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Boundedness of (ϵ, n)-complements for projective generalized pairs of Fano type

Guodu Chen [a,*], Qingyuan Xue [b]

[a] Institute for Theoretical Sciences, Westlake Institute for Advanced Study, Westlake University, Hangzhou, Zhejiang, 310024, China
[b] Department of Mathematics, The University of Uath, Salt Lake City, UT 84112, USA

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ABSTRACT

We show the existence of (ϵ, n)-complements for $(\epsilon, \mathbb{R})$-complementary projective generalized pairs $(X, B+M)$ of Fano type, when either the coefficients of B and μ_j belong to a finite set, or the coefficients of B belong to a DCC set and $M' \equiv 0$, where $M' = \sum \mu_j M'_j$ and M'_j are nef Cartier divisors.

Contents

1. Introduction

We work over the field of complex numbers $\mathbb{C}$.

Shokurov introduced the theory of complements to investigate log flips for threefolds [13], and it turns out that the theory plays an important role in birational geometry. The theory of complements has been studied

* Corresponding author.
E-mail addresses: chenguodu@westlake.edu.cn (G. Chen), xue@math.utah.edu (Q. Xue).

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in recent years; for example, see [1,9] for the boundedness of log canonical complements, and [14,11,12] for more on the boundedness of complements. We refer the readers to [9,7] for the brief history and applications of the theory of complements.

In recent years a new concept of space, generalized pairs, has evolved. Generalized pairs appear naturally in the study of birational geometry in higher dimensions. They were first introduced in [5], and we refer the readers to [2] for more motivations and applications. It is natural to consider the boundedness of complements for generalized pairs. Indeed the boundedness of generalized log canonical complements is established; see [1,6]. It is worth mentioning that in [6], the first author has studied the complements in a more general setting. More precisely, the nef part of the generalized pair is allowed to have irrational coefficients; see Definition 2.7.

However, the above results mainly focus on the boundedness of generalized log canonical complements. It is very natural to consider the boundedness of complements with good singularities, that is, generalized ϵ-log canonical complements. In this paper, we deal with the following deep conjecture regarding the boundedness of (ϵ, n)-complements for generalized pairs, which is an analog of [7, Conjecture 1.1] in the context of generalized pairs.

Conjecture 1.1. *Let d, p be two positive integers, ϵ a non-negative real number, and $\Gamma \subseteq [0,1]$ a DCC set. Then there exists a positive integer n divisible by p depending only on d, p, ϵ and Γ satisfying the following.*

Assume that $(X/Z, B+M)$ is a generalized pair of dimension d, $X \to Z$ a contraction and $z \in Z$ a (not necessarily closed) point such that

(1) *X is of Fano type over Z,*
(2) *$M' = \sum \mu_j M'_j$, where M'_j are nef/Z Cartier divisors and $\mu_j \in \Gamma$,*
(3) *$B \in \Gamma$, that is, the coefficients of B belong to Γ, and*
(4) *$(X/Z \ni z, B+M)$ is $(\epsilon, \mathbb{R})$-complementary.*

Then there is an (ϵ, n)-complement $(X/Z \ni z, B^+ + M^+)$ of $(X/Z \ni z, B+M)$. Moreover, if $\mathrm{Span}_{\mathbb{Q}_{\geq 0}}(\bar{\Gamma} \cup \{\epsilon\} \backslash \mathbb{Q}) \cap (\mathbb{Q} \backslash \{0\}) = \emptyset$, then we may pick $B^+ \geq B$ and $\mu_j^+ \geq \mu_j$, where $M^{+\prime} = \sum \mu_j^+ M'_j$.

Remark 1.2. In Conjecture 1.1, M' is allowed to have irrational coefficients while in [1] M' is a $\mathbb{Q}$-Cartier divisor whose Cartier index is fixed. The "Moreover" part is about the monotonicity property of complements which is useful in applications especially when $\bar{\Gamma} \subseteq \mathbb{Q}$ and does not hold in general.

When $\epsilon = 0$, the conjecture is proved in [6]. When ϵ is positive, we have some partial results.

Theorem 1.3. *Conjecture 1.1 holds in the following cases:*

(1) *$\epsilon = 0$;*
(2) *$\epsilon > 0$, $\dim Z = 0$ and Γ is a finite set; and*
(3) *$\epsilon > 0$, $\dim Z = 0$ and $M' \equiv 0$.*

In order to show Theorem 1.3, we study a new class of complements, namely $(\boldsymbol{\epsilon}, n, \Gamma_0)$-decomposable $(\epsilon, \mathbb{R})$-complements. Note that when $\epsilon = \epsilon_i = 0$, $(\boldsymbol{\epsilon}, n, \Gamma_0)$-decomposable $(\epsilon, \mathbb{R})$-complements are the same as [6, Definition 1.2].

