THE MINIMAL MODEL PROGRAM FOR THREEFOLDS IN CHARACTERISTIC 5

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Abstract
We show the validity of the minimal model program (MMP) for threefolds in characteristic 5.

1. Introduction

One of the fundamental goals of algebraic geometry is to classify all algebraic varieties which, conjecturally, can be achieved by means of the minimal model program (MMP). A major part of the MMP is now known to hold in characteristic 0 (see [3]) and in the last few years substantial progress has been achieved in positive characteristic as well. Indeed, it has been shown that the program is valid for surfaces over excellent base schemes (see [24], [25]) and for three-dimensional varieties of characteristic $p > 5$ (see [15]; see also [2], [4], [9], [10], [12]).

However, little is known beyond these cases and new phenomena discovered by Cascini and Tanaka [5] suggest that the low characteristic MMP is much more subtle. Moreover, in view of [8], it has become apparent that understanding the geometry of low characteristic threefolds is the most natural step towards tackling the MMP in higher dimensions.

In [14], following some ideas of [12], we shed some light on the geometry of threefolds in all characteristics $p \leq 5$. In particular, we show that the relative MMP can be run over $\mathbb{Q}$-factorial singularities and in families. As a consequence, we establish, among other things, inversion of adjunction, normality of purely log terminal (plt) centers up to a universal homeomorphism, as well as the existence of Kollár's components and divisorial log terminal (dlt) modifications.

The goal of this article is to extend the MMP for threefolds to characteristic $p = 5$ in full generality. We believe that the methods developed in this paper will be useful in tackling the MMP for threefolds in characteristics 2 and 3 as well as the MMP in higher dimensions.

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Our main result is the following.

THEOREM 1.1
Let (X, Δ) be a $\mathbb{Q}$-factorial three-dimensional dlt pair over a perfect field k of characteristic $p = 5$. If $f: X \to Z$ is a $(K_X + \Delta)$-flipping contraction, then the flip $f^+: X^+ \to Z$ exists.

Note that this result is known for $p > 5$ by [2], [12], and [15]. As a corollary of Theorem 1.1, we get the following results on the MMP in positive characteristic.

THEOREM 1.2 (MMP with scaling)
Let (X, Δ) be a $\mathbb{Q}$-factorial three-dimensional Kawamata log terminal (klt) pair over a perfect field k of characteristic $p > 3$, and let $f: X \to Z$ be a projective contraction. Then we can run an MMP with scaling for $K_X + \Delta$ over Z. If $K_X + \Delta$ is relatively pseudoeffective, then the MMP terminates with a log minimal model over Z. Otherwise, the MMP terminates with a Mori fiber space.

In particular, Theorem 1.2 shows that Zariski's conjecture on finite generatedness of the canonical ring of smooth varieties is valid for threefolds in characteristic 5 (see [27]). Also note that Theorem 1.2 may be extended to the dlt case (see Remark 7.3).

THEOREM 1.3 (Basepoint-free theorem)
Let (X, Δ) be a three-dimensional klt pair over a perfect field k of characteristic $p > 3$, and let $f: X \to Z$ be a projective contraction. Let D be a relatively nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor such that $D - (K_X + \Delta)$ is nef and big over Z. Then D is semiample over Z.

THEOREM 1.4 (Cone theorem)
Let (X, Δ) be a projective $\mathbb{Q}$-factorial three-dimensional dlt pair over a perfect field k of characteristic $p > 3$. Then there exists a countable number of rational curves Γ_i such that

- $\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(X)_{K_X+\Delta\geq 0} + \sum_i \mathbb{R}[\Gamma_i]$,
- $-6 \leq (K_X + \Delta) \cdot \Gamma_i < 0$,
- *for any ample $\mathbb{R}$-divisor A,*

$$(K_X + \Delta + A) \cdot \Gamma_i \geq 0$$

  holds for all but a finitely many Γ_i, and so
- *the rays $\mathbb{R}[\Gamma_i]$ do not accumulate inside $\overline{\mathrm{NE}}(X)_{K_X+\Delta<0}$.*

The above results, in this generality, were proved in [2], [4], and [12] (cf. [9], [15], [17]) contingent upon the existence of flips with standard coefficients. Hence, they follow immediately from Theorem 1.1. There are many other results around the MMP (cf. [2], [4], [12], [16], [28]) that generalize to characteristic 5 in view of Theorem 1.1. For example, every projective three-dimensional geometrically connected normal variety of Fano type defined over a finite field of characteristic $p = 5$ admits a rational point (cf. [12, Theorem 1.2]).

1.1. The idea of the proof of Theorem 1.1

For simplicity, we suppose in this subsection that the divisorial centers of the dlt pairs we consider are normal. This is not far from the truth, as these divisorial centers are normal up to a universal homeomorphism (see [14, Theorem 1.2]).

By the same argument as in [2, Theorem 6.3], we can suppose that the coefficients of Δ are standard. By perturbation and reduction to prelimiting flips (pl-flips), we can assume that $\Delta = S + B$ and $(X, S + B)$ is plt, where S is an irreducible divisor. Let $f: X \to Z$ be a pl-flipping contraction. The proof of the existence of flips for threefolds in characteristic $p > 5$ (see [15]) consists of two steps:

(1) showing that the flip of f exists if $(X, S + B)$ is relatively purely F-regular, and

(2) showing that $(X, S + B)$ is relatively purely F-regular when $p > 5$.

The first step holds in every characteristic. Unfortunately, the second statement is false for $p \leq 5$ in general. To circumvent this problem, we construct pl-flips by a mix of blowups, contractions, and pl-flips admitting dlt 6-complements.

PROPOSITION 1.5 (cf. Proposition 5.1)
Let $(X, S + B)$ be a $\mathbb{Q}$-factorial three-dimensional plt pair with standard coefficients over a perfect field k of characteristic $p > 3$, where S is an irreducible divisor, and let $f: X \to Z$ be a pl-flipping contraction. Assume that there exists a dlt 6-complement $(X, S + B^c)$ of $(X, S + B)$. Then the flip $f^+: X^+ \to Z$ exists.

Let $C = \mathrm{Exc}(f)$. For simplicity, assume that C is irreducible. We split the proof of this proposition into three cases:

(1) $(X, S + B^c)$ is plt in a neighborhood of C, or

(2) $C \cdot E < 0$ for a divisor $E \subseteq \lfloor B^c \rfloor$, or

(3) $C \cdot E \geq 0$ for a divisor $E \subseteq \lfloor B^c \rfloor$ intersecting C.

In case (1), write $K_S + B_S = (K_X + S + B)|_S$ and

$$K_S + B_S^c = (K_X + S + B^c)|_S.$$

Since $(X, S + B^c)$ is plt along C, we get that (S, B_S^c) is klt along C. Our key observation is the following: *if a birational log Fano contraction of a surface pair with* *standard coefficients in characteristic* $p > 3$ *admits a klt* 6*-complement, then it is relatively* F*-regular* (see Proposition 3.1). Therefore, $(X, S + B)$ is relatively purely F-regular by F-adjunction, and so the flip exists by [15] (see the aforementioned step (1)). This is the main part of our arguments which we are unable to generalize to characteristic 3. On the other hand, one might expect some analogue of this statement to hold in higher dimensions for all bounded complements and $p \gg 0$.

In case (2), we can construct the flip explicitly as the closure of X under the rational map defined by a pencil of sections spanned by kS and lE for some $k, l \in \mathbb{N}$ such that $kS \sim lE$.

In case (3), assume for simplicity that S and E are the only log canonical (lc) divisors of $(X, S + B^c)$. Then we can show that $(X, S + B^c - \epsilon E)$ is relatively F-split over Z for $0 < \epsilon < 1$ by F-adjunction applied to S and $S \cap E$. In fact, with a bit more work one can show that it is relatively purely F-regular, and thus $(X, S + B)$ is so as well. Hence the flip exists by [15] as in case (1).

In view of Proposition 1.5, it is important to construct complements of pl-flipping contractions. By standard arguments, (S, B_S) admits an m-complement (S, B_S^c) for some $m \in \{1, 2, 3, 4, 6\}$, and the following result shows that we can lift it to an m-complement of $(X, S + B)$.

THEOREM 1.6

Let $(X, S + B)$ *be a* $\mathbb{Q}$*-factorial three-dimensional plt pair with standard coefficients defined over a perfect field of characteristic* $p > 2$*, and let* $f : X \to Z$ *be a flipping contraction such that* $-(K_X + S + B)$ *and* $-S$ *are* f*-ample.*

Then there exists an m*-complement* $(X, S + B^c)$ *of* $(X, S + B)$ *in a neighborhood of* Exc f *for some* $m \in \{1, 2, 3, 4, 6\}$*.*

If $m \in \{1, 2, 3, 4\}$, then one can show that the flipping contraction is relatively purely F-regular in characteristic $p = 5$ (cf. Remark 4.8), and so it exists by [15]. In what follows, we focus on the case of $m = 6$.

Although $(X, S + B)$ need not necessarily be relatively purely F-regular in general, we can still apply F-splitting techniques to prove Theorem 1.6 as we do not need to lift all the sections, but just some very special ones. Note that this result is new even for $p > 5$.

In order to construct the flip of f from the flips of Proposition 1.5, we argue as follows. Let $(X, S + B^c)$ be an m-complement of $(X, S + B)$ for $m \in \{1, 2, 3, 4, 6\}$ which exists by Theorem 1.6. Take a dlt modification $\pi : Y \to X$ of $(X, S + B^c)$ with an exceptional divisor E (see [14, Corollary 1.4]). Write $K_Y + S_Y + B_Y^c = \pi^*(K_X + S + B^c)$, $K_Y + S_Y + B_Y = \pi^*(K_X + S + B)$, and run a $(K_Y + S_Y + B_Y)$-MMP over Z. Note that it could happen that B_Y is not effective, but we can rectify this situation by taking a linear combination of B_Y and B_Y^c (see the proof of Theorem 1.1 for details). By the negativity lemma, if this MMP terminates, then its output is the flip of X. Therefore, it is enough to show that all the steps of this MMP can be performed.

The first step of this MMP definitely exists. Indeed, either it is a divisorial contraction which can be shown to exist by [14] and [17], or it is a flipping contraction followed by a flip with a dlt m-complement which exists by Proposition 1.5 and the sentence after Theorem 1.6 (the $\mathbb{Q}$-divisor B_Y may not have standard coefficients, so one needs to be a bit more careful; see the proof for details). However, each step of this $(K_Y + S_Y + B_Y)$-MMP is $(K_Y + S_Y + B_Y^c)$-relatively trivial, and so the dlt-ness of $(Y, S_Y + B_Y^c)$ need not be preserved.

To rectify this problem, we employ the notion of *qdlt singularities*, that is, log canonical pairs which are quotient singularities at log canonical centers (at least in characteristic 0; see Definition 2.3 for the formal statement). In fact, Proposition 1.5 holds for qdlt-flipping contractions (Proposition 5.1), and we can show the existence of a qdlt modification $\pi : Y \to X$ with irreducible exceptional locus (Corollary 6.2). Therefore, the output of any divisorial contraction in the $(K_Y + S_Y + B_Y)$-MMP is automatically the flip of $(X, S + B)$. Moreover, the qdlt-ness of $(Y, S_Y + B_Y^c)$ is preserved by flops (Lemma 2.7) except in one special case in which we can construct the flip of $(X, S + B)$ directly.

2. Preliminaries

A scheme X will be called a *variety* if it is integral, separated, and of finite type over a field k. Throughout this paper, k is a perfect field of characteristic $p > 0$. We refer the reader to [20] for basic definitions in birational geometry and to [13] for a brief introduction to F-splittings. We remark that in this paper, unless otherwise stated, if (X, B) is a pair, then B is a $\mathbb{Q}$-divisor. For two $\mathbb{Q}$-divisors A and B, we denote by $A \wedge B$ the maximal $\mathbb{Q}$-divisor smaller or equal to both A and B. We say that (X, Δ^c) is an *m-complement* of (X, Δ) if (X, Δ^c) is log canonical, $m(K_X + \Delta^c) \sim 0$, and $\Delta^c \geq \Delta^*$, where $\Delta^* := \frac{1}{m}\lfloor (m+1)\Delta \rfloor$. If Δ has standard coefficients, then $\Delta^* = \frac{1}{m}\lceil m\Delta \rceil$, and so the last condition is equivalent to $\Delta^c \geq \Delta$. We say that a morphism $f : X \to Y$ is a *projective contraction* if it is a projective morphism of quasiprojective varieties and $f_*\mathcal{O}_X = \mathcal{O}_Y$.

Since the existence of resolutions of singularities is not known in positive characteristic in general, the classes of singularities are defined with respect to all birational maps. For example, a log pair (X, Δ) is *klt* if and only if the log discrepancies are positive for every birational map $Y \to X$. Similarly, *log canonical centers* are defined as images of divisors of log discrepancy zero under birational maps $Y \to X$. These definitions coincide with the standard ones up to dimension 3, as log resolutions of singularities are known to exist in this case.

The starting point for the construction of flips is the following result from [15]. We say that a projective birational morphism $f: X \to Z$ for a $\mathbb{Q}$-factorial plt pair $(X, S+B)$, with S irreducible, is a *pl-flipping contraction* if f is small, $-(K_X + S + B)$ and $-S$ are relatively ample, and $\rho(X/Z) = 1$.

THEOREM 2.1
Let $(X, S+B)$ be a $\mathbb{Q}$-factorial three-dimensional plt pair defined over a perfect field of characteristic $p > 0$ with S irreducible. Let $f: X \to Z$ be a pl-flipping contraction. Let $g: \tilde{S} \to S$ be the normalization of S, and write $K_{\tilde{S}} + B_{\tilde{S}} = (K_X + S + B)|_{\tilde{S}}$. If $(\tilde{S}, B_{\tilde{S}})$ is relatively globally F-regular over $f(S) \subseteq Z$, then the flip of f exists.

Note that the condition on the relative global F-regularity of $(\tilde{S}, B_{\tilde{S}})$ is equivalent to the relative pure F-regularity of $(X, S+B)$ by F-adjunction.

