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RESEARCH ARTICLE

On the relative minimal model program for fourfolds in positive and mixed characteristic

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Abstract

We show the validity of two special cases of the four-dimensional minimal model program (MMP) in characteristic $p > 5$: for contractions to $\mathbb{Q}$-factorial fourfolds and in families over curves ('semistable MMP'). We also provide their mixed characteristic analogues. As a corollary, we show that liftability of positive characteristic threefolds is stable under the MMP and that liftability of three-dimensional Calabi–Yau varieties is a birational invariant. Our results are partially contingent upon the existence of log resolutions.

1. Introduction

In recent years, there has been much progress in understanding the birational geometry of threefolds over algebraically closed fields of characteristic $p > 0$ by using methods inspired by the theory of F-singularities. One of the main achievements in this area is the proof of the minimal model program (MMP) in dimension three in wide generality (see, e.g., [HX15], [CTX15], [Bir16], [BW17], [HW21], [HW22]).

Long before these developments, Kawamata has established the validity of the *semistable* MMP in dimension three based on classification results ([Kaw94]). Although weaker than the general form of the MMP, this has nevertheless led to plenty of applications, especially concerning Tate's conjecture and criteria for the existence of good reductions for K3 surfaces and other classes of varieties, generalising Serre–Tate's and Néron-Ogg–Shafarevich results for abelian varieties (see, e.g., [Mau14, LM18, CLL22, CL16]). The purpose of our paper is to extend the semistable MMP (and also the MMP for resolutions of singularities) to dimension four and derive applications thereof. This is the first step towards establishing the MMP in full generality for fourfolds in high characteristic. In view of the recent developments in mixed characteristic ([BMPSTWW20]), we are also able to provide results on the MMP for arithmetic fourfolds. In particular, we can address questions about the MMP of lifts of positive characteristic three-dimensional varieties which arose after the work of Kawamata.

For the convenience of the reader, we state the simplified versions of our main theorems and refer to Theorems 4.1, 5.1, 6.1, 6.2 for the precise statements. In particular, in all the results below we assume that log resolution of four-dimensional varieties exists.

Theorem 1.1 (Theorem 4.1). *Every log resolution of a four-dimensional $\mathbb{Q}$-factorial klt singularity with perfect residue field of characteristic $p > 5$ decomposes into a sequence of steps of the MMP.*

More generally, we show that we can run a dlt MMP on a log resolution of every $\mathbb{Q}$-factorial singularity.

Theorem 1.2 (Theorem 5.1). *The MMP may be run on every four-dimensional semistable variety of nonnegative Kodaira dimension which is projective over a discrete valuation ring with perfect residue field of characteristic $p > 5$.*

The assumption on the residue field being perfect is necessary as the three-dimensional base point free theorem is not known in full generality in a more general setting (for example, when the residue field is F-finite). Note that the assumptions on the Kodaira dimension and the base of the MMP have been weakened after our article was made available by improving on our termination results (see [XX21a]).

In order to show the above theorems, we find a new way of constructing flips when the boundary contains an ample divisor (see Theorem 1.6). This key technical result is explained in the next subsection of the introduction.

As in [HW21], Theorem 1.1 allows us to derive a series of results for $\mathbb{Q}$-factorial four-dimensional singularities, which in characteristic 0 are proven using vanishing theorems. In particular, we show that positive characteristic klt singularities are $W\mathcal{O}_X$-rational, dlt modifications exist and the inversion of adjunction holds (Corollaries 4.7, 4.8, 4.9). The first result implies the following generalisation of the Chevalley–Warning criterion (see [Esn03, Corollary 1.3] for the smooth rationally chain connected case in any dimension and its generalisation to the klt case in dimension three by [GNH19]).

Corollary 1.3. *Every four-dimensional geometrically connected $\mathbb{Q}$-factorial klt variety X which is projective over a finite field $\mathbb{F}_q$ of characteristic $p > 5$ and which admits a rationally chain connected log resolution satisfies $|X(\mathbb{F}_q)| \equiv 1 (\mathrm{mod}\ q)$.*

In particular, X admits a rational point. For the definition of rational chain connectedness, we refer to [GNH19, Definition 4.5].

Similarly, one should be able to extend other statements from dimension three in positive characteristic to dimension four, such as finite generatedness of the local Picard group (cf. [CSK20]) or, assuming the Borisov-Alexeev-Borisov (BAB) conjecture in dimension three, finiteness of the tame étale fundamental group. To avoid making our article too long, we do not write down proofs of these results here.

Theorem 1.2 can be used, for example, to construct canonical skeleta of degenerations of varieties with nonnegative Kodaira dimension ([NX16]) and allows for descending deformations of threefolds under operations of the MMP. In particular, we are able to provide a positive answer to [HW19, Question 5.2] for lifts to characteristic 0 contingent upon the resolutions of singularities. This gives a new insight into the results of [LS14] on the behaviour of liftability under birational maps.

Corollary 1.4 (Theorem 6.1). *Let X be a $\mathbb{Q}$-factorial terminal projective three-dimensional variety over a perfect field of characteristic $p > 5$. Suppose that K_X is pseudo-effective and that X lifts to characteristic 0. Then a (possibly non-$\mathbb{Q}$-factorial) minimal model of X lifts to characteristic 0 as well.*

As a corollary to the full statement of the above result (Theorem 6.1), we get that liftability of three-dimensional terminal Calabi–Yau varieties is a birational invariant. Here, we say that a projective variety X is *Calabi–Yau* if ω_X is trivial and $H^i(X, \mathcal{O}_X) = 0$ for $0 < i < \dim X$.

Corollary 1.5 (Theorem 6.2). *Let X and Y be three-dimensional terminal $\mathbb{Q}$-factorial projective Calabi–Yau varieties defined over a perfect field of characteristic $p > 5$. Suppose that X and Y are birational and that X lifts to characteristic 0. Then Y lifts to characteristic 0 as well.*

We hope that Theorem 1.2 will also find applications in the study of Néron–Ogg–Shafarevich-type criteria for good reductions of three-dimensional varieties.

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The idea of the proof of Theorem 1.1

Here, we explain the proof in positive characteristic. The mixed characteristic case is analogous with S^0 and F-regularity replaced by B^0 and +-regularity, respectively, as developed in [BMPSTWW20].

By the same arguments as in [Bir16], we can assume that Δ has standard coefficients. Our strategy follows the ideas developed in [HW21]. The main new difficulty is to show the existence of pl-flips occurring in the above MMP. As observed in [HW21], the corresponding pl-flipping contractions contain a relatively ample divisor in the boundary. In particular, by applying inversion of F-adjunction twice and using the F-regularity of relative log Fano surfaces in the birational case (see [HX15, Theorem 3.1]) we can deduce that these pl-flipping contractions are relatively purely F-regular (up to a small perturbation); see Theorem 4.3.

Unluckily, in contrast to the three-dimensional case ([HX15]), this is not enough to conclude the existence of flips. Indeed, [HX15] employs a strategy of Shokurov for constructing flips which is not valid in higher dimensions. On the other hand, it is not clear how to generalise the higher-dimensional proof of the existence of flips ([HMK10]) from characteristic 0 to positive characteristic in this setting, as the proof calls for applying vanishing theorems on a log resolution, in which case the relative pure F-regularity is lost.

As mentioned earlier, we address this problem by finding a new way of constructing flips when the boundary contains a relatively ample divisor.

Theorem 1.6 (cf. Theorem 4.3). *Let $(X, S+A+B)$ be an n-dimensional $\mathbb{Q}$-factorial dlt pair defined over an F-finite field of characteristic $p > 0$ and let $f : X \to Z$ be a $(K_X + S + A + B)$-flipping contraction with $\rho(X/Z) = 1$, where S is an f-anti-ample divisor, A is an f-ample divisor and $\lfloor B \rfloor = 0$. Suppose that the MMP is valid in dimension $n-1$. If $(X, S + (1-\epsilon)A + B)$ is relatively purely F-regular for all $0 < \epsilon < 1$, then the flip of f exists.*

In fact, we show that $H^0(X, m(K_X+S+A+B)) \to H^0(S, m(K_S+A_S+B_S))$ is surjective for all $m > 0$ sufficiently divisible, where $K_S + A_S + B_S = (K_X + S + A + B)|_S$ (Proposition 3.1). In particular, the finite generatedness of canonical rings in dimension $n-1$ implies that the canonical ring of $(X, S+A+B)$ is finitely generated as well, and so its projectivisation is the sought-for flip. Note that this surjectivity is false without the ample divisor in the boundary (see [dFH11, Remark 4.9]).

Resolutions of singularities

It is natural to wonder if the assumption on the existence of log resolutions can be dropped. In general, this seems to be a very difficult problem in view of the fact that alterations or quasi-resolutions do not behave nicely with respect to the canonical divisor. However, one may hope to remove this assumption if we start to run the MMP on a regular variety, and this is, to some extent, the case (see Proposition 5.8) provided a much weaker assertion: that embedded log resolutions of subschemes of smooth fourfolds exist. Note, however, that the current proofs of Theorem 1.1 and of Theorem 1.2 (and so that of Corollary 1.3, Corollary 1.4 and Corollary 1.5 as well) require additionally that the log resolutions are constructed by a sequence of blowups along the non-simple normal crossing locus (non-snc locus) (see the second paragraph of the proof of Theorem 4.1 and of the proof of Claim 5.5).

Mixed charactieristic

The original version of this article covered the positive characteristic case only. A few months after its submission to the arxiv, [BMPSTWW20] developed the mixed characteristic analogues of S^0 and F-regularity and proved the validity of the three-dimensional MMP for arithmetic threefolds. In view of this and recent results on relative semiampleness ([Wit21b]), the original results of our article generalise to mixed characteristic with small modifications. The current updated version of our paper incorporates these generalisations and several applications thereof.

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We also note that results on the three-dimensional MMP in mixed characteristic in the semistable case (in a wider generality than that of Kawamata) were obtained in [TY20] independently of [BMPSTWW20]. The geometric part of their proof is based on the ideas of the original version of our article and on [HW21]. In particular, [TY20] generalised our Theorem 1.6 to mixed characteristic before this updated version of our article was submitted.

The notions of +-regularity and +-stable sections (B^0) work in both positive and mixed characteristic, and so we could restate the whole article to avoid the use of F-regularity and Frobenius stable sections (S^0). We decided against doing so, instead treating positive and mixed characteristic cases separately, to prevent our article from being unnecessarily technical for the readers interested in the positive characteristic case only. The notion of F-regularity has been in use for a long time, in contrast to +-regularity which was introduced recently and is more technically involved.

In the mixed characteristic case, some of our results are valid beyond the case of schemes defined over discrete valuation rings (DVRs) (see, e.g., the setting of [BMPSTWW20]). For the sake of readability and in order to avoid dealing with unnecessary technicalities, we do not strive to provide the most general version. Note, however, that there is a fundamental obstacle to generalising Theorem 5.1 to the case of Dedekind domains such as $\operatorname{Spec}\mathbb{Z}$ (or $\operatorname{Spec} k[t]$)) due to the issues with termination of flips; indeed, over such a base it could *a priori* happen that there is an infinite sequence of flipping curves contained in fibres over different prime numbers (fortunately, this has been now resolved in [XX21a]). Furthermore, Theorem 5.1 needs the residue field to be perfect so that we can invoke the three-dimensional base point free theorem as well as [NT20], and the positive characteristic case thereof requires essentially that the base spreads out over an algebraically closed field so that we can apply positive characteristic Bertini theorems from [SZ13].

To align with the notation in the positive characteristic case, we use the word *variety* (as defined in the preliminaries below) even for schemes over DVRs. We hope that this will not cause any confusion for the reader.

2. Preliminaries

A scheme X will be called a *variety* if it is integral, separated and of finite type over a field k or a divisorial valuation ring R. Throughout this paper, unless otherwise stated, we work over F-finite fields k of characteristic $p > 0$ or DVRs R of characteristic $(0, p)$ for $p \geq 0$.

We refer the reader to [KM98] for the standard definitions and results of the MMP, to [HW21] for a brief introduction to F-singularities and to [BMPSTWW20] for the results on +-regularity. Further, we refer to [GNH19, Remark 2.7] for a discussion on properties of varieties which are independent of a base change from a perfect field. We warn the reader that $\mathbb{Q}$-factoriality is not such a property.

In this paper, a *pair* (X, B) consists of a normal variety X and an effective $\mathbb{Q}$-divisor B such that $K_X + B$ is $\mathbb{Q}$-Cartier. The pair (X, B) is *kawamata log terminal* (*klt*) (resp. *log canonical* (*lc*)) if for any proper birational morphism $f\colon Y \to X$ and any prime divisor E on Y we have $\operatorname{mult}_E(B_Y) < 1$ (resp. $\operatorname{mult}_E(B_Y) \leq 1$), where $K_Y + B_Y = f^*(K_X + B)$. If (X, B) admits a log resolution $f\colon Y \to X$, then it suffices to check the above condition for all prime divisors E on Y.

In the article, we often consider log resolutions of singularities $f\colon Y \to X$ of dlt pairs (X, Δ) which are isomorphisms over the generic points of the strata of (X, Δ). Every log resolution f constructed by a sequence of blowups along the non-snc locus satisfies this property. Note that using the notation from [Kol13, Definition 2.79 and Theorem 2.87], such f is a morphism which is both a log resolution and a thrifty resolution.

The definitions of singularities in birational geometry may be found in [Kol13] and [BMPSTWW20, Section 2.5]. We say that a pair (X, B) such that $B = \sum b_i B_i$ with $0 \leq b_i \leq 1$ is *divisorially log terminal* (*dlt*) if there exists an open subset $U \subset X$ such that U is regular and $B|_U$ has simple normal crossings support and for every proper birational morphism $f\colon Y \to X$ and any prime divisor E on Y with centre Z contained in $X \setminus U$, we have $\operatorname{mult}_E(B_Y) < 1$. We say that $a_E(X, B) := 1 - \operatorname{mult}_E(B_Y)$ is *the log discrepancy* of (X, B) along E. A pair $(X, S + B)$ with $\lfloor S + B \rfloor = S$ irreducible is *purely log terminal* (*plt*) if $a_E(X, S + B) > 0$ for any $E \neq S$.

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A morphism of schemes $f\colon X \to Y$ is a *universal homeomorphism* if for any morphism $Y' \to Y$, the induced morphism $X' = X \times_Y Y' \to Y'$ is a homeomorphism. In positive characteristic, it is known that a finite morphism is a universal homeomorphism if and only if it factors a sufficiently high power of the Frobenius morphism [Kol97, Proposition 6.6]. We say that a variety X is *normal up to universal homeomorphism* if its normalisation $X^\nu \to X$ is a universal homeomorphism. We learned the following result from János Kollár.

Lemma 2.1. *Let $(X, D+\Delta)$ be a dlt pair with D prime and $\mathbb{Q}$-Cartier. Then D is normal up to universal homeomorphism.*

Proof. By [Kol16, Theorem 41], it is enough to show that $D^{sh} \setminus \{x^{sh}\}$ is connected for the Henselisation D^{sh} of D at every point $x \in D$. If x is a point of codimension at most two in X, then this follows by surface theory (cf. [Kol13]) as $(X, D+\Delta)$ is dlt. Thus, we may assume that x is of codimension at least three in X. Let $X^\wedge$ and $D^\wedge$ be the completions of X and D at x, respectively. It is enough to show that $D^\wedge \setminus \{x\}$ is connected. Note that $X^\wedge$ is normal ([Sta14, Tag 033C, 07GC, 037D] and [Pop00, Lemma 2.7]), hence S_2, and so $X^\wedge \setminus \{x\}$ is connected. Moreover, $X^\wedge \setminus \{x\}$ is S2 at every closed point $x' \in X^\wedge \setminus \{x\}$ (as the scheme is Catenary, the codimension of x' in $X^\wedge$ is at least two). Since $D^\wedge$ is of codimension one ([Sta14, Tag 07NV]) and $\mathbb{Q}$-Cartier, we can apply the local Lefschetz property of [Gro62, XIII.2, Theorem 2.1, p. 139] to a Cartier multiple of $D^\wedge$ to conclude the proof. □

We refer to [GNH19] and [HW21, Subsection 2.3] for the notion of $W\mathcal{O}$-rational singularities in positive characteristic. Since log resolutions of singularities are not known to exist beyond dimension three, a positive characteristic singularity X is called *$W\mathcal{O}$-rational* if $R^i f_* W\mathcal{O}_{V,\mathbb{Q}} = 0$ for $i > 0$ and every quasi-resolution $f\colon V \to X$ (see [GNH19, Section 3]). Note that it is enough to verify this condition on a single quasi-resolution ([CR12, Corollary 4.5.1]). In particular, if X admits a usual resolution of singularities, we can verify this condition on it.

Lemma 2.2. *Let (X, Δ) be a $\mathbb{Q}$-factorial dlt pair and S an irreducible component of $\lfloor \Delta \rfloor$. If $S^\nu \to S$ is the normalisation and $K_{S^\nu} + \Delta_{S^\nu} = (K_X + \Delta)|_{S^\nu}$, then (S^ν, Δ_{S^ν}) is dlt and there is a bijection between the strata of $\lfloor \Delta_{S^\nu} \rfloor$ and the strata of $\lfloor \Delta \rfloor$ that are contained in S.*

Let $P \in S$ be any codimension 1 point on S and m the Cartier index of K_X at P. Then on a neighbourhood of $P \in X$, mD is Cartier for any divisor D, and S is normal, and $(K_X + S)|_S = K_S + (1 - \frac{1}{m})P$.

Proof. By assumption, there is an open subset $U \subset X$ containing the generic points of every strata of $\lfloor \Delta \rfloor$ such that $(U, \lfloor \Delta \rfloor|_U)$ has simple normal crossings and the complement $Z = X \setminus U$ contains no non-klt centres. Then, it is easy to see that $(S \cap U, \lfloor \Delta \rfloor|_{S\cap U}) = (S^\nu \cap U, \lfloor \Delta_{S^\nu} \rfloor|_{S^\nu \cap U})$ has simple normal crossings. Let E be an exceptional divisor over S with centre c_E whose generic point is not contained in $\lfloor \Delta \rfloor|_{S\cap U}$, and let I be an ideal sheaf on S, the blowup of which contains E. By blowing up the pushforward of I on X and taking the normalisation, we may construct a morphism $X' \to X$ such that if $S' \subset X'$ is the strict transform of S, then E is a divisor on S'. After possibly further blowups of X, we may assume that there is an exceptional divisor $F \subset X'$ such that $S' \cap F = E$ and S', F intersect transversely at the generic point of E. But then the log discrepancies satisfy $a_E(S^\nu, \Delta_{S^\nu}) = a_F(X, \Delta) > 0$. The first part of the lemma now follows easily.

The second part of the lemma follows by standard results for surfaces once we localise at $P \in X$. □

We state an application of [Wit22].

Theorem 2.3 [Wit21b, Theorem 2.22]. *Let $f\colon Y \to X$ be a finite universal homeomorphism of schemes which are proper over a Noetherian base scheme S. Let $\mathcal{L}$ be a nef line bundle on X such that $\mathcal{L}|_{X_\mathbb{Q}}$ and $f^*\mathcal{L}$ are semiample, where $X_\mathbb{Q}$ is the characteristic 0 fibre of $X \to \operatorname{Spec} \mathbb{Z}$. Then $\mathcal{L}$ is semiample.*

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2.1. F-regularity and +-regularity

Since the work of Hacon and Xu [HX15], (global) F-regularity has been one of the main tools in the MMP in positive characteristics. In the local setting, it gives a good analog of (and often coincides with) klt singularities and in the global setting it provides us with a log Fano structure and implies vanishing theorems. For the convenience of the reader, we recall several definitions and key results.

Definition 2.4. Let X be an F-finite $\mathbb{F}_p$-scheme. Given an effective $\mathbb{Q}$-divisor B, we say that (X, B) is *globally F-split* if for every integer $e > 0$, the natural homomorphism of $\mathcal{O}_X$-modules

$$\mathcal{O}_X \to F^e_* \mathcal{O}_X(\lfloor (p^e - 1)B \rfloor)$$

has an $\mathcal{O}_X$-module splitting. We say that (X, B) is *globally F-regular* (resp. *purely globally F-regular*) if for every effective divisor D on X (resp. every $D \geq 0$ intersecting $\lfloor B \rfloor$ properly) and every integer $e \gg 0$, the natural homomorphism of $\mathcal{O}_X$-modules

$$\mathcal{O}_X \to F^e_* \mathcal{O}_X(\lfloor (p^e - 1)B \rfloor + D)$$

splits.