Definition 1.4. Let n be a positive integer, ϵ, ϵ_i non-negative real numbers, and $\Gamma_0 \subseteq (0,1]$ a finite set. We say that $(X/Z \ni z, B^+ + M^+)$ is an $(\boldsymbol{\epsilon}, n, \Gamma_0)$-*decomposable* $(\epsilon, \mathbb{R})$-*complement* of $(X/Z \ni z, B+M)$ if

(1) $(X/Z \ni z, B^+ + M^+)$ is an $(\epsilon, \mathbb{R})$-complement of $(X/Z \ni z, B+M)$,

(2) $K_X + B^+ + M^+ = \sum a_i \left(K_X + B_i^+ + M_i^+\right)$ for some boundaries B_i^+, nef parts $M_i^{+\prime}$ and $a_i \in \Gamma_0$ with $\sum a_i = 1$ and $\sum a_i\epsilon_i \geq \epsilon$, and
(3) $\left(X/Z \ni z, B_i^+ + M_i^+\right)$ is an (ϵ_i, n)-complement of itself for any i.

As an important step in the proof of Theorem 1.3, we show the existence of $(\boldsymbol{\epsilon}, n, \Gamma_0)$-decomposable $(\epsilon, \mathbb{R})$-complements under the conditions of Theorem 1.3. More precisely, we show the following result.

Theorem 1.5. *Let d be a positive integer, ϵ a positive real number and $\Gamma \subseteq [0,1]$ a DCC set. Then there exist a positive integer n, positive real numbers ϵ_i and a finite set $\Gamma_0 \subseteq (0,1]$ depending only on d, ϵ and Γ satisfying the following.*

Assume that $(X, B+M)$ is a generalized pair of dimension d such that

(1) *$B \in \Gamma$,*
(2) *X is of Fano type,*
(3) *$M' = \sum \mu_j M_j'$, where M_j' are nef Cartier divisors and $\mu_j \in \Gamma$,*
(4) *$(X, B+M)$ is $(\epsilon, \mathbb{R})$-complementary, and*
(5) *either Γ is a finite set or $M' \equiv 0$.*

Then there is an $(\boldsymbol{\epsilon}, n, \Gamma_0)$-decomposable $(\epsilon, \mathbb{R})$-complement $(X, B^+ + M^+)$ of $(X, B+M)$. Moreover, if $\bar{\Gamma} \subseteq \mathbb{Q}$, then we may pick $\Gamma_0 = \{1\}$, $\epsilon_i = \epsilon$, $B^+ \geq B$ and $\mu_j^+ \geq \mu_j$ for any j.

Since we work on (ϵ, n)-complements, we should be careful with the singularities when we prove Theorem 1.3. The key observation is the *uniform linearity of minimal log discrepancies* (MLDs for short) for generalized pairs.

Theorem 1.6. *Let ϵ be a positive real number, d, c, m, l positive integers, $\boldsymbol{r}_0 = (r_1, \ldots, r_c) \in \mathbb{R}^c$ a point such that $r_0 = 1, r_1, \ldots, r_c$ are linearly independent over $\mathbb{Q}$, and $s_1, \ldots, s_{m+l} : \mathbb{R}^c \to \mathbb{R}$ $\mathbb{Q}$-linear functions. Then there exist a positive real number δ and a $\mathbb{Q}$-linear function $f(\boldsymbol{r}) : \mathbb{R}^c \to \mathbb{R}$ depending only on $\epsilon, d, \boldsymbol{r}_0$ and s_i satisfying the following.*

Assume that $(X, B(\boldsymbol{r}_0) + M(\boldsymbol{r}_0))$ is a projective generalized pair of dimension d such that

(1) *X is of Fano type,*
(2) *$B(\boldsymbol{r}) = \sum_{i=1}^m s_i(\boldsymbol{r}) B_i$, where $B_i \geq 0$ are Weil divisors,*
(3) *$M'(\boldsymbol{r}) = \sum_{j=1}^l s_{m+j}(\boldsymbol{r}) M_j'$, where M_j' are nef Cartier divisors, and*
(4) *$(X, B(\boldsymbol{r}_0) + M(\boldsymbol{r}_0))$ is $(\epsilon, \mathbb{R})$-complementary.*

Then there exists a prime divisor E over X such that

$$\operatorname{mld}(X, B(\boldsymbol{r}) + M(\boldsymbol{r})) = a(E, X, B(\boldsymbol{r}) + M(\boldsymbol{r})) \geq f(\boldsymbol{r})$$

for any $\boldsymbol{r} \in \mathbb{R}^c$ satisfying $||\boldsymbol{r} - \boldsymbol{r}_0||_\infty \leq \delta$.

Structure of the paper. We outline the organization of the paper. In Section 2, we recall some definitions, introduce the tools and prove certain basic results that will be used in this paper. In section 3, we prove Theorem 1.6. In section 4, we prove Theorem 1.5. In section 5, we prove Theorem 1.3.