Proof
This follows from [15] as explained in [14, Remark 3.6]. □

Remark 2.2
By [15, Theorem 3.1] (cf. Proposition 2.9), the above assumption on F-regularity is always satisfied when $p > 5$ and B has standard coefficients.

Note that [15] assumes that the base field is algebraically closed, but their results extend to perfect fields as explained in [12] (cf. [14]).

2.1. Qdlt pairs
Qdlt singularities will play an important role in this article.

Definition 2.3 ([11, Definition 35])
Let (X, Δ) be a log canonical pair. We say that (X, Δ) is *qdlt* if for every log canonical center $x \in X$ of codimension $k > 0$, there exist distinct irreducible divisors $D_1, \ldots, D_k \subseteq \Delta^{=1}$ such that $x \in V := D_1 \cap \cdots \cap D_k$.

Remark 2.4
Note that if (X, Δ) is log canonical and x is a generic point of a stratum $V := D_1 \cap \cdots \cap D_k$ of $\Delta^{=1}$, then $\operatorname{codim} x = k$. Indeed, let $\tilde{D}_1 \to D_1$ be the normalization of D_1. Then, by adjunction, $(\tilde{D}_1, \Delta_{\tilde{D}_1})$ is log canonical where $K_{\tilde{D}_1} + \Delta_{\tilde{D}_1} = (K_X + \Delta)|_{\tilde{D}_1}$. Moreover, by localizing at generic points of $D_1 \cap D_l$ and using surface theory, we see that $D_l|_{\tilde{D}_1} \subseteq \Delta^{=1}_{\tilde{D}_1}$ have no mutually common components for $2 \leq l \leq k$. Therefore,

x is a generic point of $E_2 \cap \cdots \cap E_k$, where the E_l are some irreducible components of $D_l|_{\tilde{D}_1}$. Now the claim follows by induction.

By [11, Proposition 34], in characteristic 0 the above definition of qdlt singularities is equivalent to saying that (X, Δ) is locally a quotient of a dlt pair by a finite abelian group preserving the divisorial centers. In positive characteristic, we know the following.

LEMMA 2.5

Let (X, Δ) be a $\mathbb{Q}$-factorial qdlt pair of dimension $n \leq 3$ defined over a perfect field of characteristic $p > 0$. Then

(1) *$(\tilde{D}, \Delta_{\tilde{D}})$ is qdlt, where $g\colon \tilde{D} \to D$ is the normalization of a divisor $D \subseteq \Delta^{=1}$ and $K_{\tilde{D}} + \Delta_{\tilde{D}} = (K_X + \Delta)|_{\tilde{D}}$,*

(2) *the strata of $\Delta^{=1}$ are normal up to a universal homeomorphism,*

(3) *the log canonical centers of (X, Δ) coincide with the generic points of the strata of $\Delta^{=1}$.*

Proof

We work in a sufficiently small neighborhood of a point of X.

If $n \leq 2$, then the lemma follows by standard results on surface pairs (cf. [19]). Indeed, a two-dimensional pair (X, Δ) is qdlt if either it is plt, or $\Delta = C_1 + C_2$, X is an A_m-singularity, (X, Δ) is simple normal crossing (snc) when $m = 1$, and, when $m > 1$, the strict transforms of C_1 and C_2 intersect the exceptional locus of the minimal resolution of X transversally at single points on the first and the last curve, respectively. Thus, we may assume that $n = 3$.

First, note that irreducible divisors in $\Delta^{=1}$ are normal up to a universal homeomorphism. Indeed, if $D \subseteq \Delta^{=1}$ is an irreducible divisor, then $(X, \Delta - \lfloor \Delta \rfloor + D)$ is plt and we can apply [14, Theorem 1.2].

Let $x \in \tilde{D}$ be a log canonical center of $(\tilde{D}, \Delta_{\tilde{D}})$. Then $g(x)$ is a log canonical center of (X, Δ). Indeed, otherwise there exists an nonzero divisor H passing through $g(x)$ and $\epsilon > 0$ such that $(X, \Delta + \epsilon H)$ is lc at $g(x)$. Thus, by adjunction, $(\tilde{D}, \Delta_{\tilde{D}} + \epsilon H|_{\tilde{D}})$ is lc at x, which is impossible.

Let k be the codimension of $g(x)$ in X. By definition of qdlt pairs, there exist divisors $D_1, \ldots, D_k \subseteq \Delta^{=1}$, with $D_1 = D$, such that

$$g(x) \in D_1 \cap \cdots \cap D_k.$$

Then $x \in D_2|_{\tilde{D}} \cap \cdots \cap D_k|_{\tilde{D}}$, where $D_i|_{\tilde{D}} \subseteq \Delta_{\tilde{D}}^{=1}$ for $i \geq 2$ have no mutually common components (cf. Remark 2.4). Since x is of codimension $k - 1$ in $\tilde{D}$, this shows that $(\tilde{D}, \Delta_{\tilde{D}})$ is qdlt at x. Hence (1) holds.

As for (2), pick a stratum $V = D_1 \cap \cdots \cap D_k$ of $\Delta^{=1}$. If $k = 1$, then we are done by the first paragraph. Otherwise,

$$g^{-1}(V) = D_2|_{\tilde{D}_1} \cap \cdots \cap D_k|_{\tilde{D}_1}$$

is a stratum of $\Delta^{=1}_{\tilde{D}_1}$, where $g\colon \tilde{D}_1 \to D_1$ is the normalization of D_1 and $K_{\tilde{D}_1} + \Delta_{\tilde{D}_1} = (K_X + \Delta)|_{\tilde{D}_1}$. Note that each $D_l|_{\tilde{D}_1}$ is irreducible, as otherwise (X, Δ) admits a log canonical center of codimension 3 which is contained in only two divisors, D_1 and D_l, of $\Delta^{=1}$. By the surface case, $g^{-1}(V)$ is normal up to a universal homeomorphism, and hence so is V, as g is a universal homeomorphism.

Now, we deal with (3). Since the images of log canonical centers of the surface pair $(\tilde{D}, \Delta_{\tilde{D}})$ in X, for the normalization $\tilde{D}$ of a divisor $D \subseteq \Delta^{=1}$, are log canonical centers of (X, Δ), we see that the generic points of the strata of $\Delta^{=1}$ are log canonical centers. If $x \in X$ is a log canonical center of (X, Δ) of codimension k, then by definition $x \in V := D_1 \cap \cdots \cap D_k$ for $D_1, \ldots, D_k \subseteq \Delta^{=1}$ and $\operatorname{codim}_X(V) = k$ (cf. Remark 2.4). Thus, x is a generic point of V. □

The following lemma generalizes the inversion of adjunction from [13, Corollary 1.5].

LEMMA 2.6 (Inversion of adjunction)
Consider a $\mathbb{Q}$-factorial three-dimensional log pair $(X, S + E + B)$ defined over a perfect field of characteristic $p > 0$, where S, E are irreducible divisors and $\lfloor B \rfloor = 0$. Write $K_{\tilde{S}} + C_{\tilde{S}} + B_{\tilde{S}} = (K_X + S + E + B)|_{\tilde{S}}$, where $\tilde{S}$ is the normalization of S, the divisor $C_{\tilde{S}} = E|_{\tilde{S}}$ is irreducible, and $\lfloor B_{\tilde{S}} \rfloor = 0$. Assume that $(\tilde{S}, C_{\tilde{S}} + B_{\tilde{S}})$ is plt. Then $(X, S + E + B)$ is qdlt in a neighborhood of S.

Proof
Assume by contradiction that $(X, S + E + B)$ admits a log canonical center Z of codimension at least 2, different from $C = S \cap E$, and intersecting S. Let H be a general Cartier divisor containing Z. Then for any $0 < \delta \ll 1$, we can find $0 < \epsilon \ll 1$ such that $(X, S + (1-\epsilon)E + B + \delta H)$ is not lc at Z. On the other hand, $(\tilde{S}, (1-\epsilon')C_{\tilde{S}} + B_{\tilde{S}} + \delta H|_{\tilde{S}})$ is klt for any $0 < \epsilon' \ll 1$ and $0 < \delta \ll 1$. This contradicts [14, Corollary 1.5]. □

We will use qdlt singularities for log pairs with two divisorial centers. In this case, the qdlt-ness is preserved under flops as long as the divisorial centers intersect each other.

LEMMA 2.7

Let $(X, S_1 + S_2 + B)$ be a $\mathbb{Q}$-factorial three-dimensional qdlt pair, where S_1, S_2 are irreducible divisors and $\lfloor B \rfloor = 0$. Let

$$f\colon (X, S_1 + S_2 + B) \dashrightarrow (X', S_1' + S_2' + B')$$

be a $(K_X + S_1 + S_2 + B)$-flop of a curve Σ for a relative Picard rank-one flopping contraction $g\colon X \to Z$. Suppose that $\Sigma \cdot S_1 < 0$. Then $(X', S_1' + S_2' + B')$ is qdlt or $S_1' \cap S_2' = \emptyset$ in a neighborhood of $\operatorname{Exc}(g')$, *where $g'\colon X' \to Z$ is the flopped contraction.*

Proof

In proving the proposition, we can assume that X and X' are sufficiently small neighborhoods of $\operatorname{Exc}(g)$ and $\operatorname{Exc}(g')$, respectively. Further, we can assume that the flop is nontrivial, and so a strict transform of a g-ample divisor is g'-antiample.

First, consider the case when $\Sigma \cdot S_2 \geq 0$. Pick a connected component $C \subseteq S_1 \cap S_2$, and let $\tilde{S}_1 \to S_1$ be the normalization of S_1. Since $(X, S_1 + S_2 + B)$ is qdlt, C is an irreducible curve. We claim that C is not g-exceptional. Indeed, otherwise, in view of $\rho(X/Z) = 1$, we have $C \cdot S_2 \geq 0$, which contradicts the following calculation:

$$C \cdot S_2 = C|_{\tilde{S}_1} \cdot S_2|_{\tilde{S}_1} = C|_{\tilde{S}_1} \cdot (\lambda C|_{\tilde{S}_1}) < 0,$$

where $\lambda > 0$. As a consequence, no component of $S_1 \cap S_2$ is contained in $\operatorname{Exc} g$, and so divisorial places over $\operatorname{Exc} g$ have log discrepancy greater than zero with respect to $(X, S_1 + S_2 + B)$. Since flops preserve discrepancies, we get that the codimension-2 log canonical centers of $(X', S_1' + S_2' + B')$ are images of the generic points of $(S_1 \cap S_2) \setminus \operatorname{Exc} g$, and so they are generic points of $S_1' \cap S_2'$. Hence, the pair $(X', S_1' + S_2' + B')$ is qdlt.

Therefore, we can assume that $\Sigma \cdot S_2 < 0$. In particular,

$$\operatorname{Exc} g = S_1 \cap S_2$$

up to replacing X by a neighborhood of $\operatorname{Exc} g$. Indeed, if we pick an irreducible curve $C \subseteq \operatorname{Exc} g$, then $C \cdot S_i < 0$ for $1 \leq i \leq 2$ as $\rho(X/Z) = 1$, and so $C \subseteq S_1 \cap S_2$. To prove the inclusion in the opposite direction, assume that there exists a nonexceptional irreducible curve $C \subseteq S_1 \cap S_2$ which intersects $\operatorname{Exc} g$ at some exceptional irreducible curve C'. As above, C is a connected component of $S_1 \cap S_2$, and so $C' \nsubseteq S_1 \cap S_2$. In particular, $C' \cdot S_i \geq 0$ for some $1 \leq i \leq 2$, which is a contradiction.

We aim to show that $S_1' \cap S_2' = \emptyset$. Assume by contradiction that $S_1' \cap S_2' \neq \emptyset$. By the above paragraph, we have that $S_1' \cap S_2' \subseteq \operatorname{Exc} g'$. Since S_2 is g-antiample, S_2' is g'-ample and $S_2'|_{\tilde{S}_1'}$ is an exceptional effective relatively ample divisor, where $\tilde{S}_1'$ is the normalization of S_1'. This is easily seen to contradict the negativity lemma. □

2.2. Surface lemmas

We prove a slightly stronger variant of the construction explained in the proof of [15, Theorem 3.2].

LEMMA 2.8

Let (X, B) be a two-dimensional klt pair defined over a perfect field of characteristic $p > 0$, and let $f: X \to Z$ be a projective birational map to a normal surface germ (Z, z) such that $-(K_X + B)$ is relatively nef. Then there exist an f-exceptional irreducible curve C on a blowup of X and projective birational maps $g: Y \to X$ and $h: Y \to W$ over Z such that

(1) *g extracts C or is the identity if $C \subseteq X$,*

(2) *$(Y, C + B_Y)$ is plt,*

(3) *$(W, C_W + B_W)$ is plt and $-(K_W + C_W + B_W)$ is ample over Z,*

(4) *$h^*(K_W + C_W + B_W) - (K_Y + C + B_Y) \geq 0$,*

where $K_Y + bC + B_Y = g^*(K_X + B)$ for $C \nsubseteq \operatorname{Supp} B_Y$, $C_W := h_*C \neq 0$, and $B_W := h_*B_Y$.

The variety W is the canonical model of $-(K_Y + C + B_Y)$ over Z. By saying that g *extracts* C, we mean that $\operatorname{Exc}(g) = C$.

Proof

Let Δ be as in [15, Claim 3.3], that is, such that $(X, B + \Delta)$ is lc and admits a unique non-klt place C exceptional over Z and $K_X + B + \Delta \sim_{\mathbb{Q},Z} 0$. Let $g: Y \to X$ be the extraction of the unique non-klt place C of $(X, B + \Delta)$, or the identity if C is a divisor on X (see the proof of [15, Theorem 3.2]). By construction, (1) and (2) hold.

Let $G := g^*\Delta - g^*\Delta \wedge C$. Note that $g^*\Delta \wedge C = (1 - b)C$. Let $h: Y \to W$ be the output of a G-MMP over Z (which is equivalent to a $-(K_Y + C + B_Y)$-MMP; also note that it is equivalent to a $(g^*(K_X + B + \Delta) + \epsilon G)$-MMP for $0 < \epsilon \ll 1$ and G has no common components with C, which justifies the existence of this MMP). Let $G_W := h_*G$. Now, (4) follows by the negativity lemma.