If the above splittings hold locally on X, then we refer to these notions as *F-purity*, *strong F-regularity*, and *pure F-regularity*, respectively. Given a morphism $f\colon X \to Y$, we say that (X, B) is relatively (over Y) F-split, F-regular, or purely F-regular, if the above splittings hold locally over Y.

Let L be a divisor and (X, B) be a log pair where p does not divide the index of B. We let

$$S^0(X, B; L) := \bigcap \operatorname{Im}\left(H^0(X, F^e_* \mathcal{O}_X((1 - p^e)(K_X + B) + p^e L)) \to H^0(X, \mathcal{O}_X(L))\right),$$

where the intersection is taken over $e > 0$ sufficiently divisible so that $(p^e - 1)B$ is integral. Note that if $f\colon X \to U$ is a projective morphism to an affine variety such that L is $\mathbb{Q}$-Cartier and $L - (K_X + B)$ is f-ample, then (identifying $f_* \mathcal{O}_X(L)$ with $H^0(X, \mathcal{O}_X(L))$) the subsheaf $S^0(X, B; L) \subset f_* \mathcal{O}_X(L)$ is coherent and equal to the single image for a sufficiently big and divisible $e > 0$ (see the proof of [HX15, Proposition 2.15]).

We quickly review the theory of +-regularity ([BMPSTWW20]). Let X be a normal integral scheme which is proper over a *complete* Noetherian local domain $(R, \mathfrak{m})$ with characteristic $p > 0$ residue field. Let $B \geq 0$ be a $\mathbb{Q}$-divisor on X. For a Weil divisor L on X, we define the subspace of +-stable sections $B^0(X, B; \mathcal{O}_X(L))$ to be

$$\bigcap_{\substack{f\colon Y \to X \\ \text{finite}}} \operatorname{Im}\left(H^0(Y, \mathcal{O}_Y(K_Y + \lceil f^*(L - K_X - B) \rceil)) \to H^0(X, \mathcal{O}_X(L))\right),$$

where the intersection taken over all finite covers $f\colon Y \to X$ by a normal integral scheme Y (more precisely, here and throughout the article, this means that we take the intersection over the category of all finite covers equipped with an embedding of $K(Y)$ into a fixed algebraic closure $\overline{K(X)}$ of $K(X)$; see [BMPSTWW20, Convention 4.1]). When X is not irreducible, but still normal, we define $B^0(X, B; \mathcal{O}_X(L))$ to be the direct sum of B^0 for each connected (thus irreducible) component of X.

Similarly, for a reduced divisor S having no common components with an effective $\mathbb{Q}$-divisor B, we define the adjoint variant $B^0_S(X, S + B; \mathcal{O}_X(L))$ to be equal to

$$\bigcap_{\substack{f\colon Y \to X \\ \text{finite}}} \operatorname{Im}\left(H^0(Y, \bigoplus_{i=1}^{t} \mathcal{O}_Y(K_Y + S_{i,Y} + \lceil f^*(L - K_X - S - B) \rceil)) \to H^0(X, \mathcal{O}_X(L))\right),$$

where $S_{i,Y}$ are compatibly chosen prime divisors lying over S_i for $S = \sum_{i=1}^t S_i$ and S_i prime. The choice of such divisors $S_{i,Y}$ is equivalent to choosing prime divisors S_i^+ in X^+ lying over S_i. Note that the definition of B^0_S is independent of the choice of these S_i^+ (see [BMPSTWW20, Lemma 4.23]).

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Definition 2.5. Let X be a normal, integral, excellent scheme with a dualising complex and with every closed point having positive characteristic residue field. Further, let $B \geq 0$ be a $\mathbb{Q}$-divisor on X.

We say that (X, B) is *globally +-regular* if for every finite dominant map $f : Y \to X$ with Y normal, the morphism

$$\mathcal{O}_X \to f_* \mathcal{O}_Y(\lfloor f^* B \rfloor)$$

splits. When X is not integral, but still normal with all connected components of the same dimension, we say that X is *globally +-regular* if also all of its connected components are.

Let $(X, S + B)$ be a log pair such that S is a reduced divisor having no common components with an effective $\mathbb{Q}$-divisor B. We say that $(X, S + B)$ such that $\lfloor B \rfloor = 0$, is *purely globally +-regular* (along S), if for every finite dominant map $f : Y \to X$ with Y normal, the morphism

$$\mathcal{O}_X \to f_* \bigoplus_{i=1}^{t} \mathcal{O}_Y(\lfloor f^*(S + B) \rfloor - S_{i,Y})$$

splits, where $S_{i,Y}$ are chosen as in the definition of B^0_S above.

In what follows, we recall the relative analogs of the above notions. Note that the completion $\hat{R}$ of an excellent normal local ring R is automatically normal (see [Sta14, Tag 0C23]). Similarly, if X is a normal excellent scheme over R, then the base change $X_{\hat{R}}$ is also normal (see [Sta14, Tag 0C22]).

Definition 2.6. Let Z be a normal integral excellent scheme with a dualising complex. Let $f : X \to Z$ be a proper morphism for a normal integral scheme X.

We say that (X, B) for a $\mathbb{Q}$-divisor $B \geq 0$ is *completely relatively +-regular* over Z if for every closed point $z \in Z$ with positive residue characteristic its base change $(X_{\widehat{Z}_z}, B_{\widehat{Z}_z})$ to the completion of $\mathcal{O}_{Z,z}$ at z is globally +-regular.

We say that $(X, S+B)$ for a reduced divisor S having no common components with a $\mathbb{Q}$-divisor $B \geq 0$ is *completely relatively purely +-regular* over Z if for every closed point $z \in Z$ with positive residue characteristic its base change $(X_{\widehat{Z}_z}, S_{\widehat{Z}_z} + B_{\widehat{Z}_z})$ to the completion of $\mathcal{O}_{Z,z}$ at z is purely globally +-regular.

Note that in contrast to [BMPSTWW20], we do not assume that Z has all closed points of positive residue characteristics. In particular, our definition is *meaningless* in a neighbourhood of *closed* points of characteristic 0. However, such a formulation of the definition allows us to simplify some of the statements later on.

The scheme X proper over $Z = \operatorname{Spec} R$ having all closed points of positive residue characteristics is globally +-regular if and only if it is completely relatively +-regular over R ([BMPSTWW20, Corollary 6.9]). However, this is not known for pure global +-regularity (unless $H^0(X, \mathcal{O}_X) = R$ and $-(K_X + S + B)$ is big and semiample; see [BMPSTWW20, Corollary 7.6]), and so for coherence of notation, we shall always add the prefix *completely* when talking about any of these notions in the relative setting.

We also point out to the reader that, with notation as above, if X is proper over a complete Noetherian local domain R of positive residue characteristic, then global +-regularity and pure global +-regularity of (X, B) and $(X, S + B)$, respectively, are equivalent to $B^0(X, B; \mathcal{O}_X) = H^0(X, \mathcal{O}_X)$ and $B^0_S(X, S + B; \mathcal{O}_X) = H^0(X, \mathcal{O}_X)$ by [BMPSTWW20, Proposition 6.8 and Proposition 6.26].

Lemma 2.7. *Let X and X' be two normal varieties defined over an F-finite field of characteristic $p > 0$, and let $\phi : X \dashrightarrow X'$ be a birational map which is an isomorphism at any codimension one point. Let $B \geq 0$ be a $\mathbb{Q}$-divisor on X and $B' = \phi_* B$ the corresponding $\mathbb{Q}$-divisor on X'. Then (X, B) is globally F-split (resp. globally F-regular or purely globally F-regular) if and only if (X', B') is globally F-split (resp. globally F-regular or purely globally F-regular).*

The same result holds for global +-regularity and global pure +-regularity when X and X' are as in Definition 2.5.

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Proof. This follows from the fact that a splitting of structure sheaves for a finite map of normal varieties may be checked on an open subset with complement of codimension at least two. In positive characteristic, this lemma has been observed in [SS10, Proof of Proposition 6.3]. In mixed characteristic, the proof is analogous to [BMPSTWW20, Proposition 6.18]. □

Proposition 2.8. *Suppose that $(X, S+B)$ is a purely F-regular pair over an F-finite field of characteristic $p > 0$ (or a purely +-regular pair as in Definition 2.5), where $\lfloor S+B \rfloor = S$ is a prime divisor. Then S is normal.*

Proof. In positive characteristic, see [MSTWW22, Theorem A] or the proof of [Das15, Theorem A]. In mixed characteristic, this is [BMPSTWW20, Corollary 7.9]. □

Further, recall the following result known as inversion of F-adjunction (+-adjunction, resp.).

Lemma 2.9. *Let $(X, S+B)$ be a plt pair, where $S = \lfloor S+B \rfloor$ is a prime divisor and $B \geq 0$ has no common components with S, and let $f : X \to Z$ be a projective birational morphism of normal varieties over an F-finite field of characteristic $p > 0$ (resp. over a DVR of characteristic $(0, p)$ for $p > 0$). Assume that $-(K_X + S + B)$ is f-ample and write $K_{S^\nu} + B_{S^\nu} = (K_X + S + B)|_{S^\nu}$ for the normalisation $S^\nu \to S$.*

Then $(X, S + B)$ is relatively purely F-regular (resp. completely relatively purely +-regular) over a neighbourhood of $f(S) \subset Z$ if and only if (S^ν, B_{S^ν}) is relatively F-regular (resp. completely relatively +-regular) over $f(S)$. Under these equivalent assumptions, S is normal.

Proof. In positive characteristic, this follows by the same proof as [HW19, Lemma 2.10]. It uses the equality between the different and the F-different. For this, note that [Das15, Theorem B] assumes that the base field is algebraically closed, but the proof goes through for every F-finite base field.

In mixed characteristic, this is [BMPSTWW20, Corollary 7.5] (after completing or invoking [BMPSTWW20, Corollary 7.6]).

Finally, the normality of S around positive characteristic closed points follows from Proposition 2.8, and around characteristic 0 closed points by standard results ([KM98, Theorem 5.50]). □

We also have the following immediate consequence of [Sch14, 5.3].

Proposition 2.10. *Let $f : X \to U$ be a projective morphism of normal varieties over an F-finite field of positive characteristic, where U is affine, L a $\mathbb{Q}$-Cartier Weil divisor and $(X, S + B)$ a log pair such that*

(1) *$\lfloor S + B \rfloor = S$ is normal and integral;*
(2) *p does not divide the index of B;*
(3) *$L - (K_X + S + B)$ is f-ample;*
(4) *L is Cartier on an open neighbourhood of $S \setminus Z$, where $Z \subseteq S$ is a closed subset of codimension at least two;*
(5) *X is $\mathbb{Q}$-factorial, strongly F-regular and of dimension at least three.*

Then $S^0(X, S + B; L) \to S^0(S, B_S; L|_S)$ is surjective, where $(K_X + S + B)|_S = K_S + B_S$.

Since $Z \subseteq S$ is a closed subset of codimension at least two, $L|_S$ is well defined as a Weil divisor. If L is Cartier, then the assumption that X is $\mathbb{Q}$-factorial, strongly F-regular and of dimension at least three is not needed (cf. [Sch14, 5.3]).

Proof. This follows easily from the following commutative diagram, which exists as the different is equal to the F-different

$$\begin{array}{ccc} F^e_* \mathcal{O}_X((1-p^e)(K_X + S + B) + p^e L) & \longrightarrow & F^e_* \mathcal{O}_S((1-p^e)(K_S + B_S) + p^e L|_S) \\ \downarrow & & \downarrow \\ \mathcal{O}_X(L) & \longrightarrow & \mathcal{O}_S(L|_S). \end{array}$$

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Here, we use that $(X, S+B)$ is $\mathbb{Q}$-factorial, strongly F-regular and of dimension at least three to guarantee that divisorial sheaves on X are S_3 (cf. [HW19, Proposition 2.7]), which implies that the upper horizontal arrow is a surjection of sheaves (cf. [HW19, Lemma 2.4]). This is turn gives the surjection on global sections for $e \gg 0$ by Serre's vanishing. □

In the case of B^0, the assumption on +-regularity of singularities when L is not Cartier is not needed. This discrepancy stems from X^+ being cohomologically Cohen–Macaulay in positive and mixed characteristic (informally speaking, the 'non-Cohen–Macauliness' is not necessarily killed by Frobenii but is always killed by finite covers).

Proposition 2.11. *Let X be a projective normal integral scheme over a complete local Noetherian domain $(R, \mathfrak{m})$ of characteristic $(0, p)$ for $p > 0$. Let L be a $\mathbb{Q}$-Cartier Weil divisor, and let $(X, S+B)$ a log pair such that*

(1) *$\lfloor S+B \rfloor = S$ is normal and integral,*
(2) *$L - (K_X + S + B)$ is f-ample,*
(3) *L is Cartier on an open neighbourhood of $S \setminus Z$, where $Z \subseteq S$ is a closed subset of codimension at least two.*

Then $B^0_S(X, S+B; L) \to B^0(S, B_S; L|_S)$ is surjective, where $(K_X + S + B)|_S = K_S + B_S$.

The assumption that S is connected is unnecessary, but we kept it for simplicity as in our applications the connectedness will be preserved under completion. In general, however, this is not always the case.

Proof. The proof is the same as in [BMPSTWW20, Theorem 7.2]. We recall it for the convenience of the reader. We refer to [BMPSTWW20, Subsection 2.1] for a primer on Matlis duality.

Set $\mathcal{N} = \mathcal{O}_{X^+}(N)$ for $N = \pi^*(K_X + S + B - L)$, where $\pi\colon X^+ \to X$ is the natural map. Note that $\mathcal{N}$ is a line bundle as every $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor is rendered Cartier by some finite cover. Let S^+ be the chosen prime divisor on X^+ such that $\pi(S^+) = S$, and consider the following diagram wherein the left square exists as B is effective:

$$\begin{array}{ccccccccc}
0 & \to & \mathcal{O}_X(K_X - L) & \to & \mathcal{O}_X(K_X + S - L) & \to & \mathcal{O}_X(K_X + S - L) \,/\, \mathcal{O}_X(K_X - L) & \to & 0 \\
 & & \downarrow & & \downarrow & & \vdots & & \\
0 & \to & \pi_*\mathcal{O}_{X^+}(N - S^+) & \to & \pi_*\mathcal{O}_{X^+}(N) & \to & \pi_*(\mathcal{O}_{S^+} \otimes \mathcal{N}) & \to & 0.
\end{array}$$

Here, the right vertical dashed arrow exists by an easy diagram chase. Moreover, this arrow factorises through the S_2-fification $\mathcal{O}_S(K_S - L|_S)$ of the upper term as S^+ is normal, and so $\mathcal{O}_{S^+} \otimes \mathcal{N}$ is S_2. Now, apply local cohomology with $d = \dim X$ to get:

$$\begin{array}{ccc}
H^{d-1}R\Gamma_{\mathfrak{m}}R\Gamma(S, \mathcal{O}_X(K_X + S - L) \,/\, \mathcal{O}_X(K_X - L)) & \to & H^d R\Gamma_{\mathfrak{m}}R\Gamma(X, \mathcal{O}_X(K_X - L)) \\
\downarrow\!\!\downarrow & & \\
H^{d-1}R\Gamma_{\mathfrak{m}}R\Gamma(S, \mathcal{O}_S(K_S - L|_S)) & & \downarrow (**) \\
\downarrow (*) & & \\
H^{d-1}R\Gamma_{\mathfrak{m}}R\Gamma(S^+, \mathcal{O}_{S^+} \otimes \mathcal{N}) & \hookrightarrow & H^d R\Gamma_{\mathfrak{m}}R\Gamma(X^+, \mathcal{O}_{X^+}(N - S^+)).
\end{array}$$

Here, the upper left vertical arrow is surjective as the cokernel of

$$\mathcal{O}_X(K_X + S - L) \,/\, \mathcal{O}_X(K_X - L) \to \mathcal{O}_S(K_S - L|_S)$$

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has codimension at least 1 in S, and so the top local cohomology ($H^{d-1}R\Gamma_{\mathfrak{m}}R\Gamma$) thereof vanishes. The bottom horizontal arrow is injective as

$$H^{d-1}R\Gamma_{\mathfrak{m}}R\Gamma(X^+, \mathcal{O}_{X^+}(N)) = 0$$

by [BMPSTWW20, Corollary 3.7].

The upshot is that the image of $(*)$ injects into the image of $(**)$. Note that

$$\mathcal{O}_{S^+} \otimes \mathcal{N} \simeq \mathcal{O}_{S^+}((\pi|_{S^+})^*(K_S + B_S - L|_S))$$

by definition of the different (cf. the second part of [MSTWW22, Subsection 2.1]), and so by taking Matlis duality we get that

$$B^0_S(X, S + B; L) \to B^0(S, B_S; L|_S)$$

is surjective. Here, the left-hand side is Matlis dual to the image of $(**)$ by [BMPSTWW20, Definition 4.21 and Lemma 4.24], while the right-hand side is Matlis dual to the image of $(*)$ by [BMPSTWW20, Lemma 4.8]. □

2.2. *Special termination*

In this section, X is a normal variety defined over a field k of characteristic $p > 0$ or a DVR R of characteristic $(0, p)$ for $p > 0$. Recall the following result known as special termination [Fuj07, Theorem 4.2.1].

Theorem 2.12. *Assume that the log MMP for $\mathbb{Q}$-factorial dlt pairs holds in dimension $\leq n-1$ (including the termination of all flips). Let X be a normal $\mathbb{Q}$-factorial n-dimensional variety, let B be an effective $\mathbb{R}$-divisor such that (X, B) is dlt and let*

$$(X, B) \dashrightarrow (X_1, B_1) \dashrightarrow (X_2, B_2) \dashrightarrow \dots$$

be a sequence of $(K_X + B)$-flips. Then after finitely many steps, the flipping loci are disjoint from $\lfloor B_i \rfloor$.

Since the MMP for surfaces is known in full generality, the above result implies special termination for threefolds and in some special cases for fourfolds over perfect F-finite fields of characteristic $p > 0$. Note that it is not true that in positive characteristic log canonical centres are normal. However, by Lemma 2.2, we know that if $W^\nu \to W$ is the normalisation of a log canonical centre of a dlt pair (X, B), then the induced pair $K_{W^\nu} + B_{W^\nu} = (K_X + B)|_{W^\nu}$ is also dlt, its log canonical centres are in bijection with the log canonical centres of (X, B) contained in W and the coefficients of B_{W^ν} are the same as those given by the usual adjunction in characteristic 0.

Theorem 2.13. *The statement of Theorem 2.12 holds in dimension three over all F-finite fields k of characteristic $p > 5$ or DVRs R of characteristic $(0, p)$ for $p > 5$.*

Proof. The three-dimensional case of Theorem 2.12 is an immediate consequence of the proof of [Fuj07, Theorem 4.2.1] and the two-dimensional MMP [Tan18]. □

Theorem 2.14. *Let (X, B) be a $\mathbb{Q}$-factorial four-dimensional dlt pair defined over a field of characteristic $p \geq 0$ or a DVR of characteristic $(0, p)$ for $p > 0$, where B is an effective $\mathbb{R}$-divisor. Let $\pi\colon X \to U$ be a projective morphism, and suppose that*

$$(X, B) =: (X_0, B_0) \dashrightarrow (X_1, B_1) \dashrightarrow (X_2, B_2) \dashrightarrow \dots$$

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is a sequence of $(K_X + B)$-flips and divisorial contractions over U. Then after finitely many steps, the intersections of the flipping and flipped loci with the non-klt locus are at most one dimensional. In particular, the flipping and flipped loci for $X_i \dashrightarrow X_{i+1}$ cannot be both contained in $\lfloor B_i \rfloor$ and $\lfloor B_{i+1} \rfloor$.

Moreover, if $\dim U = 1$, $\kappa(K_X + B/U) \geq 0$ and $\lfloor B \rfloor$ contains the fibre X_u for some point $u \in U$, then after finitely many steps, the flipping loci are disjoint from X_u.

Proof. Since divisorial contractions decrease the Picard rank, we may assume that the above sequence consists only of flips. Denote the flips by $\phi_i \colon X_i \dashrightarrow X_{i+1}$, and denote the flipping and the flipped contractions by $f_i \colon X_i \to Z_i$ and $f_i^+ \colon X_{i+1} \to Z_i$, respectively. Let S be an irreducible component of $\lfloor B \rfloor$, let S_i be its strict transform on X_i and let $T_i := f_i(S_i)$. Note that the normalisation of S_i need not be $\mathbb{Q}$-factorial.

