

# Vanishing Theorems for Three-folds in Characteristic $p > 5$

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We prove Grauert–Riemenschneider–type vanishing theorems for excellent divisorially log terminal threefolds pairs whose closed points have perfect residue fields of positive characteristic  $p > 5$ . Then we discuss applications to dlt singularities and to Mori fiber spaces of three-folds.

## 1 Introduction

The Grauert–Riemenschneider vanishing theorem says that if  $g: Y \rightarrow X$  is a proper birational morphism over  $\mathbb{C}$  and  $Y$  is smooth then  $R^i g_* \omega_Y = 0$  for  $i > 0$ . This is known to fail in positive characteristic, even for three-folds. However, Grauert–Riemenschneider vanishing holds if  $Y, X$  are both regular excellent schemes by [7, Theorem 1], or if  $Y$  is an excellent Cohen–Macaulay scheme and  $X$  has rational singularities by [25, Theorem 1.4].

In many applications one would need a similar vanishing theorem where  $\omega_Y$  is replaced by some other line bundle. Our main technical result, Theorem 1.3, gives such a vanishing for excellent dlt three-folds whose closed points have perfect residue fields of positive characteristic  $p > 5$ .

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Then in Section 3 we use Theorem 1.3 to derive local rationality properties of divisorially log terminal (dlt) three-fold pairs in positive and mixed characteristic. Finally we use the same techniques to obtain liftability and rationality results for Mori fiber spaces in dimension 3 over perfect fields of positive characteristic; see Section 5.

**Definition 1.1** (G-R vanishing). Let  $(X, \Delta)$  be a pair, where  $X$  is a normal, excellent scheme with a dualizing complex and  $\Delta$  is a boundary on  $X$  (i.e., an  $\mathbb{R}$ -divisor whose coefficients are in  $[0, 1]$ ). We assume from now on that log resolutions of singularities exist for the pairs that we work with.

We say that (strong) *G-R vanishing* holds over  $(X, \Delta)$  if the following is satisfied for every log resolution  $g: (X', E + g_*^{-1}\Delta) \rightarrow (X, \Delta)$ , where  $E$  is the exceptional divisor of  $g$ .

Let  $D'$  be a Weil  $\mathbb{Z}$ -divisor and  $\Delta'$  an effective  $\mathbb{R}$ -divisor on  $X'$ . Assume that  $g_*\Delta' \leq \Delta$ ,  $[\text{Ex}(\Delta')] = 0$  (where  $\text{Ex}(\Delta')$  denotes the  $g$ -exceptional part of  $\Delta'$ ), and

- (a) (for G-R vanishing)  $D' \sim_{g, \mathbb{R}} K_{X'/X} + \Delta'$ ,
- (b) (for strong G-R vanishing)  $D' \sim_{g, \mathbb{R}} K_{X'/X} + \Delta' + (g\text{-nef } \mathbb{R}\text{-divisor})$ .

Then  $R^i g_* \mathcal{O}_{X'}(D') = 0$  for  $i > 0$ .

We check in Section 2.3 that if G-R vanishing holds for one log resolution of  $X$ , then it holds for every log resolution.

**Remark 1.2.** G-R vanishing is simpler to handle since the condition  $D' \sim_{g, \mathbb{R}} K_{X'} + \Delta'$  is preserved by contractions and flips. By contrast, being  $g$ -nef is not preserved. However, the two versions might be almost equivalent, though we cannot formulate a precise statement or conjecture.

If  $X$  is essentially of finite type over a field of characteristic 0, then strong G-R vanishing is a special case of the general Kodaira-type vanishing theorems; see [17, 2.68]. Recently these vanishing theorems have been extended to general excellent  $\mathbb{Q}$ -schemes by Murayama; see [28]. Strong G-R vanishing also holds over 2-dimensional, excellent schemes by [27]; see [20, 10.4]. In particular, if  $X$  is any normal excellent scheme, then the support of  $R^i g_* \mathcal{O}_{X'}(D') = 0$  has codimension  $\geq 3$  for  $i > 0$ .

However, G-R vanishing is known to fail for three-folds in every positive characteristic, as shown by affine cones over smooth projective surfaces violating the Kodaira vanishing theorem. Thus we need some restrictions on the singularities of the base for G-R vanishing to hold in positive and mixed characteristic. The following is our main result:

**Theorem 1.3.** Let  $(X, \Delta)$  be a 3-dimensional dlt pair, where  $X$  is an excellent scheme with a dualizing complex whose closed points have perfect residue fields of characteristic  $p > 5$ . Then G-R vanishing holds over  $(X, \Delta)$ .