To prove (3), notice that since C is not contained in the support of G, then $G \cdot C \geq 0$ and so C is not contracted by $Y \to W$. Since

$$K_Y + C + B_Y + G = g^*(K_X + B + \Delta) \sim_{\mathbb{Q},Z} 0$$

is plt, it follows that $(W, C_W + B_W + G_W)$ is plt, and hence so is $(W, C_W + B_W)$. Since W is a G-minimal model over Z, then $-(K_W + C_W + B_W) \sim_{\mathbb{Q},Z} G_W$ is nef and in particular semiample over Z. To conclude the proof of (3), we need to show that $(K_W + C_W + B_W) \cdot C_W < 0$. Indeed, if this is true, then the associated semiample fibration does not contract C_W and so we can replace W by the image of the associated semiample fibration to make $-(K_W + C_W + B_W)$ ample without giving up the plt-ness of $(W, C_W + B_W)$.

Assume by contradiction that $(K_W + C_W + B_W) \cdot C_W = 0$. Let Γ be an effective $\mathbb{Q}$-divisor constructed as a connected component of

$$h^*(K_W + C_W + B_W) - (K_Y + bC + B_Y)$$

containing C. Since Γ is exceptional over Z, we have $\Gamma^2 < 0$. This contradicts the following calculation:

$$\begin{aligned}\Gamma^2 &= \Gamma \cdot \big(h^*(K_W + C_W + B_W) - (K_Y + bC + B_Y)\big)\\ &\geq \Gamma \cdot h^*(K_W + C_W + B_W)\\ &= h_*\Gamma \cdot (K_W + C_W + B_W) = 0,\end{aligned}$$

as $\operatorname{Supp} h_*\Gamma = C_W$. □

The above result allows for a shorter proof of [15, Theorem 3.1].

PROPOSITION 2.9 ([15, Theorem 3.1])
With notation as in the above lemma, suppose that B has standard coefficients and $p > 5$. Then (X, B) is globally F-regular over Z.

Proof
By [15, Proposition 2.11, Lemma 2.12], it is enough to show that $(W, C_W + B_W)$ is purely globally F-regular over Z, and so, by F-adjunction (see [13, Lemma 2.10]), it is enough to show that (C, B_C) is globally F-regular, where $K_C + B_C = (K_W + C_W + B_W)|_C$ and C is identified with C_W. Since $-(K_C + B_C)$ is ample and B_C has standard coefficients, this follows from [29, Theorem 4.2]. □

Remark 2.10
If $p = 5$, then the above proposition holds true unless $B_C = \frac{1}{2}P_1 + \frac{2}{3}P_2 + \frac{4}{5}P_3$ for three distinct points P_1, P_2, and P_3 (see [29, Theorem 4.2]).

The following result will be needed below.

LEMMA 2.11
With notation as in Lemma 2.8, suppose that $p > 3$ and (X, B) admits a 6-complement $(X, E + B^c)$, where E is a nonexceptional irreducible curve intersecting the exceptional locus over Z. Then (X, B) is globally F-regular over Z.

Note that we do not assume that B has standard coefficients.

Proof
As in the proof of Proposition 2.9, it is enough to show that $(W, C_W + B_W)$ is purely globally F-regular over Z, and so, by F-adjunction (see [13, Lemma 2.10]), it is enough to show that (C, B_C) is globally F-regular, where $K_C + B_C = (K_W + C_W + B_W)|_C$ and C is identified with C_W.

By pulling back the complement to Y and pushing down on W, we obtain a sub-lc pair $(W, aC_W + E_W + B^c_W)$ for a (possibly negative) number $a \in \mathbb{Q}$ such that $6(K_W + aC_W + E_W + B^c_W) \sim_Z 0$, a nonexceptional irreducible curve E_W intersecting the exceptional locus over Z, and an effective $\mathbb{Q}$-divisor B^c_W such that $E_W + B^c_W \geq B_W$. Let T_W be an effective exceptional antiample $\mathbb{Q}$-divisor on W, and let $\lambda \geq 0$ be such that the coefficient of C_W in $aC_W + \lambda T_W$ is 1. By the Kollár–Shokurov connectedness theorem (see, e.g., [25, Theorem 5.2]), the pair $(W, aC_W + \lambda T_W + E_W + B^c_W)$ is not plt along C_W (otherwise, E_W is disjoint from C_W and moreover, by connectedness, C_W must be the whole non-klt locus of $(W, aC_W + \lambda T_W + E_W + B^c_W)$ which is impossible as it is also non-klt at the intersection of E_W with the exceptional locus). In particular, B^c_C contains a point with coefficient at least 1, where

$$(K_W + aC_W + \lambda T_W + E_W + B^c_W)|_C = K_C + B^c_C.$$

Since T_W is antiample over Z, we have that $K_C + B^c_C$ is antinef. In particular, there exists a $\mathbb{Q}$-divisor $B_C \leq B'_C \leq B^c_C$ such that (C, B'_C) is plt (but not klt) and $-(K_C + B'_C)$ is nef.

If $-(K_C + B'_C)$ is ample, then (C, B'_C) is purely F-regular by [7, Lemma 2.9] (applied to perturbations of (C, B'_C)), and so (C, B_C) is globally F-regular. If $-(K_C + B'_C)$ is trivial, then $a = 1$, $\lambda = 0$, $6(K_C + B^c_C) \sim 0$, and (C, B^c_C) is plt (but not klt). Since $\mathrm{GCD}(p, 6) = 1$, [7, Lemma 2.9] implies that (C, B^c_C) is globally F-split, and so (C, B_C) is globally F-regular by [23, Corollary 3.10]. □

2.3. *Dual complexes*
Let (X, Δ) be a three-dimensional dlt pair. Its dual complex $D(\Delta^{=1})$ is a simplex with nodes corresponding to irreducible divisors of $\Delta^{=1}$ and k-simplices between $k + 1$ nodes corresponding to $k + 1$ divisors containing a common codimension-$(k + 1)$ locus.

Let $\pi\colon Y \to X$ be a projective birational morphism such that (Y, Δ_Y) is dlt, where $K_Y + \Delta_Y = \pi^*(K_X + \Delta)$. In characteristic 0 one can show, using the weak factorization theorem, that $D(\Delta_Y^{=1})$ is homotopy equivalent to $D(\Delta^{=1})$. In characteristic $p > 0$, the weak factorization theorem is not known to hold, but a similar result may be obtained by running an MMP and using the proof of [11, Theorem 19] (cf. [21, Section 2.3]).

For the convenience of the reader, we give a direct proof of a consequence of the above result, one that we will need later. Here we say that an irreducible divisor D in $\Delta^{=1}$ is an *articulation point* if $\Delta^{=1} - D$ is disconnected.

LEMMA 2.12
Let (X, Δ) be a $\mathbb{Q}$-factorial dlt threefold over a perfect field, and let $\pi : Y \to X$ be a projective birational morphism such that $(Y, \pi_*^{-1}\Delta + E)$ is dlt, where E is the exceptional locus of π. Write $K_Y + \Delta_Y = \pi^*(K_X + \Delta)$. Let S be an irreducible divisor in $\Delta^{=1}$, and let S_Y be its strict transform. If S_Y is an articulation point, then so is S.

Proof
Assume that S_Y is an articulation point of $D(\Delta_Y^{=1})$, and let $h\colon Y \dashrightarrow X'$ be the output of a $(K_Y + \pi_*^{-1}\Delta + E)$-MMP over X (which we can run by [13, Theorem 1.1]). Further, let

$$\Delta_{X'} := h_*\Delta_Y = h_*(\pi_*^{-1}\Delta + E)$$

and $S_{X'} := h_*S_Y$. First, we show that $S_{X'}$ is an articulation point of $D(\Delta_{X'}^{=1})$. To this end, we claim that there is a natural inclusion of dual complexes

$$D(\Delta_{X'}^{=1}) \subseteq D(\Delta_Y^{=1})$$

which identifies the nodes of these dual complexes. Indeed, decompose $h\colon Y \dashrightarrow X'$ into flips and divisorial contractions of the $(K_Y + \pi_*^{-1}\Delta + E)$-MMP:

$$Y =: Y_1 \overset{h_1}{\dashrightarrow} Y_2 \overset{h_2}{\dashrightarrow} \cdots \overset{h_{k-1}}{\dashrightarrow} Y_k := X'.$$

Denote the strict transforms of Δ_Y by $\Delta_{Y_1}, \dots, \Delta_{Y_k}$ and $\Delta_{X'}$, respectively, and denote the projections to X by $\pi_i : Y_i \to X$. Note that $K_Y + \pi_*^{-1}\Delta + E \sim_{X,\mathbb{Q}} a_1E_1 + \cdots + a_mE_m$ for all exceptional divisors $E_1, \dots, E_m \not\subseteq \Delta_Y^{=1}$ and $a_1, \dots, a_m > 0$, and so, by the negativity lemma, this MMP contracts exactly those divisors in E which are not contained in $\Delta_Y^{=1}$. In particular, it preserves the nodes of $D(\Delta_Y^{=1})$.

Set $\overline{\Delta}_{Y_l} = (\pi_l^{-1})_*\Delta + \mathrm{Exc}(\pi_l)$. Note that there is no log canonical center of (Y_l, Δ_{Y_l}) contained in $\mathrm{Exc}((h_{l-1})^{-1})$ by the negativity lemma. Indeed, suppose that there is such a center Z. Then Z is also a log canonical center of $(Y_l, \overline{\Delta}_{Y_l})$, and there exists an exceptional divisorial place E_Z over Y_l with center Z such that $a_{E_Z}(Y_l, \overline{\Delta}_{Y_l}) = 0$. Since h_{l-1} is not an isomorphism over the generic point of Z, [20, Lemma 3.38] implies that

$$0 \leq a_{E_Z}(Y_{l-1}, \overline{\Delta}_{Y_{l-1}}) < a_{E_Z}(Y_l, \overline{\Delta}_{Y_l}) = 0,$$

which is a contradiction.

Now, projecting by h_{l-1} provides a bijection

$$\{Z_{l-1} \in \mathrm{LCC}(Y_{l-1}, \Delta_{Y_{l-1}}) \mid Z_{l-1} \nsubseteq \mathrm{Exc}(h_{l-1})\} \leftrightarrow \{Z_l \in \mathrm{LCC}(Y_l, \Delta_{Y_l})\}$$

for any $1 < l \leq k$. In particular, this induces an inclusion $D(\Delta_{Y_l}^{=1}) \subseteq D(\Delta_{Y_{l-1}}^{=1})$, and so the claim holds and $S_{X'}$ is an articulation point.

Let $\pi' \colon X' \to X$ be the induced morphism. Note that $K_{X'} + \Delta_{X'} = (\pi')^*(K_X + \Delta)$ and that the divisor $\mathrm{Exc}(\pi')$ is contained in $\Delta_{X'}^{=1}$. First, we show that

$$\pi'(\Delta_{X'}^{=1} - S_{X'}) \subseteq \mathrm{Supp}(\Delta^{=1} - S).$$

To this end, pick an irreducible divisor $D \subseteq \mathrm{Supp}(\Delta_{X'}^{=1} - S_{X'})$. Then $\pi'(D)$ is a log canonical center of (X, Δ), and so, since (X, Δ) is dlt, there exists a divisor $S' \subseteq \mathrm{Supp}(\Delta^{=1} - S)$ such that $\pi'(D) \subseteq S'$. This shows the above inclusion.

Now, note that

$$\pi'|_{\mathrm{Supp}(\Delta_{X'}^{=1} - S_{X'})} \colon \mathrm{Supp}(\Delta_{X'}^{=1} - S_{X'}) \to \mathrm{Supp}(\Delta^{=1} - S)$$

has connected fibers. Indeed, $\mathrm{Exc}(\pi') \subseteq \mathrm{Supp}(\Delta_{X'}^{=1} - S_{X'})$ and π' has connected fibers. Therefore, $\Delta_{X'}^{=1} - S_{X'}$ is disconnected if and only if so is $\Delta^{=1} - S$. In particular, S is an articulation point. □

3. Complements on surfaces

The following proposition is fundamental in showing that flips admitting a qdlt 6-complement exist. Note that every two-dimensional log pair with standard coefficients and which is log Fano with respect to a projective birational map admits a relative m-complement for $m \in \{1, 2, 3, 4, 6\}$ (cf. [15, Theorem 3.2]).

PROPOSITION 3.1

Let (S, B) be a two-dimensional klt pair with standard coefficients defined over a perfect field of characteristic $p > 3$, and let $S \to T$ be a birational contraction such that $-(K_S + B)$ is relatively nef but not numerically trivial. Assume that (S, B) is not relatively globally F-regular over T.

Then every 6-complement of (S, B) is non-klt and has a unique non-klt valuation which is exceptional over T.

Proof

We work over a sufficiently small neighborhood of a point $t \in T$. By Lemma 2.8, there exist an irreducible, exceptional over T, curve C on a blowup of S and projective birational maps $g \colon Y \to S$ and $h \colon Y \to W$ over T such that

(1) g extracts C or is the identity if $C \subseteq S$,

(2) $(Y, C + B_Y)$ is plt,

(3) $(W, C_W + B_W)$ is plt and $-(K_W + C_W + B_W)$ is ample over T,

where $C_W := h_*C \neq 0$, $B_W := h_*B_Y$, and $K_Y + bC + B_Y = g^*(K_S + B)$ for $C \not\subseteq \operatorname{Supp} B_Y$.

By Proposition 2.9 and Remark 2.10, $(K_W + C_W + B_W)|_{C_W} = K_{C_W} + \frac{1}{2}P_1 + \frac{2}{3}P_2 + \frac{4}{5}P_3$ for some three distinct points P_1, P_2, and P_3.