By [Fuj07, Proposition 4.2.14], we may assume that $f_i^+|_{S_{i+1}} \colon S_{i+1} \to T_i$ is small. Moreover, if $f_i|_{S_i} \colon S_i \to T_i$ is divisorial, then the Picard rank as defined in [AHK07, Lemma 1.6] satisfies $\rho(S_i^\nu/U) > \rho(S_{i+1}^\nu/U)$, where S_i^ν and S_{i+1}^ν are appropriate normalisations. Thus, again we may assume $f_i|_{S_i}$ is small, that is, $\phi_i|_{S_i} \colon S_i \dashrightarrow S_{i+1}$ is of type (SS). The sum of the dimensions of the flipping and the flipped locus of $X_i \dashrightarrow X_{i+1}$ is at least three, so one of them must be two-dimensional (in fact, they must be of type $(2,1)$, $(2,2)$ or $(1,2)$ [KMM87, Lemma 5.1.17] and [AHK07, Lemma 1.2]), and so they cannot be both contained in the non-klt locus.

Now, assume that $\dim U = 1$, $\kappa(K_X + B/U) \geq 0$, and $\lfloor B \rfloor$ contains the support of the fibre X_u for some point $u \in U$. As observed above, after finitely many steps, $\lfloor B_i \rfloor$ contains no two-dimensional component of the flipping or flipped loci. Suppose that the flipping locus (and thus also the flipped locus) is contained in the fibre over u and hence in the non-klt locus. Since either the flipping or flipped locus is two dimensional, this can not happen infinitely often. Therefore, we may assume that for $i \gg 0$ the flipping and flipped loci dominate U.

Therefore, we can replace (X, B) and (X_i, B_i) by (X_η, B_η) and $((X_i)_\eta, (B_i)_\eta)$, respectively, so that these are three-dimensional $\mathbb{Q}$-factorial pairs defined over a field of characteristic 0 or an F-finite field $k(\eta)$ for the generic point $\eta \in U$. In particular, we can assume that log resolutions and terminalisations exist (see [DW19b] when η is of positive characteristic and [BCHM10] when it is of characteristic 0).

First, we will show that $\lfloor B_i \rfloor$ is disjoint from the flipping, and so the flipped, locus. By what we have proven above, we may also assume that the intersection of the non-klt locus with the flipping and the flipped locus is at most zero dimensional. Now, suppose that a flipping curve C intersects $\lfloor B_i \rfloor$. Then $C \cdot \lfloor B_i \rfloor > 0$, and so the flipped locus is contained in $\lfloor B_{i+1} \rfloor$ which is impossible. Now, replacing B_i by $\{B_i\}$, we may assume that (X_i, B_i) are klt, and so the MMP should terminate by Proposition 2.15 for $U = \operatorname{Spec} k(\eta)$.

However, the above replacement may render the assumption $\kappa(K_X + B) \geq 0$ invalid, so we need to be more careful. Precisely, write $K_X + B \sim_{\mathbb{Q}} M \geq 0$, where the supports of the strict transforms M_i on X_i contain the flipping loci for all i. Since the flipping loci are disjoint from $\lfloor B_i \rfloor$, the above $(K_X + B)$-MMP (equivalently, a $(K_X + B + \epsilon M)$-MMP), is also a $(K_X + \{B\} + \epsilon M)$-MMP for $0 < \epsilon \ll 1$. Since the flipping loci are contained in $\operatorname{Supp}(\{B\} + \epsilon M)$, this MMP terminates by Proposition 2.15. □

Proposition 2.15. *Let (X, B) be a $\mathbb{Q}$-factorial klt log pair of dimension n defined over a field of characteristic $p \geq 0$, where B is an effective $\mathbb{R}$-divisor. Let $\pi \colon X \to U$ be a projective morphism and suppose that*

$$(X, B) =: (X_0, B_0) \dashrightarrow (X_1, B_1) \dashrightarrow (X_2, B_2) \dashrightarrow \dots$$

is a sequence of $(K_X + B)$-flips over U.

Assume that log resolutions of all n-dimensional log pairs with the underlying variety being birational to X exist and so do terminalisations for n-dimensional $\mathbb{Q}$-factorial klt pairs. Then it can happen only finitely many times that the flipping or the flipped locus has a component of codimension two in X_n and which is contained in $\operatorname{Supp} B_n$. In particular, if $\kappa(K_X + B/U) \geq 0$, then the flipping locus is of codimension at least three for $n \gg 0$.

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This result holds in a more general setting (that is, as in [Kol13] or [BMPSTWW20]).

Proof. This follows by the same argument as [AHK07, Theorem 2.15]. For the convenience of the reader, we recall the argument briefly. Define the following functions $w^-_\alpha, w^+_\alpha, W^-_\alpha, W^+_\alpha \colon (-\infty, 1) \to \mathbb{R}_{\geq 0}$ for $\alpha \in (0,1)$ by

$$\begin{aligned} w^-_\alpha(x) &= 1-x \text{ for } x \leq \alpha, \text{ and } w^-_\alpha(x) = 0 \text{ otherwise,} \\ w^+_\alpha(x) &= 1-x \text{ for } x < \alpha, \text{ and } w^+_\alpha(x) = 0 \text{ otherwise,} \\ W^-_\alpha(b) &= \textstyle\sum_{k=1}^{\infty} w^-_\alpha(k(1-b)), \\ W^+_\alpha(b) &= \textstyle\sum_{k=1}^{\infty} w^+_\alpha(k(1-b)). \end{aligned}$$

For any subterminal log pair (Y, B_Y) and $B_Y = \sum b_i B_i$ with B_i irreducible, we define the difficulty ([AHK07, Definition 2.3])

$$d^+_\alpha(Y, B_Y) = \sum_{b_i \leq 0} W^+_\alpha(b_i)\rho(\tilde{B}_i) + \sum_v w^+_\alpha(a_v) - \sum_{\tilde{C}_{i,j}} W^+_\alpha(b_i), \text{ where}$$

- ρ is as in [AHK07, Lemma 1.6] (in particular, stable under small birational maps),
- v is taken over all valuations which are not echoes on Y (here, a_v denotes the discrepancy of (Y, B_Y) at v) and
- $\tilde{C}_{i,j}$ are codimension one irreducible components in $\tilde{B}_i$ of the inverse image of a codimension two integral subscheme $C \subseteq Y$ under the normalisation $\bigsqcup \tilde{B}_i \to \bigcup B_i$. Here, we exclude those C which are centres of echoes.

We say that an irreducible $C \subseteq Y$ is a *centre of an echo* if it is of codimension two, contained in exactly one B_i, but not in $\operatorname{Sing} B_i$. Here, a valuation is called a (*k-th*) *echo* for such C if it corresponds to the last exceptional divisor of a sequence of k blowups at strict transforms of C (see [AHK07, Example 1.4]). The discrepancy of the k-th echo is $k(1-b_i)$. The difficulty is well defined and finite because there are only finitely many valuations with discrepancy in $(-1, 0]$ and also finitely many in $(0, 1)$ if we exclude echoes ([AHK07, Lemma 1.5]; here, we use log resolutions).

We can similarly define d^-_α. The difficulties are stable under log pullbacks ([AHK07, Lemma 2.7]), and so we extend these definitions to arbitrary $\mathbb{Q}$-factorial klt pairs by taking the difficulty of a terminalisation ([AHK07, Definition 2.8]). The difficulties are nonnegative for klt pairs ([AHK07, Lemma 2.9]) and decreasing for a sequence of flips $(X_n, B_n) \dashrightarrow (X_{n+1}, B_{n+1})$ if we start with $n \gg 0$ ([AHK07, Theorem 2.12]).

We are ready to give the proof of the proposition. First, we show that there are only finitely many flips with flipping or flipped loci admitting a component C of codimension two and contained in $\operatorname{Sing} X_n$ (this is [AHK07, Lemma 2.14] with the use of Bertini replaced by localisation at C). By taking $n \gg 0$, we can assume that the places with discrepancies smaller or equal to 0 are stable under flips: Let $E_1, \ldots, E_m$ be such places over X_n with discrepancies $a_1, \ldots, a_m \leq 0$. In particular, the minimal discrepancy of the two-dimensional singularity obtained by localisation at C must be equal to a_i for some i. It is known by experts that minimal log discrepancies (mlds) for klt surface singularities satisfy the ascending chain condition, and since flips increase discrepancies, the statement follows. As this result is unpublished, we refer to Lemma 2.16 instead.

We are left to show that there are only finitely many flips with flipping or flipped loci admitting a component C of codimension two and contained in $\operatorname{Supp} B_n$ but not $\operatorname{Sing} X_n$. The blowup along C produces a divisor with discrepancy $1 - \sum m_i b_i$ for $m_i \in \mathbb{Z}_{\geq 0}$. There are only finitely many such discrepancies, and, thus, it is enough to prove the following claim: Given $\alpha \in (-1, 1)$, there cannot be infinitely many flips for which there exists a valuation v with $a_v = \alpha$ and centre contained in the flipping or the flipped locus ([AHK07, Theorem 2.13]). The case $\alpha \in (-1, 0]$ follows by finiteness, while for $\alpha \in (0, 1)$ we notice that d^-_α drops by at least $1 - \alpha$ if the flipping locus admits such a valuation and so

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does d_α^+ for the flipped locus. Since the difficulties are positive and finite, this can only happen finitely many times.

The last assertions follows by replacing B by $B + \epsilon M$ for $0 \leq M \sim_{\mathbb{Q},U} K_X + B$ and $0 < \epsilon \ll 1$ since then every flipping curve Σ satisfies $\Sigma \cdot M < 0$, and so is contained in M. □

Lemma 2.16. *Fix $m \in \mathbb{N}$ and $\epsilon > 0$. Consider all ϵ-lc excellent two-dimensional pairs (S, B) with the minimal log resolution having at most m distinct discrepancies. Then the set of all these discrepancies for such surfaces is finite.*

This result holds in a more general setting (that is, as in [Kol13] or [BMPSTWW20]).

Proof. Let $f\colon S' \to S$ be the minimal resolution at a point $s \in S$ with k exceptional curves C_i (which are $k(s)$-schemes), and let Γ be the corresponding graph. Write $K_{S'} + B_{S'} = f^*(K_S + B_S)$, and let a_i be the log discrepancy of C_i in $(S', B_{S'})$. We will use the results of [Kol13]. Set $r_i := \dim_{k(s)} H^0(C_i, \mathcal{O}_{C_i})$. Then $C_i^2 = -r_i c_i$ for $c_i \in \mathbb{Z}$. Note that C_i are conics over the field $H^0(C_i, \mathcal{O}_{C_i})$ and in particular $\deg \omega_{C_i} = -2r_i$ ([Kol13, Reduction 3.30.2]). By adjunction,

$$a_i C_i^2 = (K_{S'} + B_{S'} + a_i C_i) \cdot C_i \geq -2r_i,$$

and so $-c_i \leq \frac{2}{a_i}$ is bounded. By classification ([Kol13, 3.31 and 3.41]), $r_i \leq 4$, and so $-C_i^2$ is bounded. Further, $C_i \cdot C_j$ is bounded as well for every $i \neq j$ (see [Kol13, 3.41]).

Recall that Γ is a tree with at most one fork and three legs ([Kol13, 3.31]), there are no K_X-negative curves and the convexity of discrepancies holds ([Kol13, Proposition 2.37]), that is, given three consecutive curves C_1, C_2 and C_3 on a leg, we have $2a_2 \leq a_1 + a_3$. In fact, it follows easily from the same proof that if $2a_2 = a_1 + a_3$, then $c_2 = -2$, $f_*^{-1}B \cdot C_2 = 0$ and $C_2 \cdot C_1 = C_2 \cdot C_3 = r_2$.

By the negativity lemma, the k equations

$$C_i^2 = -2r_i + B_{S'} \cdot C_i$$

are linearly independent and uniquely determine the discrepancies. Given a maximal sequence of successive curves $C_1, \ldots, C_r$ on one leg with the same discrepancies a_i and parameters r_i, we remove the equations corresponding to $C_2, \ldots, C_{r-1}$ and replace all occurrences of $a_2, \ldots, a_r$ by a_1. This operation does not change the set of solutions (indeed, this is equivalent to adding equations $a_1 = a_2 = \ldots = a_r$ which does not change the set of solutions but renders the equations corresponding to $C_2, \ldots, C_{r-1}$ trivial), and so, eventually, given the shape of Γ ([Kol13, 3.31]), we are left with bounded-in-m number of equations with bounded coefficients and in the same or lower number of variables. Therefore, the set of discrepancies is bounded in terms of m as well. □

Remark 2.17. Special termination of sequences of the fourfold MMP with scaling now holds by [XX21a, Remark 2.17].

3. Extending sections

In this section, we prove an extension result for purely F-regular (and purely +-regular) pairs. In particular, the following proposition implies Theorem 1.6. Our original arXiv submission covered the positive characteristic case only. Its generalisation to mixed characteristic was then obtained in [TY20, Proposition 3.28]. For the convenience of the reader, we append the proof of the mixed characteristic case at the end of the argument using the techniques of [BMPSTWW20].

Proposition 3.1. *Let $(X, S + A + B)$ be a $\mathbb{Q}$-factorial dlt pair of dimension at least three defined over an F-finite field of characteristic $p > 0$ (resp. over a DVR of characteristic $(0, p)$ for $p > 0$), where $\lfloor S + A + B \rfloor = S + A$, the $\mathbb{Q}$-Cartier Weil divisor A is ample, and the Weil divisor S is irreducible. Further, let $f\colon X \to Z$ be a contraction with Z an affine variety such that $(X, S + (1 - \epsilon)A + B)$ is*

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relatively purely F-regular (resp. completely relatively purely +-regular) over Z for any $\epsilon > 0$. Write $K_S + B_S = (K_X + S + B)|_S$ and $A_S = A|_S$.

In the mixed characteristic case, we also assume that $-(K_X + S + B + A)$ is relatively ample and that closed points of Z are of positive residue characteristic. Then

$$H^0(X, \mathcal{O}_X(k(K_X + S + A + B))) \to H^0(S, \mathcal{O}_S(k(K_S + A_S + B_S)))$$

is surjective for every $k \geq 1$ such that $k(K_X + S + B + A)$ is Cartier.

Note that S is normal by Lemma 2.9. Also, observe that the assumption on $\mathbb{Q}$-factoriality and the dimension being at least three is unnecessary in mixed characteristic. Further, the statement is in fact true over characteristic 0 closed points by Remark 3.3 or by an analogous argument with S^0 replaced by H^0 using Nadel vanishing.

Proof. We tackle the positive characteristic case first. Let $F \in |k(K_S + B_S + A_S)|$. We will construct divisors $G_m \in |k(K_X + S + B + A) + mA|$ such that $G_m|_S = F + mA_S$ by descending induction. First, we take $M \gg 0$ for which we can construct G_M by Serre vanishing. Now, assume that we constructed G_{m+1} as above. To construct G_m we proceed as follows. Write

$$\begin{aligned} L &= k(K_X + S + B + A) + mA \\ &= K_X + S + B + (k-1)(K_X + S + B + A) + (m+1)A \\ &\sim_{\mathbb{Q}} K_X + S + B + \tfrac{k-1}{k}G_{m+1} + \tfrac{m+1}{k}A. \end{aligned}$$

Since $L - (K_X + S + B + \frac{k-1}{k}G_{m+1})$ is ample, up to a small perturbation of the coefficients, by Proposition 2.10 we have a surjection (the assumption on the non-Cartier locus of A, and hence of L, is satisfied as $(X, S + A + B)$ is dlt)

$$S^0(X, S + B + \tfrac{k-1}{k}G_{m+1}; L) \to S^0(S, B_S + \tfrac{k-1}{k}G_{m+1}|_S; L|_S).$$

Since $(S, B_S + \frac{k-1}{k}A_S)$ is relatively F-regular and

$$F + mA_S \geq \tfrac{k-1}{k}G_{m+1}|_S - \tfrac{k-1}{k}A_S = \tfrac{k-1}{k}(F + mA_S),$$

Lemma 3.2 implies that the section corresponding to $F + mA_S$ lies in the image of the above map. This concludes the construction of G_m, and so we obtain a lift $G_0 \in |k(K_X + S + B + A)|$ of F.

In mixed characteristic, since surjectivity of maps of coherent sheaves may be checked on completions of closed points (see [Sta14, Tag 00MC and Tag 00MA(3)]), we can complete at a closed point $z \in Z$ of positive residue characteristic and assume that $Z = \operatorname{Spec} R$ for a complete Noetherian local domain R (note that this need not preserve $\mathbb{Q}$-factoriality). By [BMPSTWW20, Corollary 7.8] (and [BMPSTWW20, Corollary 7.5]), S stays prime after the base change to the completion (here, we use that $-(K_X + S + B + A)$ is relatively ample). Now the proof follows the positive characteristic case word for word with F-regular replaced by +-regular, Proposition 2.10 replaced by Proposition 2.11 and the surjectivity of S^0 replaced by the surjectivity of

$$B^0_S(X, S + B + \tfrac{k-1}{k}G_{m+1}; L) \to B^0(S, B_S + \tfrac{k-1}{k}G_{m+1}|_S; L|_S).$$ □

Lemma 3.2. *Let (X, B) be a projective globally F-regular pair defined over an affine variety over an F-finite field of characteristic $p > 0$ (resp. a globally +-regular pair defined over a complete local Noetherian domain of characteristic $(0, p)$ for $p > 0$), let L be a Weil divisor and let Γ be an effective $\mathbb{Q}$-divisor. If $g \in H^0(X, \mathcal{O}_X(L))$ corresponds to a divisor $G \in |L|$ such that $G \geq \Gamma$, then $g \in S^0(X, B+\Gamma; L)$ (resp. $g \in B^0(X, B + \Gamma; L)$).*

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Proof. First, we deal with the positive characteristic case. By replacing X by its regular locus, we can assume that it is regular and hence factorial. Further, we perturb B so that its index is not divisible by p. Since (X, B) is globally F-regular, we have a splitting of

$$F_*^e \mathcal{O}_X((1-p^e)(K_X+B)) \to \mathcal{O}_X,$$

and so $H^0(X, \mathcal{O}_X) = S^0(X, B; \mathcal{O}_X)$.

Thus, we have the following diagram

$$\begin{array}{ccc} H^0(X, F_*^e \mathcal{O}_X((1-p^e)(K_X+B))) & \xrightarrow{\cdot g} & H^0(X, F_*^e \mathcal{O}_X((1-p^e)(K_X+B)+p^e L)) \\ \downarrow & & \downarrow \\ H^0(X, \mathcal{O}_X) & \xrightarrow{\cdot g} & H^0(X, \mathcal{O}_X(L)). \end{array}$$

Therefore, g is in the image of the map on global sections induced by

$$H^0(X, F_*^e \mathcal{O}_X((1-p^e)(K_X+B)+p^e(L-G))) \to H^0(X, \mathcal{O}_X(L)). \tag{$\dagger$}$$

Since $G \geq \Gamma$, we have the following sequence of homomorphisms

$$\mathcal{O}_X(-G) \to F_*^e \mathcal{O}_X((p^e-1)(B+\Gamma) - p^e G) \to F_*^e \mathcal{O}_X((p^e-1)B).$$

Applying Grothendieck duality to these maps shows that ($\dagger$) factors through

$$H^0(X, F_*^e \mathcal{O}_X((1-p^e)(K_X+B+\Gamma)+p^e L)) \to H^0(X, \mathcal{O}_X(L)),$$

concluding the proof.

In mixed characteristic, take a finite cover $f\colon Y \to X$ with Y normal. By global +-regularity of (X, B), the trace map

$$\mathrm{Tr}\colon f_* \mathcal{O}_Y(K_Y + \lceil -f^*(K_X+B) \rceil) \to \mathcal{O}_X$$

is split surjective (see the proof of [BMPSTWW20, Proposition 6.8]). Hence, by $B^0(X, B; \mathcal{O}_X) = H^0(X, \mathcal{O}_X)$ (see also [BMPSTWW20, Lemma 6.11]). By the same argument as above, g is in the image of the map on global sections induced by

$$H^0(Y, \mathcal{O}_Y(K_Y + \lceil f^*(L-K_X-B-G) \rceil)) \to H^0(X, \mathcal{O}_X(L)),$$

and as $G \geq \Gamma$, this homomorphism factors through

$$H^0(Y, \mathcal{O}_Y(K_Y + \lceil f^*(L-K_X-B-\Gamma) \rceil)) \to H^0(X, \mathcal{O}_X(L)),$$

concluding the proof. □

Remark 3.3. (C. Xu) Let k be a field of characteristic 0 or any field of positive characteristic, respectively. Suppose that $f\colon X \to Z$ is a flipping contraction and $\phi\colon X \dashrightarrow X^+$ is the flip of a dlt pair $(X, S+A+B)$, where A is an f-ample Weil divisors, S is irreducible and $\lfloor S+A+B \rfloor = S+A$. Write $K_S+B_S = (K_X+S+B)|_S$ and $A_S = A|_S$. Assume for simplicity that S is normal.