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2. Preliminaries

In this section, we collect some definitions and preliminary results which will be used in this paper.

2.1. Generalized pairs

We always assume that all varieties are normal and quasi-projective. For an $\mathbb{R}$-divisor $D = \sum d_i D_i$ on X, we define $\lfloor D \rfloor = \sum \lfloor d_i \rfloor D_i$, and $\{D\} = \sum \{d_i\} D_i$. Assuming that $\Gamma \subseteq [0, +\infty)$ is a set, then by $D \in \Gamma$ we mean that $d_i \in \Gamma$ for any i.

Definition 2.1. We say $\pi : X \to Z$ is a *contraction* if π is a projective morphism and $\pi_* \mathcal{O}_X = \mathcal{O}_Z$. In particular, π is surjective and has connected fibers.

Definition 2.2 *(Generalized pairs).* A *generalized sub-pair* $(X/Z, B + M)$ consists of a normal variety X equipped with projective morphisms

$$X' \xrightarrow{f} X \longrightarrow Z,$$

where f is birational and X' is normal, an $\mathbb{R}$-divisor B on X with coefficients ≤ 1, and an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor M' on X' which is nef$/Z$ such that $f_* M' = M$ and $K_X + B + M$ is $\mathbb{R}$-Cartier. We call B the *sub-boundary part* and M the *nef part*.

For simplicity, we denote such a generalized pair only by $(X/Z, B+M)$, but we implicitly remember the whole generalized pair structure. If $\dim Z = 0$, the generalized sub-pair is called *projective*, and we will omit Z. We omit the prefix "sub" everywhere if $B \geq 0$.

Note that the definition is flexible with respect to X' and M'. More precisely, if $g : Y \to X'$ is a projective birational morphism from a normal variety Y, then we can replace X' and M' by Y and $g^* M'$ respectively.

Let $(X/Z, B + M)$ be a generalized pair. We may write

$$K_{X'} + B' + M' = f^*(K_X + B + M)$$

for some uniquely determined $\mathbb{R}$-divisor B'. The *generalized log discrepancy* of a prime divisor E on X' with respect to $(X, B + M)$ is defined as

$$a(E, X, B + M) = 1 - \operatorname{mult}_E B.$$

The *minimal log discrepancy* of the generalized pair $(X/Z, B + M)$ is

$$\operatorname{mld}(X/Z, B + M) = \inf\{a(E, X, B + M) \mid E \text{ is a prime divisor}/X\}.$$

We say that $(X/Z, B + M)$ is *generalized ϵ-lc* (resp. *generalized klt, generalized lc*) for some $\epsilon \geq 0$ if $\operatorname{mld}(X, B + M) \geq \epsilon$ (resp. > 0, ≥ 0).

Definition 2.3. We say that X is of *Fano type* over Z if (X, B) is klt and $-(K_X + B)$ is ample over Z for some boundary B.

Remark 2.4. Assume that X is of Fano type over Z. Then we can run the MMP/Z on any $\mathbb{R}$-Cartier $\mathbb{R}$-divisor D on X which terminates with some model Y (cf. [12, Corollary 2.7], [4]).

Definition 2.5 *(Generalized dlt).* Let $(X/Z, B+M)$ be a generalized pair. We say that $(X/Z, B+M)$ is *generalized dlt* if it is generalized lc and there is a closed subset $V \subseteq X$ such that

(1) $X \setminus V$ is smooth and $B|_{X\setminus V}$ is an snc divisor, and
(2) if $a(E, X, B+M) = 0$ for some prime divisor E over X, then $\mathrm{Center}_X\, E \not\subseteq V$ and $\mathrm{Center}_X\, E \setminus V$ is a non-klt center of $(X, B)|_{X\setminus V}$.

We remark that if $(X/Z, B+M)$ is a $\mathbb{Q}$-factorial generalized dlt pair, then X is klt. For any generalized lc pair, there always exists a generalized dlt modification (see [8, Proposition 3.9]).

Definition 2.6 *(Generalized a-lc thresholds).* Let $(X/Z, B+M)$ be a generalized pair which is generalized a-lc for some $a \geq 0$. Assume that D is an effective $\mathbb{R}$-divisor and N' is an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X' which is nef/Z, such that $D+N$ is $\mathbb{R}$-Cartier, where $N = f_*N'$. The *generalized a-lc threshold of* $D+N$ *with respect to* $(X/Z, B+M)$ is defined as

$$a\text{-}\mathrm{lct}(X/Z, B+M; D+N) := \sup\{t \geq 0 \mid (X/Z, (B+tD)+(M+tN)) \text{ is generalized } a\text{-lc}\}.$$

When $M' = N' = 0$, we just call $a\text{-}\mathrm{lct}(X/Z, B; D)$ the a-lc threshold of D with respect to (X, B).

2.2. Complements

We then recall the definition of complements for generalized pairs.