Now, let (S, B^c) be any 6-complement of (S, B). By the negativity lemma, $\operatorname{Supp}(B^c - B)$ contains a nonexceptional curve. Let $K_Y + aC + B^c_Y = g^*(K_S + B^c)$, where $C \not\subseteq \operatorname{Supp} B^c_Y$, and let $B^c_W := h_*B^c_Y$. Since $6(K_S + B^c) \sim_T 0$ is lc, we get that

$$(W, aC_W + B^c_W)$$

is sub-lc and $6(K_W + aC_W + B^c_W) \sim_T 0$. In particular, $6B^c_W$ is an integral divisor. Moreover, $B^c_W \geq B_W$ as $B^c \geq B$.

To prove the proposition, it is now enough to show that $a = 1$. Indeed, in this case $-(K_W + C_W + B^c_W) \sim_{\mathbb{Q},T} 0$ and, by the Kollár–Shokurov connectedness lemma, the non-klt locus of $(W, C_W + B^c_W)$ is connected (note that $W \to T$ is birational so $-(K_W + C_W + B^c_W)$ is relatively nef and big). The only 6-complement of

$$\Big(C_W, \frac{1}{2}P_1 + \frac{2}{3}P_2 + \frac{4}{5}P_3\Big)$$

is $(C_W, \frac{1}{2}P_1 + \frac{2}{3}P_2 + \frac{5}{6}P_3)$, so $(W, C_W + B^c_W)$ is plt along C_W by adjunction, and connectedness of the non-klt locus implies that $(W, C_W + B^c_W)$ is in fact plt everywhere. In particular, (S, B^c) admits a unique exceptional over T non-klt valuation.

In order to prove the proposition, we assume that $a < 1$ and derive a contradiction. We will not need to refer to (S, B) or $(Y, aC + B_Y)$ anymore, so, for ease of notation, we replace C_W, B_W, and B^c_W by C, B, and B^c, respectively.

If $(B^c - B) \cdot C \neq 0$, then Lemma 3.2 applied to $(W, C + B^c)$ implies that $(K_W + C + B^c) \cdot C = 0$. This is impossible because

$$(K_W + C + B^c) \cdot C < (K_W + aC + B^c) \cdot C = 0.$$

Hence, we can assume that $(B^c - B) \cdot C = 0$. Since $\operatorname{Supp}(B^c - B)$ contains a nonexceptional curve, the exceptional locus over T cannot be irreducible (otherwise it is equal to C and $(B^c - B) \cdot C > 0$), and so as the exceptional locus is connected we can pick an irreducible exceptional curve $E \neq C$ such that $E \cap C \neq \emptyset$. Since $E \cong \mathbb{P}^1$ and $E^2 < 0$, we may contract E over T. Let $f : W \to W_1$ be a contraction of E, and let C_1, B^c_1 be the strict transforms of C and B^c. We have that

$$(K_W + C + B^c) \cdot E > (K_W + aC + B^c) \cdot E = 0,$$

and hence for some $t > 0$ and with the natural identification $C \simeq C_1$,

$$\begin{aligned}(K_{W_1} + C_1 + B_1^c)|_{C_1} &= f^*(K_{W_1} + C_1 + B_1^c)|_C \\ &= (K_W + C + B^c + tE)|_C \\ &\geq K_C + \frac{1}{2}P_1 + \frac{2}{3}P_2 + \frac{4}{5}P_3 + tE|_C.\end{aligned}$$

As before, $(K_{W_1} + C_1 + B_1^c) \cdot C_1 < (K_{W_1} + aC_1 + B_1^c) \cdot C_1 = 0$. By applying Lemma 3.2 to $(W_1, C_1 + B_1^c)$, we again obtain a contradiction. □

In the following result, it is key that Δ is nonzero.

LEMMA 3.2
Let $(S, C + B)$ be a two-dimensional log pair, and let $f : S \to Z$ be a projective birational morphism such that the irreducible normal divisor C is exceptional and $(K_S + C + B) \cdot C \leq 0$. Assume that $6B$ is an integral divisor and that

$$B_C = \frac{1}{2}P_1 + \frac{2}{3}P_2 + \frac{4}{5}P_3 + \Delta$$

for distinct points $P_1, P_2, P_3 \in C$ and a nonzero effective $\mathbb{Q}$-divisor Δ, where $(K_S + C + B)|_C = K_C + B_C$. Then $(K_S + C + B) \cdot C = 0$.

Proof
Assume by contradiction that $(K_S + C + B) \cdot C < 0$. Since $\frac{1}{2} + \frac{2}{3} + \frac{4}{5} = 2 - \frac{1}{30}$, we obtain

$$-\frac{1}{30} < (K_S + C + B) \cdot C < 0.$$

Set $y_i := \frac{1}{2}, \frac{2}{3}, \frac{4}{5}$ for $i = 1, 2, 3$, respectively, and write

$$B_C = x_1 P_1 + x_2 P_2 + x_3 P_3 + \Delta',$$

where $\Delta' \geq 0$ and $P_i \not\subseteq \operatorname{Supp} \Delta'$. We have that

$$y_i \leq x_i < y_i + 1/30,$$

and so $x_i < \frac{4}{5} + \frac{1}{30} = \frac{5}{6}$. Further, $\deg \Delta' < \frac{1}{30}$. By adjunction, $(S, C + B)$ is plt along C.

Let Γ_i be the intersection matrix of the singularity of S at P_i. Recall that $\det \Gamma_i$ is the $\mathbb{Q}$-factorial index of P_i; that is, for any Weil divisor D, it holds that $(\det \Gamma_i) D$

is Cartier at P_i (see [7, Lemma 2.2]). By [19, Corollary 3.45],

$$x_i = 1 - \frac{1}{\det \Gamma_i} + \frac{k}{6 \det \Gamma_i}$$

for some integer $k \geq 0$; in particular, it is of the form $\frac{m}{6 \det \Gamma_i}$. Moreover, $\det \Gamma_i \leq 5$, as otherwise $x_i \geq \frac{5}{6}$.

We claim that $x_i = y_i$. If $i \in \{1, 2\}$, then $6(\det \Gamma_i) y_i \in \mathbb{N}$ and so either $x_i = y_i$ or

$$x_i \geq y_i + \frac{1}{6 \det \Gamma_i} \geq y_i + \frac{1}{30},$$

which is a contradiction. If $i = 3$, then since $\det \Gamma_3 \leq 5$, it is easy to see that

$$\frac{5 \det \Gamma_3 - 1}{6 \det \Gamma_3} \leq \frac{4}{5} \leq x_3 < \frac{5 \det \Gamma_3}{6 \det \Gamma_3} = \frac{5}{6}.$$

Since $x_i = \frac{m}{6 \det \Gamma_3}$, it follows that $x_3 = y_3 = \frac{4}{5}$ and $\det \Gamma_3 = 5$.

Hence, $x_i = y_i$ for all $i \in \{1, 2, 3\}$ and $\Delta' = \Delta$. In particular, either $\operatorname{Supp} \Delta$ is contained in the smooth locus of S and $\deg \Delta \geq \frac{1}{6} \geq \frac{1}{30}$, or $\deg \Delta$ is bounded from below by the smallest standard coefficient: $1/2$. In either case, this is a contradiction. □

4. Lifting complements

The new building blocks for the low characteristic MMP are flips admitting a qdlt 6-complement. Therefore, it is fundamental to construct 6-complements of flipping contractions. This is done by lifting them from divisorial centers as described by Theorem 1.6. Before we move on to the proof of this result, we need to show some results about Frobenius-stable sections for $\mathbb{Q}$-divisors.

4.1. Frobenius-stable sections and integral adjunction

In this subsection, we assume the existence of log resolutions of singularities admitting relatively antiample effective exceptional divisors. In particular, the results of this section are valid up to dimension 3. Further, we denote the Frobenius-stable sections of a line bundle L with respect to the Frobenius trace map associated to (X, Δ) by $S^0(X, \Delta; L)$. Note that this space is often denoted by $S^0(X, \sigma(X, \Delta) \otimes L)$. We refer to [22] and [15] for the definition and a comprehensive treatment of S^0.

Let (X, Δ) be a positive characteristic log Fano pair. Fix $m \in \mathbb{N}$, and set $A := -(K_X + \Delta)$. We want to study the sections in $H^0(X, \lfloor mA \rfloor)$ which are Frobenius stable with respect to a carefully chosen boundary.

If Δ has standard coefficients, then the theory of complements gives a natural candidate: $\Phi := \{(m+1)\Delta\}$. Indeed, in this case,

$$\lfloor mA \rfloor - (K_X + \Phi) = -(m+1)(K_X + \Delta)$$

is ample (see (1)), which suggests that one should look at the subspace

$$S^0\big(X, \Phi; \lfloor mA \rfloor\big) \subseteq H^0\big(X, \lfloor mA \rfloor\big).$$

Since standard coefficients are not stable under log pullbacks or perturbations, we need to work in a more general setting.

SETTING 4.1
As mentioned before, we assume the existence of log resolutions of singularities admitting relatively antiample effective exceptional divisors.

Fix a natural number $m \in \mathbb{N}$ and a perfect field k of characteristic $p > 0$. Let $(X, S + B)$ be a sublog pair which is projective over an affine k-variety Z and such that S is a (possibly empty) reduced Weil divisor, $\lfloor B \rfloor \leq 0$, and $A := -(K_X + S + B)$ is nef and big.

We are ready to define

$$\begin{aligned} \Phi &:= S + \big\{(m+1)B\big\}, \\ D &:= \lceil mB \rceil - \big\lfloor (m+1)B \big\rfloor, \qquad \textit{and} \\ L &:= \lfloor mA \rfloor + D. \end{aligned}$$

For the sake of future perturbations, we choose an effective $\mathbb{Q}$-divisor Λ with sufficiently small coefficients, no common components with S, and such that $K_X + S + B + \Lambda^m$ is of Cartier index nondivisible by $p > 0$, where $\Lambda^m := \frac{1}{m+1}\Lambda$. Such Λ exists by Remark 4.3.

We call D the *defect* divisor and say that $(X, S + B)$ has *zero defect* if $D = 0$. Note that $(X, S + B)$ has zero defect when B has standard coefficients. In general, since $\lfloor B \rfloor \leq 0$, we have

$$\begin{aligned} D &= \lceil mB \rceil - \big\lfloor (m+1)B \big\rfloor \\ &= \big\lceil mB - (m+1)B + \big\{(m+1)B\big\}\big\rceil \\ &= \big\lceil -B + \big\{(m+1)B\big\}\big\rceil \geq 0. \end{aligned}$$

Moreover,

$$\begin{aligned} \lfloor mA \rfloor &= -m(K_X + S) - \lceil mB \rceil \\ &= -m(K_X + S) - \big\lfloor (m+1)B \big\rfloor - D \\ &= K_X + \Phi - (m+1)(K_X + S + B) - D, \end{aligned}$$

and so

$$L - (K_X + \Phi) = -(m+1)(K_X + S + B) = (m+1)A \tag{1}$$

is nef and big. In particular,

$$L - (K_X + \Phi + \Lambda) = -(m+1)(K_X + S + B + \Lambda^m), \tag{2}$$

and so the Weil index of $K_X + \Phi + \Lambda$ is not divisible by p.

Definition 4.2
With notation as above, define $C^0_\Lambda(X, S+B; L) := S^0(X, \Phi + \Lambda; L) \subseteq H^0(X, L)$.

By Noetherianity and the fact that Λ is assumed to have sufficiently small coefficients, we can replace Λ by any Λ' satisfying the assumptions of Setting 4.1, having sufficiently small coefficients, and such that $\operatorname{Supp}\Lambda' = \operatorname{Supp}\Lambda$.

Remark 4.3
There always exists Λ as in Setting 4.1. Indeed, we can assume that K_X is such that $S \not\subseteq \operatorname{Supp}(A)$. Pick a sufficiently ample Cartier divisor M, use Serre vanishing to find $M' \sim M$ vanishing along $\operatorname{Supp}(A)$ with high multiplicity but without vanishing along S, and set $\Lambda := (m+1)(M' + A)$. Moreover, given such Λ, we can replace it by $\epsilon\Lambda$ for some $0 < \epsilon \ll 1$ by the same argument as in [30, Lemma 2.10].

The following lemma allows for calculating C^0 on a log resolution.

LEMMA 4.4
With notation as in Setting 4.1, suppose that $(X, S+B)$ is plt and has zero defect. Let $\pi\colon Y \to X$ be a projective birational map, and set $K_Y + S_Y + B_Y = \pi^*(K_X + S + B)$ with $S_Y := \pi^{-1}_* S$. Then

$$C^0_{\Lambda_Y}(Y, S_Y + B_Y; L_Y) = C^0_\Lambda(X, S+B; L),$$

where L_Y is defined for $(Y, S_Y + B_Y)$ as in Setting 4.1, and $\Lambda_Y := \pi^*\Lambda$.

Note that $\pi_* L_Y = L$, but L_Y is rarely the pullback of L. This lemma holds for any $\mathbb{Q}$-divisor Λ_Y satisfying the assumptions of Setting 4.1 and such that $\operatorname{Supp}\Lambda_Y = \operatorname{Supp}\pi^*\Lambda$.

Proof
Set $\Lambda^m_Y := \frac{1}{m+1}\Lambda_Y$. Since $(X, S+B)$ is plt, we have that $\lfloor B_Y \rfloor \leq 0$. The subspace $S^0(Y, \Phi_Y + \Lambda_Y; L_Y)$ is given as the image of

$$H^0\big(Y, F^e_*\mathcal{O}_Y\big((1-p^e)(K_Y+\Phi_Y+\Lambda_Y)+p^eL_Y\big)\big) \to H^0\big(Y, \mathcal{O}_Y(L_Y)\big)$$

for a sufficiently divisible integer $e > 0$. Therefore, it is enough to show the following two identities: $\pi_*\mathcal{O}_Y(L_Y) = \mathcal{O}_X(L)$ and

$$\pi_*\mathcal{O}_Y\big((1-p^e)(K_Y+\Phi_Y+\Lambda_Y)+p^eL_Y\big) = \mathcal{O}_X\big((1-p^e)(K_X+\Phi+\Lambda)+p^eL\big).$$

We begin by checking the first one. Since $\pi_*L_Y = L$, there is an inclusion $\pi_*\mathcal{O}_Y(L_Y) \subset \mathcal{O}_X(L)$. Since $mA_Y + D_Y = \pi^*(mA) + D_Y$, where D_Y is an effective Weil divisor, we have

$$L_Y = \lfloor mA_Y \rfloor + D_Y = \big\lfloor \pi^*(mA) \big\rfloor + D_Y \geq \pi^*\big(\lfloor mA \rfloor\big) + D_Y = \pi^*L + D_Y.$$

Here we used the fact that the defect $D = 0$. Since D_Y is effective and exceptional, $\pi_*\mathcal{O}_Y(D_Y) = \mathcal{O}_X$. The inclusion $\pi_*\mathcal{O}_Y(L_Y) \supset \mathcal{O}_X(L)$ now follows from the projection formula.