We claim that $|k(K_X+S+A+B)|_S = |k(K_S+A_S+B_S)|$ for every integer $k > 1$ such that $k(K_X+S+A+B)$ is integral (resp. any sufficiently divisible integer). This explains why we can obtain such strong liftability results, which are normally false when there is no ample divisor in the boundary (cf. [HMK10]).

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To see the claim, notice that if $S^+ = \phi_* S$, then $S \dashrightarrow S^+$ extracts no divisors. In fact, if $P \subset S^+$ is an extracted divisor, then since $A^+ = \phi_* A$ is $f^+ \colon X^+ \to Z$ antiample (as A is f-ample), it contains P and hence there is a divisor F over X^+ with centre P such that the log-discrepancy $a_F(X^+, S^+ + A^+ + B^+) = 0$ which is impossible, as $(X, S + A + B)$ is dlt and discrepancies improve after flips.

Now, since $K_{X^+} + S^+ + A^+ + B^+$ is f^+-ample, by Kawamata–Viehweg vanishing (resp. Serre vanishing)

$$H^0(X^+, k(K_{X^+} + S^+ + A^+ + B^+)) \to H^0(S^+, k(K_{S^+} + A_{S^+} + B_{S^+}))$$

is surjective. Since ϕ is a small birational morphism, $H^0(X^+, k(K_{X^+} + S^+ + A^+ + B^+)) \cong H^0(X, k(K_X + S + A + B))$ and since $S \dashrightarrow S^+$ is a $(K_S + A_S + B_S)$-nonpositive birational contraction,

$$H^0(S^+, k(K_{S^+} + A_{S^+} + B_{S^+})) \cong H^0(S, k(K_S + A_S + B_S)).$$

4. Relative MMP over $\mathbb{Q}$-factorial fourfolds

In this section, we prove Theorem 1.1 whose statement in full generality is as follows.

Theorem 4.1. *Let (Y, Δ) be a four-dimensional $\mathbb{Q}$-factorial dlt pair defined over a perfect field of characteristic $p > 5$ or a DVR of characteristic $(0, p)$ for $p > 5$ with a perfect residue field. Suppose that Δ has standard coefficients or that log resolutions of all log pairs with the underlying variety being birational to Y exist (and are given by a sequence of blowups along the non-snc locus). Let $\pi \colon Y \to X$ be a projective birational morphism to a normal $\mathbb{Q}$-factorial variety X such that $\mathrm{Ex}(\pi) \subset \lfloor \Delta \rfloor$. Then we may run a $(K_Y + \Delta)$-MMP over X, which terminates with a minimal model.*

4.1. Existence of pl-flips with ample divisor in the boundary

In order to show Theorem 4.3, we need the following lemma.

Lemma 4.2. *Let $(S, C + B)$ be a three-dimensional plt pair with standard coefficients defined over an F-finite field of characteristic $p > 5$ (resp. a DVR of characteristic $(0, p)$ for $p > 5$), where $f \colon S \to T$ is a projective birational morphism, C is a prime divisor with $f|_C \colon C \to f(C)$ birational, and $-(K_S + C + B)$ is an f-ample $\mathbb{Q}$-divisor.*

Then $(S, C + B)$ is relatively purely F-regular (resp. completely relatively purely $+$-regular) over a neighbourhood of $f(C) \subset T$.

Proof. Let $\tilde{C}$ be the normalisation of C, and write $K_{\tilde{C}} + B_{\tilde{C}} = (K_S + C + B)|_{\tilde{C}}$. Then $(\tilde{C}, B_{\tilde{C}})$ is klt with standard coefficients (see Lemma 2.2). Since $\tilde{C} \to f(C)$ is birational, $(\tilde{C}, B_{\tilde{C}})$ is relatively globally F-regular ([DW19a, Theorem 5.1], cf. [HX15, Theorem 3.1], [HW22, Proposition 2.9]) or completely relatively globally $+$-regular ([BMPSTWW20, Theorem 7.14]), respectively. Lemma 2.9 implies that $(S, C + B)$ is relatively purely F-regular (resp. completely relatively purely $+$-regular) over a neighbourhood of $f(C)$. □

Theorem 4.3. *Let (X, Δ) be a four-dimensional $\mathbb{Q}$-factorial dlt pair with standard coefficients defined over an F-finite field k of characteristic $p > 5$ (resp. a DVR of characteristic $(0, p)$ for $p > 5$) and $\phi \colon X \to Z$ a flipping contraction of a $(K_X + \Delta)$-negative extremal ray R with $\rho(X/Z) = 1$. Suppose that there exist irreducible divisors $S, A \subset \lfloor \Delta \rfloor$ such that $R \cdot S < 0$ and $R \cdot A > 0$.*

Then the flip (X^+, Δ^+) exists. Moreover, both X and X^+ are relatively F-regular (resp. completely relatively $+$-regular) over a neighbourhood of the image of the flipping locus in Z.

Proof. The statement is local, and hence we may assume that Z is affine and we work in a neighbourhood of some closed point $z \in Z$ of positive residue characteristic in the image of the flipping locus. Replacing $\lfloor \Delta \rfloor - S - A$ by $(1 - \frac{1}{m})(\lfloor \Delta \rfloor - S - A)$ for some $m \gg 0$, we may assume that $\lfloor \Delta \rfloor = S + A$. Let $K_{S^\nu} + A_{S^\nu} + B_{S^\nu} = (K_X + S + A + B)|_{S^\nu}$ with $A_{S^\nu} = A|_{S^\nu}$ and S^ν being the normalisation of S. Then $(S^\nu, A_{S^\nu} + B_{S^\nu})$ is a plt threefold with standard coefficients. Since A_{S^ν} is relatively ample, it is not

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exceptional, and so it is birational over its image in Z. As $-(K_{S^\nu} + A_{S^\nu} + B_{S^\nu})$ is relatively ample, Lemma 4.2 implies that $(S^\nu, A_{S^\nu} + B_{S^\nu})$ is purely F-regular (resp. completely purely +-regular) over a neighbourhood of $f(A_{S^\nu})$. By Lemma 2.9, $(X, S + (1-\epsilon)A + B)$ is purely F-regular (resp. completely purely +-regular) over a neighbourhood of $f(A_S)$ for any $\epsilon > 0$. Since every flipping curve is contained in S and intersects A, $(X, S + (1-\epsilon)A + B)$ is purely F-regular (resp. completely purely +-regular) over a neighbourhood of the image of the flipping locus. In particular, S is normal (Lemma 2.9).

By Shokurov's reduction to pl-flips (see, e.g., [Cor07, Lemma 2.3.6]), it suffices to show that the restricted algebra

$$R_S(K_X + S + A + B) := \operatorname{Im}\left(R(K_X + S + A + B) \to R(K_S + A_S + B_S)\right)$$

is finitely generated. By Proposition 3.1 in fact $R_S(K_X + S + A + B) = R(K_S + A_S + B_S)$ (in divisible enough degrees). Since $(S, (1-\epsilon)A_S + B_S)$ is a klt threefold for $0 < \epsilon \leq 1$, the ring $R(K_S + A_S + B_S)$ is finitely generated (see [DW19b, Theorem 1.4 and 1.6] when S is purely positive characteristic and [BMPSTWW20, Theorem F and G] when S is of mixed characteristic).

If $\phi\colon X \dashrightarrow X^+$ is the corresponding flip, then since the flipping contraction $f\colon X \to Z$ is relatively F-regular (resp. completely relatively +-regular) over a neighbourhood of the image of the flipping locus and ϕ is an isomorphism in codimension two, the flipped contraction $f^+\colon X^+ \to Z$ is also relatively F-regular (resp. completely relatively +-regular); see Lemma 2.7. □

Remark 4.4. The same argument shows that Lemma 4.2 holds in dimension two for every F-finite field of characteristic $p > 0$ without the assumption that B has standard coefficients, and so Theorem 4.3 holds in dimension three for $p > 0$ without the assumption that Δ has standard coefficients (the required result on the canonical ring being finitely generated in dimension two follows from [Tan18, Theorem 1.1 and 4.2]). This gives a direct proof of the existence of flips needed in [HW21] without referring to [HX15].

4.2. Proof of Theorem 4.1

We start by constructing contractions.

Proposition 4.5. *Let $(X, S+B)$ be a $\mathbb{Q}$-factorial dlt fourfold defined over perfect field of characteristic $p > 5$ (or a DVR of characteristic $(0,p)$ for $p > 5$ with perfect residue field) and equipped with a projective morphism $\pi\colon X \to U$ to a quasi-projective variety. Let Σ be a $(K_X + S + B)$-negative extremal ray over U such that $S \cdot \Sigma < 0$. Further, suppose the Mori cone $\overline{\mathrm{NE}}(X/U)$ is generated by $\overline{\mathrm{NE}}(X/U)_{K_X+S+B\geq 0}$ and countably many extremal rays which do not accumulate in $\overline{\mathrm{NE}}(X/U)_{K_X+S+B<0}$. Then the contraction $f\colon X \to Z$ of Σ exists so that f is a projective morphism with $\rho(X/Z) = 1$.*

Proof. Perturbing the coefficients, we may assume that $(X, S+B)$ is plt. We may pick an ample over U $\mathbb{Q}$-divisor H such that $L = K_X + S + B + H$ is nef over U, and $L^\perp$ is spanned by Σ. Let A be an ample $\mathbb{Q}$-divisor such that $(S + A) \cdot \Sigma = 0$, then for any $0 < \epsilon \ll 1$ we may pick an ample $\mathbb{Q}$-divisor $H_\epsilon \sim_{\mathbb{Q}} H + \epsilon(S + A)$ such that $L_\epsilon = K_X + S + B + H_\epsilon$ is nef over U, $(L_\epsilon)^\perp = \mathbb{R}[\Sigma]$. Let $\mathbb{E}(L_\epsilon)$ be the exceptional locus of the nef line bundle L_ϵ given by the union of the integral subvarieties $V \subset X$ such that $(L_\epsilon|_V)$ is not relatively big (cf. [kee99, Definition 0.1] and [Wit22, Lemma 2.2]). Then $\mathbb{E}(L_\epsilon) \subset S$. To verify the last inclusion, notice that if $V \subset X$ is a subvariety not contained in S, then $L_\epsilon|_V = (L + \epsilon(S + A))|_V$ is nef and big over U, and so $L_\epsilon^{\dim V} \cdot V > 0$. Replacing L by L_ϵ, we may assume that $\mathbb{E}(L) \subset S$.

By adjunction, (S^ν, B_{S^ν}) is klt, where $K_{S^\nu} + B_{S^\nu} = (K_X + S + B)|_{S^\nu}$, the map $S^\nu \to S$ is the normalisation, and $H_{S^\nu} = H|_{S^\nu}$. By [HNT20, Theorem 1.4(1)] and [BMPSTWW20, Theorem G], $L|_{S^\nu}$ is semiample over U (note that in the mixed characteristic case, by [HNT20, Theorem 1.4(1)], we may assume that $\dim \pi(S) \geq 1$, and so [BMPSTWW20, Theorem G] applies). Since S is normal up to universal morphism, $L|_S$ is semiample by [kee99, Lemma 1.4] and Theorem 2.3. Since $\mathbb{E}(L) \subseteq S$,

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[kee99, Theorem 0.2] and [Wit22, Theorem 1.2] imply that L is semiample over U, and thus it induces a projective morphism $f\colon X \to Z$ contracting Σ and such that $f_*\mathcal{O}_X = \mathcal{O}_Z$ and $\rho(X/Z) = 1$. □

Theorem 4.6. *Let (Y, Δ) be a four-dimensional $\mathbb{Q}$-factorial dlt pair with standard coefficients defined over a perfect field of characteristic $p > 5$ (or a DVR of characteristic $(0, p)$ for $p > 5$ with perfect residue field). Assume that there exists a projective birational morphism $\pi\colon Y \to X$ over a normal $\mathbb{Q}$-factorial variety X such that $\mathrm{Ex}(\pi) \subset \lfloor \Delta \rfloor$. Then π-relative divisorial and flipping contractions and flips exist for $K_Y + \Delta$, and we can run a $(K_Y + \Delta)$-MMP over X which terminates with a minimal model $\phi\colon Y \dashrightarrow Y'$.*

Proof. If $K_Y + \Delta$ is π-nef, then we are done. Otherwise, we run a $(K_Y + \Delta)$-MMP over X. Let $\phi_l\colon Y = Y_0 \dashrightarrow Y_1 \dashrightarrow \ldots \dashrightarrow Y_l$ be a sequence of flips and contractions, and let Δ_l be the strict transforms of Δ on Y_l. Since X is $\mathbb{Q}$-factorial, there exists an effective exceptional divisor E on Y_l such that $-E$ is ample over X. In particular, $\lfloor \Delta_l \rfloor$ contains all $\pi_l\colon Y_l \to X$ exceptional curves. We must show that we can continue the MMP. The termination will then follow by Theorem 2.14 as both the flipping and flipped locus must be contained in $\mathrm{Ex}(\pi) \subseteq \lfloor \Delta \rfloor$. Note that if $K_{Y_l} + \Delta_l$ is nef over X, then Y_l is the required minimal model. Therefore, we may assume that $K_{Y_l} + \Delta_l$ is not nef over X.

We start by establishing the cone theorem. Let S_i be all the π_l-exceptional divisors for $1 \leq i \leq r$. Restricting to the normalisation S_i^ν of S_i, we get that $(S_i^\nu, \Delta_{S_i^\nu})$ is dlt, where $K_{S_i^\nu} + \Delta_{S_i^\nu} = (K_{Y_l} + \Delta_l)|_{S_i^\nu}$. The three-dimensional cone theorem ([DW19b, Theorem 1.1] and [BMPSTWW20, Theorem H]) states that

$$\mathrm{NE}(S_i^\nu/X) = \mathrm{NE}(S_i^\nu/X)_{K_{S_i^\nu}+\Delta_{S_i^\nu}\geq 0} + \sum_{j\geq 1} \mathbb{R}_{\geq 0}[\overline{\Gamma}_{i,j}],$$

where $0 > (K_{S_i^\nu} + \Delta_{S_i^\nu}) \cdot \overline{\Gamma}_{i,j} = (K_{Y_l} + \Delta_l) \cdot \Gamma_{i,j}$. Here $\{\overline{\Gamma}_{i,j}\}_{j\geq 1}$ is a countable collection of curves on S_i^ν and $\Gamma_{i,j}$ denote their images on $S_i \subseteq Y_l$. Since $\mathrm{Ex}\,(\pi_l) \subset \mathrm{Supp}\,(E)$, there is a surjection $\sum_{i=1}^r \mathrm{NE}(S_i^\nu/X) \to \mathrm{NE}(Y_l/X)$ and hence

$$\mathrm{NE}(Y_l/X) = \mathrm{NE}(Y_l/X)_{K_{Y_l}+\Delta_l\geq 0} + \sum_{1\leq i\leq r, 1\leq j} \mathbb{R}_{\geq 0}[\Gamma_{i,j}].$$

Note that for any ample $\mathbb{Q}$-divisor H_l on Y_l, the set $\{\Gamma_{i,j} \mid (K_{Y_l} + \Delta_l + H_l) \cdot \Gamma_{i,j} \leq 0\}$ is finite, and so extremal rays do not accumulate in $\mathrm{NE}(Y_l/X)_{K_{Y_l}+\Delta_l<0}$.

Now, pick an extremal curve $C = \Gamma_{i,j}$ for some i, j. Then, there is an effective π_l-exceptional divisor S such that $S \cdot C < 0$, and hence a contraction $f\colon Y_l \to Z$ of C exists by Proposition 4.5 applied to (Y_l, Δ_l). If f is a divisorial contraction, we let $Y_{l+1} = Z$. If f is a flipping contraction, then by [HW21, Lemma 3.1] (this lemma is stated over a field but it holds in much wider generality such as over DVRs), there exists an f-ample irreducible divisor A which is π-exceptional and hence contained in $\lfloor \Delta_l \rfloor$. By Theorem 4.3, the flip $f^+\colon Y^+ \to Z$ exists. We let $Y_{l+1} = Y^+$, $\psi\colon Y_l \dashrightarrow Y_{l+1}$, and $K_{Y_{l+1}} + \Delta_{l+1} = \psi_*(K_{Y_l} + \Delta_l)$. □

Proof of Theorem 4.1. Note that the proof of Theorem 4.6 implies the existence of all relevant divisorial and flipping contractions and the termination of the relevant flips. It suffices therefore to show the existence of flips. Suppose that $f\colon Y \to Z$ is a flipping contraction over X. We must show that the corresponding flip $Y^+ \to Z$ exists. Let $\zeta(\Delta)$ be the number of components of Δ whose coefficient is not contained in the standard set $\{1 - \frac{1}{m} \mid m \in \mathbb{N}\} \cup \{1\}$. We will prove the result by induction on $\zeta(\Delta)$.

Note that if $\zeta(\Delta) = 0$, then by Theorem 4.6 the required flip exists. Therefore, we assume that $\zeta(\Delta) > 0$, and so we may write $\Delta = aS + B$, where $a \notin \{1 - \frac{1}{m} \mid m \in \mathbb{N}\} \cup \{1\}$. Note that S is not exceptional over X (as the exceptional divisors occur with coefficient one). Let $\nu\colon Y' \to Y$ be a log resolution which is an isomorphism at the generic points of strata of $\lfloor \Delta \rfloor$, and let $B' = \nu_*^{-1}B + \mathrm{Ex}(\nu)$, $S' = \nu_*^{-1}S$. Since $\zeta(S' + B') < \zeta(\Delta)$, we may run the $(K_{Y'} + S' + B')$-MMP over Z. We obtain a birational contraction $\phi\colon Y' \dashrightarrow Y''$ such that $K_{Y''} + S'' + B'' = \phi_*(K_{Y'} + S' + B')$ is nef over Z and all components of $\lfloor S'' + B'' \rfloor$ are normal up to universal homeomorphism. We now run a $(K_{Y''} + aS'' + B'')$-MMP over

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Z with scaling of $(1-a)S''$. Note that this is also a $(K_{Y''}+B'')$-MMP and $\zeta(B'') < \zeta(\Delta)$. Therefore, the required minimal model $Y'' \dashrightarrow Y^+$ exists. Since the log resolution was an isomorphism at the generic points of strata of $\lfloor \Delta \rfloor$, it is easy to see that this is the required flip (cf. [HW22, Section 7.1]). □

4.3. Applications

We begin by proving that fourfold $\mathbb{Q}$-factorial klt singularities in positive characteristic are $W\mathcal{O}$-rational.

Corollary 4.7. *Let X be a $\mathbb{Q}$-factorial klt four-dimensional variety over a perfect field k of characteristic $p > 5$ admitting a log resolution of singularities. Then X has $W\mathcal{O}$-rational singularities.*

Proof. Let $\pi\colon Y \to X$ be a log resolution, and run the $(K_Y + \mathrm{Ex}(\pi))$-MMP over X. By standard arguments, this MMP contracts $\mathrm{Ex}(\pi)$, and so its output $Y' \to X$ is a small birational morphism and hence an isomorphism as X is $\mathbb{Q}$-factorial. We shall show that if $g\colon Y \dashrightarrow Y_n$ is a sequence of steps of the MMP, with the induced map $\pi_n\colon Y_n \to X$, then $R^i\pi_* W\mathcal{O}_{Y,\mathbb{Q}} = R^i(\pi_n)_* W\mathcal{O}_{Y_n,\mathbb{Q}}$. As a consequence, $R^i\pi_* W\mathcal{O}_{Y,\mathbb{Q}} = 0$ for $i > 0$, and so X has $W\mathcal{O}$-rational singularities. Thus, inductively, after replacing Y by some steps of the MMP, we may assume that $(Y, \mathrm{Ex}(f))$ is dlt. Let $f\colon Y \to Z$ be a Mori contraction over X with $\rho(Y/Z) = 1$.