Definition 2.7 *(Complements).* Let ϵ be a non-negative real number and n a positive integer. Let $(X/Z, B+M)$ be a generalized pair and $z \in Z$ a point such that $M' = \sum \mu_j M'_j$, where $\mu_j \geq 0$ and M'_j is a nef/Z Cartier divisor for any j. We say that $(X/Z \ni z, B^+ + M^+)$ is an $(\epsilon, \mathbb{R})$-*complement* of $(X/Z \ni z, B+M)$ if $(X/Z, B^+ + M^+)$ is generalized ϵ-lc, $B^+ \geq B, \mu_j^+ \geq \mu_j$, and $K_X + B^+ + M^+ \equiv 0$ over a neighborhood of z, where $M^{+\prime} = \sum_j \mu_j^+ M'_j$.

We say that $(X/Z \ni z, B^+ + M^+)$ is an (ϵ, n)-*complement* of $(X/Z \ni z, B+M)$, if over a neighborhood of z, we have

(1) $(X, B^+ + M^+)$ is generalized ϵ-lc,
(2) $nB^+ \geq n\lfloor B \rfloor + \lfloor (n+1)\{B\} \rfloor$,
(3) $n\mu_j^+ \geq n\lfloor \mu_j \rfloor + \lfloor (n+1)\{\mu_j\} \rfloor$ for any j, and
(4) $n(K_X + B^+ + M^+) \sim 0$ and $nM^{+\prime}$ is Cartier,

where $M^{+\prime} = \sum \mu_j^+ M'_j$. If additionally we have $B^+ \geq B$ and $\mu_j^+ \geq \mu_j$ for any j, then we say that $(X/Z \ni z, B^+ + M^+)$ is a *monotonic* (ϵ, n)-*complement* of $(X/Z \ni z, B+M)$.

We say that $(X/Z \ni z, B+M)$ is $(\epsilon, \mathbb{R})$-*complementary* (resp. (ϵ, n)-*complementary*) if it has an $(\epsilon, \mathbb{R})$-complement (resp. (ϵ, n)-complement). If $\dim Z = 0$, we will omit Z and z. If for any $z \in Z$, $(X/Z \ni z, B+M)$ is $(\epsilon, \mathbb{R})$-complementary, then we say that $(X/Z, B+M)$ is $(\epsilon, \mathbb{R})$-complementary.

The following lemma is well-known to experts (cf. [7, Lemma 3.13]). We will use the lemma frequently without citing it in this paper.

Lemma 2.8. *Let ϵ be a non-negative real number, and $(X, B+M)$ a generalized pair as in Definition 2.7. Assume that $g : X \dashrightarrow X''$ is a birational contraction and B'', M'' are the strict transforms of B, M on X'' respectively.*

(1) *If $(X/Z \ni z, B+M)$ is $(\epsilon, \mathbb{R})$-complementary, then $(X''/Z \ni z, B''+M'')$ is $(\epsilon, \mathbb{R})$-complementary.*
(2) *Let n be a positive integer. If g is $-(K_X+B+M)$-non-positive and $(X''/Z \ni z, B''+M'')$ is $(\epsilon, \mathbb{R})$-complementary (resp. monotonically (ϵ, n)-complementary), then $(X/Z \ni z, B+M)$ is $(\epsilon, \mathbb{R})$-complementary (resp. monotonically (ϵ, n)-complementary).*

2.3. Bounded families

Definition 2.9. A *couple* (X, D) consists of a normal projective variety X and a reduced divisor D on X. Two couples (X, D) and (X', D') are *isomorphic* if there exists an isomorphism $X \to X'$ mapping D onto D'. A set $\mathcal{P}$ of couples is *bounded* if there exist finitely many projective morphisms $V^i \to U^i$ of varieties and reduced divisors C^i on V^i such that for each $(X, D) \in \mathcal{P}$, there exist i and a closed point $t \in U^i$ such that the two couples (X, D) and (V^i_t, C^i_t) are isomorphic, where V^i_t and C^i_t are the fibers over t of the morphisms $V^i \to U^i$ and $C^i \to U^i$ respectively.

A set $\mathcal{C}$ of projective pairs (X, B) is said to be *log bounded* if the corresponding set of couples $\{(X, B_{\mathrm{red}})\}$ is bounded. A set $\mathcal{D}$ of projective varieties X is said to be *bounded* if the corresponding set of couples $\{(X, 0)\}$ is bounded. A log bounded (resp. bounded) set is also called a *log bounded family* (resp. *bounded family*).

We will need the following theorem.

Theorem 2.10 *(BBAB Theorem, [3, Theorem 1.1]). Let d be a positive integer and ϵ a positive real number. Then the projective varieties X such that*

(1) *(X, B) is ϵ-lc of dimension d for some boundary B, and*
(2) *$-(K_X+B)$ is nef and big,*

form a bounded family.

The following lemma is an easy consequence of Theorem 2.10.