In Theorem 1.3 the assumption on the characteristics is optimal, as shown by the examples in [1, 2, 8].

**Sketch of the proof.** To prove Theorem 1.3, we consider a log resolution  $g: Y \rightarrow (X, \Delta)$  and we run a suitable Minimal Model Program (MMP) on  $Y$ , which will end with  $(X, \Delta)$ . Since in Section 2.3 we show that G-R vanishing holds on snc three-fold pairs, we are left to show that G-R vanishing is preserved under the steps of the MMP. This is done in Proposition 4.1 (resp. Proposition 4.2) in the case of divisorial contractions (resp. flips). The case of divisorial contractions is where we need the assumptions on the residue fields as we use the following vanishing theorem for surfaces of del Pezzo type proved in [1, Theorem 1.1]:

**Theorem 1.4.** Let  $k$  be a perfect field of characteristic  $p > 5$ . Let  $X$  be a surface of del Pezzo type over  $k$ . Let  $D$  be a Weil divisor on  $X$  and suppose that there exists an effective  $\mathbb{Q}$ -divisor  $\Delta$  such that  $(X, \Delta)$  is a klt pair and  $D - (K_X + \Delta)$  is big and nef. Then  $H^i(X, \mathcal{O}_X(D)) = 0$  for  $i > 0$ . ■

## 2 Preliminaries

In this section we gather various results that we use. Most of these have appeared in the literature, but not stated in the form or generality that we need.

We refer to [20] and [4, Section 2.5] for the basic definitions in birational geometry and of the singularities appearing in the Minimal Model Program for excellent schemes.

### 2.1 A restriction short exact sequence

The following two results are proved, though not explicitly stated, in [14, Section 3].

**Lemma 2.1.** Let  $(X, S + \Delta)$  be an excellent dlt pair where  $S$  is a prime divisor with normalization  $\nu: \bar{S} \rightarrow S$ . Then, for every Weil  $\mathbb{Z}$ -divisor class  $D_X$  on  $X$ , there are

1. a Weil  $\mathbb{Z}$ -divisor class  $D_{\bar{S}}$  on  $\bar{S}$ ,

2. a  $\mathbb{Q}$ -divisor  $0 \leq \Theta_D \leq \text{Diff}_{\bar{S}}(\Delta)$ , and
3. a natural left exact sequence

$$0 \rightarrow \omega_X(D_X) \rightarrow \omega_X(S + D_X) \xrightarrow{r_S} v_* \omega_{\bar{S}}(D_{\bar{S}}) \rightarrow 0, \quad (2.1.3.1)$$

such that

4.  $D_{\bar{S}} \sim_{\mathbb{Q}} D|_{\bar{S}} + \Theta_D$ ,
5. the sequence (2.1.3.1) is exact along the regular locus of  $S$ , and
6. if  $\omega_X(D_X)$  is  $S_3$  along  $S$ , then (2.1.3.1) is exact.

**Proof.** Note that  $S$  is regular in codimension one by [20, 2.31], but it need not be normal in positive characteristic (see [3, 8]).

We start by replacing  $D_X$  with a linearly equivalent divisor not containing  $S$ . In this way  $D_X|_{\bar{S}}$  is a well-defined  $\mathbb{Q}$ -divisor on  $\bar{S}$ . We now have the exact sequence

$$0 \rightarrow \omega_X(D_X) \rightarrow \omega_X(S + D_X) \xrightarrow{r_S} \mathcal{Q} \rightarrow 0,$$

where  $\mathcal{Q}$  is a sheaf supported on  $S$ . Note that  $\mathcal{Q}$  is a torsion free sheaf of rank one on  $S$  by [20, 2.60] and there is a natural injection into its  $S_2$ -hull  $\mathcal{Q} \hookrightarrow \mathcal{Q}^{(**)}$  (if  $S$  is normal, the  $S_2$ -hull equals the reflexive hull; see [21] for the general case.)

Let  $\mathcal{Q}|_{S_{\text{reg}}}$  be the restriction of  $\mathcal{Q}$  to the regular locus of  $S$ . Since  $S$  is regular in codimension one,  $\mathcal{Q}|_{S_{\text{reg}}}$  extends naturally to a reflexive divisorial sheaf  $\bar{\mathcal{Q}}$  on  $\bar{S}$ . Since  $v_* \bar{\mathcal{Q}}$  is  $S_2$  and coincide with  $\mathcal{Q}^{(**)}$  on a big open set, we conclude they are isomorphic.

