) is then used to uniformize $R \cap ([C]^{<\omega_1}_* \times \mathsf{club})$. Moreover, everywhere short length club uniformization is not provable under AD as it fails in $L(\mathbb{R})$ by [3] Fact 3.9. Thus the almost everywhere version is the best possible under AD.

Naturally one would like to study these properties at strong partition cardinals larger than ω_1 such as the next strong partition cardinal $\boldsymbol{\delta}^1_3$ (or more generally the odd projective ordinals $\boldsymbol{\delta}^1_{2n+1}$) or the Σ_1-stable ordinals $\boldsymbol{\delta}_A$ for $L(A, \mathbb{R})$ where $A \subseteq \mathbb{R}$. As in [3], one would like to first prove the everywhere short length club uniformization at a strong partition cardinal $\delta > \omega_1$. Numerous issues with generalization quickly arises. First, more general generic coding functions exist for many cardinals beyond ω_1; however, these require that relevant sets possess scales. The stable ordinals $\boldsymbol{\delta}_A$ generally are not associated with pointclasses with scales. The odd projective ordinals however still have generic coding functions. These generic coding functions are more technical than the simple generic coding function on ω_1, but a more substantial issue is that the generic coding function acts on ${}^\omega\delta$. Thus category and generic coding arguments of [3] would at best give an everywhere club uniformization for families indexed by countable sequences (which will be verified in this paper).

To obtain almost everywhere short length club uniformization at strong partition cardinals δ greater than ω_1 under AD (or $\mathsf{AD} + \mathsf{DC}_\mathbb{R}$), one would need to find scale-free arguments. [1] defines a notion of a good coding family for δ which augments a good coding system for ${}^\delta\delta$ with a coding of the short functions on δ which interact under strict definability conditions. Moreover, this good coding family has a continuous function which merges a code for a short function and a good code for a full function and returns a good code so the short function overrides an initial segment of the original full function. [1] show that ω_1, (and more generally for all $n \in \omega$) $\boldsymbol{\delta}^1_{2n+1}$, and the stable ordinals $\boldsymbol{\delta}_A$ all possess very good coding families. It is then shown that a cardinal δ that possesses a very good coding family is a strong partition cardinal which also satisfies the almost everywhere short length club uniformization at δ.

The goal of this paper is to verify under suitable conditions that the everywhere countable length club uniformization holds at certain cardinals κ. That is, for every relation $R \subseteq {}^{<\omega_1}\kappa \times \mathsf{club}$ which is $\subseteq$-downward closed (where club refers to the collection of club subsets of κ), there is a uniformization function $\Lambda : \mathrm{dom}(R) \to \mathsf{club}$. As mentioned above, this seems to be the best everywhere club uniformization result obtainable by the method of generic coding. In this general setting, one will encounter ordinal games so Suslin representations will be necessary to conclude the determinacy of such games. Moreover, one will need to find winning strategies uniformly which will require the ideas of the third periodicity theorem of

Moschovakis. The main theorem is the following.

Theorem 3.8. *Assume* $\mathsf{ZF} + \mathsf{AD}$ *and all sets of reals are Suslin. Let* Γ *be a boldface pointclass closed under* $\wedge$, $\vee$, *and* $\forall^{\mathbb{R}}$ *with the scale property. Then the countable length everywhere club uniformization holds for* $\delta(\Gamma)$. *In particular, for all* $n \in \omega$, *the countable length everywhere club uniformization holds for* $\boldsymbol{\delta}^1_{2n+1}$.

2. Basics

Definition 2.1. Let X be a set and κ be an ordinal. ${}^{\kappa}X$ is the set of all functions from κ into X. ${}^{<\kappa}X$ is the set of all functions $\ell : \epsilon \to X$ where $\epsilon < \kappa$. In this case, one writes $|\ell| = \epsilon$ to indicate the length of ℓ is ϵ. If $s \in {}^{<\kappa}X$ and $t \in {}^{<\kappa}X \cup {}^{\kappa}X$, one write $s \subseteq t$ to indicate that t is an extension of s.

Let X be given the discrete topology and ${}^{\omega}X$ be given the product of the discrete topology. If $s \in {}^{<\omega}X$, then let $N_s^X = \{f \in {}^{\omega}X : s \subseteq f\}$. The topology on ${}^{\omega}X$ is equivalent to the topology generated by $\{N_s^X : s \in {}^{<\omega}X\}$ as a basis.

A tree on X is a set $T \subseteq {}^{<\omega}X$ which is closed under the substring relation $\subseteq$. If T is a tree, let $[T] = \{f \in {}^{\omega}X : (\forall^{\omega} n)(f \upharpoonright n \in T)\}$. A set $A \subseteq {}^{\omega}X$ is closed if and only if there is a tree T so that $A = [T]$.

As common in descriptive set theory, $\mathbb{R}$ may be used to denote either ${}^{\omega}\omega$ or ${}^{\omega}2$.

Definition 2.2. A strategy on a set X is a function $\rho : {}^{<\omega}X \to X$. The run of a strategy ρ_1 against a strategy ρ_2 is denoted $\rho_1 * \rho_2 \in {}^{\omega}X$ and it is defined recursively as follows: Suppose $\rho_1 * \rho_2 \upharpoonright n$ has been defined, if n is even, then $(\rho_1 * \rho_2)(n) = \rho_1(\rho_1 * \rho_2 \upharpoonright n)$ and if n is odd, then $(\rho_1 * \rho_2)(n) = \rho_2(\rho_1 * \rho_2 \upharpoonright n)$.

If $A \subseteq {}^{\omega}X$, then one says that A (or the game on X with payoff set A) is determined if either there is a strategy ρ_1 so that for all strategies ρ_2, $\rho_1 * \rho_2 \in A$ or there is a strategy ρ_2 so that for all strategies ρ_1, $\rho_1 * \rho_2 \notin A$. Intuitively, the game G_A^X consists of Player 1 and Player 2 taking turns playing elements of X where Player 1 wins if and only if the joint infinite run belongs to A. Thus the determinacy of A is the existence of a winning strategy for one of the two players in this game.

AD_X is the statement that for all $A \subseteq {}^{\omega}X$, A is determined (as a game on X). The common determinacy axioms are AD_ω (which is denoted simply AD) and $\mathsf{AD}_{\mathbb{R}}$. $\mathsf{AD}_{\frac{1}{2}\mathbb{R}}$ is the determinacy of games on $\mathbb{R}$ where one player is required to play only elements of ω.

If $x \in {}^{\omega}X$, let ρ_x be the strategy such that if $s \in {}^{<\omega}\omega$ has length $2n$ or $2n+1$, then $\rho_x(s) = x(n)$. That is, ρ_x can be used as either a Player 1 or Player 2 strategy which simply outputs the bits of x on each turn.

If $x \in {}^{\omega}X$, let $x_{\text{even}}, x_{\text{odd}} \in {}^{\omega}X$ be defined by $x_{\text{even}}(k) = x(2k)$ and $x_{\text{odd}}(k) = x(2k+1)$.

Let ρ be a strategy. Define $\Sigma^1_\rho, \Sigma^2_\rho : {}^{\omega}X \to {}^{\omega}X$ by $\Sigma^1_\rho(z) = \rho * \rho_z$ and $\Sigma^2_\rho(z) = \rho_z * \rho$. Define $\Xi^1_\rho : {}^{\omega}X \to {}^{\omega}X$ by $\Xi^1_\rho(z) = (\Sigma^1_\rho(z))_{\text{even}} = (\rho * \rho_z)_{\text{even}}$. Ξ^1_ρ is a Lipschitz function which simply collects the moves of ρ (used as a Player 1 strategy) when played against ρ_z. Similar, $\Xi^2_\rho : {}^{\omega}X \to {}^{\omega}X$ is defined by $\Xi^2_\rho(z) = (\Sigma^2_\rho(z))_{\text{odd}} = (\rho_z * \rho)_{\text{odd}}$.

The article will work implicitly under $\mathsf{ZF} + \mathsf{AD}$ and additional assumptions will be made explicit.

Next, one will review the necessary concepts concerning prewellordering and scales. See [8], [10] Chapter 2, 4, and 6, and [5] Section 2.

Definition 2.3. Pointclasses are collections of subsets of spaces of the form ${}^{j}\omega \times {}^{k}\mathbb{R}$ where $j, k \in \omega$. A pointclass is boldface if it is closed under continuous substitution. If Γ is a pointclass, then $\check{\Gamma}$ denotes the dual pointclass and $\Delta = \Gamma \cap \check{\Gamma}$.

Definition 2.4. A norm on a set A is a map $\phi : A \to \mathrm{ON}$. The associated prewellordering on A is $\preceq_\phi \subseteq A \times A$ defined by $x \preceq_\phi y$ if and only if $\phi(x) \leq \phi(y)$. (One will use the term norm and prewellordering interchangeably.) The length of the prewellordering ϕ is the ordertype of $\phi[A]$.