We will now show the second one. To this end, we can use (2) to write

$$(1-p^e)(K_Y+\Phi_Y+\Lambda_Y)+p^eL_Y = (1-p^e)(m+1)(K_Y+S_Y+B_Y+\Lambda^m_Y)+L_Y.$$

Since $K_Y + S_Y + B_Y + \Lambda^m_Y = \pi^*(K_X + S + B + \Lambda^m)$ is Cartier up to multiplying by $p^e - 1$ for a sufficiently divisible integer $e > 0$, the second identity follows from the first one by the projection formula. ☐

The following lemma allows for lifting sections.

LEMMA 4.5
With notation as in Setting 4.1, suppose that $(X, S+B)$ is plt with standard coefficients, S is an irreducible divisor, and $A := -(K_X + S + B)$ is ample. Assume that Supp Λ *contains the non-snc locus of $(X, S+B)$, and write $A_{\tilde{S}} := -(K_{\tilde{S}} + B_{\tilde{S}}) = -(K_X+S+B)|_{\tilde{S}}$ for the normalization $\tilde{S}$ of S. Then, by restricting sections, we get a surjection*

$$C^0_\Lambda\big(X, S+B; \lfloor mA \rfloor\big) \to C^0_{\Lambda_{\tilde{S}}}\big(\tilde{S}, B_{\tilde{S}}; \lfloor mA_{\tilde{S}} \rfloor\big),$$

where $\Lambda_{\tilde{S}} := \Lambda|_{\tilde{S}}$.

Proof
Let $\pi : Y \to X$ be a log resolution of $(X, S+B)$ which is an isomorphism over the simple normal crossings locus. We can write

$$K_Y + S_Y + B_Y = \pi^*(K_X + S + B) \quad \text{and}$$

$$K_{S_Y} + B_{S_Y} = (K_Y + S_Y + B_Y)|_{S_Y}$$

for $S_Y := \pi^{-1}_*S$. Define L_Y, L_{S_Y}, Φ_Y, Φ_{S_Y} as in Setting 4.1.

Pick a π-exceptional effective antiample divisor E. Let

$$\Lambda_Y := \pi^*\Lambda + \epsilon E$$

for $0 < \epsilon \ll 1$ such that Λ_Y satisfies the assumptions of Setting 4.1 and $\operatorname{Supp}\Lambda_Y = \operatorname{Supp}\pi^*\Lambda$. Set $\Lambda_{S_Y} := \Lambda_Y|_{S_Y}$.

By the standard adjunction for S^0 (see, e.g., [15, Proposition 2.3]), since

$$L_Y - (K_Y + \Phi_Y + \Lambda_Y) = -(m+1)(K_Y + S_Y + B_Y + \Lambda_Y^m)$$

is ample, restricting sections induces a surjective map

$$S^0(Y, \Phi_Y + \Lambda_Y; L_Y) \to S^0(S_Y, \Phi_{S_Y} + \Lambda_{S_Y}; L_{S_Y}).$$

Indeed, $K_{S_Y} + \Phi_{S_Y} = (K_Y + \Phi_Y)|_{S_Y}$ and $L_Y|_{S_Y} = L_{S_Y}$ as $(Y, S_Y + B_Y)$ is log smooth. Thus, $C^0_{\Lambda_Y}(Y, S_Y + B_Y; L_Y) \to C^0_{\Lambda_{S_Y}}(S_Y, B_{S_Y}; L_{S_Y})$ is surjective, and the claim follows from Lemma 4.4 applied to both sides. Note that even though the hypothesis $\Lambda_Y = \pi^*\Lambda$ is not satisfied, we have $\pi^*\Lambda \le \Lambda_Y \le \pi^*\Lambda'$, where the supports of Λ and Λ' coincide and we may assume that $C^0_\Lambda(X, S + B; L) = C^0_{\Lambda'}(X, S + B; L)$. □

Finally, we show that C^0 gets smaller when the boundary gets bigger.

LEMMA 4.6
Let $(X, S + B)$ and $(X, S' + B')$ be two sublog pairs satisfying the assumptions of Setting 4.1. Suppose that $S' + B' \ge S + B$, and define L and L' for $(X, S + B)$ and $(X, S' + B')$, respectively, as in Setting 4.1.

Then $L - L' \ge 0$ and the inclusion $H^0(X, \mathcal{O}_X(L')) \subseteq H^0(X, \mathcal{O}_X(L))$ induces an inclusion

$$C^0_{\Lambda'}(X, S' + B'; L') \subseteq C^0_\Lambda(X, S + B; L),$$

where Λ, Λ' are as in Setting 4.1 and $\operatorname{Supp}\Lambda \subseteq \operatorname{Supp}\Lambda' \cup (S' - S)$.

Note that it would be too restrictive to assume that $\operatorname{Supp}\Lambda \subseteq \operatorname{Supp}\Lambda'$. Indeed, Λ' as in Setting 4.1 has no common components with S', while Λ has no common components with S but will often have common components with $S' - S$.

Proof
Let Φ and Φ' be defined for $(X, S + B)$ and $(X, S' + B')$ as in Setting 4.1. By (1), we have

$$\begin{aligned} L - L' &= \Phi - \Phi' + (m+1)(S' + B' - S - B) \\ &= S - S' + \lfloor (m+1)(S'+B') \rfloor - \lfloor (m+1)(S+B) \rfloor, \end{aligned}$$

and so $L - L' \geq 0$.

We may assume that $\Lambda \leq \Lambda' + (m+1)(S' + B' - S - B)$. Then

$$\begin{aligned} S^0(X, \Phi' + \Lambda'; L') &\subseteq S^0\big(X, \Phi' + \Lambda' + (L - L'); L\big) \\ &= S^0\big(X, \Phi + \Lambda' + (m+1)(S' + B' - S - B); L\big) \\ &\subseteq S^0(X, \Phi + \Lambda; L). \square \end{aligned}$$

4.2. The proof of Theorem 1.6

We are ready to show that m-complements of pl-flipping contractions exist for $m \in \{1,2,3,4,6\}$. With notation as in Setting 4.1, note that

$$\lfloor mA \rfloor = \lfloor -m(K_X + S + B) \rfloor = -m(K_X + S + B^*)$$

for $B^* := \frac{1}{m}\lceil mB \rceil \geq B$. When B has standard coefficients, then the defect is zero, $B^* = \frac{1}{m}\lfloor (m+1)B \rfloor$, and $L = \lfloor mA \rfloor = -m(K_X + S + B^*)$.

Proof of Theorem 1.6

We may assume that Z is affine. Let $\tilde{S}$ be the normalization of S. By Lemma 4.5, restricting sections gives a surjective map

$$C^0_{\Lambda}\big(X, S + B; -m(K_X + S + B^*)\big) \to C^0_{\Lambda_{\tilde{S}}}\big(\tilde{S}, B_{\tilde{S}}; -m(K_{\tilde{S}} + B^*_{\tilde{S}})\big),$$

where $K_{\tilde{S}} + B_{\tilde{S}} = (K_X + S + B)|_{\tilde{S}}$, $B^*_{\tilde{S}} = \frac{1}{m}\lceil mB_{\tilde{S}} \rceil$, and Λ is as in Setting 4.1 with $\operatorname{Supp} \Lambda$ containing $\operatorname{Exc}(f)$ and the non-snc locus of $(X, S + B)$. Set $\Lambda_{\tilde{S}} := \Lambda|_{\tilde{S}}$, and note that it satisfies the assumptions of Setting 4.1 for $(\tilde{S}, B_{\tilde{S}})$.

By Lemma 4.7, there exists $\Gamma_{\tilde{S}} \in |-m(K_{\tilde{S}} + B^*_{\tilde{S}})|$ such that $(\tilde{S}, B^c_{\tilde{S}})$ is an m-complement of $(\tilde{S}, B_{\tilde{S}})$ for $B^c_{\tilde{S}} = B^*_{\tilde{S}} + \frac{1}{m}\Gamma_{\tilde{S}}$ and which moreover lifts to

$$\Gamma \in \big|-m(K_X + S + B^*)\big|.$$

Set $B^c = B^* + \frac{1}{m}\Gamma$. Then $m(K_X + S + B^c) \sim 0$ and $(K_X + S + B^c)|_{\tilde{S}} = K_{\tilde{S}} + B^c_{\tilde{S}}$. By inversion of adjunction (see [14, Corollary 1.5]) applied to $(X, S + (1-\epsilon)B^c)$ for $0 < \epsilon \ll 1$, we get that $(X, S + B^c)$ is lc in a neighborhood of $\operatorname{Exc} f$, and hence it is an m-complement of $(X, S + B)$. $\square$

In the above proof, we used the following lemma.

LEMMA 4.7
Let (X, B) *be a two-dimensional klt pair with standard coefficients defined over a perfect field of characteristic* $p > 2$*, and let* $f: X \to Z$ *be a projective birational map such that* $-(K_X + B)$ *is ample. Then there exists* $m \in \{1, 2, 3, 4, 6\}$ *and*

$$s \in C^0_\Lambda\big(X, B; -m(K_X + B^*)\big) \subseteq H^0\big(X, -m(K_X + B^*)\big)$$

such that $(X, B^* + \frac{1}{m}\Gamma)$ *is an* m*-complement of* (X, B) *in a neighborhood of* $\mathrm{Exc}(f)$*, where* $B^* := \frac{1}{m}\lceil mB \rceil$ *and* Γ *is the divisor corresponding to* s*. Here* Λ *is as in Setting 4.1.*

Proof
By Lemma 2.8, there exist an irreducible, exceptional over Z, curve C on a blowup of X and projective birational maps $g: Y \to X$ and $h: Y \to W$ over Z such that
(1) g extracts C or is the identity if $C \subseteq X$,
(2) $(Y, C + B_Y)$ is plt,
(3) $(W, C_W + B_W)$ is plt and $-(K_W + C_W + B_W)$ is ample over Z,
(4) $B_Y^+ - B_Y \geq 0$,
where $K_Y + bC + B_Y = g^*(K_X + B)$ for $C \not\subseteq \mathrm{Supp}\, B_Y$, $C_W := h_*C \neq 0$, $B_W := h_*B_Y$, and $K_Y + C + B_Y^+ = h^*(K_W + C_W + B_W)$.

We have

$$\begin{aligned} C^0_\Lambda(X, B; L) &= C^0_{\Lambda_Y}(Y, bC + B_Y; L_Y) \\ &\supseteq C^0_{\Lambda_Y^+}(Y, C + B_Y^+; L_Y^+) \\ &= C^0_{\Lambda_W}(W, C_W + B_W; L_W), \end{aligned}$$

where L, L_Y, L_Y^+, and L_W are defined as in Setting 4.1 and the defects D and D_W vanish as B and $C_W + B_W$ have standard coefficients. The first and third equalities hold by Lemma 4.4, and the middle inclusion holds by Lemma 4.6 since $C + B_Y^+ \geq bC + B_Y$. Here, the perturbation divisors were chosen in the following way. First, we set $\Lambda_Y := g^*\Lambda$. Second, we pick Λ_W for $(W, C_W + B_W)$ as in Setting 4.1. By the construction in Remark 4.3, we can assume that Λ_W contains $g(\mathrm{Supp}(\Lambda_Y - \Lambda_Y \wedge C) \cup \mathrm{Exc}(h))$ and the non-snc locus of $(W, C_W + B_W)$ in its support. Lastly, we set $\Lambda_Y^+ := h^*\Lambda_W$.

Note that $L = -m(K_X + B^*)$ and $L_W = -m(K_W + C_W + B_W^*)$ for $B_W^* = \frac{1}{m}\lceil mB_W \rceil$. By Lemma 4.5, restricting sections thus gives a surjective map

$$C^0_{\Lambda_W}\big(W, C_W + B_W; -m(K_W + C_W + B_W^*)\big) \to C^0_{\Lambda_C}\big(C, B_C; -m(K_C + B_C^*)\big),$$

where C is identified with C_W and $K_C + B_C = (K_W + C_W + B_W)|_C$. As usual, $B_C^* := \frac{1}{m}\lceil mB_C \rceil$ and $\Lambda_C := \Lambda_W|_C$.

Let $m \in \{1,2,3,4,6\}$ be the minimal number such that (C, B_C) admits an m-complement. By Lemma 4.9, $(C, \{(m+1)B_C\})$ is globally F-regular, and so

$$C^0_{\Lambda_C}\big(C, B_C; -m(K_C + B^*_C)\big) = H^0\big(C, -m(K_C + B^*_C)\big).$$

In particular, there exists an lc m-complement (C, B^c_C) of (C, B_C) for some $m \in \{1,2,3,4,6\}$ (and hence of (C, B^*_C) as $mB^*_C = \lceil mB_C \rceil$) which can be lifted to W. More precisely, there exists a nonzero section

$$s \in C^0_{\Lambda_W}\big(W, C_W + B_W; -m(K_W + C_W + B^*_W)\big)$$

with associated divisor Γ such that $m(K_W + C_W + B^c_W) \sim 0$ and

$$(K_W + C_W + B^c_W)|_C = K_C + B^c_C,$$

where $B^c_W := B^*_W + \frac{1}{m}\Gamma$. By inversion of adjunction, $(W, C_W + B^c_W)$ is lc along C_W. Note that

$$K_W + C_W + \epsilon B_W + (1-\epsilon)B^c_W$$

is thus plt along C_W and $\mathbb{Q}$-equivalent over Z to $\epsilon(K_W + C_W + B_W)$, and hence by the Kollár–Shokurov connectedness principle (cf. [25, Theorem 5.2]), it is plt for any $0 < \epsilon < 1$. Hence $(W, C_W + B^c_W)$ is lc and thus an m-complement of $(W, C_W + B_W)$.