We are left to show that  $\bar{\mathcal{Q}}$  is isomorphic to  $\mathcal{O}_{\bar{S}}(K_{\bar{S}} + D_{\bar{S}})$ , where  $D_{\bar{S}} := \lfloor D_X|_{\bar{S}} \rfloor$ . It is enough to prove this after localization at generic points of  $\text{Supp } \bar{\mathcal{Q}}$ . We may thus assume that  $\dim X = 2$ , and then  $\pi: Y \rightarrow (X, S + \Delta)$  has a thrifty log resolution [20, 2.79]. The proof of [14, Proposition 3.1] now carries over.

If  $\omega_X(D_X)$  is  $S_3$ , we conclude that the sequence (2.1.3.1) is exact because  $\mathcal{Q}$  is  $S_2$  by [20, 2.60]. ■

We will use sequences as in (2.1.3.1) to lift vanishing statements from  $S$  to  $X$ , but we need a stronger form of vanishing of Kawamata–Viehweg type on  $S$ .

**Definition 2.2.** Let  $(X, S + \Delta)$  be an excellent dlt pair and  $g: X \rightarrow Z$  a proper morphism. By  $K$ - $V$  vanishing for  $g: (X, S + \Delta) \rightarrow Z$  we mean the following property.

1.  $R^i g_* \mathcal{O}_X(D) = 0$  for  $i > 0$ , for every Weil divisor  $D$  such that  $D \sim_{g, \mathbb{R}} K_X + S + \Delta' + L$  for some  $0 \leq \Delta' \leq \Delta$ , where  $L$  is  $g$ -ample.

K-V vanishing is known to hold if  $Z$  is essentially of finite type over a field of characteristic 0, if  $\dim X = 2$  and  $g$  has relative dimension  $\leq 1$ , or if  $(X, S + \Delta)$  is a log del Pezzo surface pair over a perfect field and the characteristic is  $\geq 7$  by [1].

**Proposition 2.3.** Let  $(X, S + \Delta)$  be an excellent dlt pair,  $g: X \rightarrow Z$  a proper morphism, and  $D$  a Weil  $\mathbb{Z}$ -divisor on  $X$ . Assume that

1.  $D \sim_{g, \mathbb{R}} K_X + S + \Delta' + L$  for some  $0 \leq \Delta' \leq \Delta$ ,
2.  $\mathcal{O}_X(D - mS)$  is  $S_3$  along  $S$  for every  $m \geq 1$ , and
3.  $-S$  is  $g$ -nef.

Assume further that one of the following is satisfied.

4.  $L$  is an ample  $\mathbb{Q}$ -Cartier divisor and K-V vanishing holds for  $g|_{\bar{S}}$ , or
5.  $L$  is a nef  $\mathbb{Q}$ -Cartier divisor,  $g|_{\bar{S}}$  is birational, and strong G-R vanishing holds for  $g|_{\bar{S}}$ .

Then  $R^i g_* \mathcal{O}_X(D) = 0$  for  $i > 0$  near  $g(S)$ .

**Proof.** Set  $J_m := \text{im}[\mathcal{O}_X(D - mS) \rightarrow \mathcal{O}_X(D)]$ . Applying Lemma 2.1 to  $D_m := D - K_X - (m + 1)S$  we get a short exact sequence:

$$0 \rightarrow \mathcal{O}_X(D - (m + 1)S) \rightarrow \mathcal{O}_X(D - mS) \rightarrow v_* \mathcal{O}_{\bar{S}}(G_m) \rightarrow 0,$$

where  $G_m \sim_{\mathbb{R}} -mS|_{\bar{S}} + D|_{\bar{S}} - \Delta_m$  for some  $0 \leq \Delta_m \leq \text{Diff}_{\bar{S}}(\Delta')$  and  $v: \bar{S} \rightarrow S$  is the normalisation morphism. Thus by the snake lemma we get the following short exact sequence:

$$0 \rightarrow v_* \mathcal{O}_{\bar{S}}(G_m) \rightarrow \mathcal{O}_X(D)/J_{m+1} \rightarrow \mathcal{O}_X(D)/J_m \rightarrow 0.$$

Consider the induced exact sequence for  $i > 0$ :

$$R^i g_* v_* \mathcal{O}_{\bar{S}}(G_m) \rightarrow R^i g_* (\mathcal{O}_X(D)/J_{m+1}) \rightarrow R^i g_* (\mathcal{O}_X(D)/J_m).$$