Now suppose X is a set. Let $P \subseteq {}^{\omega}X$ and $\phi : P \to \mathrm{ON}$ be a norm on P. Define a relation $\leq^*_\phi \subseteq {}^{\omega}X \times {}^{\omega}X$ by $f \leq^*_\phi g$ if and only if $f \in P \wedge (g \notin P \vee \phi(f) \leq \phi(g))$. Similarly, one defines $<^*_\phi$ by replacing $\leq$ with $<$ in the definition of $\leq^*_\phi$.

Let Γ be a pointclass closed under $\wedge$ and $\vee$. Suppose $P \subseteq {}^{\omega}\omega$ and $\phi : P \to \mathrm{ON}$ is a prewellordering. ϕ is a Γ-norm if and only if $P \in \Gamma$ and $\leq^*_\phi, <^*_\phi \in \Gamma$.

Let $\delta(\Gamma)$ be the supremum of the length of all prewellorderings ϕ on ${}^{\omega}\omega$ such that $\preceq_\phi \in \Delta$. $\delta(\Gamma)$ is called the prewellordering ordinal of Γ. Let Θ be the supremum of the length of all prewellorderings on $\mathbb{R}$. (Every ordinal considered in this article will be below Θ.)

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Definition 2.5. Fix a recursive bijection $\mathsf{pair} : \omega \times \omega \to \omega$. If $x \in {}^\omega\omega$ and $n \in \omega$, let $\widehat{x}_n \in {}^\omega\omega$ be defined by $\widehat{x}_n(k) = x(\mathsf{pair}(n,k))$. $\widehat{x}_n$ is the n^{th}-section of x. If $Z \subseteq {}^\omega\omega$, let $\widehat{Z} = \{x \in {}^\omega\omega : (\forall n)(\widehat{x}_n \in Z)\}$.

If $x \in {}^\omega 2$, let $\mathcal{R}_x \subseteq \omega \times \omega$ be defined by $\mathcal{R}_x(a,b)$ if and only if $x(\mathsf{pair}(a,b)) = 1$.

Let WO be the Π^1_1-complete set of $w \in \mathbb{R}$ so that $\mathcal{R}_x$ is a wellordering. Let $\text{ot} : \text{WO} \to \omega_1$ be the ordertype function. ot is a Π^1_1-norm on WO. If $w \in \text{WO}$ and $n \in \text{field}(\mathcal{R}_w)$, then let $\text{ot}(w,n)$ denote the ordertype of n in $\mathcal{R}_w$. If $w \in \text{WO}$ and $\alpha < \text{ot}(w)$, then let $\mathsf{num}(w,\alpha)$ be the unique element of ω with ordertype α according to $\mathcal{R}_w$.

Definition 2.6. Let X be a set. A set $A \subseteq {}^\omega X$ is Suslin if and only if there is an ordinal δ and a tree T on $X \times \delta$ so that $A = \pi_1[[T]]$, where $\pi_1 : {}^\omega X \times {}^\omega\delta \to {}^\omega X$ is the projection onto the first coordinate. T is called a Suslin representation for A. A set A is coSuslin if and only if ${}^\omega X \setminus A$ is Suslin.

Let X be a set and $A \subseteq {}^\omega X$. A sequence of norms on A, $\bar{\phi} = \langle \phi_n : n \in \omega\rangle$, is a semiscale if and only if for all $f \in {}^\omega X$ and sequence $\bar{f} = \langle f_n : n \in \omega\rangle$ of elements in A so that

(1) $f = \lim_{n\in\omega} f_n$ (in the natural topology on ${}^\omega X$).
(2) For all $n \in \omega$, there is a $\lambda_n \in \text{ON}$ so that $\lim_{i\in\omega} \phi_n(f_i) = \lambda_n$ (i.e. is eventually constant taking value λ_n).

one has that $f \in A$.

A semiscale $\bar{\phi}$ on A is good if and only if for any sequence $\bar{f}$ which satisfies just (2) above, there is an $f \in {}^\omega X$ so that $f = \lim_{n\in\omega} f_n$. A semiscale is very good if and only it is good and for all $x, y \in A$ and $n \in \omega$, $\phi_n(x) \leq \phi_n(y)$ implies that for all $m \leq n$, $\phi_m(x) \leq \phi_m(y)$.

A semiscale $\bar{\phi}$ on A is a scale if and only if if it satisfies the lower semicontinuity property: Using the notation of (1) and (2) above, for all $n \in \omega$, $\phi_n(f) \leq \lambda_n$.

Every semiscale $\bar{\phi}$ on A yields a Suslin representation for A. Suppose a tree T on $X \times \delta$ is a Suslin representation for A. If $f \in A$, then the tree $T_f = \{u \in {}^{<\omega}\delta : (f \restriction |u|, u) \in T\}$ has an infinite path so let L^T_f denote the leftmost path of T_f. The canonical semiscale for A derived from T is $\bar{\varphi}^T = \langle \varphi^T_n : n \in \omega\rangle$ defined by $\varphi^T_n(f) = L^T_f(n)$. (See [10] Theorem 2B.1.) Also every semiscale on A can be converted into a very good semiscale by a standard procedure (see [10] Lemma 4E.2).

Definition 2.7. Let $R \subseteq \mathbb{R} \times \mathbb{R}$. A uniformization for R is a function $\Phi : \text{dom}(R) \to \mathbb{R}$ so that for all $x \in \text{dom}(R)$, $R(x, \Phi(x))$. Let $\mathsf{Uniformization}$ be the statement that every relation $R \subseteq \mathbb{R} \times \mathbb{R}$ has a uniformization.

If R has a Suslin representation, then R has a uniformization. By a game argument, $\mathsf{AD}_{\frac{1}{2}\mathbb{R}}$ implies $\mathsf{Uniformization}$. Kechris [6] showed that over AD, $\mathsf{AD}_{\frac{1}{2}\mathbb{R}}$ and $\mathsf{Uniformization}$ are equivalent. $L(\mathbb{R}) \models \neg\mathsf{AD}_{\frac{1}{2}\mathbb{R}}$ since the relation $S(x,y)$ if and only if y is not OD_x cannot be uniformized in $L(\mathbb{R})$.

In this article, one will be concerned about the determinacy of certain games on ordinals. Generally, the determinacy of all games on uncountable ordinals is not consistent. However, the following result states that games with Suslin and coSuslin payoff sets are determined.

Fact 2.8. *([7] Theorem 2.8) Suppose $\kappa < \Theta$. Let $A \subseteq {}^\omega\kappa$ and suppose that A is Suslin and coSuslin. Then the game on κ with payoff set A is determined.*

To apply Fact 2.8, one will need to show some relevant ordinal games have Suslin and coSuslin payoff sets. Moreover, it will be very important in certain instances to have that the Suslin representations are obtained uniformly from certain objects. Next, one will give some closure properties of Suslin representations with a particular focus on uniformity.

Fact 2.9. *Let κ be a cardinal. Let $\epsilon < \omega_1$, $w \in \text{WO}$ with $\text{ot}(w) = \epsilon$, $\nu < \kappa$, and $\ell : \epsilon \to \nu$. Define a relation $R^\nu_\ell \subseteq {}^\omega\nu$ by $R^\nu_\ell(g)$ if and only if $\text{rang}(\ell) \subseteq \text{rang}(g)$. Then R^ν_ℓ is Suslin and coSuslin uniformly in ℓ, ν, and w. The term "uniformly" means there is a function $\mathfrak{T}$ and $\mathfrak{U}$ so that whenever ℓ, w, and ν have the above property, $\mathfrak{T}(\ell, w, \nu)$ and $\mathfrak{U}(\ell, w, \nu)$ are trees on $\nu \times \kappa$, $R^\nu_\ell = \pi_1[[\mathfrak{T}(\ell, w, \nu)]]$, and ${}^\omega\nu \setminus R^\nu_\ell = \pi_1[[\mathfrak{U}(\ell, w, \nu)]]$, where $\pi_1 : {}^\omega\nu \times {}^\omega\kappa \to {}^\omega\nu$ is the projection onto the first coordinate.*

Proof. Fix a bijection $\Upsilon : \kappa \to {}^{<\omega}\kappa$ with the property that for all $s, t \in {}^{<\omega}\kappa$, if $s \subseteq t$, then $\Upsilon^{-1}(s) \leq \Upsilon^{-1}(t)$. If $s, t \in {}^{<\omega}\nu$, then say that s is compatible with t if $s \subseteq t$ or $t \subseteq s$. Since $w \in \text{WO}$ with $\text{ot}(w) = \epsilon$, let $B_w : \omega \to \epsilon$ be the canonical bijection obtained from w.