Let $K_Y + C + B^c_Y = h^*(K_W + C_W + B^c_W)$ and $B^c := g_*(C + B^c_Y)$. Then (X, B^c) is an m-complement of (X, B) which, by the above inclusions of C^0, corresponds to a section in $C^0_\Lambda(X, B; -m(K_X + B^*))$. □

Remark 4.8

With notation as in Theorem 1.6, if $(X, S + B)$ is not purely relatively F-regular and $p = 5$, then $m = 6$. Indeed, under these assumptions, $(\tilde{S}, B_{\tilde{S}})$ is not relatively F-regular by F-adjunction, and hence, in the proof of Lemma 4.7, we have that $B_C = \frac{1}{2}P_1 + \frac{2}{3}P_2 + \frac{4}{5}P_3$ for distinct points P_1, P_2, and P_3, by Remark 2.10. The smallest m for which this (C, B_C) admits an m-complement is $m = 6$.

LEMMA 4.9

Let $(\mathbb{P}^1, B)$ be a log pair with standard coefficients and $\deg B < 2$ defined over a perfect field of characteristic $p > 2$. Let $m \in \{1,2,3,4,6\}$ be the minimal number such that $(\mathbb{P}^1, B)$ admits an m-complement. Then $(\mathbb{P}^1, \{(m+1)B\})$ is globally F-regular.

Proof

If B is supported at two or fewer points, then so is $\{(m+1)B\}$, and hence $(\mathbb{P}^1, \{(m+$

$1)B\})$ is globally F-regular. Indeed, one can always increase one of the coefficients to 1 and apply global F-adjunction.

Thus, we can assume that $B = a_1P_1 + a_2P_2 + a_3P_3$ for distinct points P_1, P_2, P_3 and $(a_1, a_2, a_3) \in \{(\frac{1}{2}, \frac{1}{2}, 1 - \frac{1}{n}), (\frac{1}{2}, \frac{2}{3}, \frac{2}{3}), (\frac{1}{2}, \frac{2}{3}, \frac{3}{4}), (\frac{1}{2}, \frac{2}{3}, \frac{4}{5})\}$, where $n \in \mathbb{N}$ is arbitrary. These are 2-, 3-, 4-, and 6-complementary, respectively.

Therefore, $\{(m+1)B\} = b_1P_1 + b_2P_2 + b_3P_3$ for

$$(b_1, b_2, b_3) \in \Big\{\Big(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\Big), \Big(\frac{1}{2}, \frac{1}{2}, \frac{n-3}{n}\Big), \Big(0, \frac{2}{3}, \frac{2}{3}\Big), \Big(\frac{1}{2}, \frac{1}{3}, \frac{3}{4}\Big), \Big(\frac{1}{2}, \frac{2}{3}, \frac{3}{5}\Big)\Big\},$$

where $n \geq 3$.

To solve the first two cases, it is enough to show that $(\mathbb{P}^1, \frac{1}{2}P_1 + \frac{1}{2}P_2 + (1 - \frac{1}{n})P_3)$ is globally F-regular which follows by [29, Theorem 4.2]. For the next two cases, we can argue as in the first paragraph: by increasing the biggest coefficient to 1 (obtaining $(0, \frac{2}{3}, 1)$, $(\frac{1}{2}, \frac{1}{3}, 1)$) and applying F-adjunction. When $p \geq 5$, the last case follows by increasing $\frac{3}{5}$ to $\frac{3}{4}$ and applying [29, Theorem 4.2] again.

We are left to show the last case for $p = 3$. By Fedder's criterion, it is enough to check that $(x+y)^{c_1}x^{c_2}y^{c_3}$ contains a monomial x^iy^j for some $i, j < p^e - 1$ and $e > 0$, where $c_r := \lceil (p^e - 1)b_i \rceil$ and $r \in \{1, 2, 3\}$. Take $e = 3$. Then we have $(c_1, c_2, c_3) = (13, 18, 16)$ and

$$(x+y)^{13}x^{18}y^{16} = \cdots + \binom{13}{9}x^{22}y^{25} + \cdots,$$

where 3 does not divide $\binom{13}{9} = \frac{10 \cdot 11 \cdot 12 \cdot 13}{4!}$. □

5. Flips admitting a qdlt complement

The goal of this section is to show the existence of flips for flipping contractions admitting a qdlt k-complement, where $k \in \{1, 2, 3, 4, 6\}$.

PROPOSITION 5.1 (cf. Proposition 1.5)
Let (X, Δ) be a $\mathbb{Q}$-factorial three-dimensional qdlt pair with standard coefficients over a perfect field k of characteristic $p > 3$. Let $f: X \to Z$ be a $(K_X + \Delta)$-flipping contraction such that $\rho(X/Z) = 1$, and let Σ be a flipping curve. Assume that there exists a qdlt 6-complement (X, Δ^c) of (X, Δ) such that $\Sigma \cdot S < 0$ for some irreducible divisor $S \subseteq \lfloor \Delta^c \rfloor$. Then the flip $f^+: X^+ \to Z$ exists.

Proof
Write $\Delta = aS + D + B$, where $1 \geq a \geq 0$, the divisor D is integral, $S \not\subseteq \operatorname{Supp}(D + B)$, and $\lfloor B \rfloor = 0$. By replacing Δ by $S + (1 - \frac{1}{k})D + B$ for $k \gg 0$, we can assume that (X, Δ) is plt. As explained in the introduction, we split the proof into three cases:

(1) (X, Δ^c) is plt along the flipping locus, or
(2) $\Sigma \cdot E < 0$ for a divisor $E \subseteq \lfloor \Delta^c \rfloor$ different from S, or
(3) $\Sigma \cdot E \geq 0$ for a divisor $E \subseteq \lfloor \Delta^c \rfloor$ intersecting the flipping locus.

Cases (1) and (3) follow from Propositions 5.2 and 5.4, respectively, applied to (X, Δ). Case (2) follows from Proposition 5.3 applied to $(X, \Delta + bE)$, where $b \geq 0$ is such that $\mathrm{mult}_E(\Delta + bE) = 1$. □

PROPOSITION 5.2

Let $(X, S + B)$ be a $\mathbb{Q}$-factorial three-dimensional plt pair over a perfect field k of characteristic $p > 3$ with S irreducible and B having standard coefficients. Let $f: X \to Z$ be a pl-flipping contraction such that $\rho(X/Z) = 1$. Assume that there exists a plt 6-complement $(X, S + B^c)$ of $(X, S + B)$ over Z. Then the flip exists.

Proof

Write $K_{\tilde{S}} + B_{\tilde{S}} = (K_X + S + B)|_{\tilde{S}}$ and $K_{\tilde{S}} + B^c_{\tilde{S}} = (K_X + S + B^c)|_{\tilde{S}}$ for the normalization $\tilde{S}$ of S. The pair $(\tilde{S}, B^c_{\tilde{S}})$ is a klt 6-complement, and so $(\tilde{S}, B_{\tilde{S}})$ is relatively F-regular by Proposition 3.1. In particular, the flip exists by Theorem 2.1. □

The following proposition addresses case (2). The idea is due to Mori [18, Theorem 20.11] and it was suggested to us by James McKernan.

PROPOSITION 5.3

Let (X, Δ) be a $\mathbb{Q}$-factorial three-dimensional qdlt pair over a perfect field k of characteristic $p > 0$, let $f: X \to Z$ be a flipping contraction such that $\rho(X/Z) = 1$, and let Σ be a flipping curve. Assume that there exist distinct irreducible divisors $S, E \subseteq \lfloor \Delta \rfloor$ such that $S \cdot \Sigma < 0$ and $E \cdot \Sigma < 0$. Then the flip of Σ exists.

Proof

Note that $\mathbb{Q}$-divisors which are numerically equivalent over Z are automatically $\mathbb{Q}$-linearly equivalent over Z by an appropriate pl-contraction theorem (see, e.g., [12, Lemma 2.4]).

We may assume that Z is a sufficiently small affine neighborhood of $Q := f(\Sigma)$. Let $k, l \in \mathbb{N}$ be such that $kS \sim_Z lE$ are Cartier, and consider a pencil $h: X \dashrightarrow \mathbb{P}^1_Z$ given by the linear system in $|kS|$ induced by these two divisors. We set X' to be the closure of the image of X under h.

Since $(X, S + E)$ is qdlt and $\mathrm{Exc}(f) \subseteq S \cap E$, we get that $S \cap E = \mathrm{Exc}(f)$. Thus the induced map $g: X' \to Z$ is an isomorphism over $Z \setminus Q$, and g is a small birational morphism. If S' is the strict transform of S, then kS' is the restriction of a section of $\mathbb{P}^1_Z$, and so S' is $\mathbb{Q}$-Cartier and relatively ample. Let $\pi\colon X^+ \to X'$ be the normalization of X'. Then $X \dashrightarrow X^+$ is a small birational morphism of normal varieties, and we have that

$$\bigoplus_{m\in\mathbb{Z}_{\geq 0}} H^0(X, mS) = \bigoplus_{m\in\mathbb{Z}_{\geq 0}} H^0(X^+, m\pi^*S')$$

is finitely generated. Since $K_X + \Delta \sim_{\mathbb{Q}} aS$ for $a \in \mathbb{Q}_{>0}$, the flip of X exists by [20, Corollary 6.4]. □

Now, we deal with case (3). Note that we will apply this proposition later in the case when B does not have standard coefficients.

PROPOSITION 5.4
Let $(X, S+B)$ be a $\mathbb{Q}$-factorial three-dimensional qdlt pair over a perfect field k of characteristic $p > 3$ with S irreducible, and let $f\colon X \to Z$ be a flipping contraction such that $\rho(X/Z) = 1$, $-(K_X + S + B)$ is relatively ample, and $-S$ is relatively ample. Let Σ be a flipping curve. Assume that there exists a 6-complement $(X, S + E + B^c)$ of $(X, S+B)$ such that E is irreducible, $E \cdot \Sigma \geq 0$, and $E \cap \Sigma \neq \emptyset$. Then the flip of Σ exists.

We remind the reader that $B^c \geq 0$ as $(X, S+B^c)$ is by definition log canonical.

Proof
Let $\tilde{S}$ be the normalization of S. By perturbing the coefficients of $\lfloor B \rfloor$, we may assume that $(X, S+B)$ is plt. The pair $(\tilde{S}, B_{\tilde{S}})$ admits a 6-complement $(\tilde{S}, E|_{\tilde{S}} + B^c_{\tilde{S}})$, where $K_{\tilde{S}} + B_{\tilde{S}} = (K_X + S + B)|_{\tilde{S}}$ and $K_{\tilde{S}} + E|_{\tilde{S}} + B^c_{\tilde{S}} = (K_X + S + E + B^c)|_{\tilde{S}}$.

We claim that $E|_{\tilde{S}}$ is not exceptional over Z. Indeed, otherwise

$$0 > (E|_{\tilde{S}})^2 = E \cdot (E \cap S) = E \cdot \sum \lambda_i \Sigma_i \geq 0$$

for some flipping curves Σ_i and some $\lambda_i > 0$, which is a contradiction. We have used the fact that as $\rho(X/Z) = 1$, if $E \cdot \Sigma \geq 0$, then $E \cdot \Sigma_i \geq 0$ for every flipping curve Σ_i.

By Lemma 2.11, the pair $(\tilde{S}, B_{\tilde{S}})$ is relatively F-regular over a neighborhood of $f(\Sigma)$ in $f(S)$, and so the flip exists by Theorem 2.1. □

6. Divisorial extractions

In [14], we have shown that dlt modifications exist. In our proof of the existence of flips, it is important to construct minimal qdlt modifications of flipping contractions.

To this end, we need to extract a single divisorial place, and the following proposition shows that this can be done for 6-complements.

PROPOSITION 6.1
Let (X, Δ) be a $\mathbb{Q}$-factorial three-dimensional lc pair defined over a perfect field of characteristic $p > 3$ such that X is klt and $6(K_X + \Delta) \sim 0$. Let E be a non-klt valuation of (X, Δ) over X. Then there exists a projective birational morphism $g\colon Y \to X$ such that E is its exceptional locus.

Proof
Let $\pi : Y \to X$ be a dlt modification of (X, Δ) such that E is a divisor on Y (see [14, Corollary 1.4]). Let $\mathrm{Exc}(\pi) = E + E_1 + \cdots + E_m$. For some $\epsilon > 0$, write

$$K_Y + \Delta_Y = \pi^*(K_X + \Delta),$$
$$K_Y + (1-\epsilon)\pi_*^{-1}\Delta + aE + a_1E_1 + \cdots + a_mE_m = \pi^*\big(K_X + (1-\epsilon)\Delta\big),$$

where $a, a_1, \ldots, a_m < 1$ as X is klt, and set

$$\Delta' = (1-\epsilon)\pi_*^{-1}\Delta + aE + E_1 + \cdots + E_m.$$

By taking $0 < \epsilon \ll 1$, we can assume that $a > 0$. Note that

$$K_Y + \Delta' \sim_{\mathbb{Q},X} (1-a_1)E_1 + \cdots + (1-a_m)E_m, \tag{3}$$

so that the $(K_Y + \Delta')$-MMP over X will not contract E and the contracted loci are always contained in the support of the strict transform of $(1-a_1)E_1 + \cdots + (1-a_m)E_m$. The negativity lemma implies that the output of a $(K_Y + \Delta')$-MMP over X is the sought-for extraction of E. Hence, it is enough to show that we can run such an MMP.

By induction, we can assume that we have constructed the nth step of the MMP $h\colon Y \dashrightarrow Y_n$ and we need to show that we can construct the $(n+1)$st step. Let $\pi_n\colon Y_n \to X$ be the induced morphism, and let $\Delta'_n := h_*\Delta'$, $\Delta_n = h_*\Delta_Y$. By abuse of notation, we denote the strict transforms of $E, E_1, \ldots, E_m$ by the same symbols.