The first term vanishes by assumptions (4–5) because  $G_m \sim_{g|_{\bar{S}}, \mathbb{R}} K_{\bar{S}} + \text{Diff}_{\bar{S}}(\Delta') - \Delta_m + L|_{\bar{S}} - mS|_{\bar{S}}$ , so we conclude by induction that  $R^i g_* (\mathcal{O}_X(D)/J_m) = 0$  for all  $m \geq 0$ . Thus  $R^i g_* \mathcal{O}_X(D) = 0$  for  $i > 0$  near  $g(S)$  by the formal function theorem.  $\blacksquare$

## 2.2 Fujita transform and CM criteria

One cannot reasonably define the pull-back of an arbitrary Weil  $\mathbb{Z}$ -divisor by a birational morphism. Nonetheless, there are two replacements—considered in [11]—that are very useful for vanishing theorems. We will use these in Section 3.

**Definition 2.4** (Fujita transforms). Let  $\pi : X \rightarrow Y$  be a proper, birational morphism of normal schemes. Let  $D$  be a Weil  $\mathbb{Z}$ -divisor on  $Y$  and  $\Delta_D$  an  $\mathbb{R}$ -divisor. (Eventually it will be a boundary, but for now this is not needed). Assume that  $D + \Delta_D$  is  $\mathbb{R}$ -Cartier.

The *Fujita transform* of  $(D, \Delta_D)$  is the unique pair  $(\pi^F(D), \Delta_{X,D})$  such that

1.  $\pi^F(D)$  is a Weil  $\mathbb{Z}$ -divisor on  $X$  such that  $\pi_*(\pi^F(D)) = D$ ,
2.  $\Delta_{X,D}$  is an  $\mathbb{R}$ -divisor on  $X$  such that  $\pi_*(\Delta_{X,D}) = \Delta_D$ ,
3.  $[\text{Ex}(\Delta_{X,D})] = 0$ , and
4.  $\pi^F(D) + \Delta_{X,D} \sim_{\pi, \mathbb{R}} 0$ .

*Warning.* The notation is slightly misleading since  $\pi^F(D)$  also depends on  $\Delta_D$ . In our applications,  $\Delta_D$  will be fixed.

Assume next that  $K_Y + \Delta$  is  $\mathbb{R}$ -Cartier and  $0 \leq \Delta_D \leq \Delta$ . Set  $\Delta_D^c := \Delta - \Delta_D$ . The *K-twisted Fujita transform* of  $(D, \Delta_D)$  is the unique pair  $(\pi^{KF}(D), \Delta_{X,D}^c)$  such that

- (1')  $\pi^{KF}(D)$  is a Weil  $\mathbb{Z}$ -divisor on  $X$  such that  $\pi_*(\pi^{KF}(D)) = D$ ,
- (2')  $\Delta_{X,D}^c$  is an  $\mathbb{R}$ -divisor on  $X$  such that  $\pi_*(\Delta_{X,D}^c) = \Delta_D^c$ ,
- (3')  $[\text{Ex}(\Delta_{X,D}^c)] = 0$ , and
- (4')  $\pi^{KF}(D) \sim_{\pi, \mathbb{R}} K_X + \Delta_{X,D}^c$ .

Note that the Fujita transform and the K-twisted Fujita transform are both functors.

The key observations of [11] are the following three claims.

*Claim 2.5.*  $\pi_*\mathcal{O}_X(\pi^F(D)) = \mathcal{O}_Y(D)$ .

**Proof.** As  $-\pi^F(D) \sim_{g, \mathbb{R}} \Delta_{X,D}$ , by Property 3 it is elementary to deduce that  $\mathcal{O}_Y(\pi_*\pi^F(D)) = \pi_*\mathcal{O}_X(\pi^F(D))$  (we refer to [20, 7.30] for details). We thus conclude by Property 1. ■

To get the second main property, write  $\pi^*(K_Y + \Delta) = K_X + \Delta_X$  where  $\pi_*(\Delta_X) = \Delta$ . Then

$$\pi^{KF}(D) - \pi^F(D) \sim_{\pi, \mathbb{R}} -\Delta_X + \Delta_{X,D}^c + \Delta_{X,D}.$$

Here  $\Delta_{X,D}^c$  and  $\Delta_{X,D}$  are effective by definition. If all divisors appear in  $\Delta_X$  with coefficients  $< 1$ , then all divisors appear in  $\pi^{\text{KF}}(D) - \pi^{\text{F}}(D)$  with coefficients  $> -1$ . Since  $\pi^{\text{KF}}(D) - \pi^{\text{F}}(D)$  is a  $\mathbb{Z}$ -divisor, it is then effective. Thus we have proved the following.