Let $S^\nu_\ell = {}^\omega\nu \setminus R^\nu_\ell$. Note that S^ν_ℓ is a countable union of closed sets in the topology of ${}^\omega\nu$ and thus has the following simple Suslin representation. Let U be a tree on ${}^\omega\nu \times \omega$ by $(s,u) \in U$ if and only if $|s| = |u| = 0$ or there exists an $m \in \omega$ so that u is the constant sequence taking value m and $\ell(B_w(m)) \notin \text{rang}(s)$. If $m \in \omega$, let $\bar{m} \in {}^\omega\omega$ be the constant infinite sequence taking value m. Note that if $g \in \pi_1[[U]]$ then there exists an $m \in \omega$ so that $(g,\bar{m}) \in [U]$. So for all $n \in \omega$, $(g \restriction n, \bar{m} \restriction n) \in U$ which implies that for all $n \in \omega$, $\ell(B_w(m)) \notin \text{rang}(g \restriction n)$. Hence $\ell(B_w(m)) \notin \text{rang}(g)$ and $\neg(\text{rang}(\ell) \subseteq \text{rang}(g))$. Thus $S^\nu_\ell(g)$. Conversely, suppose $S^\nu_\ell(g)$ which means $\neg(\text{rang}(\ell) \subseteq \text{rang}(g))$. Since B_w is a bijection between ω and $|\ell|$, there is some $m \in \omega$ so that $\ell(B_w(m)) \notin \text{rang}(g)$. Then $(g,\bar{m}) \in [U]$ and thus $g \in \pi_1[[U]]$. This shows that S^ν_ℓ is Suslin and therefore R^ν_ℓ is coSuslin.

Define a tree T on $\nu \times \kappa$ by $(s,u) \in T$ if and only if for all $k < |s|$, the following holds

- $\Upsilon(u(k)) \in {}^{<\omega}\nu$.
- $\ell(B_w(k)) \in \text{rang}(\Upsilon(u(k)))$.
- $\Upsilon(u(k))$ is compatible with s.

Suppose $g \in \pi_1[[T]]$. Then there exists an $h \in {}^\omega\kappa$ so that $(g,h) \in [T]$. Thus for all $k \in \omega$, $\Upsilon(h(k)) \subseteq g$ and $\ell(B_w(k)) \in \text{rang}(\Upsilon(h(k)))$. Hence $\text{rang}(\ell) \subseteq \text{rang}(g)$ which is equivalent to $R^\nu_\ell(g)$. Conversely, suppose $R^\nu_\ell(g)$. Then $\text{rang}(\ell) \subseteq \text{rang}(g)$. For each $n \in \omega$, let k_n be least $k \in \omega$ so that $g(k) = \ell(B_w(n))$. Let $h(n) = \Upsilon^{-1}(g \restriction k_n + 1)$. Then $(g,h) \in [T]$. Thus $g \in \pi_1[[T]]$. Observe that this explicit h is actually the left-most branch, L^T_g, of T corresponding to g. It has been shown that $g \in \pi_1[[T]]$ if and only if $R^\nu_\ell(g)$. R^ν_ℓ is Suslin.

Observe that both trees U and T are produced uniformly from ℓ, ν, and w. Let $\mathfrak{U}(\ell,w,\nu) = U$ and $\mathfrak{T}(\ell,w,\nu) = T$. ∎

Fact 2.10. *Assume the setting of Fact 2.9. Let $\bar{\varphi}^{\ell,w,\nu} = \langle \varphi^{\ell,w,\nu}_n : n \in \omega\rangle$ be the canonical semiscale derived from $\mathfrak{T}(\ell,w,\nu)$ (using the leftmost branch as in Definition 2.6) and $\bar{\varpi}^{\ell,w,\nu} = \langle \varpi^{\ell,w,\nu}_n : n \in \omega\rangle$ be the canonical semiscale derived from $\mathfrak{U}(\ell,w,\nu)$. For all $n \in \omega$, the norm relations $\leq^*_{\varphi^{\ell,w,\nu}_n}$, $<^*_{\varphi^{\ell,w,\nu}_n}$, $\leq^*_{\varpi^{\ell,w,\nu}_n}$, and $<^*_{\varpi^{\ell,w,\nu}_n}$ are Suslin and coSuslin.*

Proof. The notation from the statement and proof of Fact 2.9 will be used. Note that from the definition of $\mathfrak{U}(\ell,w,\nu)$ from Fact 2.9, the leftmost branch $L^{\mathfrak{U}(\ell,w,\nu)}_g$ is simply $\bar{m}$ where m is least so that $\ell(B_w(m)) \notin \text{rang}(g)$.

For each $m \in \omega$, let $E_m = \{g \in {}^\omega\nu : (\forall k < m)(\ell(B_w(k)) \in \text{rang}(g)) \wedge \ell(B_w(m)) \notin \text{rang}(g)\}$. One can check that E_m is Suslin and coSuslin using arguments similar to Fact 2.9 (as it is an intersection of an open and a closed set). Observe that for any $k \in \omega$

$$\leq^*_{\varpi^{\ell,w,\nu}_k} = \left(\bigcup_{m \leq n} E_m \times E_n\right) \cup \left(\bigcup_{m \in \omega} E_m \times R^\nu_\ell\right)$$

and

$$<^*_{\varpi^{\ell,w,\nu}_k} = \left(\bigcup_{m < n} E_m \times E_n\right) \cup \left(\bigcup_{m \in \omega} E_m \times R^\nu_\ell\right).$$

These norm relations are Suslin and coSuslin by Fact 2.9 and the earlier observations.

In the proof of Fact 2.9, one showed that if $f \in \pi_1[[\mathfrak{T}(\ell,w,\nu)]]$, then the left-most branch $L^{\mathfrak{T}(\ell,w,\nu)}_f$ is explicitly given by the following: For each $n \in \omega$, let k_n be the least $k \in \omega$ such that $f(k) = \ell(B_w(n))$. Then $L^{\mathfrak{T}(\ell,w,\nu)}_f(n) = \Upsilon^{-1}(f \restriction k_n + 1)$.

Let $A_n \subseteq \kappa$ be the collection of γ so that $|\Upsilon(\gamma)| > 0$, $\Upsilon(\gamma)(|\Upsilon(\gamma)|-1) = \ell(B_w(n))$ and for all $i < |\Upsilon(\gamma)|-1$, $\Upsilon(\gamma)(i) \neq \ell(B_w(n))$. Define a tree K_n on $\nu \times \nu \times \kappa \times \kappa$ by $(s,t,u,v) \in K_n$ if and only if $u(0) \leq v(0)$, $u(0), v(0) \in A_n$, $\Upsilon(u(0))$ is compatible with s, and $\Upsilon(v(0))$ is compatible with t. Define a tree J_n similarly with $u(0) \leq v(0)$ replaced with $u(0) < v(0)$.

Note that if $f, g \in R^\nu_\ell$, then $\varphi^{\ell,w,\nu}_n(f) \leq \varphi^{\ell,w,\nu}_n(g)$ if and only if $L^{\mathfrak{T}(\ell,w,\nu)}_f(n) \leq L^{\mathfrak{T}(\ell,w,\nu)}_g(n)$ if and only if $(\exists x, y)((f,g,x,y) \in [K_n])$. Similarly, if $f, g \in R^\nu_\ell$, then $\varphi^{\ell,w,\nu}_n(f) < \varphi^{\ell,w,\nu}_n(g)$ if and only if $L^{\mathfrak{T}(\ell,w,\nu)}_f(n) < L^{\mathfrak{T}(\ell,w,\nu)}_g(n)$ if and only if $(\exists x, y)((f,g,x,y) \in [J_n])$.

18 Aug 2021 22:34:40 PDT
210818-Chan Version 1 - Submitted to J. Symb. Logic

Note that $f \leq^*_{\varphi_n^{\ell,w,\nu}} g$ if and only if

$$f \in R^\nu_\ell \wedge (g \notin R^\nu_\ell \vee (\exists x, y)((f, g, x, y) \in [K_n])).$$

Also $\neg(f \leq^*_{\varphi_n^{\ell,w,\nu}} g)$ if and only if

$$f \notin R^\nu_\ell \vee (g \in R^\nu_\ell \wedge (\exists x, y)((g, f, x, y) \in [J_n])).$$

This shows that $\leq^*_{\varphi_n^{\ell,w,\nu}}$ is Suslin and coSuslin. A similar argument shows that $<^*_{\varphi_n^{\ell,w,\nu}}$ is also Suslin and coSuslin. □

Fact 2.11. *([10] 6E; Moschovakis Third Periodicity Theorem) Assume $\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}_\mathbb{R}$. Let $\nu < \Theta$. Let $A \subseteq {}^\omega\nu$ be Suslin and $\bar{\varphi} = \langle \varphi_n : n \in \omega \rangle$ be a very good semiscale on A. For each $n \in \omega$, odd $m \in \omega$, and $s, t \in {}^m\nu$, define the game ${}^nH^s_t$ on ν as in the following diagram.*

$$\begin{array}{c|cccccccccccc} & s & F\ a_0 & & & S\ a_1 & F\ a_2 & & & F\ a_3 & \cdots & a \\ {}^nH^s_t & & & & & & & & & & & \\ & t & & S\ b_0 & F\ b_1 & & & S\ b_2 & F\ b_3 & & \cdots & b \end{array}$$

The games has two players called the first and second player making move in ν as indicated in the diagram. Say that the second player wins if and only if $s\hat{\ }a \leq^*_{\varphi_n} t\hat{\ }b$.