Lemma 2.11. *Let d be a positive integer and ϵ a positive real number. Then the projective varieties X such that*

(1) $\dim X = d$,
(2) *X is of Fano type, and*
(3) *$(X, B+M)$ is an $(\epsilon, \mathbb{R})$-complementary projective generalized pair with data $X' \xrightarrow{f} X$ and M', for some boundary B and nef part M',*

form a bounded family.

Proof. Possibly replacing $(X, B+M)$ by an $(\epsilon, \mathbb{R})$-complement, we may assume that $(X, B+M)$ is an $(\epsilon, \mathbb{R})$-complement of itself. Since X is of Fano type, there exists a boundary C such that (X, C) is klt and $-(K_X+C)$ is ample. It follows that $(X, D+M/2)$ is generalized $\frac{\epsilon}{2}$-lc, and $-(K_X+D+M/2)$ is ample, where $D = \frac{B+C}{2}$. Let $A \sim_{\mathbb{R}} -(K_X+D+M/2)/2$ be a general ample $\mathbb{R}$-divisor such that $(X, D+A+M/2)$ is generalized $\frac{\epsilon}{2}$-lc. We may write

$$K_{X'} + D' + M'/2 + f^*A = f^*(K_X + D + M/2 + A)$$

for some $\mathbb{R}$-divisor D' on X'. As $\frac{M'}{2} + f^*A$ is big and nef, there exists an effective $\mathbb{R}$-divisor E such that for each positive integer k, we have

$$M'/2 + f^*A \sim_{\mathbb{R}} E/k + A_k$$

for some ample $\mathbb{R}$-divisor A_k. We can choose k sufficiently large and A_k general enough such that (X', D'_k) is sub-$\frac{\epsilon}{4}$-lc, where $D'_k = D' + E/k + A_k$. Let $K_X + D_k = f_*(K_{X'} + D'_k)$. Then (X, D_k) is $\frac{\epsilon}{4}$-lc and

$$\begin{aligned} -(K_X + D_k) &\sim_{\mathbb{R}} -f_*(K_{X'} + D' + M'/2 + f^*A) \\ &\sim_{\mathbb{R}} -(K_X + D + M/2 + A) \sim_{\mathbb{R}} A \end{aligned}$$

is ample. According to Theorem 2.10, X belongs to a bounded family. □

3. Uniform linearity for minimal log discrepancies

3.1. Uniform linearity for MLDs

We show the following result on the uniform linearity for MLDs for generalized pairs. For any point $\boldsymbol{v} = (v_1, \ldots, v_m) \in \mathbb{R}^m$, we define $||\boldsymbol{v}||_\infty = \max_{1 \le i \le m}\{|v_i|\}$.

Theorem 3.1. *Let ϵ be a non-negative real number, d, c, m, l positive integers and $\boldsymbol{r}_0 = (r_1, \ldots, r_c) \in \mathbb{R}^c$ a point such that $1, r_1, \ldots, r_c$ are linearly independent over $\mathbb{Q}$. Let $s_1, \ldots, s_{m+l} : \mathbb{R}^c \to \mathbb{R}$ be $\mathbb{Q}$-linear functions. Then there exist a positive real number δ and a $\mathbb{Q}$-linear function $f(\boldsymbol{r}) : \mathbb{R}^c \to \mathbb{R}$ depending only on $\epsilon, d, \boldsymbol{r}_0$ and s_i satisfying the following.*

(1) *$f(\boldsymbol{r}_0) \ge \epsilon$, and if $\epsilon \in \mathbb{Q}$, then $f(\boldsymbol{r}) = \epsilon$ for any $\boldsymbol{r} \in \mathbb{R}^c$.*
(2) *Assume that $(X, B(\boldsymbol{r}_0) + M(\boldsymbol{r}_0))$ is a projective generalized pair of dimension d such that*
   - *X is of Fano type,*
   - *$B(\boldsymbol{r}) = \sum_{i=1}^m s_i(\boldsymbol{r}) B_i$, where $B_i \ge 0$ are Weil divisors,*
   - *$M'(\boldsymbol{r}) = \sum_{j=1}^l s_{m+j}(\boldsymbol{r}) M'_j$, where M'_j are nef Cartier divisors, and*
   - *$(X, B(\boldsymbol{r}_0) + M(\boldsymbol{r}_0))$ is $(\epsilon, \mathbb{R})$-complementary.*

   Then for any $\boldsymbol{r} \in \mathbb{R}^c$ satisfying $||\boldsymbol{r} - \boldsymbol{r}_0||_\infty \le \delta$, the following hold:
   (a) *$(X, B(\boldsymbol{r}) + M(\boldsymbol{r}))$ is $(f(\boldsymbol{r}), \mathbb{R})$-complementary.*
   (b) *If ϵ is positive, then there exists a prime divisor E such that*

$$\operatorname{mld}(X, B(\boldsymbol{r}) + M(\boldsymbol{r})) = a(E, X, B(\boldsymbol{r}) + M(\boldsymbol{r})) \ge f(\boldsymbol{r}).$$

Proof. If $\epsilon = 0$, then we can take $f(\boldsymbol{r}) = 0$ and the existence of δ follows from [6, Theorem 3.17]. In the following, we may assume that ϵ is positive.