**Claim 2.6.** If  $(Y, \Delta)$  is klt or if  $(Y, \Delta)$  is dlt and  $\pi$  is thrifty, then  $\pi^{\text{KF}}(D) - \pi^{\text{F}}(D)$  is an effective  $\pi$ -exceptional divisor. Thus  $\pi_* \mathcal{O}_X(\pi^{\text{KF}}(D)) = \mathcal{O}_Y(D)$ .

We get the third property by comparing the definitions.

**Claim 2.7.**  $(K_X - \pi^{\text{F}}(D))$  and  $\pi^{\text{KF}}(D)$  satisfy the assumptions of G-R vanishing (1.1.1.a). Thus if G-R vanishing holds over  $(Y, \Delta)$  for some  $\Delta \geq \Delta_D$ , then  $R^i \pi_* \mathcal{O}_X(K_X - \pi^{\text{F}}(D)) = 0$  and  $R^i \pi_* \mathcal{O}_X(\pi^{\text{KF}}(D)) = 0$  for  $i > 0$ .

To get a CM criterion out of the above claims, we use the method of two spectral sequences (see [19] or [20, 7.27]) combined with the duality between local cohomology and higher direct images. Note that [20, 10.44] states the needed duality for locally free sheaves. In our case one needs to use the Cohen–Macaulay version as in [17, 5.71]. The end result is the following.

**Theorem 2.8** (CM-criterion). Let  $\pi: X \rightarrow Y$  be a proper birational morphism between integral schemes. Let  $D$  be a Weil  $\mathbb{Z}$ -divisor and  $\Delta$  an  $\mathbb{R}$ -divisor on  $Y$  such that  $D + \Delta$  is  $\mathbb{R}$ -Cartier. Assume that

1.  $X$  is normal and CM,
2.  $\mathcal{O}_X(\pi^{\text{F}}(D))$  and  $\mathcal{O}_X(\pi^{\text{KF}}(D))$  are CM.
3.  $R^i \pi_* \mathcal{O}_X(K_X - \pi^{\text{F}}(D)) = 0$  for  $i > 0$ ,
4.  $R^i \pi_* \mathcal{O}_X(\pi^{\text{KF}}(D)) = 0$  for  $i > 0$ , and
5.  $\pi^{\text{KF}}(D) - \pi^{\text{F}}(D)$  is effective.

Then  $\mathcal{O}_Y(D)$  is CM.

Note that assumptions (1–2) are automatic if  $X$  is a regular scheme.

**Proof.** Let  $p \in Y$  be a closed point and let  $W := \pi^{-1}(p)$ . We have the following commutative diagram of local cohomology groups:

$$\begin{array}{ccc}
 H_p^i(Y, \pi_* \mathcal{O}_X(\pi^{\text{KF}}(D))) & \xrightarrow{\alpha_i} & H_W^i(X, \mathcal{O}_X(\pi^{\text{KF}}(D))) \\
 \uparrow \beta_i & & \uparrow \\
 H_p^i(Y, \pi_* \mathcal{O}_X(\pi^{\text{F}}(D))) & \longrightarrow & H_W^i(X, \mathcal{O}_X(\pi^{\text{F}}(D)))
 \end{array}$$

Here  $\alpha_i$  is an isomorphism by hypothesis (4) and the Leray spectral sequence for local cohomology, and  $\beta_i$  is an isomorphism by hypothesis (5). By [20, 10.44] we have  $H_W^i(X, \mathcal{O}_X(\pi^F(D))) \simeq (R^{n-i}\pi_*\mathcal{O}_X(K_X - \pi^F(D)))_p$ , which vanishes by (3) for  $i < n$ . From these we deduce that  $H_p^i(Y, \pi_*\mathcal{O}_X(\pi^F(D))) = 0$ , and we conclude by noting that  $\pi_*\mathcal{O}_X(\pi^{KF}(D)) = \pi_*\mathcal{O}_X(\pi^F(D)) = \mathcal{O}_Y(D)$  by hypothesis (4) and Claim 2.5. ■

### 2.3 Birational invariance of higher direct images

**Conjecture 2.9.** Let  $Y$  be a normal excellent scheme,  $D$  a Weil  $\mathbb{Z}$ -divisor on  $Y$ , and  $\Delta_D$  a Weil  $\mathbb{R}$ -divisor whose coefficients are in  $[0, 1]$ . Assume that  $D + \Delta_D$  is  $\mathbb{R}$ -Cartier. Let  $\pi: X \rightarrow Y$  be a proper, birational log resolution of  $(Y, D + \Delta_D)$ , and  $\pi^{KF}(D)$  the K-twisted Fujita transform.