Assume that Player 1 has a winning strategy in the game G^ν_A on ν with payoff set A. Assume for all odd m and integer n, the games ${}^nH^s_t$ are determined. Then uniformly from ν, A, and the very good semiscale $\bar{\varphi}$, one can obtain a strategy σ for Player 1 in G^ν_A. (This means there is a function Φ so that whenever ν, A, and $\bar{\varphi}$ has the above property, $\Phi(\nu, A, \bar{\varphi})$ is a Player 1 winning strategy for G^ν_A.)

Proof. This result is essentially a coarse form of the Moschovakis third periodicity theorem for ordinal value games using the idea of the "best" strategy. (The definability estimates for the strategy will not be relevant here.) The uniformity statement will be essential so an explicit definition of the Player 1 winning strategy will be provided. The reader can see [10] 6D and 6E or [5] Section 2 for the details.

For each odd m, let W_m be the set of $s \in {}^m\nu$ so that Player 1 can win in the game G^ν_A when s is an initial partial run (m being odd implies that Player 2 is the next player to respond). Since Player 1 is assumed to have a winning strategy, for all odd $m \in \omega$, W_m is nonempty. For $s, t \in W_m$, define $s \preceq^m_n t$ if and only if the second player has a winning strategy in ${}^nH^s_t$. It can be shown that $\preceq^m_n$ is a prewellordering on W^m.

If $k \in \omega$ and $u \in {}^{2k}\nu$, then define $\sigma(u)$ to be the least $\gamma < \nu$ so that $u\hat{\ }\gamma \in W_{2k+1}$ and for all η with $u\hat{\ }\eta \in W_{2k+1}$, $u\hat{\ }\gamma \preceq^{2k+1}_k u\hat{\ }\eta$. (If u is odd length, then let $\sigma(u) = 0$ as this case is irrelevant because σ is intended to be used as a Player 1's strategy.) It can be shown that σ is a Player 1's winning strategy in G^ν_A and is produced uniformly from ν, A, and $\bar{\varphi}$. □

Fact 2.12. *Let $\nu < \Theta$. Let $\Xi : {}^\omega\nu \to {}^\omega\omega$ be a Lipschitz continuous function. Suppose Γ is a (boldface) pointclass and $\Delta = \Gamma \cap \check{\Gamma}$. Suppose there is a norm $\varphi : W \to \nu$ so that $W \in \Delta$ and the associated prewellordering $\preceq_\varphi$ on W is also in Δ. Then $\Xi[{}^\omega\nu]$ is $\exists^\mathbb{R}\Delta$.*

Proof. First, a simple coding of ${}^{<\omega}\nu$ by reals will be developed. Let finS consists of reals z so that $(\forall i < \widehat{z}_0(0))(\widehat{z}_{i+1} \in W)$. Let $\mathsf{finseq} : \mathsf{finS} \to {}^{<\omega}\nu$ by $\mathsf{finseq}(z)$ is a sequence of length $\widehat{z}_0(0)$ and for all $i < \widehat{z}_0(0)$, $\mathsf{finseq}(z)(i) = \varphi(\widehat{z}_{i+1})$. Note that finseq is a surjection of finS onto ${}^{<\omega}\nu$. Note that the relation $u, v \in \mathsf{finS}$ and $\mathsf{finseq}(u) = \mathsf{finseq}(v)$ is Δ since φ is a norm in Δ. Also the relation $u, v \in \mathsf{finS}$ and $\mathsf{finseq}(u) \subsetneq \mathsf{finseq}(v)$ is Δ.

Fix a bijection $\Upsilon : \nu \to {}^{<\omega}\nu$. Next one will show that Υ has a coded version which is $\exists^\mathbb{R}\Delta$. Define $Z \subseteq W \times \mathsf{finS}$ by $Z(w, u)$ if and only if $\Upsilon(\varphi(w)) = \mathsf{finseq}(u)$. By the Moschovakis coding lemma ([10] Section 7D or [5] Theorem 2.12), there is a $\bar{Z} \in \exists^\mathbb{R}\Delta$ so that $\bar{Z} \subseteq Z$ and for all $\alpha < \nu$, $\bar{Z} \cap (\varphi^{-1}[\{\alpha\}] \times \mathbb{R}) \neq \emptyset$ if and only if $Z \cap (\varphi^{-1}[\{\alpha\}] \times \mathbb{R}) \neq \emptyset$.

Since Ξ is Lipschitz, there is a function $\tau : {}^{<\omega}\nu \to \omega$ so that $\Xi = \Xi^2_\tau$ using the notation of Defintion 2.2. Define $Y \subseteq W \times \omega$ by $Y(w, n)$ if and only if $\tau(\Upsilon(\varphi(w))) = n$. By the Moschovakis coding lemma, there is a $\bar{Y} \in \exists^\mathbb{R}\Delta$ so that $\bar{Y} \subseteq Y$ and for all $\alpha < \nu$, $\bar{Y} \cap (\varphi^{-1}[\{\alpha\}] \times \omega) \neq \emptyset$ if and only if $Y \cap (\varphi^{-1}[\{\alpha\}] \times \omega) \neq \emptyset$.

Now observe that $x \in \Xi[{}^\omega\nu]$ if and only if there exists an $y \in {}^\omega\omega$ so that the conjunction of the following holds

18 Aug 2021 22:34:40 PDT
210818-Chan Version 1 - Submitted to J. Symb. Logic

(1) For all $n \in \omega$, $\widehat{y}_n \in W$.
(2) For all $n \in \omega$, there exist $v, w \in W$ and there exists a $u \in \mathsf{finS}$ so that $\varphi(v) = \varphi(w)$, for all $i < |\mathsf{finseq}(u)|$, $\mathsf{finseq}(u)(i) = \varphi(\widehat{y}_i)$, $\bar{Z}(v, u)$, and $\bar{Y}(w, x(n))$.

Intuitively, if one lets $f \in {}^\omega\nu$ be defined by $f(n) = \varphi(\widehat{y}_n)$, then the above asserts that $x = \Xi^2_\tau(f)$. The above expression is $\exists^{\mathbb{R}}\Delta$. □

Fact 2.13. *(Boundedness Principle) Suppose Γ is a (boldface) pointclass closed under $\forall^{\mathbb{R}}$. Suppose $W \in \Gamma$ is a Γ-complete set with a surjective Γ-norm $\varphi : W \to \kappa$. If $A \subseteq W$ and $A \in \check{\Gamma}$, then there is a $\delta < \kappa$ so that $\varphi[A] \subseteq \delta$.*

Fact 2.14. *(Moschovakis [5] Theorem 2.6 and Lemma 2.13) Let Γ be a pointclass closed under $\wedge$, $\vee$, and $\forall^{\mathbb{R}}$ with a Γ-complete set W and a Γ-norm φ on W. Then the length of φ is $\delta(\Gamma)$ and $\delta(\Gamma)$ is a regular cardinal.*

Recall that under AD, Wadge's lemma implies that every nonselfdual boldface pointclass has complete sets. Solovay club coding techniques for ω_1 under AD can be extended to $\delta(\Gamma)$ when Γ is a nonselfdual pointclass closed under $\forall^{\mathbb{R}}$.

Fact 2.15. *Let Γ be a nonselfdual boldface pointclass closed under $\forall^{\mathbb{R}}$. Let $\kappa = \delta(\Gamma)$. Let W be a Γ-complete set with surjective Γ-norm $\varphi : W \to \kappa$. Let $\mathsf{clubcode} \subseteq \mathbb{R}$ consists of the strategies τ with the property*

$$(\forall w)(w \in W \Rightarrow (\Xi^2_\tau(w) \in W \wedge \varphi(\Xi^2_\tau(w)) > \varphi(w))).$$

If $\tau \in \mathsf{clubcode}$, then let

$$\mathfrak{C}_\tau = \{\eta < \kappa : (\forall w \in W)(\varphi(w) < \eta \Rightarrow \varphi(\Xi^2_\tau(w)) < \eta)\}.$$

$\mathfrak{C}_\tau$ is a club. If $C \subseteq \kappa$ is club, then there is a $\tau \in \mathsf{clubcode}$ so that $\mathfrak{C}_\tau \subseteq C$.

If $A \subseteq \mathsf{clubcode}$ is $\check{\Gamma}$, then uniformly in A, one can produce a club C so that for all $\tau \in A$, $C \subseteq \mathfrak{C}_\tau$. (Uniformly here means there is a function Υ so that whenever $A \subseteq \mathsf{clubcode}$ is $\check{\Gamma}$, $\Upsilon(A)$ is club with the property that for all $\tau \in A$, $\Upsilon(A) \subseteq \mathfrak{C}_\tau$.)

Proof. These are proved using the boundedness principle (Fact 2.13). See [2] Fact 4.7 for a similar argument. □

The simplest example of the Kechris-Woodin generic coding function occurs at ω_1. In this case, the generic coding function is very explicitly defined with no use of scale concepts.

Definition 2.16. Let $\alpha \in \omega_1$. If $s \in {}^{<\omega}\alpha$, let $N^\alpha_s = \{f \in {}^\omega\alpha : s \subseteq f\}$. The topology on ${}^\omega\alpha$ generated by $\{N^\alpha_s : s \in {}^{<\omega}\alpha\}$ as a basis is homeomorphic to ${}^\omega\omega$. Thus the familiar category notion can be formulated for ${}^\omega\alpha$ in this topology. Let surj_α be the collection of $f \in {}^\omega\alpha$ such that $f[\omega] = \alpha$, i.e. f is a surjection onto α. surj_α is comeager in ${}^\omega\alpha$.