Assume that f is divisorial. There exists an f-antiample irreducible divisor $S \subseteq \mathrm{Ex}(f)$ with the induced map $f_S\colon S \to X$. Set $K_{S^\nu} + \Delta_{S^\nu} = (K_Y + \mathrm{Ex}(f))|_{S^\nu}$, where S^ν is the normalisation of S with the induced map $f_{S^\nu}\colon S^\nu \to X$. Then (S^ν, Δ_{S^ν}) is dlt by Lemma 2.2 and $-(K_{S^\nu} + \Delta_{S^\nu})$ is f_{S^ν}-ample. Up to perturbation, we can assume that (S^ν, Δ_{S^ν}) is klt, and so, by [NT20, Theorem 3.11],

$$R^i(f_{S^\nu})_* W\mathcal{O}_{S^\nu,\mathbb{Q}} = 0$$

for $i > 0$. Since S is normal up to universal homeomorphism (Lemma 2.1), [NT20, Lemma 2.17] and [GNH19, Lemma 2.20 and 2.21] together with the Leray spectral sequence imply that $R^i(f_S)_* W\mathcal{O}_{S,\mathbb{Q}} = 0$ for $i > 0$. Let $u\colon S \to Y$ be the inclusion, and let I_S be the ideal sheaf defining S. Then, we get the following short exact sequence (cf. [GNH19, Proof of Proposition 3.4])

$$0 \to WI_{S,\mathbb{Q}} \to W\mathcal{O}_{Y,\mathbb{Q}} \to u_* W\mathcal{O}_{S,\mathbb{Q}} \to 0.$$

By [GNH19, Proposition 2.23], $R^i f_* WI_{S,\mathbb{Q}} = 0$ for $i > 0$, and so

$$R^i f_* W\mathcal{O}_{Y,\mathbb{Q}} = R^i f_*(u_* W\mathcal{O}_{S,\mathbb{Q}}) = R^i(f_S)_* W\mathcal{O}_{S,\mathbb{Q}} = 0,$$

where the second equality follows from [GNH19, Lemma 2.20 and Lemma 2.21]. Moreover, $f_* W\mathcal{O}_{Y,\mathbb{Q}} = W\mathcal{O}_{Z,\mathbb{Q}}$ by [NT20, Lemma 2.17], and so we can conclude the proof when f is divisorial by the Leray spectral sequence.

Suppose that f is a flipping contraction, and let $\pi_Z\colon Z \to X$ be the induced map. Let $\phi\colon Y \dashrightarrow Y^+$ be the flip, let $f^+\colon Y^+ \to Z$ be the flipped contraction and let $\pi^+\colon Y^+ \to X$ be the induced map. As before, by [HW21, Lemma 3.1] and Theorem 4.3, both Y and Y^+ are relatively F-regular over Z. In particular, $Rf_*\mathcal{O}_Y = \mathcal{O}_Z$ and $Rf^+_*\mathcal{O}_{Y^+} = \mathcal{O}_Z$ (see [SS10, Theorem 6.8] and the proof of [HW22, Proposition 3.4]). By [NT20, Lemma 2.17] and [GNH19, Lemma 2.19 and Lemma 2.21], $f_* W\mathcal{O}_{Y,\mathbb{Q}} = f^+_* W\mathcal{O}_{Y^+,\mathbb{Q}} = W\mathcal{O}_{Z,\mathbb{Q}}$ and $R^i f_* W\mathcal{O}_{Y,\mathbb{Q}} = R^i f^+_* W\mathcal{O}_{Y^+,\mathbb{Q}} = 0$ for $i > 0$. By the Leray spectral sequence:

$$R^i\pi_* W\mathcal{O}_{Y,\mathbb{Q}} = R^i(\pi_Z)_* W\mathcal{O}_{Z,\mathbb{Q}} = R^i\pi^+_* W\mathcal{O}_{Y^+,\mathbb{Q}}.$$ □

We can now prove Corollary 1.3.

Proof of Corollary 1.3. Let $\pi\colon Y \to X$ be a resolution of singularities such that Y is rationally chain connected. Since X has $W\mathcal{O}_X$-rational singularities by the above corollary, [GNH19, Remark 3.1] and [BBE07, Proposition 6.9] imply that $|X(\mathbb{F}_q)| \equiv |Y(\mathbb{F}_q)| (\mathrm{mod}\ q)$.

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We can conclude the proof as $|Y(\mathbb{F}_q)| \equiv 1 \pmod q$ by [Esn03, Theorem 1.1] and the argument explained between this theorem and [Esn03, Corollary 1.3]. Specifically, this follows from the Lefschetz trace formula for crystalline cohomology

$$|Y(\mathbb{F}_q)| = \sum_i (-1)^i \mathrm{Trace}(F \mid H^i_{\mathrm{crys}}(X/K))$$

given that Y is rationally chain connected, and so all the slopes of Frobenius are ≥ 1 by [Esn03, Theorem 1.1]. □

Next, we show the existence of dlt modifications.

Corollary 4.8. *Let (X, Δ) be a four-dimensional $\mathbb{Q}$-factorial log pair with standard coefficients admitting a log resolution and defined over a perfect field of characteristic $p > 5$ or a DVR of characteristic $(0, p)$ for $p > 5$ with perfect residue field. Then a dlt modification of (X, Δ) exists, that is, a birational morphism $\pi\colon Y \to X$ such that $(Y, \pi_*^{-1}\Delta + Ex(\pi))$ is dlt, $\mathbb{Q}$-factorial and minimal over X.*

Proof. Let $\pi\colon Y \to X$ be a log resolution of (X, Δ). Then, a dlt modification is a minimal model of $(Y, \pi_*^{-1}\Delta + \mathrm{Ex}(\pi))$ over X (see Theorem 4.1). □

Finally, we prove that inversion of adjunction holds.

Corollary 4.9. *Consider a $\mathbb{Q}$-factorial four-dimensional log pair $(X, S + B)$ with standard coefficients defined over a perfect field of characteristic $p > 5$ or a DVR of characteristic $(0, p)$ for $p > 5$ with perfect residue field, where S is a prime divisor with no common component with $B \geq 0$. Assume that $(X, S + B)$ admits a log resolution.*

Then $(X, S+B)$ is plt on a neighbourhood of S if and only if $(\tilde{S}, B_{\tilde{S}})$ is klt, where $\tilde{S}$ is the normalisation of S and $B_{\tilde{S}}$ is the different.

Proof. By considering a log resolution of $(X, S + B)$, it is easy to see that if $(X, S + B)$ is plt, then $(\tilde{S}, B_{\tilde{S}})$ is klt. Thus, we can assume that $(\tilde{S}, B_{\tilde{S}})$ is klt and aim to show that $(X, S + B)$ is plt near S.

Let $\pi\colon Y \to X$ be a dlt modification of $(X, S + B)$ (see Corollary 4.8), and write $K_Y + S_Y + B_Y = \pi^*(K_X + S + B)$. By definition (of a dlt modification) for any π-exceptional irreducible divisor E, we have that $E \subseteq \lfloor B_Y \rfloor$. Write

$$(\pi|_{S_Y})^*(K_{\tilde{S}} + B_{\tilde{S}}) = (K_Y + S_Y + B_Y)|_{\tilde{S}_Y} = K_{\tilde{S}_Y} + B_{\tilde{S}_Y},$$

where $\tilde{S}_Y \to S_Y$ is the normalisation of S_Y, and $B_{\tilde{S}_Y}$ is the different. Let E be a π-exceptional divisor intersecting S_Y. Since $E \subseteq \lfloor B_Y \rfloor$ and $(Y, S_Y + \mathrm{Ex}(\pi))$ is dlt, we must have that $E \cap S_Y \subseteq \lfloor B_{\tilde{S}_Y} \rfloor$. Since $E \cap S_Y \neq \emptyset$, this contradicts $(\tilde{S}, B_{\tilde{S}})$ being klt.

Therefore, we may assume that $E \cap S_Y = \emptyset$ so that $Y = X$ near S, and hence $(X, S + B)$ is dlt on a neighbourhood of S. Since S is irreducible, $(X, S + B)$ is in fact plt. □

5. Relative MMP over a DVR

Throughout this section, we assume that log resolutions of all log pairs with the underlying variety being birational to X as below exist (and are given by a sequence of blowups along the non-snc locus). The assumption on the field (in the positive characteristic case) being algebraically closed is necessary to invoke Bertini's theorem for strongly F-regular singularities.

In what follows, given a scheme X over an integral scheme S and a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor L on X, we define $\kappa(L/S) := \kappa(L|_{X_\eta})$, where X_η is the fibre over the generic point $\eta \in S$.

Our goal is to show Theorem 1.2, whose statement in full generality is as follows. Note that when ϕ is smooth or X is semistable, the K_X-MMP and the $(K_X + \mathrm{Supp}\, \phi^{-1}(s))$-MMP coincide.

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Theorem 5.1. *Let (X, Δ) be a four-dimensional $\mathbb{Q}$-factorial dlt pair with standard coefficients projective over a DVR R with perfect residue field of characteristic $p > 5$. Let $s \in \operatorname{Spec} R$ be its special point, and let $\phi\colon X \to \operatorname{Spec} R$ be the natural projection. When R is purely positive characteristic, we also assume that it is a local ring of a curve C defined over an algebraically closed field and that $(X, \Delta) := (\mathcal{X}, \Phi) \times_C \operatorname{Spec}(R)$ for a $\mathbb{Q}$-factorial four-dimensional dlt pair $(\mathcal{X}, \Phi)$ which is projective over C.*

Further, suppose that log resolutions of all log pairs with the underlying variety being birational to X exist (and are constructed by a sequence of blowups along the non-snc locus).

If $\kappa(K_X + \Delta / Spec(R)) \geq 0$ and $Supp(\phi^{-1}(s)) \subset \lfloor \Delta \rfloor$, then we can run an arbitrary $(K_X + \Delta)$-MMP over $\operatorname{Spec} R$ which terminates with a minimal model. In particular, every sequence of steps of the MMP terminates.

In mixed characteristic, it is enough to assume that $K_X + \Delta$ is pseudo-effective, as then $\kappa(K_X + \Delta) \geq 0$ holds by the nonvanishing theorem for characteristic 0 threefolds. Similarly, in the equicharacteristic case $p > 5$, if the generic fibre is geometrically klt, then nonvanishing follows from [XZ19, Theorem 1.1] and [Wit21a].

We start by proving the following base point free theorem. This result would follow easily from [CT20] and [Wit21b] if we knew the abundance for slc threefolds.

Proposition 5.2. *Let (X, Δ) be a four-dimensional $\mathbb{Q}$-factorial dlt pair with standard coefficients projective over a DVR R with perfect residue field of characteristic $p > 5$. Let $s, \eta \in \operatorname{Spec} R$ be the special and the generic point, respectively, and let $\phi\colon X \to \operatorname{Spec} R$ be the natural projection. When R is purely positive characteristic, we also assume that it is a local ring of a curve C defined over an algebraically closed field and that $(X, \Delta) := (\mathcal{X}, \Phi) \times_C \operatorname{Spec}(R)$ for a $\mathbb{Q}$-factorial four dimensional dlt pair $(\mathcal{X}, \Phi)$ which is projective over C.*

Suppose that $K_X + \Delta$ is nef and big, and $\lfloor \Delta \rfloor = Supp(\phi^{-1}(s))$. Then $K_X + \Delta$ is semiample.

Proof. Write $\operatorname{Supp} X_s = \sum_{i=1}^{r} E_i$ for irreducible divisors E_i, and $L := K_X + \Delta$. Since (X, Δ) is klt over η, we get that $L|_{X_\eta}$ is semiample by [DW19b, Theorem 1.4] or the base point free theorem in characteristic 0 [KM98]. By [CT20, Theorem 1.1] and [Wit21b, Theorem 1.2], it thus suffices to show that $L|_{X_s}$ is semiample.

Since L is big, we may assume that $\Delta + L \sim_{\mathbb{Q}} H + F + G$, where H is ample, $F \geq 0$ is supported on X_s and the support of $G \geq 0$ contains no divisors of X_s. In this paragraph, we reduce the proposition to the case when G does not contain any log canonical centres of (X, Δ). To this end, let $\pi\colon Y \to X$ be a dlt modification of $(X, \Delta + \delta G)$ for some $0 < \delta \ll 1$ (see Corollary 4.8). Then we may assume that $K_Y + \Delta_Y = \pi^*(K_X + \Delta)$, where $\Delta_Y = \pi_*^{-1}\Delta + \operatorname{Ex}(\pi)$ and $\pi_*^{-1}G$ contains no strata of $\lfloor \Delta_Y \rfloor$; indeed as (X, Δ) is dlt, the divisorial non-klt places of $(X, \Delta + \delta G)$ and (X, Δ) coincide on a given resolution (on which we run the MMP to construct the dlt modification) for $0 < \delta \ll 1$. Let P be an effective π-exceptional divisor such that $-P$ is π-ample. Then the support of P is contained in Y_s and we may assume that $\pi^*H - P$ is ample over $\operatorname{Spec} R$. Note that $\Delta_Y - \pi^*\Delta$ is supported on $\operatorname{Ex}(\pi) \subseteq Y_s$. We have

$$\begin{aligned}\Delta_Y + \pi^*L &= \Delta_Y - \pi^*\Delta + \pi^*(H + F + G)\\ &= (\pi^*H - P) + (P + \Delta_Y - \pi^*\Delta + \pi^*F + \pi^*G - \pi_*^{-1}G) + \pi_*^{-1}G\\ &= H_Y + F_Y + G_Y,\end{aligned}$$

where $G_Y = \pi_*^{-1}G$, $H_Y = \pi^*H - P - aY_s$ is ample over $\operatorname{Spec} R$, and $F_Y = P + (\Delta_Y - \pi^*\Delta) + \pi^*F + (\pi^*G - \pi_*^{-1}G) + aY_s$ is supported on Y_s (we choose $a \gg 0$ so that $F_Y \geq 0$). Replacing X, Δ, L, H, F, G by $Y, \Delta_Y, \pi^*L, H_Y, F_Y, G_Y$, we may assume that $(X, \Delta + \delta G)$ is dlt, $\lfloor \Delta \rfloor = \operatorname{Supp} X_s$, F is supported on X_s, and H is ample.

Since H is ample, we may further assume that the support of $F = \sum f_i F_i$ equals X_s, where the f_i are chosen generically (here, F_i are distinct prime divisor). Moreover, we have that

$$(X, \Delta_{\epsilon,\delta} := (1 - \delta)\Delta + \epsilon X_s + \delta(H + F + G))$$

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is klt for $0 < \delta \ll 1$ and some possibly negative ϵ such that $|\epsilon| \ll 1$. Note that $K_X + \Delta_{\epsilon,\delta} \sim_{\mathbb{Q}} (1+\delta)L$.

Fixing δ and increasing ϵ, since the f_i are chosen generically, we obtain a sequence of rational numbers $\epsilon < \epsilon_1 < \epsilon_2 < \ldots < \epsilon_r$ such that $\lfloor \Delta_{\epsilon_i,\delta} \rfloor = U_i := \sum_{j=1}^{i} E_j$ and E_i occurs with coefficient one in $\Delta_{\epsilon_i,\delta}$. Here, of course, we have reindexed the E_i accordingly. Note that all E_i are defined over a perfect field of positive characteristic. We claim that

Claim 5.3. *$(K_X+\Delta)|_{E_i^\nu}$ is semiample, where $E_i^\nu \to E_i$ is the normalisation, and hence also $(K_X+\Delta)|_{E_i}$ is semiample as the E_i are normal up to universal homeomorphism (see Theorem 2.3).*

Granting the claim and proceeding by induction, we may assume that $(K_X+\Delta)|_{U_{i-1}}$ is semiample and we must show that $(K_X+\Delta)|_{U_i}$ is semiample. By [kee99, Corollary 2.9], it suffices to show that $g_2|_{U_{i-1}\cap E_i^\nu}$ has connected geometric fibres, where $g_2\colon E_i^\nu \to V$ is the morphism associated to the semiample $\mathbb{Q}$-divisor $(K_X+\Delta)|_{E_i^\nu}$. Note that $(K_X+\Delta)|_{E_i^\nu} \equiv_V 0$, hence $-(K_{E_i^\nu}+\Delta'_{E_i^\nu}) := -(K_X+\Delta_{\epsilon_i,\delta}-\delta H)|_{E_i^\nu}$ is ample over V. By [NT20, Theorem 1.2] (this requires the base field to be perfect), the fibres of the non-klt locus of $(E_i^\nu, \Delta'_{E_i^\nu})$ are geometrically connected. Since this non-klt locus coincides with $U_{i-1}\cap E_i$ (as $K_X+\Delta+\delta(H+G)$ is dlt), the statement of the proposition follows.

Proof of Claim 5.3. We proceed by induction. We warn the reader that E_i^ν need not be $\mathbb{Q}$-factorial. Set $K_{E_i^\nu}+\Delta_{E_i^\nu} = (K_X+\Delta)|_{E_i^\nu}$.

First, if $L|_{E_i^\nu}$ is big, then it is semiample by [Wal18, Theorem 1.1] and base change to an algebraically closed field. Hence, we can assume that it is not big. In particular, $L|_{E_i^\nu} - \gamma\lfloor\Delta_{E_i^\nu}\rfloor$ is not pseudo-effective for any $\gamma > 0$ (note that $\lfloor\Delta_{E_i^\nu}\rfloor = \lfloor\Delta - E_i\rfloor|_{E_i^\nu}$ is $\mathbb{Q}$-Cartier). Indeed, assume otherwise and write $L \sim_{\mathbb{Q}} H'+F'+G'$, where H' is ample, $F' \geq 0$ is supported on X_s and the support of $G' \geq 0$ contains no divisors of X_s. By shifting F' by a multiple of $\phi^{-1}(s)$, we may assume that F' does not contain E_i in its support (but it is not necessarily effective any more). Thus, we can write

$$\begin{aligned}(m+1)L|_{E_i^\nu} &\sim_{\mathbb{Q}} H'|_{E_i^\nu} + mL|_{E_i^\nu} + F'|_{E_i^\nu} + G'|_{E_i^\nu}\\ &\sim_{\mathbb{Q}} H'|_{E_i^\nu} + (mL|_{E_i^\nu} - t\lfloor\Delta_{E_i^\nu}\rfloor) + (F'|_{E_i^\nu} + t\lfloor\Delta_{E_i^\nu}\rfloor) + G'|_{E_i^\nu}\end{aligned}$$

for $m,t \in \mathbb{N}$ such that $mL|_{E_i^\nu} - t\lfloor\Delta_{E_i^\nu}\rfloor$ and $F'|_{E_i^\nu} + t\lfloor\Delta_{E_i^\nu}\rfloor$ are pseudo-effective. This shows that $L|_{E_i^\nu}$ is big, which is the required contradiction.

Let $(\overline{E}_i, \Delta_{\overline{E}_i})$ be a dlt modification of $(E_i^\nu, \Delta_{E_i^\nu})$ so that $\overline{E}_i$ is $\mathbb{Q}$-factorial (see [HNT20, Corollary 3.6]). Let r be the Cartier index of $K_{\overline{E}_i}+\Delta_{\overline{E}_i}$, and let L_i be the pullback of $L|_{E_i^\nu}$. Note that $L_i = K_{\overline{E}_i}+\Delta_{\overline{E}_i}$. We will run the $(K_{\overline{E}_i}+\Delta_{\overline{E}_i} - \gamma\lfloor\Delta_{\overline{E}_i}\rfloor)$-MMP for $\gamma = \frac{1}{6r+1}$, which by the above must terminate with a Mori fibre space (cf. [GNH19, Theorem 2.13]).

We claim that each step of the MMP is $(K_{\overline{E}_i}+\Delta_{\overline{E}_i})$-trivial. We start by showing this for the first step. Note that $(K_{\overline{E}_i}+\Delta_{\overline{E}_i} - \lfloor\Delta_{\overline{E}_i}\rfloor)\cdot\Gamma \geq -6d_\Gamma$ for any $(K_{\overline{E}_i}+\Delta_{\overline{E}_i} - \gamma\lfloor\Delta_{\overline{E}_i}\rfloor)$-negative extremal ray Γ (see [DW19b, Theorem 1.1]; here, d_Γ satisfies $d_\Gamma \mid \Gamma\cdot M$ for every Cartier divisor M). But if $(K_{\overline{E}_i}+\Delta_{\overline{E}_i})\cdot\Gamma > 0$, then $(K_{\overline{E}_i}+\Delta_{\overline{E}_i})\cdot\Gamma \geq d_\Gamma/r$ by the definition of r, and hence

$$(K_{\overline{E}_i}+\Delta_{\overline{E}_i} - \gamma\lfloor\Delta_{\overline{E}_i}\rfloor)\cdot\Gamma \geq \frac{(1-\gamma)d_\Gamma}{r} - 6\gamma d_\Gamma = 0$$

which is a contradiction. In particular, $(K_{\overline{E}_i}+\Delta_{\overline{E}_i})\cdot\Gamma = 0$. Hence, the nefness of $K_{\overline{E}_i}+\Delta_{\overline{E}_i}$ is preserved by the first step of the MMP and so is its Cartier index r by [Ber21, Theorem 1.1], [ABL22, Corollary 1.5] (precisely, if $\overline{E}_i \to Z$ is the contraction of Γ given by $L' = K_{\overline{E}_i}+\Delta_{\overline{E}_i} - \gamma\lfloor\Delta_{\overline{E}_i}\rfloor + A$ for some ample $\mathbb{Q}$-divisor A, then we apply this result to $rL_i \sim_{\mathbb{Q},Z} K_{\overline{E}_i}+\Delta_{\overline{E}_i} - \gamma\lfloor\Delta_{\overline{E}_i}\rfloor + A + (rL_i - L')$ over Z). Hence, we may repeat this procedure for all steps of the $(K_{\overline{E}_i}+\Delta_{\overline{E}_i} - \gamma\lfloor\Delta_{\overline{E}_i}\rfloor)$-MMP.