The cone theorem is valid by [17] (cf. [14, Theorem 2.4]). Let R be a $K_{Y_n} + \Delta'_n$ negative extremal ray. By (3), we have that $R \cdot E_i < 0$ for some $i \geq 1$. Thus the contraction $f : Y_n \to Y'_n$ of R exists by [14, Theorem 1.2, Proposition 2.6].

If f is divisorial, then we set $Y_{n+1} := Y'_n$. If f is a flipping contraction, then the proof of [14, Lemma 3.1] applied to (Y_n, Δ_n) over X implies the existence of a divisor $E' \subseteq \mathrm{Exc}(\pi_n)$ such that $R \cdot E' > 0$. Since (Y_n, Δ'_n) is dlt, (Y_n, Δ_n) is lc, $6(K_{Y_n} + \Delta_n) \sim_{\pi_n} 0$, and $E' \leq \Delta_n$, we can apply Proposition 5.4 to infer the existence of the flip of f.

The termination of this MMP follows by the usual special termination argument. □

Let $(X, S + B)$ be a three-dimensional plt pair with different B_S, and let $(X, S + B^c)$ be a k-complement with different B^c_S. Assume for simplicity that S is normal. Then (S, B^c_S) is a k-complement of (S, B_S). Assume that (S, B^c_S) admits a unique non-klt place; that is, it has a dlt modification with an irreducible exceptional curve. Such complements are of fundamental importance in this article due to Proposition 3.1. By inversion of adjunction, $(X, S + B^c)$ has a unique log canonical center strictly contained in S, but infinitely many log canonical places over this center. Thus, its dlt modifications might be very complicated with many exceptional divisors. The following corollary shows that this problem may be solved by allowing qdlt singularities: under the above assumptions it stipulates that there exists a qdlt modification with an irreducible exceptional divisor.

COROLLARY 6.2
Let $(X, S + B)$ be a $\mathbb{Q}$-factorial three-dimensional plt pair defined over a perfect field of characteristic $p > 3$, where X is klt and S is a prime divisor. Assume that $(X, S + B)$ admits a 6-complement $(X, S + B^c)$ such that $(\tilde{S}, B^c_{\tilde{S}})$ has a unique non-klt place, where $K_{\tilde{S}} + B^c_{\tilde{S}} = (K_X + S + B^c)|_{\tilde{S}}$ and $\tilde{S}$ is the normalization of S.

Then $(X, S + B^c)$ is qdlt in a neighborhood of S, or $\lfloor B^c \rfloor$ is disjoint from S and there exists a projective birational map $\pi \colon Y \to X$ such that $(Y, S_Y + B^c_Y)$ is qdlt over a neighborhood of S, the exceptional divisor E is irreducible, and $E \subseteq \lfloor B^c_Y \rfloor$, where $K_Y + S_Y + B^c_Y = \pi^*(K_X + S + B^c)$.

In particular, this corollary implies that if $(X, S + B^c)$ is not qdlt, then the log canonical centers in a neighborhood of S are the generic points of $\pi(S_Y \cap E)$, $\pi(E)$, and S itself. Note that $S_Y \cap E$ must be irreducible as $(\tilde{S}, B^c_{\tilde{S}})$ has a unique log canonical place. Now there are two possibilities: either $\pi(E) \subseteq S$, in which case $(X, S + B)$ admits a unique log canonical center $\pi(E) = \pi(S_Y \cap E)$ (a point or a curve), or $\pi(E) \not\subseteq S$ is a curve intersecting S at the point $\pi(S_Y \cap E)$. Moreover, if $(X, S + B^c)$ is qdlt, then the proof below shows that $\lfloor B^c \rfloor$ is irreducible in a neighborhood of S and intersects S at its unique non-klt place (which is a curve).

Proof
We work in a sufficiently small open neighborhood of S. First, suppose that $\lfloor B^c \rfloor$ is nonempty and intersects S. Under this assumption, the unique log canonical center of $(\tilde{S}, B^c_{\tilde{S}})$ must be an irreducible curve given as $\lfloor B^c \rfloor|_{\tilde{S}}$. In particular, $\lfloor B^c \rfloor$

is irreducible (cf. Remark 2.4), the pair $(\tilde{S}, B^c_{\tilde{S}})$ is plt, and $(X, S + B^c)$ is qdlt by Lemma 2.6.

Thus, we can assume that $\lfloor B^c \rfloor = 0$, and so the dlt modification $\pi\colon Y \to X$ is nontrivial. Set $K_Y + \Delta^c_Y = \pi^*(K_X + S + B^c)$, and pick an irreducible exceptional divisor E_1 which is not an articulation point of $D(\Delta^{c,=1}_Y)$ (e.g., pick any divisor with the farthest distance edgewise in $D(\Delta^{c,=1}_Y)$ from the node corresponding to S). Let $g\colon X_1 \to X$ be the extraction of E_1 (see Proposition 6.1), and write

$$K_{X_1} + S_1 + E_1 + B^c_1 = g^*(K_X + S + B^c),$$

where S_1, B^c_1 are the strict transforms of S, B^c, respectively. Note that S_1 intersects E_1.

We claim that $(X_1, S_1 + E_1 + B^c_1)$ is qdlt in a neighborhood of S_1. To this end, we note that

$$K_{\tilde{S}_1} + B^c_{\tilde{S}_1} := (K_{X_1} + S_1 + E_1 + B^c_1)|_{\tilde{S}_1} = (g|_{\tilde{S}_1})^*(K_{\tilde{S}} + B^c_{\tilde{S}}),$$

where $\tilde{S}_1$ is the normalization of S_1. Since $(\tilde{S}, B^c_{\tilde{S}})$ admits a unique non-klt place, we obtain that $(\tilde{S}_1, B^c_{\tilde{S}_1})$ is plt. In particular, Lemma 2.6 implies that $(X_1, S_1 + E_1 + B^c_1)$ is qdlt in a neighborhood of S_1.

Therefore, it is enough to show that $(X_1, S_1 + E_1 + B^c_1)$ does not admit a log canonical center which is disjoint from S_1 and intersects E_1. Assume by contradiction that it does admit such a log canonical center. Let $h\colon W \to X_1$ be a projective birational morphism which factors through Y,

$$g \circ h\colon W \xrightarrow{h_Y} Y \xrightarrow{\pi} X,$$

and such that $g \circ h$ is a log resolution of $(X, S + B)$. Write $K_W + \Delta^c_W = h^*(K_{X_1} + S_1 + E_1 + B^c_1)$. Since $S_1 \cap E_1$ is disjoint from the other log canonical centers, the strict transform $E_{W,1}$ of E_1 is an articulation point of $D(\Delta^{c,=1}_W)$. Since $K_W + \Delta^c_W = h^*_Y(K_Y + \Delta^c_Y)$, Lemma 2.12 implies that E_1 is an articulation point of $D(\Delta^{c,=1}_Y)$, which is a contradiction. In particular, S_1, E_1, and the irreducible curve $S_1 \cap E_1$ are the only log canonical centers of $(X_1, S_1 + E_1 + B^c_1)$. □

7. Existence of flips

In this section, we prove the main theorem. We start by showing the following result.

THEOREM 7.1

Let (X, Δ) be a $\mathbb{Q}$-factorial three-dimensional klt pair with standard coefficients defined over a perfect field k of characteristic $p = 5$. If $f\colon X \to Z$ is a flipping contraction, then the flip $f^+\colon X^+ \to Z$ exists.

Proof

We will assume throughout that Z is a sufficiently small affine neighborhood of $Q := f(\mathrm{Exc}(f))$. We say that a $\mathbb{Q}$-Cartier divisor D is ample if it is relatively ample over Z.

By Shokurov's reduction to pl-flips, it suffices to show the existence of pl-flips. Let $(X, S+B)$ be a plt pair with standard coefficients, and let $f\colon X \to Z$ be a pl-flipping contraction. In particular, $-S$ and $-(K_X + S + B)$ are f-ample, and so $\mathrm{Exc}(f) \subseteq S$. By Theorem 2.1, the flip exists unless $(\tilde{S}, B_{\tilde{S}})$ is not globally F-regular over $T = f(S)$, where $K_{\tilde{S}} + B_{\tilde{S}} = (K_X + S + B)|_{\tilde{S}}$ and $\tilde{S}$ is the normalization of S. Thus, we can assume that $(\tilde{S}, B_{\tilde{S}})$ is not globally F-regular over T.

Theorem 1.6 shows the existence of an m-complement $(X, S + B^c)$ of $(X, S + B)$ for $m \in \{1,2,3,4,6\}$. Since $(X, S+B)$ is not relatively purely F-regular, Remark 4.8 implies that $m = 6$. Let $(\tilde{S}, B^c_{\tilde{S}})$ be the induced 6-complement of $(\tilde{S}, B_{\tilde{S}})$. By Proposition 3.1, the pair $(\tilde{S}, B^c_{\tilde{S}})$ has a unique place C of log discrepancy zero which is exceptional over T.

If $(X, S + B^c)$ is qdlt, then the flip exists by Proposition 5.1. Thus, by Corollary 6.2, we may assume that $\lfloor B^c \rfloor = 0$ and there exists a qdlt modification $g\colon X_1 \to X$ of $(X, S + B^c)$ with an irreducible exceptional divisor E_1. Let S_1 be the strict transform of S, let $f_1\colon X_1 \to Z$ be the induced map to Z, and write $K_{X_1} + S_1 + B_1 + aE_1 = g^*(K_X + S + B)$ and $K_{X_1} + S_1 + B_1^c + E_1 = g^*(K_X + S + B^c)$. In particular, $S_1 \cap E_1$ is the unique log canonical place of $(\tilde{S}, B_{\tilde{S}})$, and so there are two possibilities: either $g(E_1) \subseteq S$ and $f_1(E_1) = Q$, or $g(E_1) \not\subseteq S$ is a curve intersecting S.

We would like to run a $(K_{X_1} + S_1 + B_1 + aE_1)$-MMP. It could possibly happen that $a < 0$, so we take $0 < \lambda \ll 1$ and set

$$\Delta_1 := \lambda(S_1 + B_1 + aE_1) + (1-\lambda)(S_1 + E_1 + B_1^c)$$

so that $K_{X_1} + \Delta_1 \sim_{\mathbb{Q},Z} \lambda(K_{X_1} + S_1 + B_1 + aE_1)$, and (X_1, Δ_1) is plt.

Since $\rho(X/Z) = 1$ and both $-(K_X + S + B)$ and $-S$ are ample over Z, it follows that $K_X + S + B \sim_{Z,\mathbb{Q}} \mu S$ for some $\mu > 0$ and so

$$K_{X_1} + \Delta_1 \sim_{Z,\mathbb{Q}} \lambda(K_{X_1} + S_1 + aE_1 + B_1) \sim_{Z,\mathbb{Q}} \lambda\mu S_1 + \lambda' E_1, \tag{4}$$

where $\lambda' \geq 0$. Note that $\lambda' > 0$ if $g(E_1) \subseteq S$ and $\lambda' = 0$ if $g(E_1) \not\subseteq S$.

CLAIM 7.2

There exists a sequence of $(K_{X_1} + \Delta_1)$-flips $X_1 \dashrightarrow \cdots \dashrightarrow X_n$ over Z such that either X_n admits a $(K_{X_n} + \Delta_n)$-negative contraction of E_n of relative Picard rank one, or $K_{X_n} + \Delta_n$ is semiample with the associated fibration contracting E_n. Here Δ_n and E_n are strict transforms of Δ_1 and E_1, respectively.

In the course of the proof we will show that the qdlt-ness of $(X_1, S_1 + E_1 + B_1^c)$ is preserved (see Lemma 2.7) except possibly at the very last step before the contraction takes place. Therefore, all the flips in this MMP exist by Proposition 5.1.

Proof
Let $f_i\colon X_i \to Z$ be the induced map to Z. Since we work over a sufficiently small neighborhood of $Q \in Z$, we can assume that all the flipped curves are contracted to Q under f_i, and so $X_1 \dashrightarrow X_n$ is an isomorphism over $Z \setminus \{Q\}$. Let (X_i, Δ_i) and $(X_i, S_i + E_i + B_i^c)$ be the appropriate strict transforms. The latter pair is a 6-complement of $(X_i, S_i + E_i + B_i)$, where the strict transforms B_i of B_1 have standard coefficients. Note that E_1 is not contracted as $X_1 \dashrightarrow \cdots \dashrightarrow X_n$ is a sequence of flips, thus inducing an isomorphism on the generic point of E_1.

Suppose that $K_{X_n} + \Delta_n$ is nef. There are two cases: either $g(E_1) \subseteq S$ and $f_1(E_1) = Q$, or $g(E_1) \nsubseteq S$. We claim that the former cannot happen. Indeed, assume that $f_1(E_1) = Q$, and let $\pi_1\colon W \to X_1$ and $\pi_n\colon W \to X_n$ be the normalization of the graph of $X_1 \dashrightarrow X_n$ so that π_1 and π_n are isomorphisms over $Z \setminus \{Q\}$. Since $K_{X_n} + \Delta_n$ is nef and $K_{X_1} + \Delta_1$ is antinef (but not numerically trivial) over Z,

$$\pi_n^*(K_{X_n} + \Delta_n) - \pi_1^*(K_{X_1} + \Delta_1)$$

is exceptional, nef, and antieffective over Z by the negativity lemma. Moreover, its support must be equal to the whole exceptional locus over Z as it is nonempty and contracted to Q under the map to Z (cf. [20, Lemma 3.39(2)]). This is impossible, because E_1 is not contained in its support while $f_1(E_1) = Q$.