Recall that under AD, the category ideal has full wellordered additivity. That is, if δ is an ordinal and $\langle X_\alpha : \alpha < \delta\rangle$ is a collection of meager subsets of $\mathbb{R}$, then $\bigcup_{\alpha<\delta} X_\alpha$ is a meager subset of $\mathbb{R}$. Thus the meager ideal on ${}^\omega\alpha$ also has the full wellordered additivity.

Fact 2.17. *There is a function $\mathfrak{G} : {}^\omega\omega_1 \to \mathrm{WO}$ so that for all $\alpha < \omega_1$, if $f \in \mathsf{surj}_\alpha$, then $\mathrm{ot}(\mathfrak{G}(f)) = \alpha$.*

Proof. Let $A_f = \{n \in \omega : (\forall m)(m < n \Rightarrow f(m) \neq f(n))\}$. Define $\mathfrak{G}(f) \in \mathbb{R}$ so that $\mathcal{R}_{\mathfrak{G}(f)}(a, b) = 1 \Leftrightarrow a, b \in A_f \wedge f(a) < f(b)$. Note that the domain of $\mathcal{R}_{\mathfrak{G}(f)}$ is A_f and $\mathfrak{G}(f) \in \mathrm{WO}$. If $f \in \mathsf{surj}_\alpha$, then $(A_f, \mathcal{R}_{\mathfrak{G}(f)})$ is order-isomorphic to α. □

The following results are generalizations of the category boundedness arguments found in the proof of the main theorems in [3].

Fact 2.18. *Let $\nu < \omega_1$ and κ be a cardinal with $\mathrm{cof}(\kappa) > \omega$. Suppose $A \subseteq {}^\omega\nu$ is comeager in ${}^\omega\nu$ and $\Phi : A \to \kappa$. Then there is a $\delta < \kappa$ and a comeager $B \subseteq A$ so that $\Phi[B] \subseteq \delta$.*

Proof. For each $\alpha < \kappa$, let $A_\alpha = \{f \in A : \Phi(f) = \alpha\}$. Note that $A = \bigcup_{\alpha<\kappa} A_\alpha$. Let $T = \{\alpha < \kappa : A_\alpha \text{ is nonmeager}\}$. AD implies that a wellordered union of meager sets in ${}^\omega\nu$ is meager and since A is not meager, $T \neq \emptyset$. Since AD implies that all sets of reals have the Baire property and there are no uncountable sets of disjoint open subsets of ${}^\omega\nu$, one has that T must be countable. Since $\mathrm{cof}(\kappa) > \omega$, $\sup(T) < \kappa$. Let $\delta = \sup(T) + 1 < \kappa$. Note that $B = \bigcup_{\alpha<\delta} A_\alpha$ is comeager. By definition of B, $\Phi[B] \subseteq \delta$. □

18 Aug 2021 22:34:40 PDT
210818-Chan Version 1 - Submitted to J. Symb. Logic

Fact 2.19. *Let $\nu < \omega_1$ and κ be cardinal with $\mathrm{cof}(\kappa) > \omega$. Let* club *be the set of club subsets of κ. Suppose $A \subseteq {}^\omega\nu$ is comeager and $\Phi : A \to \mathsf{club}$. Then uniformly from Φ, there is a club $C \subseteq \kappa$ and a comeager set $B \subseteq A$ so that for all $f \in B$, $C \subseteq \Phi(f)$.*

Proof. If $X \subseteq \kappa$ and $|X| = \kappa$, then let $\mathsf{enum}_X : \kappa \to X$ be the increasing enumeration of X. For $\alpha < \kappa$, let $E_\alpha : A \to \kappa$ be defined by $E_\alpha(f) = \mathsf{enum}_{\Phi(f)}(\alpha)$. For $\alpha < \delta < \kappa$, let $Y^\delta_\alpha = \{f \in A : E_\alpha(f) < \delta\}$. Define $K : \kappa \to \kappa$ by letting $K(\alpha)$ be the least δ so that Y^δ_α is comeager. Note that for each $\alpha < \kappa$, $K(\alpha)$ is well defined by Fact 2.18 applied to the function E_α. Since for any $f \in A$, $E_\alpha(f) \geq \alpha$, one has that $K(\alpha) > \alpha$. Also note that for any $\alpha_0 \leq \alpha_1$ and $f \in A$, $E_{\alpha_0}(f) \leq E_{\alpha_1}(f)$ and thus $K(\alpha_0) \leq K(\alpha_1)$.

Let $C = \{\eta < \kappa : (\forall \xi < \eta)(K(\xi) < \eta)\}$. (Note that C is produced uniformly from Φ.) First, to show C is unbounded. Let $\alpha < \kappa$. Let $\alpha_0 = \alpha$. If α_n has been defined, then let $\alpha_{n+1} = K(\alpha_n)$. By the property of K mentioned above, $\langle \alpha_n : n \in \omega\rangle$ is a strictly increasing sequence in κ. Let $\eta = \sup\{\alpha_n : n \in \omega\}$ and note that $\alpha < \eta < \kappa$ since $\mathrm{cof}(\kappa) > \omega$. Let $\xi < \eta$ be arbitrary. There is an $n \in \omega$ so that $\xi < \alpha_n$. Since $K(\xi) \leq K(\alpha_n) = \alpha_{n+1} < \eta$, one has that $K(\xi) < \eta$. Since $\xi < \eta$ was arbitrary, $\eta \in C$. Next to show C is closed. Suppose η is a limit point of C. Let $\xi < \eta$ be arbitrary. Then there is an $\eta' \in C$ with $\xi < \eta' < \eta$. Thus $K(\xi) < \eta' < \eta$. Since $\xi < \eta$ was arbitrary, one has that $\eta \in C$. It has been established that C is a club subset of κ.

Fix $\eta \in C$. For all $\xi < \eta$, $K(\xi) < \eta$ so Y^η_ξ is comeager. Let $Y^\eta = \bigcap_{\xi<\eta} Y^\eta_\xi$. Since wellordered intersection of comeager sets are comeager under AD, Y^η is comeager. Note that for each $f \in Y^\eta$ and $\xi < \eta$, $\xi \leq E_\xi(f) < \eta$. Since $\xi < \eta$ is arbitrary, $E_\xi(f) = \mathsf{enum}_{\Phi(f)}(\xi) \in \Phi(f)$, and $\Phi(f) \subseteq \kappa$ is a club, one must have that $\eta \in \Phi(f)$. Thus for all $f \in Y^\eta$, $\eta \in \Phi(f)$. Now let $Y = \bigcap_{\eta \in C} Y^\eta$. Again since a wellordering intersection of comeager sets is comeager under AD, Y is comeager. Take any $f \in Y$. For any $\eta \in C$, $f \in Y^\eta$. By the previous observation, $\eta \in \Phi(f)$. Since $\eta \in C$ was arbitrary, one has shown that $C \subseteq \Phi(f)$. □

Next, the more general notion of a reliable ordinal and its associated Kechris-Woodin generic coding function will be defined.

Definition 2.20. ([9]) An ordinal λ is reliable if and only if there is a $W \subseteq \mathbb{R}$ and a scale $\bar{\varphi} = \langle \varphi_i : i \in \omega\rangle$ on W with the following properties.

- $\varphi_0 : W \to \lambda$ is surjective.
- The norm relations $\leq^*_{\varphi_0}$ and $<^*_{\varphi_0}$ are Suslin and coSuslin.

$(W, \bar{\varphi})$ is called the witness to the reliability of λ.

For $\xi < \lambda$, say that $S \subseteq \lambda$ is ξ-honest if and only if there is a $w \in W$ such that $\varphi_0(w) = \xi$ and for all $n \in \omega$, $\varphi_n(w) \in S$. A set $S \subseteq \lambda$ is honest if and only if for all $\xi \in S$, S is ξ-honest. (Note that the notion of honest and ξ-honest depend on the witness to reliability.)

Fact 2.21. *Let λ be a regular reliable cardinal as witnessed by $(W, \bar{\varphi})$. Then for any $\alpha < \lambda$, there exists an α' such that $\alpha \leq \alpha' < \lambda$ and α' is honest.*

Proof. For each $\xi < \lambda$, there is a $\xi' \geq \xi$ so that ξ' is ξ-honest. To see this, pick any $w \in W$ so that $\varphi_0(w) = \xi$. Let $\xi' = \sup\{\varphi_n(w) + 1 : n \in \omega\}$. ξ' is ξ-honest. Since $\mathrm{cof}(\lambda) > \omega$, $\xi' < \lambda$. Let $\Lambda : \lambda \to \lambda$ be defined by $\Lambda(\xi)$ is the least ξ' with $\xi \leq \xi' < \lambda$ and ξ' is ξ-honest.