By replacing $\overline{E}_i$ by the output of this MMP and $\Delta_{\overline{E}_i}$ together with L_i by their pushforwards, we can assume that we have a $(K_{\overline{E}_i}+\Delta_{\overline{E}_i} - \gamma\lfloor\Delta_{\overline{E}_i}\rfloor)$-Mori fibre space $\psi\colon \overline{E}_i \to Z$ (in particular Z is normal). Since ψ is L_i-trivial, $\lfloor\Delta_{\overline{E}_i}\rfloor$ is relatively ample so that one of its components, say W, dominates

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Z. We claim that $L_i|_{\lfloor \Delta_{\overline{E}_i} \rfloor}$ (and so $L_i|_W$) is semiample. This can be checked after a base change to an algebraically closed field and on a dlt modification of $(\overline{E}_i, \Delta_{\overline{E}_i})$ (cf. [CT20, Lemma 2.11(3)]) in which case it follows from [Wal18, Theorem 1.3]. Now, $L_i \sim_{\mathbb{Q}} \psi^* D$ for some $\mathbb{Q}$-divisor D on Z. But then $(\psi|_W)^* D \sim_{\mathbb{Q}} L_i|_W$, which is semiample, and hence both D and L_i are semiample as Z is normal (cf. [CT20, Lemma 2.11(4)]). □

□

Proposition 5.4. *With notation as in the first paragraph of Proposition 5.2, suppose that* $Supp(\phi^{-1}(s)) \subset \lfloor \Delta \rfloor$. *Let* L *be a nef and big* $\mathbb{Q}$*-Cartier* $\mathbb{Q}$*-divisor on* X *such that* $L - (K_X + \Delta)$ *is ample. Then* L *is semiample and induces a morphism* $f : X \to Z$ *over* Spec R. *In particular, every* f*-numerically trivial* $\mathbb{Q}$*-Cartier divisor descends to a* $\mathbb{Q}$*-Cartier divisor on* Z.

Proof. This follows from Proposition 5.2 by perturbation. □

Proof of Theorem 5.1. The proof is similar to that of [HW21, Proposition 4.1]. We will run an arbitrary $(K_X + \Delta)$-MMP.

In the positive characteristic case, we will implicitly work with the spread-out $\mathcal{X}$ of X and repeatedly replace C by an appropriate neighbourhood of the special point $s \in U \subset C$ and $\mathcal{X}$ by $\mathcal{X} \times_C U$ so that we can apply results on the MMP for finite type schemes over a field. Let η be the generic point of Spec R.

Following the proof of Theorem 4.6, we will run a $(K_X + \Delta)$-MMP $X = X_0 \dashrightarrow X_1 \dashrightarrow \ldots \dashrightarrow X_l$. We will denote the central fibre by $X_{l,s}$ and the pushforward of Δ on X_l by Δ_l. Note that $X_{l,s} \sim_{\mathrm{Spec}\, R} 0$. By the same argument as that in Theorem 4.6, the cone theorem is valid for $K_{X_l} + \Delta_l$. Hence, if $K_{X_l} + \Delta_l$ is not nef over Spec R, then there exists a $(K_{X_l} + \Delta_l)$-negative extremal ray over Spec R spanned by a curve $\Sigma \subset X_{l,s}$.

Let G be an ample $\mathbb{Q}$-divisor such that $L = K_{X_l} + \Delta_l + G$ is nef and $L^\perp = \mathbb{R}[\Sigma]$. Then L is semiample by Proposition 5.4. Let $f : X_l \to Z$ be the corresponding contraction, which is birational as $\kappa(K_X + \Delta \,/\, \mathrm{Spec}(R)) \geq 0$. There are two cases. If $f : X_l \to Z$ is a divisorial contraction (over Spec R), then we may set $X_{l+1} := Z$ and continue our MMP over Spec R. Thus, we must show that if the induced morphism $f : X_l \to Z$ (over Spec R) is a flipping contraction, then the flip $f^+ : X_{l+1} \to Z$ exists. Further, we must show that there is no infinite sequence of flips. The latter follows from Theorem 2.14. As for the former, by means of perturbation, we may assume that $\lfloor \Delta_l \rfloor = \mathrm{Supp}\, X_{l,s}$, and consider two cases.

Claim 5.5. *If the flipping locus is contained in the special fibre* $X_{l,s}$, *then the flip* $X_l \dashrightarrow X_{l+1}$ *exists.*

Proof. Let Σ be a flipping curve. If $\Sigma \cdot S \neq 0$ for some component S of $X_{l,s}$, then since $\Sigma \cdot X_{l,s} = 0$, we may assume that there is another component E of $X_{l,s}$ such that $\Sigma \cdot S < 0$ and $\Sigma \cdot E > 0$ (up to swapping S and E). Thus, the flip exists by Theorem 4.3. We may therefore assume that $\Sigma \cdot S = 0$ for every component of $\mathcal{X}_{l,s}$. We follow the proof of [HW21, Proposition 4.1]. Now, let S be a component of $X_{l,s}$ such that $\Sigma \subseteq S$. Since $S \cdot \Sigma = 0$, we may assume that $S' := f_* S$ is also $\mathbb{Q}$-Cartier (as f is a flip, then $\rho(X/Z) = 1$ and so every divisor that intersects Σ trivially is pulled back from Z, equivalently apply Proposition 5.4). Let H' be a reduced $\mathbb{Q}$-Cartier divisor on Z such that

(1) $H = f_*^{-1} H'$ contains $\mathrm{Ex}(f)$,
(2) $(X_l, \Delta_l + H)$ is dlt over the generic point $\eta \in$ Spec R,
(3) For any proper birational morphism $h : Y \to Z$ such that Y is $\mathbb{Q}$-factorial, we have that $N^1(Y/Z)$ is generated by the irreducible components of the strict transform of H' and the h-exceptional divisors, and
(4) H and Δ_l have no common components.

Explicitly, we take H_1 and H_2 as in Lemma 5.6. Then $H = H_1 + H_2$ descends to a $\mathbb{Q}$-Cartier divisor H' on Z by Proposition 5.4. We leave the verification of (3) to the reader (here, we use that $\rho(X_l/Z) = 1$, otherwise we would need to enlarge H').

Let $p : Y \to X_l$ be a dlt modification of $(X_l, \Delta_l + H)$ (see Corollary 4.8). Since the generic points of $H_1 \cap H_2$ are simple normal crossing in $X_{l,\eta}$, we may assume that the dlt modification is constructed

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from a log resolution which is an isomorphism over the generic points of $(H_1 \cap H_2)|_{X_{l,\eta}}$. In particular, p must be an isomorphism over the generic point $\eta \in R$. Set $h = f \circ p \colon Y \to Z$.

First, we will run a $(K_Y + \Delta_Y + H_Y)$-MMP over Z, where $H_Y = p_*^{-1}H$ and $\Delta_Y = p_*^{-1}\Delta_l + \mathrm{Ex}(p)$. All extremal rays R are contained in the support of h^*H' and some component of h^*H' has a nonzero intersection number with R (by condition (3) above). Since $h^*H' \cdot R = 0$, there are components E, E' of $\mathrm{Supp}\, h^*H'$ such that $E \cdot R < 0$ and $E' \cdot R > 0$. Since the support of h^*H' is contained in the support of $\lfloor \Delta_Y + H_Y \rfloor$, the necessary flips exist by Theorem 4.3 and we may run the required MMP. Note that by Theorem 2.14, there is no infinite sequence of such flips and hence we may assume that, up to replacing Y by the output of the MMP, $K_Y + \Delta_Y + H_Y$ is nef over Z. We now run a $(K_Y + \Delta_Y)$-MMP with scaling of H_Y over Z. If R is a corresponding $(K_Y + \Delta_Y)$-negative extremal ray, then $H_Y \cdot R > 0$. Since

$$H_Y \equiv_h -\sum b_j E_j, \qquad b_j \geq 0$$

for some h-exceptional divisors E_j, it follows that $R \cdot E_j < 0$ for some j. Since E_j is contained in Y_s, it is contained in the support of h^*Z_s. Since $R \cdot h^*Z_s = 0$, there is a component E' of $\lfloor \Delta_Y \rfloor$ such that $R \cdot E' > 0$ and the necessary flip exists by Theorem 4.3.

By Theorem 2.14, there is no infinite sequence of such flips and hence we may assume that $K_Y + \Delta_Y$ is nef over Z. By Proposition 5.2 applied to $K_Y + \Delta_Y + h^*A$ for a sufficiently ample divisor A on Z, we get that $K_Y + \Delta_Y$ is semiample over Z, and thus by replacing Y by the image of the associated semiample fibration, we can assume that $K_Y + \Delta_Y$ is relatively ample over Z. We claim that $Y \to Z$ is small, and hence $X_{l+1} := Y$ is the required flip (the canonical ring $R(K_{X_l} + \Delta_l) = R(K_Z + f_*\Delta_l) = R(K_Y + \Delta_Y)$ is finitely generated). To this end, note that if $p \colon W \to X_l$ and $q \colon W \to Y$ is the normalisation of the graph of the rational map $X_l \dashrightarrow Y$, then we may write

$$p^*(K_{X_l} + \Delta_l) - q^*(K_Y + \Delta_Y) = E,$$

where $p_*E = 0$ (as $X_l \to Z$ is small) and $-E$ is relatively ample over Z. By the negativity lemma, $E \geq 0$. On the other hand, by construction, if F is a p-exceptional but not q-exceptional divisor on W, then by definition of Δ_Y we have

$$\mathrm{mult}_F(E) = \mathrm{mult}_F(p^*(K_{X_l} + \Delta_l)) - \mathrm{mult}_F(q^*(K_Y + \Delta_Y)) \leq 1 - 1 = 0,$$

and so, in fact, $\mathrm{mult}_F(E) = 0$. This contradicts $-E$ being relatively ample over Z. □

Let $f_\eta \colon (X_{l,\eta}, \Delta_{l,\eta}) \to Z_\eta$ be the restriction of f to X_η. By the above claim, we may assume that f_η is not the identity. Since f is a flipping contraction, it is easy to see that f_η is a flipping contraction and in particular $X_{l,\eta}$ is $\mathbb{Q}$-factorial, $\rho(X_{l,\eta}/Z_\eta) = 1$, $X_{l,\eta} \to Z_\eta$ is small and $-(K_{X_{l,\eta}} + \Delta_{l,\eta})$ is ample over Z_η. By [DW19b, Theorem 1.3] in positive characteristic (and [BCHM10] in characteristic 0), the corresponding flip $f_\eta^+ \colon X_{l,\eta}^+ \to Z_\eta$ exists. Let $f' \colon X_l' \to Z$ be the closure of the projective morphism f_η^+. Note that the fibre of X_l' over s may be highly singular and in particular nonnormal and not even R_1; therefore, to construct the flip we will consider an appropriate log resolution and then run an MMP over Z.

Let Δ_l' be a divisor on X_l' given as the sum of the closure of $\Delta_{l,\eta}^+$ and the support of $X_{l,s}'$. Let $p \colon Y \to X_l'$ be a log resolution of (X_l', Δ_l'), and set $\Delta_Y' = p_*^{-1}\Delta_l' + \mathrm{Ex}(p)$. We now run a $(K_Y + \Delta_Y')$-MMP over X_l'. We must show that the necessary flips exist and that the corresponding MMP terminates. Let $v \colon Y \to V$ be a flipping contraction. Note that if the flipping locus is contained in the central fibre Y_s, then the flip exists by Claim 5.5. If the flipping locus dominates $\mathrm{Spec}\, R$, then as $X_{l,\eta}'$ is $\mathbb{Q}$-factorial, by [HW21, Lemma 3.1] (the same argument works in mixed characteristic), there is a v-ample p-exceptional divisor $A_\eta \subset Y_\eta$. As $\rho(Y/V) = 1$, its closure $A \subset Y$ is also relatively ample and is contained in the support of $\lfloor \Delta_Y' \rfloor$. Similarly, we also have an effective p-exceptional divisor $F_\eta \subset Y_\eta$ which is p-antiample. Therefore, there is a component S_η of F_η such that its closure $S \subset Y$ is v-antiample. Thus, the flip exists by Theorem 4.3. By Theorem 2.14 the above sequence terminates with

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the required dlt model which we again denote by (Y, Δ'_Y). Since (X'_l, Δ'_l) is klt and $\mathbb{Q}$-factorial over η, we can assume that p is an isomorphism over η. Set $h = f' \circ p \colon Y \to Z$.

We will run a $(K_Y + \Delta'_Y)$-MMP over Z, where $(Y, \Delta'_Y = p_*^{-1}\Delta'_l + \mathrm{Ex}(p))$ is dlt. Contractions exist for the same reason as before (see Proposition 5.4). Since $K_{Y_\eta} + \Delta'_{Y_\eta}$ is ample over Z_η, the contracted locus is contained in the fibre over $s \in \operatorname{Spec} R$. In particular, the necessary flips exist by Claim 5.5. The MMP terminates by Theorem 2.14.

Hence, we can assume that $K_Y + \Delta'_Y$ is nef over Z. By Proposition 5.2, $K_Y + \Delta'_Y$ is semiample over $\operatorname{Spec} R$. Replacing Y by the image of the associated semiample fibration, we can assume that $K_Y + \Delta'_Y$ is relatively ample. We claim that $Y \to Z$ is small, and hence, $X_{l+1} := Y$ is the required flip. To this end, note that if $p \colon W \to X_l$ and $q \colon W \to Y$ is the normalisation of the graph of the rational map $X_l \dashrightarrow Y$, then we may write

$$p^*(K_{X_l} + \Delta_l) - q^*(K_Y + \Delta'_Y) = E,$$

where $p_*E = 0$ (as $X_l \to Z$ is small) and $-E$ is relatively ample over Z. By the negativity lemma, $E \geq 0$. On the other hand, by construction, if F is p-exceptional but not q-exceptional, then by definition of Δ'_Y we have

$$\operatorname{mult}_F(E) = \operatorname{mult}_F(p^*(K_{X_l} + \Delta_l)) - \operatorname{mult}_F(q^*(K_Y + \Delta'_Y)) \leq 1 - 1 = 0,$$

and so, in fact, $\operatorname{mult}_F(E) = 0$. This contradicts $-E$ being relatively ample over Z. □

In the proof we used the following technical lemma.

Lemma 5.6. *Let C be a regular affine curve defined over an algebraically closed field k of positive characteristic $p > 0$. Let $s \in C$ be a closed point, and let $\eta \in C$ be the generic point. Let $(X, S + B)$ be a $\mathbb{Q}$-factorial dlt pair of absolute dimension four, projective and dominant over the curve C such that $S = \lfloor S + B \rfloor = \operatorname{Supp} X_s \neq 0$ and B has standard coefficients. Let $f \colon X \to Z$ be a small contraction with $\rho(X/Z) = 1$ and exceptional locus $\mathrm{Ex}(f)$ contained in X_s. Suppose that every irreducible component S_i of S is f-numerically trivial.*

Then there exist $\mathbb{Q}$-Cartier Weil divisors H_i for $i \in \{1, 2\}$ such that H_1 and H_2 are f-ample and f-antiample, respectively, $H_1 + H_2$ is f-numerically trivial, and $(X, S + B + H_1 + H_2)$ is dlt over η.

The same statement holds if X is defined over $C = \operatorname{Spec} R$ for a DVR R of mixed characteristic $(0, p > 0)$.

Proof. Pick an f-ample Cartier divisor D on X and an ample Cartier divisor A on Z. Set $D_i = (-1)^{i+1}D$ and $L_i := D_i + m_i f^*A$ for $m_2 \gg m_1 \gg 0$ and $i \in \{1, 2\}$. We claim that there exist effective divisors $H_{i,\eta} \in |L_i|_{X_\eta}|$ such that $(X_\eta, B|_{X_\eta} + H_{1,\eta} + H_{2,\eta})$ is dlt. Assuming this claim, we pick H_i to be the closures of $H_{i,\eta}$ in X. Then $H_i \sim_{\mathbb{Q}} L_i + \sum a_i S_i$ for irreducible divisors S_i of S and some $a_i \in \mathbb{Q}$. Since S_i are f-numerically trivial, we get that H_1 is f-ample, H_2 is f-antiample and $H_1 + H_2$ is f-numerically trivial.

Thus, we are left to show the claim. In the mixed characteristic case, it follows automatically from the standard Bertini results in characteristic 0. Hence, we can assume that we are in the purely positive characteristic case. First, we replace C by $C \setminus \{s\}$, and so we can assume that $S = 0$ and L_i are very ample. Let $V \subseteq X$ be the subset of points on which (X, B) is strongly F-regular. By standard argument, V is open. Now, by localising at codimension two points and applying [ST18, Theorem 5.7], we get that $Z := X \setminus V$ is at most one dimensional.

Let H_1 be a general member of $|L_1|$. By [SZ13, Corollary 6.10], $(V, B|_V + H_1|_V)$ is purely F-regular (note that [SZ13] uses a different name: divisorially F-regular), and hence plt with $H_1|_V$ normal. Since $H_1 \cap Z$ is 0-dimensional, we can replace C by an open subset so that $(X, B + H_1)$ is plt and H_1 is normal.

Let H_2 be a general member of $|L_2|$. Since $m_2 \gg m_1$, Serre's vanishing implies that $H^0(X, L_2) \to H^0(H_1, L_2|_{H_1})$ is surjective, and so $H_2|_{H_1}$ is a general member of $L_2|_{H_1}$. By the same argument as above, we have that $(X, B + H_1 + H_2)$ is plt outside $H_1 \cap H_2$ up to replacing C by an open subset. Write

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$K_{H_1} + B_{H_1} = (K_X + B + H_1)|_{H_1}$. By [ST18, Theorem 5.9] and inversion of adjunction for threefolds, we get that $(H_1, B_{H_1} + H_2|_{H_1})$ is plt. In particular, the inversion of adjunction for fourfolds (Corollary 4.9) and the standard argument implies that the only log canonical centres of $(X, B + H_1 + H_2)$ are the generic points of $H_1 \cap H_2$. The generic points of $H_1 \cap H_2$ are contained in the smooth locus $X^{\mathrm{sm}} \subseteq X$, and hence $(X, B + H_1 + H_2)$ is simple normal crossing at them by the standard Bertini theorem applied to the smooth locus of H_2 (cf. [JS12, Theorem 3]). □

Let us point out that some of the assumptions on the existence of log resolutions of singularities may be weakened.

Conjecture 5.7 (Embedded resolution of singularities). *Let X be a regular quasi-projective variety and Z a closed subscheme, then there exists a projective birational morphism $\nu\colon X' \to X$ such that the schemes associated to $Ex(\nu)$ and $\nu^{-1}(Z)$ are divisors (in particular, $\nu^{-1}I_Z$ is locally free) and $\nu^{-1}Z \cup Ex(\nu)$ has simple normal crossing support.*

From this conjecture, we can deduce the existence of log resolutions for varieties birational to a regular variety.

Proposition 5.8. *Assume that Conjecture 5.7 holds for a regular projective variety X and all of its subschemes. Let Y be a quasi-projective variety birational to X. Then for any scheme $W \subset Y$, there exists a proper birational morphism from a regular variety $\mu\colon Y' \to Y$ such that schemes associated to $Ex(\mu)$ and $\mu^{-1}(W)$ are divisors and $Ex(\mu) \cup \mu^{-1}(W)$ has simple normal crossing support.*

Proof. Let $f\colon X \dashrightarrow Y$ be a birational map, and let Z be the indeterminacy locus of f. More precisely, let V be the normalisation of the graph of f with projections $p\colon V \to X$ and $q\colon V \to Y$. If H is a very ample divisor on Y and $H_1, \dots, H_{d+1} \in |H|$ are general elements, where $d = \dim Y$, then Z is cut out by $H_i' = p_*q^*H_i \in |H'|$, where H' is the strict transform of H.