By [6, Theorem 3.15], there exists a positive real number δ_0 depending only on $d, \boldsymbol{r}_0$ and s_i such that $(X, B(\boldsymbol{r}) + M(\boldsymbol{r}))$ is generalized lc for any $\boldsymbol{r} \in \mathbb{R}^c$ satisfying $||\boldsymbol{r} - \boldsymbol{r}_0||_\infty \le \delta_0$. In particular, there exist finite sets $\Gamma_1 = \{a_i\} \subseteq (0, 1]$ and $\Gamma_2 \subseteq [0, 1] \cap \mathbb{Q}$ such that

$$K_X + B(\boldsymbol{r}_0) + M(\boldsymbol{r}_0) = \sum a_i \left(K_X + B^i + M^i\right),$$

and $\left(X, B^i + M^i\right)$ is generalized lc for some $B^i \in \Gamma_2$ and $(M^i)' = \sum_j \mu_{ij} M'_j$ with $\mu_{ij} \in \Gamma_2$ for any i, j. Moreover, by [6, Theorem 1.1] and [1, Lemma 2.24], there exists a positive integer I_0 such that $I_0 \left(K_X + B^i + M^i\right)$

is Cartier for any i. Then by the same arguments as in [7, Lemma 4.7] and [6, Theorem 3.17], one can find a positive real number $\delta < \delta_0$ with the required properties. □

Proof of Theorem 1.6. The statement follows from Theorem 3.1(2). □

3.2. Han type polytopes for $(\epsilon, \mathbb{R})$-complementary generalized pairs

We will need the following result on Han type polytopes for $(\epsilon, \mathbb{R})$-complementary generalized pairs.

Theorem 3.2. *Let ϵ be a positive real number, d, m, l positive integers, and $\boldsymbol{v}_0 = \left(v_1^0, \ldots, v_{m+l}^0\right) \in \mathbb{R}^{m+l}$ a point. Then there exist positive real numbers a_k, positive real numbers ϵ_k and points $\boldsymbol{v}_k = \left(v_1^k, \ldots, v_{m+l}^k\right) \in \mathbb{R}^{m+l}$ depending only on ϵ, d and $\boldsymbol{v}_0$ satisfying the following.*

(1) *$\sum a_k = 1$, $\sum a_k \boldsymbol{v}_k = \boldsymbol{v}_0$, and $\sum a_k \epsilon_k \geq \epsilon$. Moreover, if $\epsilon \in \mathbb{Q}$, then $\epsilon_k = \epsilon$ for any k.*
(2) *Assume that $\left(X, \left(\sum_{i=1}^m v_i^0 B_i\right) + \left(\sum_{j=1}^l v_{m+j}^0 M_j\right)\right)$ is a projective generalized pair of dimension d such that*
   - *X is of Fano type,*
   - *$B_1, \ldots, B_m \geq 0$ are Weil divisors on X,*
   - *M_j' is nef Cartier for any $1 \leq j \leq l$, and*
   - *$\left(X, \left(\sum_{i=1}^m v_i^0 B_i\right) + \left(\sum_{j=1}^l v_{m+j}^0 M_j\right)\right)$ is $(\epsilon, \mathbb{R})$-complementary.*

   Then $\left(X, \left(\sum_{i=1}^m v_i^k B_i\right) + \left(\sum_{j=1}^l v_{m+j}^k M_j\right)\right)$ is $(\epsilon_k, \mathbb{R})$-complementary for any k.

Proof. The result follows from the same arguments as in [9, Theorem 5.16] but with [9, Theorem 5.15] replaced by Theorem 3.1. □

4. Decomposable complements

4.1. Reduce DCC sets to finite sets

The proof of Theorem 4.1 is quite similar to the proof of [9, Theorem 5.18 and Theorem 5.20].

Theorem 4.1. *Let ϵ be a non-negative real number, d a positive integer and $\Gamma \subseteq [0, 1]$ a DCC set. Then there exist a finite set $\Gamma' \subseteq \bar{\Gamma}$, and a projection $g : \bar{\Gamma} \to \Gamma'$ (i.e., $g \circ g = g$) depending only on ϵ, d and Γ satisfying the following.*

(1) *$g(\gamma') \geq g(\gamma) \geq \gamma$ for any $\gamma, \gamma' \in \Gamma$ with $\gamma' \geq \gamma$.*
(2) *Assume that $(X, \sum b_i B_i)$ is a d-dimensional $(\epsilon, \mathbb{R})$-complementary pair such that X is of Fano type, and $B_i \geq 0$ is a $\mathbb{Q}$-Cartier Weil divisor and $b_i \in \Gamma$ for any i. Then $(X, \sum g(b_i) B_i)$ is $(\epsilon, \mathbb{R})$-complementary.*