Then  $R\pi_*\mathcal{O}_X(\pi^{KF}(D))$  is independent of  $X$ .

It is possible that this can be proved by the methods of [25]. Here we have a more modest aim, to prove that resolution of singularities implies Conjecture 2.9. Let us first state what we need about the existence of resolutions.

**2.10. (Resolution assumption).** Fix an integral scheme  $Y$ . Let  $g: X \rightarrow Y$  be a proper, birational morphism. Assume that  $(X, D)$  is an simple normal crossing (snc) pair. Let  $g': X' \rightarrow Y$  be a proper, birational morphism. Then there is a sequence of blow-ups

$$X_m \xrightarrow{\tau_m} X_{m-1} \rightarrow \cdots \rightarrow X_1 \xrightarrow{\tau_1} X_0 = X$$

such that

1. the center of each blow-up is a non-singular subvariety that has simple normal crossing with the total transform of  $D$ , (these are called *non-singular blow-ups*), and
2. the induced map  $X_m \dashrightarrow X'$  is a morphism.

The usual Leray spectral sequence argument reduces Conjecture 2.9 to the case when  $(Y, D + \Delta_D)$  is an snc pair.

**Conjecture 2.11.** G-R vanishing holds over snc pairs. That is, using the notation of (2.9), assume in addition that  $(Y, D + \Delta_D)$  is snc. Then

$$R^i\pi_*\mathcal{O}_X(\pi^{KF}(D)) = 0 \quad \text{for } i > 0. \quad (2.11.1)$$

**Theorem 2.12.** Assume the resolution assumption 2.10. Then Conjectures 2.9 and 2.11 are true. In particular, the conjectures hold in the case where  $Y$  is an excellent  $\mathbb{Q}$ -scheme or  $Y$  is an excellent scheme of dimension  $d \leq 3$ .

**Proof.** By an elementary computation, (2.11.1) holds if  $\pi$  is a non-singular blow-up. Thus it also holds for compositions of non-singular blow-ups, more generally, for any birational map that is a composition of non-singular blow-ups and blow-downs. The Weak Factorization Conjecture asserts that every birational map between non-singular varieties is such, but this is not known in positive characteristic. We go around this using an unpublished argument of Hironaka (cf. [13, page 153]).

To simplify notation, let  $D_X$  denote the  $K$ -twisted Fujita transform of  $D$  on  $X$ . By induction on  $j \geq 1$  we prove that

$$R^i \sigma_* \mathcal{O}_X(D_X) = 0 \quad \text{for } j \geq i > 0, \quad (2.12.1)$$

for any birational morphism  $\sigma: X \rightarrow X'$  between log resolutions of  $(Y, D + \Delta_D)$ .

For  $j = 1$ , given any  $\sigma: X \rightarrow X'$ , we use our Resolution assumption 2.10 to get

$$\tau: Z \xrightarrow{\pi} X \xrightarrow{\sigma} X',$$

where  $\tau$  is a composition of non-singular blow-ups.

We know that (2.11.1) holds for  $\tau$ . The Leray spectral sequence

$$R^p \sigma_* (R^q \pi_* \mathcal{O}_Z(D_Z)) \Rightarrow R^{p+q} \tau_* \mathcal{O}_Z(D_Z)$$

gives the inclusion  $R^1 \sigma_* (\pi_* \mathcal{O}_Z(D_Z)) = R^1 \sigma_* (\mathcal{O}_X(D_X)) \hookrightarrow R^1 \tau_* \mathcal{O}_Z(D_Z) = 0$ . So we prove 2.12.1 for  $j = 1$ .

Now assume 2.12.1 holds for  $j$  and we prove it for  $j + 1$ . Again by the Leray spectral sequence we have the inclusion

$$R^{j+1} \sigma_* \mathcal{O}_X(D_X) = R^{j+1} \sigma_* (\pi_* \mathcal{O}_Z(D_Z)) \hookrightarrow R^{j+1} \tau_* \mathcal{O}_Z(D_Z) = 0.$$

Thus (2.12.1) also holds for  $j + 1$ .

Finally, Assumption 2.10 is known to hold for varieties in characteristic 0 by [13] and for excellent three-folds [6]. ■

## 2.4 Rational singularities

We recall the definition of rational singularities, following [25].