Let $\alpha_0 = \alpha$. If $\alpha_n < \lambda$ has been defined, let $\alpha_{n+1} = \sup \Lambda[\alpha_n]$. Note that $\alpha_{n+1} < \lambda$ since λ is regular. Let $\alpha' = \sup\{\alpha_n : n \in \omega\}$ and note that $\alpha' < \lambda$ since λ is regular. Now suppose $\xi < \alpha'$. There is some $n \in \omega$ so that $\xi < \alpha_n$. Since $\Lambda(\xi) \leq \alpha_{n+1}$, one has that α_{n+1} is ξ-honest. Since $\alpha_{n+1} \subseteq \alpha'$, one has that α' is ξ-honest. Since $\xi < \alpha'$ was arbitrary, this shows that α' is honest. □

Fact 2.22. *([9] Lemma 1.1) Let λ be a reliable ordinal with witness $(W, \bar{\varphi})$. Then there is a Lipschitz continuous function $\mathfrak{F} : {}^\omega\lambda \to \widehat{W}$ so that for all $f \in {}^\omega\lambda$, if $f[\omega]$ is honest, then for all $n \in \omega$, $\varphi_0(\widehat{\mathfrak{F}(f)}_n) = f(n)$.*

Definition 2.23. Let BS consists of the collection of $z \in \mathbb{R}$ so that $\widehat{z}_0 \in \mathrm{WO}$ and $\widehat{z}_1 \in \widehat{\mathrm{WO}}$, i.e. for all $n \in \omega$, $\widehat{(\widehat{z}_1)}_n \in \mathrm{WO}$. If $z \in \mathsf{BS}$, then let $\mathsf{seq}(z) \in {}^{<\omega_1}\omega_1$ be the sequence of length $\mathrm{ot}(\widehat{z}_0)$ such that for all $\alpha < \mathrm{ot}(\widehat{z}_0)$, $\mathsf{seq}(z)(\alpha) = \mathrm{ot}(\widehat{(\widehat{z}_1)}_{\mathsf{num}(\widehat{z}_0,\alpha)})$. Note that for all $\ell \in {}^{<\omega_1}\omega_1$, there is a $z \in \mathsf{BS}$ so that $\mathsf{seq}(z) = \ell$.

Definition 2.24. Let λ be an ordinal, $W \subseteq \mathbb{R}$, and $\varphi : W \to \lambda$ be a surjective norm. Let CS consists of those $z \in {}^\omega\omega$ so that $\widehat{z}_0 \in \mathrm{WO}$ and $\widehat{z}_1 \in \widehat{W}$, i.e. for all $n \in \omega$, $\widehat{(\widehat{z}_1)}_n \in W$.

For each $z \in \mathsf{CS}$, let $\mathsf{cseq}(z)$ be the sequence in λ of length $\mathrm{ot}(\widehat{z}_0)$ defined by $\mathsf{cseq}(z)(\alpha) = \varphi(\widehat{(\widehat{z}_1)}_{\mathsf{num}(\widehat{z}_0,\alpha)})$. Note that for all $\ell \in {}^{<\omega_1}\lambda$, there is some $z \in \mathsf{CS}$ so that $\mathsf{cseq}(z) = \ell$.

3. Countable Length Everywhere Club Uniformization

Definition 3.1. If κ is a cardinal, let club denote the collection of club subsets of κ. A relation $R \subseteq {}^{<\omega_1}\kappa \times \mathsf{club}$ is $\subseteq$-downward closed in the club-coordinate if and only if for all $\ell \in {}^{<\omega_1}\kappa$ and clubs $C \subseteq D$, $R(\ell, D)$ implies $R(\ell, C)$. Let $\mathrm{dom}(R) = \{\ell \in {}^{<\omega_1}\omega_1 : (\exists C \in \mathsf{club})R(\ell, C)\}$. A uniformization for R is a function $\Lambda : \mathrm{dom}(R) \to \mathsf{club}$ so that for all $\ell \in \mathrm{dom}(R)$, $R(\ell, \Lambda(\ell))$.

Countable length everywhere club uniformization for κ is the statement that for all $R \subseteq {}^{<\omega_1}\kappa \times \mathsf{club}$ which is $\subseteq$-downward closed in the club-coordinate, there is a uniformization for R.

Fact 3.2. *([3] Countable Length Everywhere Club Uniformization for ω_1) Assume $\mathsf{ZF} + \mathsf{AD}$. Let $R \subseteq {}^{<\omega_1}\omega_1 \times \mathsf{club}$ be $\subseteq$-downward closed in the club-coordinate. Let $\tilde{R} \subseteq \mathsf{BS} \times \mathsf{clubcode}$ be the coded version of R defined by $\tilde{R}(z, e)$ if and only if $R(\mathsf{seq}(z), \mathfrak{C}_e)$. Assume $\tilde{R}$ has a uniformization (i.e. a function $\Phi : \mathrm{dom}(\tilde{R}) \to \mathbb{R}$ so that for all $z \in \mathrm{dom}(\tilde{R})$, $\tilde{R}(z, \Phi(e))$). Then R has a uniformization.*

Thus, under $\mathsf{ZF} + \mathsf{AD}_{\frac{1}{2}\mathbb{R}}$, countable length everywhere club uniformization for ω_1 holds.

$L(\mathbb{R}) \models \neg\mathsf{AD}_{\frac{1}{2}\mathbb{R}}$. [3] gives an example to show that countable length everywhere club uniformization for ω_1 cannot hold in $L(\mathbb{R})$ and thus it is not provable under AD alone.

The main result is a generalization of the countable length everywhere club uniformization for ω_1.

Theorem 3.3. *Assume $\mathsf{ZF}+\mathsf{AD}+\mathsf{DC}_\mathbb{R}$. Let Γ be a nonselfdual boldface pointclass closed under $\wedge$, $\vee$, and $\forall^\mathbb{R}$. Let $\kappa = \delta(\Gamma)$ and assume that κ is reliable with witness $(W, \bar{\varphi})$ such that W is Γ-complete and $\varphi_0 : W \to \kappa$ is a surjective Γ-norm. Let CS and cseq be the coding of ${}^{<\omega_1}\kappa$ from Definition 2.24 defined relative to φ_0. Let $\mathsf{clubcode}$ and $\mathfrak{C}_e$ (for each $e \in \mathsf{clubcode}$) be the coding of club subsets of κ from Definition 2.15 relative to Γ and the Γ-norm φ_0 on the Γ-complete set W.*

Let $R \subseteq [\kappa]^{<\omega_1} \times \mathsf{club}$ be a $\subseteq$-downward closed relation in the club-coordinate. Let $\tilde{R} \subseteq \mathsf{CS} \times \mathsf{clubcode}$ be the coded version of R defined by $\tilde{R}(z, e)$ if and only if $R(\mathsf{cseq}(z), \mathfrak{C}_e)$. Assume that $\tilde{R}$ is Suslin and coSuslin meaning there are trees $\mathcal{T}$ on $\omega \times \omega \times \zeta_0$ and $\mathcal{S}$ on $\omega \times \omega \times \zeta_1$ so that $\tilde{R} = \{(z, e) : (\exists f \in {}^\omega\zeta_0)((z, e, f) \in [T])\}$ and $\mathbb{R} \times \mathbb{R} \setminus \tilde{R} = \{(z, e) : (\exists g \in {}^\omega\zeta_1)((z, e, f) \in [S])\}$. Let $\bar{\phi} = \langle \phi_n : n \in \omega \rangle$ be the canonical semiscale on $\tilde{R}$ derived from the Suslin representation $\mathcal{T}$ for $\tilde{R}$ as in Definition 2.6. Assume each norm relation $\leq^*_{\phi_n}$ is Suslin and coSuslin. Then there is a $\Lambda : \mathrm{dom}(R) \to \mathsf{club}$ so that for all $\ell \in \mathrm{dom}(R)$, $R(\ell, \Lambda(\ell))$.

Thus assuming $\mathsf{ZF} + \mathsf{AD}$ and all sets of reals are Suslin, countable length everywhere club uniformization holds for κ with the above properties.

Proof. By the hypothesis, each norm relation $\leq^*_{\phi_n}$ for $\tilde{R}$ is assumed to be Suslin and coSuslin so there are trees $\mathcal{P}$ on $\omega \times \omega \times \lambda_0$ and $\mathcal{Q}$ on $\omega \times \omega \times \lambda_1$ with $\lambda_0, \lambda_1 < \Theta$ which project onto $\leq^*_{\phi_n}$ and its complement, respectively. By the Moschovakis coding lemma and $\mathsf{AC}^\mathbb{R}_\omega$, one may find a sequence $\langle \mathcal{P}_n : n \in \omega \rangle$ and $\langle \mathcal{Q}_n : n \in \omega \rangle$ so that for each $n \in \omega$, $\pi_1[[\mathcal{P}_n]] = \leq^*_{\phi_n}$ and $\pi_1[[\mathcal{Q}_n]] = \mathbb{R} \times \mathbb{R} \setminus \leq^*_{\phi_n}$.