Now, assuming that $g(E_1) \nsubseteq S$ is a curve intersecting S, we will show that $K_{X_n} + \Delta_n \sim_{\mathbb{Q},Z} \lambda\mu S_n$ is semiample. Let $G := f_n^{-1}(P)$ for a (non-necessarily closed) point $P \in Z$. By [6, Theorem 1.1], it is enough to show that $S_n|_G$ is semiample. Since $X_1 \dashrightarrow X_n$ is an isomorphism over $Z \setminus \{Q\}$, $S_1 = g^*S$, and S is semiample over $Z \setminus \{Q\}$, we obtain that $S_n|_G$ is semiample when $P \neq Q$. Thus, we may assume that $P = Q$. By [17, Theorem], it is enough to verify that $S_n|_{\mathbb{E}(S_n|_G)}$ is semiample. Since G is one-dimensional, every connected component of $\mathbb{E}(S_n|_G) \subseteq G$ is either entirely contained in S_n or is disjoint from it. In particular, it is enough to show that $S_n|_{S_n}$, or equivalently $(K_{X_n} + \Delta_n)|_{S_n}$, is semiample. Recall that $S_n \subseteq \lfloor \Delta_n \rfloor$, and so $K_{\tilde{S}_n} + \Delta_{\tilde{S}_n} = (K_{X_n} + \Delta_n)|_{\tilde{S}_n}$ is semiample by [26, Theorem 1.1], where $\tilde{S}_n$ is the normalization of S_n. Since $\tilde{S}_n \to S_n$ is a universal homeomorphism (see [14, Theorem 1.2]), $(K_{X_n} + \Delta_n)|_{S_n}$ is semiample and so is $K_{X_n} + \Delta_n$. Since $(K_{X_n} + \Delta_n)|_{E_n}$ is relatively numerically trivial over $Z \setminus \{Q\}$ (as so is $(K_{X_1} + \Delta_1)|_{E_1}$), we get that the associated semiample fibration contracts E_n.

From now on, $K_{X_n} + \Delta_n$ is not nef. In order to run the MMP, we assume that $(X_n, S_n + E_n + B_n^c)$ is qdlt by induction. The cone theorem is valid by [17] (cf. [14, Theorem 2.4]). Pick Σ_n a $(K_{X_n} + \Delta_n)$-negative extremal curve. By (4), we have $K_{X_n} + \Delta_n \sim_{Z,\mathbb{Q}} \lambda\mu S_n + \lambda' E_n$, and so $\Sigma_n \cdot S_n < 0$ or $\Sigma_n \cdot E_n < 0$. The contraction of Σ_n exists by [14, Theorem 1.2, Proposition 2.6] applied to (X_n, Δ_n) in the former case and to $(X_n, S_n + E_n + B_n)$ in the latter ([14, Theorem 1.2, Proposition 2.6] assumes that the singularities are dlt, but we can immediately reduce the qdlt case to the plt case by making the coefficients smaller).

If the corresponding contraction is divisorial, then we are done as it must contract E_n. Hence, we can assume that Σ_n is a flipping curve. If $E_n \cdot \Sigma_n \leq 0$, then $-(K_{X_n} + S_n + B_n + E_n)$ has standard coefficients, is qdlt, and is ample over the contraction of Σ_n, so the flip exists by Proposition 5.1 as $(X_n, S_n + E_n + B_n^c)$ is a 6-complement. If $E_n \cdot \Sigma_n > 0$, then the flip exists by Proposition 5.4 applied to (X_n, Δ_n).

To conclude the proof, we shall show that $(X_{n+1}, S_{n+1} + E_{n+1} + B_{n+1}^c)$ is qdlt unless X_{n+1} admits a contraction of E_{n+1}. By Lemma 2.7, we can suppose that $S_{n+1} \cap E_{n+1} = \emptyset$ and aim for showing that the sought-for contraction exists.

Let Σ' be a curve which is exceptional over $Q \in Z$, contained neither in S_{n+1} nor E_{n+1}, but intersecting S_{n+1} (it exists by connectedness of the exceptional locus over $Q \in Z$, and the fact that both S_{n+1} and E_{n+1} intersect this exceptional locus), and let $C \subseteq E_{n+1}$ be any exceptional curve such that $C \cdot E_{n+1} < 0$ (it exists by the negativity lemma as E_{n+1} is exceptional over Z). We claim that $C' \cdot S_{n+1} > 0$ for every exceptional curve $C' \nsubseteq E_{n+1}$. To this end, assume by contradiction that there exists $C' \nsubseteq E_{n+1}$ satisfying $C' \cdot S_{n+1} \leq 0$. Since $\rho(X_{n+1}/Z) = 2$, we get that

$$C' \equiv aC + b\Sigma',$$

for $a, b \in \mathbb{R}$. Given $C \cdot S_{n+1} = 0$ and $\Sigma' \cdot S_{n+1} > 0$, we have $b \leq 0$. As $C' \cdot E_{n+1} \geq 0$, $C \cdot E_{n+1} < 0$, and $\Sigma' \cdot E_{n+1} \geq 0$, we have $a \leq 0$. Therefore, for an ample divisor A we have

$$0 < C' \cdot A = (aC + b\Sigma') \cdot A \leq 0,$$

which is a contradiction.

Since $S_{n+1} \cap E_{n+1}$ is empty, S_{n+1} is thus nef and $\mathbb{E}(S_{n+1}) \subseteq E_{n+1}$ (see [6] for the definition of $\mathbb{E}$ in the relative setting). Hence S_{n+1} is semiample by [6, Proposition 2.20] and induces a contraction of E_{n+1}. It does not contract Σ', and so is of relative Picard rank one. Moreover,

$$(K_{X_{n+1}} + \Delta_{n+1}) \cdot C \sim_{Z,\mathbb{Q}} \mu\lambda S_{n+1} \cdot C + \lambda' E_{n+1} \cdot C = \lambda' E_{n+1} \cdot C \leq 0,$$

and so either $\lambda' = 0$ and $K_{X_{n+1}} + \Delta_{n+1} \sim_{Z,\mathbb{Q}} \mu\lambda S_{n+1}$ is semiample with the associated fibration contracting E_{n+1}, or $\lambda' > 0$, $(K_{X_{n+1}} + \Delta_{n+1}) \cdot C < 0$, and so the above contraction is a $(K_{X_{n+1}} + \Delta_{n+1})$-negative Mori contraction of relative Picard rank one. $\square$

Let $\phi\colon X_n \to X^+$ be the contraction of E_n as in the claim, let $\Delta^+ := \phi_*\Delta_n$, let $S^+ := \phi_* S_n$, and let $B^+ := \phi_* B_n$. The projection onto Z factors through a small contraction $\pi^+\colon X^+ \to Z$ and $\rho(X^+/Z) \le 1$. Recall that

$$K_{X_n} + \Delta_n \sim_{Z,\mathbb{Q}} \lambda(K_{X_n} + S_n + aE_n + B_n) \sim_{Z,\mathbb{Q}} \lambda\mu S_n + \lambda' E_n.$$

Since ϕ is either $(K_{X_n} + S_n + aE_n + B_n)$-negative of Picard rank one or $(K_{X_n} + S_n + aE_n + B_n)$-trivial, the discrepancies of $(X^+, S^+ + B^+)$ are not smaller than those of $(X_n, S_n + aE_n + B_n)$. Moreover, since $K_{X_1} + S_1 + aE_1 + B_1$ is antinef over Z and not numerically trivial, at least one step of the $(K_{X_1} + \Delta_1)$-MMP (equivalently, $(K_{X_1} + S_1 + aE_1 + B_1)$-MMP) has been performed (i.e., $n \ge 2$ or ϕ is a $(K_{X_n} + \Delta_n)$-negative contraction of E_n). In particular, there exists a divisorial valuation for which the discrepancy of $(X^+, S^+ + B^+)$ is higher than the discrepancy of $(X_1, S_1 + aE_1 + B_1)$, which in turn coincide with the discrepancy of $(X, S + B)$.

Therefore, $K_{X^+} + \Delta^+$ cannot be relatively antiample, because then $(X^+, S^+ + B^+)$ would be isomorphic to $(X, S + B)$, which is impossible as the MMP has increased the discrepancies. If $K_{X^+} + \Delta^+$ is relatively numerically trivial, then we claim that $K_{X^+} + \Delta^+ \sim_{Z,\mathbb{Q}} 0$. Indeed,

$$K_{X^+} + \Delta^+ \sim_{Z,\mathbb{Q}} \lambda\mu S^+,$$

for $\lambda, \mu > 0$, and since S^+ intersects the exceptional locus, we must in fact have that $\operatorname{Supp}\operatorname{Exc}(\pi^+) \subseteq S^+$. By [6, Proposition 2.20], it is thus enough to show that $K_{\tilde{S}^+} + \Delta_{\tilde{S}^+} = (K_{X^+} + \Delta^+)|_{\tilde{S}^+}$ is semiample, where $\tilde{S}^+ \to S^+$ is the normalization of S^+, which in turn follows from [26, Theorem 1.1]. Here we used the fact that $\tilde{S}^+ \to S^+$ is a universal homeomorphism (see [14, Theorem 1.2]). As a consequence, S^+ descends to Z. This is impossible as its image (equal to the image of S) in Z is not $\mathbb{Q}$-Cartier.

Therefore, $K_{X^+} + \Delta^+$ is relatively ample, and so $X^+ \to Z$ is the flip of $X \to Z$ by [20, Corollary 6.4]. □

7.1. *The proof of Theorem 1.1*

Given Theorem 7.1, the following proof follows the same strategy as in [2, Theorem 6.3]. For the convenience of the reader, we provide a brief sketch of Birkar's argument in the projective case.

Proof of Theorem 1.1

First, we can assume that every component S of $\operatorname{Supp}\Delta$ is relatively antiample. Further, let $\zeta(\Delta)$ be the number of components of Δ with coefficients not in the set $\Gamma := \{1\} \cup \{1 - \frac{1}{n} \mid n > 0\}$. If $\zeta(\Delta) = 0$, then the flip exists by Theorem 7.1. By induction, we can assume that the flip exists for all flipping contractions of log pairs (X', Δ') such that $\zeta(\Delta') < \zeta(\Delta)$.

By replacing Δ with $\Delta - \frac{1}{l}\lfloor \Delta \rfloor$ for $l \gg 0$, we can assume that (X, Δ) is klt without changing $\zeta(\Delta)$. Write $\Delta = aS + B$, where $S \not\subseteq \operatorname{Supp} B$ and $a \notin \Gamma$. Let $\pi\colon W \to X$ be a log resolution of $(X, S + B)$ with exceptional divisor E, and set $B_W := \pi_*^{-1}B + E$. Since $K_X + \Delta \equiv_Z \mu S$ for some $\mu > 0$, we have that

$$\begin{aligned} K_W + S_W + B_W &= \pi^*(K_X + \Delta) + (1-a)S_W + F \\ &\equiv_Z (1-a+\mu)S_W + F', \end{aligned}$$

where $S_W := \pi_*^{-1}S$, and F, F' are effective $\mathbb{Q}$-divisors exceptional over X.

Run a $(K_W + S_W + B_W)$-MMP over Z. By induction, all flips exist in this MMP as $\zeta(S_W + B_W) < \zeta(\Delta)$. Moreover, by the above equation, every extremal ray is negative on $(1-a+\mu)S_W + F'$ and hence on an irreducible component of $\lfloor S_W + B_W \rfloor$. In particular, all contractions exist by [14, Theorem 1.2, Proposition 2.6]. The cone theorem is valid by a result of Keel (see, e.g., [14, Theorem 2.4]). Let $h\colon W \dashrightarrow Y$ be an output of this MMP, and let S_Y, B_Y, and F_Y be the strict transforms of S_W, B_W, and F, respectively.

Now, run a $(K_Y + aS_Y + B_Y)$-MMP over Z with scaling of $(1-a)S_Y$. In particular, if R is an extremal ray, then $R \cdot S_Y > 0$ and

$$(K_Y + B_Y) \cdot R < 0.$$

As $\zeta(B_Y) < \zeta(\Delta)$, all the flips in this MMP exist by induction. By the same argument as in the above paragraph, the cone theorem is valid in this setting and all contractions exist. Let $(X^+, aS^+ + B^+)$ be an output of this MMP. We claim that this is the flip of $(X, aS + B)$.

To this end, we notice that the negativity lemma applied to a common resolution $\pi_1\colon W' \to X$ and $\pi_2\colon W' \to X^+$ implies that

$$\pi_1^*(K_X + aS + B) - \pi_2^*(K_{X^+} + aS^+ + B^+) \geq 0.$$

Since $(X, aS + B)$ is klt, this shows that $\lfloor B^+ \rfloor = 0$ and all the divisors in E were contracted. In particular, $X \dashrightarrow X^+$ is an isomorphism in codimension 1. We claim that $K_{X^+} + aS^+ + B^+$ is relatively ample over Z and so $(X^+, aS^+ + B^+)$ is the flip of X.

To this end, we note that $\rho(X^+/Z) = 1$ (cf. [1, Lemma 1.6]). Indeed,

$$\rho(W/X^+) + \rho(X^+/Z) = \rho(W/Z) = \rho(W/X) + \rho(X/Z)$$

and $\rho(W/X) = \rho(W/X^+)$ is equal to the number of exceptional divisors. Since $\rho(X/Z)$ is equal to one, so is $\rho(X^+/Z)$. In particular, to conclude the proof of the theorem, it is enough to show that $K_{X^+} + aS^+ + B^+$ cannot be relatively numerically trivial over Z. Assume by contradiction that it is relatively numerically trivial. Then

$$\pi_1^*(K_X + aS + B) - \pi_2^*(K_{X^+} + aS^+ + B^+)$$

is exceptional and relatively numerically trivial over X. Thus, it is empty by the negativity lemma which contradicts the fact that it is exceptional and non-numerically trivial over Z. □

Theorem 1.2, Theorem 1.3, and Theorem 1.4 now follow by exactly the same proof as [4, Theorems 1.5, 1.7], [4, Theorem 1.2], and [4, Theorem 1.1], respectively, in view of [12, Section 2.3].

Remark 7.3
Theorem 1.2 may be extended to the dlt case. The main issue is to show termination of flips. To this end, one can either argue as in [16], or use special termination to automatically reduce the problem to the termination of klt flips. The latter statement requires the termination of all flips (as opposed to the termination with scaling proved in [4]). Such a stronger termination follows from the argument of [1, Section 2] in view of the non-vanishing conjecture for klt pair (which can be proved by exactly the same argument as in [31, Theorem 3], now that we extended [4] to $p = 5$).

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