**Definition 2.13.** A scheme  $X$  has *rational singularities* if

1.  $X$  is a normal, excellent, Cohen–Macaulay scheme admitting a dualising complex;
2. for every locally projective, birational morphism  $\pi: \tilde{X} \rightarrow X$  where  $\tilde{X}$  is an excellent Cohen–Macaulay scheme, the natural morphism  $\mathcal{O}_X \rightarrow \mathbf{R}\pi_*\mathcal{O}_{\tilde{X}}$  is an isomorphism.

If  $X$  has a resolution of singularities, then, by [25, Corollary 9.11], it is enough to check condition (2) in Definition 2.13 for a resolution, recovering the traditional notion of rational singularities.

In positive and mixed characteristics it is necessary to assume that  $X$  is Cohen–Macaulay, but in characteristic 0 it can be deduced from the other properties as an application of the Grauert–Riemenschneider vanishing theorem.

The following descent property is very similar to [25, Theorem 9.12].

**Proposition 2.14.** Let  $\pi: Y \rightarrow X$  be a morphism of normal excellent schemes. Assume that

1.  $Y$  has rational singularities,
2. the natural morphism  $\mathcal{O}_X \rightarrow \mathbf{R}\pi_*\mathcal{O}_Y$  splits in  $D(X)$ ,
3.  $X$  is a Cohen–Macaulay scheme admitting a dualising complex.

Then  $X$  has rational singularities.

**Proof.** The proof closely follows [23, Theorem 1.1]. By [25, Corollary 9.11] it is sufficient to show that  $X$  has pseudo-rational singularities, namely that for every normal scheme  $\tilde{X}$  and every projective birational morphism  $\varphi: \tilde{X} \rightarrow X$ , the natural map  $\varphi_*\omega_{\tilde{X}} \rightarrow \omega_X$  is an isomorphism.

By the existence of a Macaulayfication (see [16, Theorem 1.1]) we can construct a projective birational morphism  $\tilde{Y} \rightarrow Y$  such that  $\tilde{Y}$  is Cohen–Macaulay and the following

diagram commutes:

$$\begin{array}{ccc} \tilde{Y} & \xrightarrow{\tilde{\pi}} & \tilde{X} \\ \downarrow \psi & & \downarrow \varphi \\ Y & \xrightarrow{\pi} & X \end{array}$$

Thus we have the following commutative diagram in  $D(X)$ :

$$\begin{array}{ccc} \mathcal{O}_X & \longrightarrow & \mathbf{R}\pi_*\mathcal{O}_Y \\ \downarrow & & \downarrow \\ \mathbf{R}\varphi_*\mathcal{O}_{\tilde{X}} & \longrightarrow & \mathbf{R}\varphi_*\mathbf{R}\tilde{\pi}_*\mathcal{O}_{\tilde{Y}} \end{array}$$

Since  $Y$  has rational singularities and  $\tilde{Y}$  is Cohen–Macaulay, the composition

$$\mathbf{R}\pi_*\mathcal{O}_Y \rightarrow \mathbf{R}\varphi_*\mathbf{R}\tilde{\pi}_*\mathcal{O}_{\tilde{Y}} \simeq \mathbf{R}\pi_*\mathbf{R}\psi_*\mathcal{O}_{\tilde{Y}} \simeq \mathbf{R}\pi_*\mathcal{O}_Y$$

is an isomorphism. Therefore by assumption (2) we have a splitting in  $D(X)$ :

$$\mathcal{O}_X \rightarrow \mathbf{R}\varphi_*\mathcal{O}_{\tilde{X}} \rightarrow \mathcal{O}_X.$$

Applying  $\mathbf{R}\mathcal{H}om_X(-, \omega_X^\bullet)$  and Grothendieck duality to the above sequence, we have the following splitting:

$$\omega_X^\bullet \rightarrow \mathbf{R}\varphi_*\omega_{\tilde{X}}^\bullet \rightarrow \omega_X^\bullet. \quad (1)$$

Since  $X$  is Cohen–Macaulay, we have  $\omega_X^\bullet \simeq \omega_X[-d]$ . Considering the  $(-d)$ -th cohomology group in sequence (1), we deduce that the composition

$$\omega_X \rightarrow \varphi_*\omega_{\tilde{X}} \rightarrow \omega_X$$

is an isomorphism. Note that  $\varphi_*\omega_{\tilde{X}}$  is a torsion-free sheaf of rank one since  $\omega_{\tilde{X}}$  is so. Therefore  $\varphi_*\omega_{\tilde{X}} \rightarrow \omega_X$  is an isomorphism. This means that  $X$  has pseudo-rational singularities.  $\blacksquare$

### 3 Consequences of Grauert–Riemenschneider Vanishing

We claim that those dlt pairs for which G-R vanishing holds satisfy all the other known local rationality properties that are known to hold in characteristic 0. This claim is not really new; the arguments below are at least implicit in many characteristic 0 papers, especially [19], and in the positive characteristic works of [1, 14]. However, the clearest statements of such principles are in [23, 25]; see also [24] and [20, Secs. 2.5 and 7.3] for more introductory treatments.