Proof. The statement follows from the same arguments as in [9, Theorem 5.18 and Theorem 5.20] but with ACC for log canonical thresholds replaced by Lemma 4.2. □

Lemma 4.2. *Let $\mathcal{S}$ be the set of pairs $(X, B = \sum b_i B_i)$ satisfying the assumptions in Theorem 4.1 and assume that ϵ is a positive real number. Then the set*

$$\Gamma'' = \left\{ \epsilon\text{-lct}\left(X, \sum_{i=1}^{j-1} b_i B_i + \sum_{i=j+1}^{s} b_i B_i; B_j\right) \mid \left(X, \sum_{i=1}^{s} b_i B_i\right) \in \mathcal{S}, 1 \leq j \leq s \right\}$$

satisfies the ACC.

Proof. We first show that (X, B) belongs to a log bounded family. Let $c = \min\{\alpha > 0 \mid \alpha \in \Gamma\}$. By Lemma 2.11, X belongs to a bounded family. In particular, there exist a very ample divisor A on X and a positive real number r depending only on d and ϵ such that $A^{d-1} \cdot (-K_X) \leq r$. Since

$$B_{\text{red}} \cdot A^{d-1} = \frac{1}{c}\left(K_X + c\sum B_i - K_X\right) \cdot A^{d-1}$$
$$\leq \frac{1}{c}\left(K_X + \sum b_i B_i - K_X\right) \cdot A^{d-1} \leq \frac{r}{c},$$

$(X, \sum B_i)$ is log bounded by [1, Lemma 2.20].

Let $(\mathcal{X}, \mathcal{D}) \to U$ be the corresponding log bounded family. Then there exists a stratification $U_1, \ldots, U_l$ of U such that, possibly after taking a finite étale cover, each restricted family $(\mathcal{X}_{U_i}, \mathcal{D}_{U_i}) \to U_i$ admits a fiberwise log resolution. Therefore there exist two finite sets $\Gamma_1 \subseteq (-\infty, 1]$ and $\Gamma_2 \subseteq [0, +\infty)$ depending only on this log bounded family such that

$$f^* K_X = K_Y + \sum_k a_k E_k \quad \text{and} \quad f^* B_i = \sum_k a_{ik} E_k$$

for some real numbers $a_k = a(E_k, X, 0) \in \Gamma_1$ and $a_{ik} \in \Gamma_2$. Hence

$$m_{jk} = \text{mult}_{E_k}\left(f^*\left(K_X + \sum_{i=1}^{j-1} b_i B_i + \sum_{i=j+1}^{s} b_i B_i\right) - K_Y\right)$$

lies in a DCC set depending only on the DCC set Γ and the log bounded family. Thus

$$\epsilon\text{-}\operatorname{lct}\left(X, \sum_{i=1}^{j-1} b_i B_i + \sum_{i=j+1}^{s} b_i B_i; B_j\right) = \min_k \left\{\frac{1 - \epsilon - m_{jk}}{a_{jk}}\right\}$$

forms an ACC set. □

4.2. Proof of Theorem 1.5

We first show that Theorem 1.3(3) holds when $\Gamma \subseteq [0, 1] \cap \mathbb{Q}$ is a finite set.

Proposition 4.3. *Theorem 1.3(3) holds when $\Gamma \subseteq [0, 1] \cap \mathbb{Q}$ is a finite set. Moreover, $(X, B + M)$ has a monotonic (ϵ, n)-complement $(X, B^+ + M)$.*

Proof. We may run a $-(K_X + B + M)$-MMP and it terminates with a model X'' on which $-(K_{X''} + B'' + M'')$ is nef, where B'' and M'' are the strict transforms of B and M on X'' respectively. It is clear that $(X'', B'' + M'')$ is generalized ϵ-lc, as $(X, B + M)$ is $(\epsilon, \mathbb{R})$-complementary. Possibly replacing $(X, B + M)$ by $(X'', B'' + M'')$, we may assume that $-(K_X + B + M)$ is nef.

By Lemma 2.11, X belongs to a bounded family. According to [1, Lemma 2.25], there exists a positive integer n_0 such that $-n_0(K_X + B + M)$ is Cartier. Hence $|-n(K_X + B + M)|$ is base point free for some positive integer n depending only on n_0 and d by Kollár's effective base point free theorem (cf. [10]). Pick a general member $G \in |-n(K_X + B + M)|$. Then $(X, B + G/n + M)$ is an (ϵ, n)-complement of $(X, B + M)$. We finish the proof. □

Proof of Theorem 1.5. We may assume that $1 \in \Gamma$. Possibly replacing $(X, B+M)$ by a generalized dlt modification, we may assume that X is $\mathbb{Q}$-factorial.