Let $\ell \in \mathrm{dom}(R)$. Recall that by Fact 2.14, $\kappa = \delta(\Gamma)$ is a regular cardinal. By Fact 2.21, let ν_ℓ be the least honest ordinal greater than $\sup(\ell)$. Let $R^{\nu_\ell}_\ell \subseteq {}^\omega\nu_\ell$ be defined by $R^{\nu_\ell}_\ell(g)$ if and only if $\mathrm{rang}(\ell) \subseteq \mathrm{rang}(g)$.

Fix $g \in {}^\omega\nu_\ell$ so that $R^{\nu_\ell}_\ell(g)$. Fix $w \in \mathrm{WO}$ with $\mathrm{ot}(w) = |\ell|$. Let $\mathfrak{r}(\ell, g, w)$ be the unique real with the following properties.

- For all $n \in \omega$, if $n \notin \mathrm{field}(\mathcal{R}_w)$, then $\widehat{\mathfrak{r}(\ell, g, w)}_n = \widehat{\mathfrak{F}(g)}_n$
- For $n \in \mathrm{field}(\mathcal{R}_w)$, let i^g_n be the least k so that $g(k) = \ell(\mathrm{ot}(w, n))$. Then one has $\widehat{\mathfrak{r}(\ell, g, w)}_n = \widehat{\mathfrak{F}(g)}_{i_n}$.

Let $\mathsf{extract}(\ell, g, w)$ be the unique real z so that $\widehat{z}_0 = w$, $\widehat{z}_1 = \mathfrak{r}(\ell, g, w)$, and for all $n > 1$, $\widehat{z}_n = \bar{0}$, the constant 0 sequence.

Lemma 3.4. *Let $E^{\ell,w} : R^{\nu_\ell}_\ell \to \mathbb{R}$ be defined by $E^{\ell,w}(g) = \mathsf{extract}(\ell, g, w)$. If $\mathrm{rang}(g)$ is honest then $\mathsf{cseq}(\mathsf{extract}(\ell, g, w)) = \ell$.*

Proof. Assume $\mathrm{rang}(\ell) \subseteq \mathrm{rang}(g)$ and g is honest. Let $\alpha < |\ell|$ and $n = \mathsf{num}(w, \alpha)$. Since $\mathrm{rang}(\ell) \subseteq \mathrm{rang}(g)$, i_n is defined with the property that $g(i_n^g) = \ell(\mathrm{ot}(w, n)) = \ell(\mathrm{ot}(w, \mathsf{num}(w, \alpha))) = \ell(\alpha)$. Since g is honest, one has that for all $n \in \omega$, $g(n) = \varphi_0(\widehat{\mathfrak{F}(g)}_n)$. This implies that $\mathsf{cseq}(\mathsf{extract}(\ell, g, w)) = \ell$. ☐

Lemma 3.5. *Let* $\mathsf{graph}(E^{\ell,w}) \subseteq R_\ell^{\nu_\ell} \times \mathbb{R}$ *be defined as the graph of* $E^{\ell,w}$. $\mathsf{graph}(E^{\ell,w})$ *is Suslin and coSuslin uniformly in* ℓ *and* w. *Moreover, the canonical semiscale derived from this Suslin representation as in Definition 2.6 has associated norm relations which are Suslin and coSuslin.*

Proof. Note that although $E^{\ell,w}$ is only defined on $R_\ell^{\nu_\ell}$, it is continuous on $R_\ell^{\nu_\ell}$. For instance: For $n \in \mathrm{field}(w)$ and $g_0, g_1 \in {}^\omega\kappa$, let $i_n^{g_0}$ and $i_n^{g_1}$ be the corresponding objects for g_0 and g_1, respectively. If $i_n^{g_0} = k$, then note that for any g_1 such that $g_0 \upharpoonright k+1 = g_1 \upharpoonright k+1$, $i_n^{g_0} = i_n^{g_1}$. The continuity of $E^{\ell,w}$ follows from this observation and the fact that the generic coding function $\mathfrak{F}$ is Lipschitz.

The domain of $E^{\ell,w}$ is $R_\ell^{\nu_\ell}$. Fact 2.9 and Fact 2.10 give an analogous result for $R_\ell^{\nu_\ell}$. The proof of the lemma is quite similar to the arguments of these two facts. The details are left to the reader. ☐

Consider the game G_ℓ^w defined as follows.

$$
\begin{array}{cccccccccc}
 & \mathrm{I} & g(0), e(0) & & g(2), e(1) & & g(4), e(2) & & & e \\
G_\ell^w & & & & & & & & \cdots & g \\
 & \mathrm{II} & & g(1) & & g(3) & & g(5) & &
\end{array}
$$

For all $n \in \omega$, $g(n) \in \nu_\ell$. Player 1 plays $g(2n)$ for all $n \in \omega$. Player 2 plays $g(2n+1)$ for all $n \in \omega$. Player 1 also plays $e(n) \in \omega$ for all $n \in \omega$. After an infinite run, Player 1 and Player 2 together produce $g \in {}^\omega\nu_\ell$ and Player 1 alone produces $e \in \mathbb{R}$. Player 1 wins G_ℓ^w if and only if $P_\ell^w(g)$, where $P_\ell^w(g)$ is defined by the conjunction of the following.

(1) $\mathrm{rang}(\ell) \subseteq \mathrm{rang}(g)$.
(2) $\mathsf{extract}(\ell, g, w) \in \mathrm{dom}(\tilde{R})$.
(3) $\tilde{R}(\mathsf{extract}(\ell, g, w), e)$.

Lemma 3.6. *The payoff set* P_ℓ^w *for the game* G_ℓ^w *is Suslin and coSuslin uniformly in* ℓ *and* w. *Moreover, the semiscale derived from this Suslin representation as in Definition 2.6 is Suslin and coSuslin.*

Proof. (1) is Suslin and coSuslin uniformly in ℓ and w by Fact 2.9 (and note that ν_ℓ is defined uniformly from ℓ). Note that $\mathrm{dom}(\tilde{R})$ is Suslin and coSusin uniformly from ℓ and w (and the fixed tree representations $\mathcal{T}$ and $\mathcal{S}$). (2) is Suslin and coSuslin uniformly from ℓ and w using this observation and Lemma 3.5. Similarly (3) is Suslin and coSuslin uniformly from ℓ and w using the tree $\mathcal{T}$ and $\mathcal{S}$ and Lemma 3.5. This establishes that the payoff set P_ℓ^w is Suslin and coSuslin uniformly in ℓ and w. Using Fact 2.10 and Lemma 3.5, one can show each norm relation of the derived semiscale is Suslin and coSuslin. ☐

The first part of Lemma 3.6 implies the ordinal game G_ℓ^w is determined by Fact 2.8.

Lemma 3.7. *Suppose* τ *is a Player 2 strategy for* G_ℓ^w *and* $e \in \mathbb{R}$. *Then there is an* $h \in {}^\omega\nu_\ell$ *with the following properties.*

- $\mathrm{rang}(\ell) \subseteq \mathrm{rang}(h)$.
- *Let* $h \oplus e \in {}^\omega\nu_\ell$ *be defined by* $(h \oplus e)(n) = \langle h(n), e(n)\rangle$. *Let* $(g, e) = \Sigma_\tau^2(h \oplus e)$. *(That is,* (g, e) *is the run of the game where Player 2 uses* τ *against Player 1 using* $\rho_{h\oplus e}$*.) Then* $\mathrm{rang}(g)$ *is honest.*

Suppose σ *is a Player 1 strategy for* G_ℓ^w. *Then there is an* $h \in {}^\omega\nu_\ell$ *with the following properties.*

- $\mathrm{rang}(\ell) \subseteq \mathrm{rang}(h)$.
- *Let* $(g, e) = \Sigma_\sigma^1(h)$. *Then* $\mathrm{rang}(g)$ *is honest.*

Proof. Fix a bijection $B : \omega \to |\ell|$. Using $\mathsf{AC}_\omega^{\mathbb{R}}$, find a sequence $\langle x_n : n \in \omega\rangle$ in W so that for all $n \in \omega$, $\varphi_0(x_n) = \ell(B(n))$. Let $\mathsf{pair} : \omega^2 \to \omega$ be a bijection with the property that for all $a, b, c \in \omega$, if $\mathsf{pair}(a, b) = c$, then $a, b \leq c$ and for all $a, b, c \in \omega$, if $b < c$, then $\mathsf{pair}(a, b) < \mathsf{pair}(a, c)$. Now define a tree K on $\mathbb{R}$ by $s \in K$ if and only if the following holds.

- For all $k < |s|$, $s(k) \in W$.
- Let $n = |s|$. Let $p_s : 2n \to \nu_\ell$ be defined as follows.

– For $k < n$,

$$p_s(2k) = \begin{cases} (\varphi_j(x_i), e(k)) & k = 2d \wedge d = \mathsf{pair}(i,j) \\ (\varphi_j(s(i)), e(k)) & k = 2d+1 \wedge d = \mathsf{pair}(i,j) \end{cases}$$

– The odd $p_s(2k+1)$ is the result of applying τ to the partial run $\langle p_s(j) : j < 2k+1 \rangle$ in G^w_ℓ.
– For each $k < n$, $p_s(2k+1) = \varphi_0(s(k))$.