Let $\nu\colon X' \to X$ be the resolution of Z given by Conjecture 5.7. Then $\nu^*H' = F + M$ and $\nu^*H_i' = F + M_i$, where F is a simple normal crossings divisor and the divisors $M_1, \dots, M_{d+1}$ give rise to a base point free linear series. Consider the corresponding morphism $f'\colon X' \to Y$.

We will now apply Conjecture 5.7 to $\mathrm{Ex}(f')$ and $(f')^{-1}(W)$. Let $g\colon Y' \to X'$ and $\mu\colon Y' \to Y$ be the induced morphisms. Since $\mathrm{Ex}(\mu) = \mathrm{Ex}\,(g) \cup g^{-1}(\mathrm{Ex}(f'))$, then $\mathrm{Ex}(\mu)$ is a divisor and so is $\mu^{-1}(W) = g^{-1}((f')^{-1}(W))$. It is clear that their union has simple normal crossings. □

6. Liftability to characteristic 0

Throughout this section, k is a perfect field of characteristic $p > 0$. Our goal is to show Corollary 1.4 and Corollary 1.5, whose statements in full generality are as follows.

Theorem 6.1. *Let $(R, \mathfrak{m})$ be a DVR of mixed characteristic $(0, p)$ for $p > 5$, with a perfect residue field k.*

Let X be a three-dimensional $\mathbb{Q}$-factorial terminal variety projective over k. Suppose that K_X is pseudo-effective and that X lifts to a scheme $\mathcal{X}$ over R. Further, assume that log resolutions of all log pairs with the underlying variety being birational to $\mathcal{X}$ exist (and are given by a sequence of blowups along the non-snc locus). Then the following hold.

(1) *A (possibly non-$\mathbb{Q}$-factorial) minimal model of X lifts over R.*
(2) *If $N^1(\mathcal{X}/\mathrm{Spec}\,R) \to N^1(X)$ is surjective, then*
   (a) *every sequence of steps of a K_X-MMP lifts over R, and*
   (b) *all $\mathbb{Q}$-factorial minimal models of X lift over R.*
(3) *If K_X is big, then the canonical model of X lifts over R.*

Here, $N^1(X) = \mathrm{Pic}(X) \otimes \mathbb{Q}/\equiv$; this is the set of $\mathbb{Q}$-line bundles up to numerical equivalence. Further, we say that a scheme X *lifts* over R if there exists a flat projective morphism $\mathcal{X} \to \mathrm{Spec}\,R$ such that the central fibre $\mathcal{X}_{\mathfrak{m}}$ is isomorphic to X. Finally, recall that a birational map of terminal

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varieties $\phi\colon X \dashrightarrow X_{\min}$ is a minimal model if $K_{X_{\min}}$ is nef [KM98, Definition 2.13]. It is easy to see that $a(X,0,E) < a(X_{\min},0;E)$ for every ϕ-exceptional divisor E. A well-known consequence of the negativity lemma then implies that $a(X,0,E) \leq a(X_{\min},0;E)$ for every divisor E over $X_{\min}$. However, since $X_{\min}$ is possibly not $\mathbb{Q}$-factorial, it is not necessarily the case that ϕ is given by an MMP.

As a corollary to (2b), we get that liftability of three-dimensional terminal Calabi–Yau varieties is a birational invariant. Recall that a projective variety X is Calabi–Yau if ω_X is trivial and $H^i(X,\mathcal{O}_X) = 0$ for $0 < i < \dim X$.

Theorem 6.2. *Let $(R,\mathfrak{m})$ be a complete DVR of mixed characteristic $(0,p)$ for $p > 5$, with a perfect residue field k.*

Let X and Y be three-dimensional terminal $\mathbb{Q}$-factorial projective Calabi–Yau varieties defined over k. Suppose that X and Y are birational and that X lifts to a scheme $\mathcal{X}$ over R. Further, assume that log resolutions of all log pairs with the underlying variety being birational to $\mathcal{X}$ exist (and are given by a sequence of blowups along the non-snc locus).

Then Y lifts over R as well.

It is necessary to assume that R is complete to guarantee that $N^1(\mathcal{X}/\operatorname{Spec} R) \to N^1(X)$ is surjective (see Lemma 6.5).

6.1. Normality of plt centres

Although we do not know that plt centres of four-dimensional pairs are normal in general, we can show this in the setting of lifting using deformation theory. This will be important in the later parts of this section.

Lemma 6.3. *Let $\mathcal{X} \to \operatorname{Spec} R$ be a projective morphism from a normal scheme to a complete DVR $(R,\mathfrak{m})$ of mixed characteristic $(0,p>5)$ with perfect residue field. Suppose that the special fibre X is three-dimensional, reduced, and $(\mathcal{X},X)$ is plt. Then X is normal.*

Moreover, if $(\mathcal{X},X)$ admits a log resolution of singularities, then the assumption that $(R,\mathfrak{m})$ is complete may be dropped.

Proof. As for the last part: The completion $\hat{R} \to R$ is faithfully flat (see [Sta14, Tag 00MC]), so we can replace R by $\hat{R}$ and assume that $(R,\mathfrak{m})$ is complete (see [Sta14, Tag 033G]). The plt-ness of $(\mathcal{X},X)$ is preserved by the same argument as in [BMPSTWW20, Lemma 2.7] (here, we use the existence of a resolution).

Let $\nu : X^\nu \to X$ be the normalisation. Since $(\mathcal{X},X)$ is plt, Lemma 2.1 shows that the special fibre $X = \mathcal{X}_{\mathfrak{m}}$ is normal up to universal homeomorphism. By localising at points of X of dimension one and applying [BMPSTWW20, Corollary 7.17], we get that that X is normal except at possibly finitely many points; explicitly let $Z \subset X^\nu$ be a finite set of points such that $\nu|_{X^\nu \setminus Z}$ is an isomorphism onto its image. We then have that

$$K_{X^\nu} = (K_{\mathcal{X}} + X)|_{X^\nu}$$

is $\mathbb{Q}$-Cartier. Moreover, as $(\mathcal{X},X)$ is plt, we get that X^ν is klt and so Cohen–Macaulay ([ABL22, Corollary 1.3], cf. [HW19, Theorem 1.1]).

Let $U = X^\nu \setminus Z$. Since X lifts to any order (to $X_i = \mathcal{X} \times_R R/\mathfrak{m}^{i+1}$) and $U \simeq \nu(U) \subseteq X$, we have that U also lifts to any order. Explicitly, let U_i be the induced compatible lifts of U to $R/\mathfrak{m}^{i+1}$.

By [Zda17, Lemma A.23], X^ν also lifts to any order. For the convenience of the reader, we explicate this result. Consider short exact sequences

$$0 \to \mathcal{O}_U \to \mathcal{O}_{U_i} \to \mathcal{O}_{U_{i-1}} \to 0$$

of sheaves of rings on $|U|$. Define a sheaf $\mathcal{O}_{X_i^\nu}$ of abelian groups on $|X^\nu|$ by the formula:

$$\mathcal{O}_{X_i^\nu} = i_*\mathcal{O}_{U_i},$$

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where $i\colon U \to X^\nu$ is the inclusion of topological spaces. Pushing forward the above short exact sequence of abelian groups by i and using normality of X^ν, we get

$$0 \to \mathcal{O}_{X^\nu} \to \mathcal{O}_{X_i^\nu} \to \mathcal{O}_{X_{i-1}^\nu} \to R^1 i_* \mathcal{O}_U = 0. \tag{1}$$

Here, $R^1 i_* \mathcal{O}_U$ is calculated in the category of abelian groups, but the answer is the same when it is calculated in the category of coherent sheaves (for example, thanks to Čech resolution, cf. [Sta14, Tag 09V2]), and so it is 0 as X^ν is S_3 and Z has codimension ≥ 3. Finally, $\mathcal{O}_{X_i^\nu}$ inherits a structure of a ring from U_i and so defines a scheme X_i^ν lifting X^ν over $R/\mathfrak{m}^{i+1}$ (here, the flatness follows from the exactness of (1)).

Let $j = \nu \circ i : U \to X$. Then we have compatible homomorphisms

$$\mathcal{O}_{X_i} \to j_* \mathcal{O}_{U_i} = \nu_* \mathcal{O}_{X_i^\nu},$$

and so compatible maps of schemes $X_i^\nu \to X_i$. Hence, we can apply [Sta14, Tag 09ZT and Tag 09ZW(8)], to get a finite map of schemes $\psi\colon \overline{\mathcal{X}} \to \mathcal{X}$ which reduces to $X_i^\nu \to X_i$ modulo $\mathfrak{m}^{i+1}$.

Moreover, $\psi\colon \overline{\mathcal{X}} \to \mathcal{X}$ is birational. Indeed, this is equivalent to showing that the coherent sheaf $\psi_* \mathcal{O}_{\overline{\mathcal{X}}}/\mathcal{O}_{\mathcal{X}}$ is not supported on all of $\mathcal{X}$, which can be checked after localising at a generic point η_X of $X \subseteq \mathcal{X}$. As $\psi_* \mathcal{O}_{\overline{\mathcal{X}}}/\mathcal{O}_{\mathcal{X}}$ is 0 on η_X, this yields a finite module M over a Noetherian R-algebra $A = \mathcal{O}_{\mathcal{X},\eta_X}$ such that $M/\mathfrak{m}M = 0$ (equivalently, $\mathfrak{m}M = M$), and so there exists $f \in 1 + \mathfrak{m}$ satisfying $fM = 0$ by Nakayama's lemma (see [Sta14, Tag 00DV(1)]). In particular, $\operatorname{Ann}(M) \neq 0$, and so $\operatorname{Supp} M \neq \operatorname{Spec} A$ as A is integral.

Since $\mathcal{X}$ is normal, $\psi\colon \overline{\mathcal{X}} \to \mathcal{X}$ must thus be an isomorphism, and so $X \simeq X^\nu$ is normal concluding the proof of the lemma. □

6.2. *Liftability of minimal models to characteristic 0*

Definition 6.4. Let $(R, \mathfrak{m})$ be a DVR of mixed characteristic and with residue field k. We say that a projective scheme X over k *lifts over* R if there exists a flat projective morphism $\mathcal{X} \to \operatorname{Spec} R$ with the central fibre $\mathcal{X}_{\mathfrak{m}}$ isomorphic to X.

We say that a projective scheme X defined over k *lifts to characteristic 0* if there exists a DVR $(R, \mathfrak{m})$ of mixed characteristic and with residue field k such that X lifts over R.

Note that if X is normal, then so is $\mathcal{X}$. Indeed, $\mathcal{X}$ is regular in neighbourhoods of regular points of X (cf. [Sta14, Tag 00NU]), and so it is G_1, and since X is S_2 and Cartier, $\mathcal{X}$ is S_3.

In what follows, we shall often write $S = \operatorname{Spec} R$.

Lemma 6.5. *Let $(R, \mathfrak{m})$ be a complete DVR of mixed characteristic $(0, p)$ with residue field k. Let X be a projective scheme over k and which lifts to $\mathcal{X}$ over R. Further, suppose that $H^2(X, \mathcal{O}_X) = 0$.*

Then any line bundle L on X lifts to a line bundle $\mathcal{L}$ on $\mathcal{X}$ such that $\mathcal{L}|_X = L$ (in particular, $N^1(\mathcal{X}/\operatorname{Spec} R) \to N^1(X)$ is surjective). If moreover L is ample, then so is $\mathcal{L}$.

Proof. Let $X_n = \mathcal{X} \times_S \operatorname{Spec}(R/\mathfrak{m}^{n+1})$, and suppose that L on X lifts to a line bundle L_n on X_n. By [Har10, Theorem 6.4], the obstruction to lifting L_n on X_n to L_{n+1} on X_{n+1} lies in $H^2(X, \mathcal{O}_X)$ and hence vanishes by assumption. By [Sta14, Tag 08BE], there is a line bundle $\mathcal{L}$ on $\mathcal{X}$ such that $\mathcal{L}|_{X_n} \cong L_n$ for every $n > 0$.

The last assertion follows from the fact that ampleness may be checked over closed points (see [Sta14, Tag 0D3A]). □

Lemma 6.6. *Let $\mathcal{X}$ be a normal scheme over a DVR $(R, \mathfrak{m})$ of characteristic $(0, p)$ for $p \geq 0$. Suppose that the central fibre X of $\mathcal{X}$ is a normal S_3 scheme.*

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Let $\mathcal{D}$ be a Weil divisor on $\mathcal{X}$ whose support does not contain X and such that $D = \mathcal{D}|_X$ is $\mathbb{Q}$-Cartier. Further, assume that there is a closed subset $Z \subset \mathcal{X}$ satisfying $\operatorname{codim}(Z \cap X, X) \geq 3$ and for which $D|_{\mathcal{X}\setminus Z}$ is $\mathbb{Q}$-Cartier. Then $\mathcal{D}$ is also $\mathbb{Q}$-Cartier.

Note that since X is normal, it is regular in codimension 1 and as observed above, so is $\mathcal{X}$ (cf. [Sta14, Tag 00NU]). Therefore, $\mathcal{D}|_X$ is defined by taking the closure of the restriction to the regular locus.

Proof. See [KM92, 12.1.8] and [dFH11, Proposition 3.1]. □

Corollary 6.7. *Let $\mathcal{X}$ be a normal scheme over a DVR $(R, \mathfrak{m})$ of characteristic $(0, p)$ for $p > 5$. Suppose that the central fibre X is a three-dimensional terminal variety. Then*

(1) *$\mathcal{X}$ is Cohen–Macaulay, $K_{\mathcal{X}}$ is $\mathbb{Q}$-Cartier and if X is $\mathbb{Q}$-factorial, then so is $\mathcal{X}$,*
(2) *If X is $\mathbb{Q}$-factorial and log resolutions of all log pairs with the underlying variety being birational to $\mathcal{X}$ exist (and are constructed by a sequence of blowups along the non-snc locus), then $(\mathcal{X}, X)$ is plt and $\mathcal{X}$ is terminal.*

Note that the terminality of $\mathcal{X}$ in (2) is not used later on, but we believe that it is of independent interest.

Proof. By [ABL22, Corollary 1.3], X is Cohen–Macaulay and as X is a Cartier divisor, $\mathcal{X}$ is also Cohen–Macaulay. Since X is terminal, it is normal and regular in codimension 2, thus so is $\mathcal{X}$ (cf. [Sta14, Tag 00NU]).

For the rest of (1), pick a divisor $\mathcal{D}$ whose support does not contain X and such that $\mathcal{D}|_X$ is $\mathbb{Q}$-Cartier. We claim that then $\mathcal{D}$ is also $\mathbb{Q}$-Cartier. Replacing $\mathcal{D}$ by a multiple, we may assume that $\mathcal{D}|_X$ is Cartier. Notice that if $U = \mathcal{X} \setminus \mathcal{Z}$ is the regular locus, then $\mathcal{D}$ is Cartier on $\mathcal{X} \setminus \mathcal{Z}$ and $\operatorname{codim}(Z, X) \geq 3$ where $Z = \mathcal{Z} \cap X$. Hence, the hypotheses of Lemma 6.6 are satisfied, and so $\mathcal{D}$ is Cartier.

In particular, the above paragraph shows that $K_{\mathcal{X}}$ is $\mathbb{Q}$-Cartier, as K_X is $\mathbb{Q}$-Cartier and $K_{\mathcal{X}}|_X = K_X$. Suppose now that X is $\mathbb{Q}$-factorial, and let $\mathcal{D}$ be a $\mathbb{Q}$-divisor on $\mathcal{X}$. We need to show that $\mathcal{D}$ is $\mathbb{Q}$-Cartier. Since $X \subseteq \mathcal{X}$ is Cartier and irreducible, we may assume that the support of $\mathcal{D}$ does not contain X. Since X is $\mathbb{Q}$-factorial, we have that $\mathcal{D}|_X$ is $\mathbb{Q}$-Cartier, and so by the above paragraph, $\mathcal{D}$ is $\mathbb{Q}$-Cartier as well, concluding the proof of the $\mathbb{Q}$-factoriality of $\mathcal{X}$.

Now, we move to (2) assuming the existence of appropriate resolutions. Note that as X is terminal, it is klt, and so $(\mathcal{X}, X)$ is plt by Corollary 4.9.

Next, we woud like to construct a terminalisation of $\mathcal{X}$. Alas, due to the limitations of our MMP results (Theorem 4.1 and Theorem 5.1), we can only construct a partial terminalisation, but fortunately that will be enough for our application.

Claim 6.8. *There exists a projective birational morphism $\phi\colon \mathcal{Y} \to \mathcal{X}$ such that*

(1) *$\mathcal{Y}$ is terminal,*
(2) *The central fibre Y of $\mathcal{Y}$ over* Spec *R is integral,*
(3) *$(\mathcal{Y}, Y)$ is plt and*
(4) *$\Delta_{\mathcal{Y}} \geq 0$, where $K_{\mathcal{Y}} + \Delta_{\mathcal{Y}} = \phi^* K_{\mathcal{X}}$.*

Proof. Let $\pi\colon \mathcal{Z} \to \mathcal{X}$ be a log resolution of $(\mathcal{X}, X)$, and set Z to be the reduced central fibre of $\mathcal{Z}$ over Spec R. Let $h\colon \mathcal{Z} \dashrightarrow \mathcal{Y}$ be the output of a $(K_{\mathcal{Z}} + Z)$-MMP over $\mathcal{X}$. We can run this MMP by Theorem 5.1 (note that we can run it *over* $\mathcal{X}$ as we can always add to the boundary a pullback of a sufficiently ample divisor from $\mathcal{X}$ and by boundedness of extremal rays, such an MMP would only contract rays over $\mathcal{X}$). Then $K_{\mathcal{Y}} + h_* Z$ is nef over $\mathcal{X}$.

Denote by ϕ the induced map $\mathcal{Y} \to \mathcal{X}$, and write

$$K_{\mathcal{Y}} + \Delta_{\mathcal{Y}} = \phi^* K_{\mathcal{X}}. \tag{2}$$

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In particular, $K_{\mathcal{Y}} + \phi^* X + \Delta_{\mathcal{Y}} = \phi^*(K_{\mathcal{X}} + X)$, and since $(\mathcal{X}, X)$ is plt, we get that all the coefficients of $\phi^* X + \Delta_{\mathcal{Y}}$, except for $\phi_*^{-1} X$ are smaller than 1. Then

$$K_{\mathcal{Y}} + h_* Z - \phi^*(K_{\mathcal{X}} + X) = h_* Z - (\phi^* X + \Delta_{\mathcal{Y}})$$

is nef and exceptional and so antieffective. As the coefficients of $\phi^* X + \Delta_{\mathcal{Y}}$, except for $\phi_*^{-1} X$, are smaller than 1, we get that $h_* Z = \phi_*^{-1} X$. Thus, the central fibre Y of $\mathcal{Y}$ is irreducible. Since the central fibres of $\mathcal{Y}$ and $\mathcal{X}$ agree generically, Y must also be reduced, and so $h_* Z = Y$. This proves (2). Since Y is irreducible and $(\mathcal{Y}, Y)$ is dlt as an output of a dlt MMP, $(\mathcal{Y}, Y)$ is plt proving (3).

Since Y is the central fibre, $K_{\mathcal{Y}} \sim K_{\mathcal{Y}} + Y$ is nef, and so by negativity lemma and (2), we have that $\Delta_{\mathcal{Y}} \geq 0$ concluding the proof of (4).

Finally, to show (1) we observe that $\mathcal{Y}_{\mathbb{Q}}$ is terminal, as $\mathcal{Z}_{\mathbb{Q}}$ is smooth and we were running a $(K_{\mathcal{Z}} + Z)$-MMP which, over $\mathbb{Q}$, is a $K_{\mathcal{Z}_{\mathbb{Q}}}$-MMP. Thus, to show that $\mathcal{Y}$ is terminal, we need to show that the log discrepancy with respect to $(\mathcal{X}, 0)$ of every divisor E lying over $\mathfrak{m}$, is greater than *one*. This is clear, because such divisors have positive log discrepancy with respect to the plt pair $(\mathcal{X}, X)$ and X is Cartier. □

Now, assume that X is terminal, but suppose by contradiction that $\mathcal{X}$ is not terminal, or equivalently (by property (4) in the above claim) that $\phi\colon \mathcal{Y} \to \mathcal{X}$ is not an isomorphism. Let $\mathcal{E}$ be a ϕ-exceptional prime divisor on $\mathcal{Y}$ (which exists as $\mathcal{X}$ is $\mathbb{Q}$-factorial). Since Y is the reduced special fibre and Spec R admits only one closed point, we have that $E = \mathcal{E} \cap Y$ is nonempty.