In the case when $M' \equiv 0$, we may apply Theorem 4.1 and thus assume that Γ is a finite set. By Theorem 3.2, we can find positive real numbers ϵ_i and finite sets $\Gamma_0 \subseteq (0,1], \Gamma_1 \subseteq [0,1] \cap \mathbb{Q}$ depending only on d and Γ such that $\left(X, B_i + M_{(i)}\right)$ is $(\epsilon_i, \mathbb{R})$-complementary, $\sum a_i = 1, \sum a_i \epsilon_i \geq \epsilon$ and

$$K_X + B + M = \sum a_i \left(K_X + B_i + M_{(i)}\right)$$

for some $a_i \in \Gamma_0, B_i \in \Gamma_1$ and $M'_{(i)} = \sum_j \mu_{ij} M'_j$ with $\mu_{ij} \in \Gamma_1$. Moreover, if $\Gamma \subseteq \mathbb{Q}$, then we may pick $\Gamma_0 = \{1\}$, $B_i = B$ and $M_{(i)} = M$. By Proposition 4.3, there exists a positive integer n depending only on Γ_1, ϵ_i and d such that $\left(X, B_i + M_{(i)}\right)$ has a monotonic (ϵ_i, n)-complement $\left(X, (B_i + G_i) + M_{(i)}\right)$ for some $G_i \geq 0$ for any i. Let $B^+ := \sum a_i (B_i + G_i)$. Then $(X, B^+ + M)$ is an $(\boldsymbol{\epsilon}, n, \Gamma_0)$-decomposable (ϵ, n)-complement of $(X, B+M)$. This completes the proof. □

5. Proof of the main result

Proof of Theorem 1.3. According to [6, Theorem 1.1], we only need to show Theorem 1.3(2) and (3). By Theorem 1.5, there exist a positive integer n_0, positive real numbers ϵ_i and a finite set $\Gamma_0 \subseteq (0,1]$ depending only on d, ϵ and Γ such that $(X, B+M)$ has an $(\boldsymbol{\epsilon}, n_0, \Gamma_0)$-decomposable $(\epsilon, \mathbb{R})$-complement $\left(X, \tilde{B} + \tilde{M}\right)$. More precisely, there exist $a_i \in \Gamma_0$, boundaries $\tilde{B}_i$ and nef parts $\tilde{M}'_i$ such that $\sum a_i = 1, \sum a_i \epsilon_i \geq \epsilon$,

$$K_X + \tilde{B} + \tilde{M} = \sum a_i \left(K_X + \tilde{B}_i + \tilde{M}_i\right),$$

and $\left(X, \tilde{B}_i + \tilde{M}_i\right)$ is an (ϵ_i, n_0)-complement of itself for any i.

By [7, Lemma 6.2], one can find a positive integer n divisible by pn_0 depending only on $\epsilon, p, n_0, \Gamma_0$ and ϵ_i such that there exist positive rational numbers a'_i with the following properties:

- $\sum a'_i = 1$, $\sum a'_i \epsilon_i \geq \epsilon$,
- $n a'_i \in n_0 \mathbb{Z}$ for any i,
- $nB^+ \geq n\lfloor \tilde{B} \rfloor + \lfloor (n+1)\{\tilde{B}\} \rfloor$, where $B^+ := \sum a'_i \tilde{B}_i$, and
- $n \sum_i a'_i \mu_{ij} = n \lfloor \sum_i a_i \mu_{ij} \rfloor + \lfloor (n+1) \{\sum_i a_i \mu_{ij}\} \rfloor$ for any j.

Let $\mu_j^+ = \sum_i a'_i \mu_{ij}$ for any j, and $M^{+\prime} = \sum_i a'_i \tilde{M}'_i = \sum_j \mu_j^+ M'_j$. Then

$$\begin{aligned} n\left(K_X + B^+ + M^+\right) &= n \sum a'_i \left(K_X + \tilde{B}_i + \tilde{M}_i\right) \\ &= \sum \frac{a'_i n}{n_0} \cdot n_0 \left(K_X + \tilde{B}_i + \tilde{M}_i\right) \sim 0. \end{aligned}$$

We conclude that $(X, B^+ + M^+)$ is an (ϵ, n)-complement of $(X, B+M)$, since $\left(X, \tilde{B} + \tilde{M}\right)$ is an $(\epsilon, \mathbb{R})$-complement of $(X, B+M)$.

Moreover, if $\mathrm{Span}_{\mathbb{Q}_{\geq 0}}\left(\bar{\Gamma} \cup \{\epsilon\} \backslash \mathbb{Q}\right) \cap (\mathbb{Q} \backslash \{0\}) = \emptyset$, then we may pick $B^+ \geq B$ and $\mu_j^+ \geq \mu_j$ by [7, Lemma 6.2] and [9, Lemma 6.3]. □

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