The tree K is ordered by proper string extension $\subsetneq$. One can check every node s of K can be strictly extended by simply choosing some $y \in W$ so that $\varphi_0(y)$ corresponds to τ's next response against the run p_s. By $\mathsf{DC}_\mathbb{R}$, there is an $f \in [K]$. Let $g = \bigcup_{n\in\omega} p_{f \restriction n}$. Note that $\mathrm{rang}(g)$ is honest by the definition of the tree K. Let $h = \langle g(2n) : n \in \omega \rangle$. Note that $\mathrm{rang}(\ell) \subseteq \mathrm{rang}(h)$ by the definition of the tree K. $\Sigma^2_\tau(h \oplus e) = (g, e)$ by the definition of K. Thus h is the desired object.

The argument for the second statement is quite similar. □

Next one seeks to show that Player 1 has the winning strategy for G^w_ℓ. Suppose τ is a Player 2 strategy. Since $\ell \in \mathrm{dom}(R)$, there is a club $C \subseteq \kappa$ so that $R(\ell, C)$. Pick any e so that $\mathfrak{C}_e \subseteq C$ which is possible by Fact 2.15. Now by Lemma 3.7 pick an h so that $\mathrm{rang}(\ell) \subseteq \mathrm{rang}(h)$ and if $(g, e) = \Sigma^2_\tau(h \oplus e)$ is the run according to τ where Player 1 plays $(h(n), e(n))$ for its n^{th}-move, then $\mathrm{rang}(g)$ is honest. Note that (1) of P^w_ℓ clearly holds. By Lemma 3.4, one has $\mathsf{cseq}(\mathsf{extract}(\ell, g, w)) = \ell$. Then $\mathsf{extract}(\ell, g, w) \in \mathrm{dom}(\tilde{R})$ and hence (2) of P^w_ℓ holds. Also (3) of P^w_ℓ is true since $\tilde{R}(\mathsf{extract}(\ell, g, w), e)$ holds by choice of e. Thus Player 1 wins and hence τ cannot be a winning strategy for Player 2.

This completes the argument that Player 2 cannot have a winning strategy in G^w_ℓ. By the determinacy of G^w_ℓ, Player 1 has a winning strategy. Next, one will need to show that a winning strategy for G^w_ℓ can be found uniformly in w and ℓ.

Lemma 3.6 implies that P^w_ℓ has a Suslin representation whose derived semiscale has norm relations which are Suslin and coSuslin. From these semiscales, one can construct a very good semiscale for P^w_ℓ (uniformly in w and ℓ) whose associated norm relations are all Suslin and coSuslin. In the notation of Fact 2.11, this can be used to show that the payoff set of each game ${}^nH^s_t$ is Suslin and coSuslin. ${}^nH^t_s$ is determined by Fact 2.8. Fact 2.11 can now be applied to find, uniformly in w and ℓ, a Player 1 winning strategy σ^w_ℓ in the game G^w_ℓ.

Let $\pi_1 : \nu_\ell \times \omega \to \nu_\ell$ and $\pi_2 : \nu_\ell \times \omega \to \omega$ be the projection onto the first and second coordinate, respectively. Let $\varpi : {}^\omega\nu_\ell \to {}^\omega\nu_\ell$ be defined by

$$\varpi(h)(n) = \begin{cases} \pi_1(\Sigma^1_{\sigma^w_\ell}(h)(n)) & n \text{ is even} \\ \Sigma^1_{\sigma^w_\ell}(h)(n) & n \text{ is odd} \end{cases}$$

Let $\vartheta : {}^\omega\nu_\ell \to {}^\omega\omega$ be defined by $\vartheta(h)(n) = \pi_2(\Sigma^1_{\sigma^w_\ell}(h)(n))$. (In the above notation, if (g, e) is the resulting run in the game G^w_ℓ where Player 1 uses σ^w_ℓ against Player 2 using ρ_h, then $\varpi(h) = g$ and $\vartheta(h) = e$.) Both ϖ and ϑ are Lipschitz continuous functions and are produced uniformly from w and ℓ (since they depend only on σ^w_ℓ).

Since σ^w_ℓ is a Player 1 winning strategy in G^w_ℓ, one has by (1) of the payoff set P^w_ℓ that for all $g \in \varpi[{}^\omega\nu_\ell]$, $\mathrm{rang}(\ell) \subseteq \mathrm{rang}(g)$ and thus $\mathsf{extract}(\ell, g, w)$ is well defined. Also since σ^w_ℓ is a Player 1 winning strategy in G^w_ℓ, one has that $\vartheta[{}^\omega\nu_\ell] \subseteq \mathsf{clubcode}$. Let $W_\ell = \varphi_0^{-1}[\nu_\ell]$ and let $\varphi_\ell : W_\ell \to \nu_\ell$ be defined by $\varphi_\ell = \varphi_0 \restriction W_\ell$. Since φ is a Γ-norm, one has that the associated prewellordering $\preceq_{\varphi_\ell}$ belongs to Δ. Fact 2.12 can now be applied to show $\vartheta[{}^\omega\nu_\ell]$ is $\exists^\mathbb{R}\Delta \subseteq \check{\Gamma}$ since Γ is closed under $\forall^\mathbb{R}$. By Fact 2.15, there is a club $D \subseteq \kappa$ (produced uniformly from the set $\vartheta[{}^\omega\nu_\ell]$) with the property that for all $e \in \vartheta[{}^\omega\nu_\ell]$, $D \subseteq \mathfrak{C}_e$. By Lemma 3.7, one can find a sequence h^* so that $\mathrm{rang}(\ell) \subseteq \mathrm{rang}(h^*)$ and if $(g, e^*) = \Sigma^1_{\sigma^w_\ell}(h^*)$ is the run according to σ^w_ℓ where Player 2 uses h^*, then $\mathrm{rang}(g)$ is honest. By Lemma 3.4, $\mathsf{extract}(\ell, g, w) = \ell$. Thus since $e^* = \vartheta(h^*)$, $\tilde{R}(\ell, \vartheta(h^*))$ and hence $R(\ell, \mathfrak{C}_{\vartheta(h^*)})$. Since $\vartheta(h^*) \in \vartheta[{}^\omega\nu_\ell]$, one has that $D \subseteq \mathfrak{C}_{\vartheta(h^*)}$. Since R is $\subseteq$-downward closed, one has that $R(\ell, D)$. Finally, observe that D is produced uniformly from w and $\ell \in \mathrm{dom}(R)$.

By the uniformity observation, it has been shown that there is a function Ψ so that whenever $\ell \in \mathrm{dom}(R)$ and $w \in \mathrm{WO}$ with $\mathrm{ot}(w) = |\ell|$, $\Psi(\ell, w) \in \mathsf{club}$ and $R(\ell, \Psi(\ell, w))$. One will need to remove the dependence on w.

18 Aug 2021 22:34:40 PDT
210818-Chan Version 1 - Submitted to J. Symb. Logic

Fix $\ell \in \mathrm{dom}(R)$. Observe that if $f \in \mathsf{surj}_{|\ell|}$, then $\mathfrak{G}(f) \in \mathrm{WO}_{|\ell|}$ by Fact 2.17. Let $\Phi_\ell : \mathsf{surj}_{|\ell|} \to \mathsf{club}$ be defined by $\Phi_\ell(f) = \Psi(\ell, \mathfrak{G}(f))$. Note that for all $f \in \mathsf{surj}_{|\ell|}$, $R(\ell, \Phi_\ell(f))$. Since $\mathsf{surj}_{|\ell|}$ is comeager in ${}^\omega|\ell|$, Fact 2.19 states that one can find uniformly from Φ_ℓ (which was constructed uniformly from ℓ), a comeager set $B_\ell \subseteq \mathsf{surj}_{|\ell|}$ and club $D_\ell \subseteq \kappa$ so that for all $f \in B_\ell$, $D_\ell \subseteq \Phi_\ell(f)$. Pick any $f \in B_\ell$. Since $R(\ell, \Phi_\ell(f))$ and R is $\subseteq$-downward closed, one has that $R(\ell, D_\ell)$.

By the uniformity of the construction, one can define $\Lambda : \mathrm{dom}(R) \to \mathsf{club}$ by $\Lambda(\ell) = D_\ell$. It has been shown that for all $\ell \in \mathrm{dom}(R)$, $R(\ell, \Lambda(\ell))$. Λ is the desired uniformization, and this completes the proof of the theorem. $\square$

Theorem 3.8. *Assume* $\mathsf{ZF} + \mathsf{AD}$ *and all sets of reals are Suslin. Let* Γ *be a boldface pointclass closed under* $\wedge$, $\vee$, *and* $\forall^{\mathbb{R}}$ *with the scale property. Then the countable length everywhere club uniformization holds for* $\delta(\Gamma)$. *In particular, for all* $n \in \omega$, *the countable length everywhere club uniformization holds for* $\boldsymbol{\delta}^1_{2n+1}$.

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18 Aug 2021 22:34:40 PDT
210818-Chan Version 1 - Submitted to J. Symb. Logic