Since the central fibre Y is integral (and so must be equal to $\phi_*^{-1} X$), our divisor $\mathcal{E}$ must be dominant over Spec R. In particular, the irreducible subscheme $\phi(\mathcal{E})$ is also dominant over Spec R and of codimension at least two in $\mathcal{X}$. Hence, $\phi|_Y(E) \subseteq X$ is of codimension at least two in X, and so $E \subseteq Y$ is exceptional over X. This contradicts the fact that X is terminal as

$$\begin{aligned} K_Y &= K_{\mathcal{Y}}|_Y = (\phi^* K_{\mathcal{X}} + G)|_Y = (\phi|_Y)^{-1} K_X + G|_Y \\ K_Y &= (\phi|_Y)^* K_X + F, \end{aligned}$$

where $G := -\Delta_{\mathcal{Y}} \leq 0$ by (4) in the above claim, and $F \geq 0$ with $\operatorname{Supp} F = \operatorname{Exc}(\phi|_Y)$ (in particular, it contains the exceptional divisor E with positive coefficient) as X is terminal. □

Proof of Theorem 6.1. By Theorem 6.7, $\mathcal{X}$ is Cohen Macaulay, $\mathbb{Q}$-factorial, terminal and $(\mathcal{X}, X)$ is plt. Let η be the generic point of $S = \operatorname{Spec} R$, then we claim that K_{X_η} is pseudo-effective.

To see this, let $\mathcal{A}$ be an ample Cartier divisor on $\mathcal{X}$, and let $t \in \mathbb{Q}$ be the pseudo-effective threshold (or 0, if $K_{\mathcal{X}}$ is already pseudo-effective) so that $0 \leq \kappa(K_{\mathcal{X}_\eta} + t\mathcal{A}_\eta)$ (the fact that t is rational follows by running a characteristic 0 K_{X_η}-MMP with scaling of $\mathcal{A}_\eta$ and the cone theorem applied to a resulting Mori fibre space if $t > 0$). We wish to show that $t = 0$. We may assume that $(\mathcal{X}, X + t\mathcal{A})$ is plt and (X, tA) is terminal, where $A = \mathcal{A}|_X$. By Theorem 5.1, we may run the $K_{\mathcal{X}} + X + t\mathcal{A}$ MMP with scaling over S which terminates with a minimal model $\phi\colon \mathcal{X} \dashrightarrow \mathcal{X}'$. Let X' be the special fibre of $\mathcal{X}'$ over Spec R; equivalently $X' = \phi_* X$. By definition, X' is Cartier and $K_{\mathcal{X}'} + X' + t\mathcal{A}'$ is nef over S. As $(\mathcal{X}', X' + t\mathcal{A}')$ is plt (and so is $(\mathcal{X}', X')$ in view of $\mathcal{X}'$ being $\mathbb{Q}$-factorial), Lemma 6.3 shows that X' is normal. Note that as X is terminal,

$$\phi|_X\colon X \dashrightarrow X'$$

does not extract any divisors. Indeed, let $\mathcal{A}' = \phi_* \mathcal{A}$ and $A' = \mathcal{A}'|_{X'}$, then every divisor on X', say $E \subseteq X'$, has log discrepancy ≤ 1 with respect to (X', tA'), and so log discrepancy ≤ 1 with respect to (X, tA) (running a $(K_{\mathcal{X}} + X + t\mathcal{A})$-MMP can only increase the discrepancies of $(\mathcal{X}, X + t\mathcal{A})$, and hence cannot decrease those of (X, tA) by adjunction). Thus, it is impossible for E to be extracted because then E is not a divisor on X, and so by definition of terminality has log discrepancy > 1.

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Note that $K_{X'} + tA'$ and $K_{\mathcal{X}'_\eta} + t\mathcal{A}'_\eta$ are nef and if $t > 0$, then $K_{X'} + tA'$ is big. It then follows that

$$\mathrm{vol}(K_{\mathcal{X}'_\eta} + t\mathcal{A}'_\eta) = (K_{\mathcal{X}'_\eta} + t\mathcal{A}'_\eta)^3 = (K_{X'} + tA')^3 = \mathrm{vol}(K_{X'} + tA') > 0$$

so that $K_{\mathcal{X}'_\eta} + t\mathcal{A}'_\eta$ is also big, a contradiction to t being positive. Therefore, we may assume that $t = 0$ and hence $K_{\mathcal{X}'} + X'$ is nef over S. Since $K_{X'} = (K_{\mathcal{X}'} + X')|_{X'}$ is $\mathbb{Q}$-Cartier and $(\mathcal{X}', X')$ is plt, we get that X' is klt and $K_{X'}$ is nef. It follows that $X \dashrightarrow X'$ is a (possibly non-$\mathbb{Q}$-factorial) minimal model concluding the proof of (1).

We move on to the proof of (2a). Suppose now that $N^1(\mathcal{X}/S) \to N^1(X)$ is surjective. In particular, $\overline{\mathrm{NE}}(\mathcal{X}/S) = \overline{\mathrm{NE}}(X)$; indeed this is true because only curves which are projective over S (and hence lie on the special fibre X) are used in the definition of $\overline{\mathrm{NE}}$ (cf. [Tan18] or [BMPSTWW20]) and numerical equivalence relations on X and $\mathcal{X}$ are the same in view of $N^1(\mathcal{X}/S) \to N^1(X)$ being surjective.

Our goal is to show that any sequence of K_X-MMP steps

$$X \overset{h}{\dashrightarrow} X_1 \dashrightarrow X_2 \dashrightarrow \ldots \dashrightarrow X_l$$

(where X_l is a minimal model of X) lifts to a sequence of $K_{\mathcal{X}}$-MMP steps

$$\mathcal{X} \overset{\phi}{\dashrightarrow} \mathcal{X}_1 \dashrightarrow \mathcal{X}_2 \dashrightarrow \ldots \dashrightarrow \mathcal{X}_l.$$

Consider the first step $h\colon X \dashrightarrow X_1$. Let A be an ample $\mathbb{Q}$-divisor on X such that $(K_X + A)^\perp \subseteq \overline{\mathrm{NE}}(X)$ is equal to the extremal ray Σ defining h. Since $N^1(\mathcal{X}/S) \to N^1(X)$ is surjective, there exists an ample $\mathbb{Q}$-divisor $\mathcal{A}$ on $\mathcal{X}$ such that $\mathcal{A}|_X \equiv A$. Then $(K_{\mathcal{X}} + X + \mathcal{A})^\perp = \Sigma \subseteq \overline{\mathrm{NE}}(\mathcal{X}/S)$.

We set $\phi\colon \mathcal{X} \dashrightarrow \mathcal{X}_1$ to be the step of a $(K_{\mathcal{X}} + X)$-MMP (equivalently, $K_{\mathcal{X}}$-MMP) corresponding to Σ. In particular, $(\mathcal{X}_1, (\mathcal{X}_1)_{\mathfrak{m}})$ is plt, where $(\mathcal{X}_1)_{\mathfrak{m}}$ is the special fibre, hence $(\mathcal{X}_1)_{\mathfrak{m}}$ is normal by Lemma 6.3. As explained before, the terminality of X ensures that $\phi|_X\colon X \dashrightarrow (\mathcal{X}_1)_{\mathfrak{m}}$ does not extract any divisors.

We claim that $\phi|_X\colon X \dashrightarrow (\mathcal{X}_1)_{\mathfrak{m}}$ and $h\colon X \dashrightarrow X_1$ coincide. To this end, let $f\colon X \to Z$ be the contraction of $\Sigma \subseteq \overline{\mathrm{NE}}(X)$ induced by $K_X + A$, and let $\psi\colon \mathcal{X} \to \mathcal{Z}$ be the contraction of $\Sigma \subseteq \overline{\mathrm{NE}}(\mathcal{X}/S)$ induced by $K_{\mathcal{X}} + X + \mathcal{A}$. Note that $\psi|_X\colon X \to \mathcal{Z}_{\mathfrak{m}}$ contracts exactly Σ and no other numerical class of a curve; hence, we have a factorisation

$$\psi|_X\colon X \overset{f}{\to} Z \to \mathcal{Z}_{\mathfrak{m}} \tag{3}$$

and Z is the normalisation of $\mathcal{Z}_{\mathfrak{m}}$.

Before proceeding with the proof of the claim, we will show that $\mathcal{Z}_{\mathfrak{m}} \simeq Z$ and $\psi|_X = f$. To this end, we may assume that (X, A) is klt (cf. [BMPSTWW20, Lemma 2.29] or [Bir16, Lemma 9.2]; this uses the existence of a resolution with an ample exceptional divisor). Hence, (Z, f_*A) is also klt, and so both X and Z have rational singularities by [ABL22, Corollary 1.3]. Therefore, $R^1 f_* \mathcal{O}_X = 0$ (equivalently, $R^1(\psi|_X)_* \mathcal{O}_X = 0$), which in turn implies that $R^1\psi_* \mathcal{O}_{\mathcal{X}} = 0$ (specifically, we apply the semicontinuity theorem, [Har77, Theorem III.2.18], to the flat morphism $X \to \mathrm{Spec}\, R$ and the sheaf $\psi^*\mathcal{L}^m$ for an ample line bundle $\mathcal{L}$ on $\mathcal{Z}$ and $m \gg 0$). Hence, by the projection formula

$$R^1\psi_* \mathcal{O}_{\mathcal{X}}(m(K_{\mathcal{X}} + \mathcal{A})) = 0$$

for m divisible enough, and so

$$\psi_* \mathcal{O}_{\mathcal{X}}(m(K_{\mathcal{X}} + \mathcal{A})) \to (\psi|_X)_* \mathcal{O}_X(m(K_X + A))$$

is surjective. In particular, $\mathcal{Z}_{\mathfrak{m}} \simeq Z$ and $\psi|_X = f$.

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Now, we consider the following cases:

(1) $\phi\colon \mathcal{X} \dashrightarrow \mathcal{X}_1$ is a divisorial contraction (that is, $\psi = \phi$): Then $\psi|_X = f$ is also divisorial by semicontinuity of fibre dimension [Gro67, Corollaire (13.1.5)]; in particular, the first step $h\colon X \dashrightarrow X_1$ of the K_X-MMP is the contraction $f\colon X \to Z$; and hence, it lifts to $\phi = \psi\colon \mathcal{X} \to \mathcal{Z}$.
(2) $\phi\colon \mathcal{X} \dashrightarrow \mathcal{X}_1$ is a flip over the flipping contraction $\psi\colon \mathcal{X} \to \mathcal{Z}$ (in particular, $\mathcal{Z}$ is not $\mathbb{Q}$-Gorenstein) and
   (a) $f\colon X \to Z$ is divisorial: Then Z is $\mathbb{Q}$-Gorenstein, Cohen–Macaulay (as it has rational singularities), and $\operatorname{codim}\operatorname{Sing}(Z) \geq 3$ (as X is terminal). Since $Z \simeq \mathcal{Z}_{\mathfrak{m}}$, we get that $\mathcal{Z}$ is $\mathbb{Q}$-Gorenstein by Lemma 6.6 (cf. the proof of Corollary 6.7) which is impossible. This shows that f can never be divisorial when ϕ is a flip.
   (b) $f\colon X \to Z$ is small: Then, $(\mathcal{X}_1)_{\mathfrak{m}}$ is normal (cf. Lemma 6.3), and since $\phi|_X\colon X \dashrightarrow (\mathcal{X}_1)_{\mathfrak{m}}$ does not extract any divisors, the induced map $(\mathcal{X}_1)_{\mathfrak{m}} \to Z$ is also small. As $(\mathcal{X}_1)_{\mathfrak{m}}$ is $\mathbb{Q}$-Gorenstein and $K_{(\mathcal{X}_1)_{\mathfrak{m}}} = K_{\mathcal{X}_1}|_{(\mathcal{X}_1)_{\mathfrak{m}}}$ is ample over Z, we have that $(\mathcal{X}_1)_{\mathfrak{m}}$ is the canonical model of X over Z. Hence it is a flip, $(\mathcal{X}_1)_{\mathfrak{m}} \simeq X_1$ and $\phi|_X = h$.

Having lifted $h\colon X \dashrightarrow X_1$, we would like to replace X and $\mathcal{X}$ by X_1 and $\mathcal{X}_1$, respectively, and repeat the above procedure. To be able to do this, we deduce the surjectivity of $N^1(\mathcal{X}_1/S) \to N^1(X_1)$ from that of $N^1(\mathcal{X}/S) \to N^1(X)$. Pick a line bundle L on X_1. We need to show that $L^m \equiv \mathcal{L}|_{X_1}$ for some line bundle $\mathcal{L}$ on $\mathcal{X}_1$ and $m \in \mathbb{N}$. This is a standard consequence of the base point free theorem as explained by the following case by case analysis:

(1) $\phi = \psi\colon \mathcal{X} \to \mathcal{Z}$ is a divisorial contraction (in particular, $Z = X_1$): Then up to replacing L by a multiple, we may pick $\mathcal{L}'$ on $\mathcal{X}$ such that $f^*L \equiv \mathcal{L}'|_X$. Since $\mathcal{L}'$ is numerically trivial over $\mathcal{Z}$, it descends, up to some multiple, to a line bundle $\mathcal{L}$ on $\mathcal{Z}$ by the base point free theorem (cf. Proposition 5.4), and $\mathcal{L}|_Z \equiv L$ by construction.
(2) $\phi\colon \mathcal{X} \dashrightarrow \mathcal{X}_1$ is small: Then since $K_{\mathcal{X}_1}|_{X_1} = K_{X_1}$ is ample over Z and $\rho(X_1/Z) = 1$, we can replace L by $L + aK_{X_1}$ for some $a \in \mathbb{Q}$ and assume that $L \equiv_Z 0$. Then by the base point free theorem (cf. Proposition 5.4), up to some multiple, L descends to a line bundle L_Z on $Z = \mathcal{Z}_{\mathfrak{m}}$. By the same argument as in (1), up to some multiple, there exists $\mathcal{L}_{\mathcal{Z}}$ on $\mathcal{Z}$ such that $\mathcal{L}_{\mathcal{Z}}|_Z \equiv L_Z$. Then $(\psi^+)^*\mathcal{L}_{\mathcal{Z}}|_{X_1} \equiv L$, where $\psi^+\colon \mathcal{X}_1 \to \mathcal{Z}$ is the flipped contraction.

Thus, we can repeat the above procedure until we get a lift $\mathcal{X}_l$ of X_l. Since K_{X_l} is nef, so is $K_{\mathcal{X}_l}$. Hence, $\mathcal{X}_l$ is a minimal model of $\mathcal{X}$.

Next, we tackle (2b). By (2a), we can replace X by the output of a K_X-MMP, and so assume that X is minimal. Let $h\colon X \dashrightarrow Y$ be a small birational map to another $\mathbb{Q}$-factorial minimal model Y. It is well known (cf. [Kaw08]) that $X \dashrightarrow Y$ is a small birational map given by a finite sequence of flops. To see this, pick A_Y an ample Cartier divisor on Y and let A be its strict transform on X. Since $X \dashrightarrow Y$ is small, A is $\mathbb{Q}$-movable and big. Pick $0 \leq D \sim_{\mathbb{Q}} A$. We may assume that $(X, \epsilon D)$ is terminal. We claim that for $0 < \epsilon \ll 1$ there is a finite sequence of $(K_X + \epsilon D)$-flips which are K_X-flops inducing a $(K_X + \epsilon D)$-minimal model $\phi : X \dashrightarrow X'$. We proceed as follows. If $K_X + tD$ is nef for some $0 < t < \epsilon/(6r+1)$, we may replace ϵ by t and we are done. If on the other hand $K_X + tD$ is not nef for any $0 < t < \epsilon/(6r+1)$, then let Σ be a negative extremal ray generated by a curve C. Note that Σ is also a $(K_X + \epsilon D)$-negative extremal ray, and so by cf. [DW19b, Theorem 1.1], we may assume that there is an integer $d = d_\Sigma$ (depending only on Σ) such that $(K_X + \epsilon D) \cdot C \geq -6d$ and if rK_X is Cartier and $K_X \cdot C \neq 0$, then we have $K_X \cdot C \geq d/r$. But then

$$(K_X + tD) \cdot C = (1 - \frac{t}{\epsilon})K_X \cdot C + \frac{t}{\epsilon}(K_X + \epsilon D) \cdot C \geq (1 - \frac{t}{\epsilon}) \cdot \frac{d}{r} - \frac{6dt}{\epsilon} > 0.$$

This is impossible and so $K_X \cdot C = 0$. It follows that Σ is a $(K_X + \epsilon D)$-negative extremal ray which is K_X trivial. Let $X \dashrightarrow X^+$ be the corresponding flop, then K_{X^+} is nef and rK_{X^+} is Cartier (cf. [ABL22, Corollary 1.5]). We may replace X by X^+ and repeat the procedure which terminates after finitely many

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steps by Proposition 2.15 (note in fact that each flip is D-negative, and hence the flipping locus is always contained in the support of D).

Let $\phi : X \dashrightarrow X'$ be the corresponding minimal model. Since D is movable, ϕ is an isomorphism in codimension 1. Since $K_Y + \epsilon A_Y$ is ample, the (unique) canonical model of $K_X + \epsilon D$ is isomorphic to Y and the induced morphism $X' \to Y$ is a small birational morphism of $\mathbb{Q}$-factorial varieties and hence an isomorphism. Therefore we may assume that $X \dashrightarrow Y$ is a K_X-flop and we must show that Y lifts to $\mathcal{Y}$ over R and $N^1(\mathcal{Y}/\mathrm{Spec}(R)) \to N^1(Y)$ is surjective.

Let $f\colon X \to Z$ be the flopping contraction and $H = f^*H_Z$ the pullback of an ample divisor H_Z on Z. We also set $\Sigma \in \overline{\mathrm{NE}}(X)$ to be the corresponding $(K_X + A)$-negative extremal ray.

Since $N^1(\mathcal{X}/S) \to N^1(X)$ is surjective, there exist Cartier divisors $\mathcal{A}$ and $\mathcal{H}$ on $\mathcal{X}$ such that $\mathcal{A}|_X = A$ and $\mathcal{H}|_X = H$. Since H is nef and big, it is easy to see that $\mathcal{H}_\eta$ is nef and big as well, and hence $\mathcal{H}$ is big over S. Replacing $\mathcal{H}$ by a multiple, we may assume that $\mathcal{H} + \mathcal{A}$ is big. Pick $0 \le \mathcal{D} \sim_{\mathbb{Q}} \mathcal{H} + \mathcal{A}$ an effective $\mathbb{Q}$-divisor whose support does not contain X. For $0 < \epsilon \ll 1$, we may assume that $(\mathcal{X}, X + \epsilon\mathcal{D})$ is plt and $(X, \epsilon D)$ is terminal, where $D = \mathcal{D}|_X$. Arguing as in the proof of (2a), we have a $(K_{\mathcal{X}} + \epsilon\mathcal{D})$-flip $\mathcal{X} \dashrightarrow \mathcal{X}^+$ corresponding to the extremal ray $\Sigma \in \overline{\mathrm{NE}}(\mathcal{X}/S)$ and $\mathcal{X}^+_{\mathfrak{m}} = Y$. The surjectivity of $N^1(\mathcal{Y}/\mathrm{Spec}(R)) \to N^1(Y)$ also follows as in the proof of (2a). This concludes the proof of (2b).

Finally, we tackle (3). By (1), we can replace X by a (possibly non-$\mathbb{Q}$-factorial) minimal model and assume that K_X is nef and big. Note, however, that the lift $\mathcal{X}$ of X constructed in (1) is $\mathbb{Q}$-factorial. Now, $K_{\mathcal{X}} + X \sim_{\mathbb{Q}} K_{\mathcal{X}}$ is also nef and big, and so it is semiample by Proposition 5.2. Let $\psi\colon \mathcal{X} \to \mathcal{X}^c$ be the induced birational morphism. Since $(\mathcal{X}^c, (\mathcal{X}^c)_{\mathfrak{m}})$ is plt, Lemma 6.3 shows that $(\mathcal{X}^c)_{\mathfrak{m}}$ is normal. In particular, $\psi|_X = f$, where $f\colon X \to X^c$ is the map induced by K_X. Therefore, $\mathcal{X}^c$ is the lift of X^c, which concludes the proof of (3). □

Proof of Theorem 6.2. By Lemma 6.5, $N^1(\mathcal{X}/S) \to N^1(X)$ is surjective. Hence, we can conclude by Theorem 6.1(2b) as Y is a minimal model of X. □

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