**Theorem 3.1.** Let  $(X, \Delta)$  be an excellent dlt pair that admits a thrifty log resolution. Suppose that G-R vanishing holds over it. Then

1.  $X$  is Cohen–Macaulay;
2. Let  $D$  be a Weil  $\mathbb{Z}$ -divisor on  $X$  such that  $D + \Delta_D$  is  $\mathbb{R}$ -Cartier for some  $0 \leq \Delta_D \leq \Delta$ . Then  $\mathcal{O}_X(D)$  is CM.
3.  $(X, B)$  is a rational pair (as in [20, 2.80]) for every  $B \subset \lfloor \Delta \rfloor$ , in particular  $X$  has rational singularities;
4. Log canonical centers of  $(X, \Delta)$  are normal and have rational singularities.

All the above properties hold for excellent dlt pairs defined over  $\mathbb{Q}$  by [28], and for dlt three-folds as in Theorem 1.3. Note that, for dlt three-folds pairs over perfect fields of characteristic at least 7, (1) was proved in [1, 14].

**Proof.** If (2) holds then applying it to  $D = \Delta_D = 0$  we conclude that  $X$  is CM.

As for the proof of (2), let  $\pi: Y \rightarrow (X, \Delta)$  be a thrifty log resolution. Then we can apply Theorem 2.8 by Claims 2.6 and 2.7 to conclude that  $\mathcal{O}_X(D)$  is CM.

Let  $g: Y \rightarrow (X, \Delta)$  be a thrifty log resolution and let  $B_Y$  denote the birational transform of  $B$  and  $\Delta'_Y$  the strict transform of  $\Delta' = \Delta - B$ . We write

$$K_Y + B_Y + \Delta'_Y + F = g^*(K_X + \Delta) + E,$$

where  $E$  and  $F$  are both effective exceptional divisor without common exceptional components and  $\lfloor F \rfloor = 0$ . Since  $\lceil E \rceil - B_Y \sim_{g, \mathbb{R}} K_Y + \Delta'_Y + F + (\lceil E \rceil - E)$ , we have that G-R vanishing applies both to  $\mathcal{O}_Y(\lceil E \rceil - B_Y)$  and to  $\omega_Y(B_Y)$ , so the same proof of [20, 2.87] works in this setting, giving (3).

We prove (4) by induction. Let  $B_1, \dots, B_r$  be the irreducible components of  $\lfloor \Delta \rfloor$ . We now prove that every irreducible component of  $B_{i_1} \cap \dots \cap B_{i_r}$  is normal and has rational

singularities. If  $B_i$  is an irreducible component of  $\lfloor \Delta \rfloor$ , applying (2) to  $\Delta_{B_i} := B_i$  shows that  $\mathcal{O}_X(-B_i)$  is CM. Now the sequence

$$0 \rightarrow \mathcal{O}_X(-B_i) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_{B_i} \rightarrow 0$$

shows that  $\mathcal{O}_{B_i}$  is CM (cf. [20, 2.60]). By [20, 2.31]  $B_i$  is regular in codimension 1, hence normal. Thus the pair  $(B_i, \text{Diff}_{B_i}(\Delta - B_i))$  is dlt by easy adjunction [20, 4.8].

The following claim is crucial for the induction argument:

*Claim 3.1.5.* Let  $g: Y \rightarrow X$  be a proper morphism and  $\{D_i : i \in I\}$  Cartier divisors on  $Y$ . Let  $\mathcal{L}$  be an invertible sheaf on  $Y$  and for  $J \subset I$  set  $D_J := \cup_{i \in J} D_i$ . Assume that  $R^m g_* \mathcal{L}(-D_J) = 0$  for every  $J \subset I$  and  $m > 0$ . Then, for every  $i \in I$ ,

$$R^m (g|_{D_i})_* (\mathcal{L}(-D_J)|_{D_i}) = 0 \quad \text{for every } J \subset I \setminus \{i\} \quad \text{and } m > 0.$$

**Proof.** Consider the short exact sequence  $0 \rightarrow \mathcal{O}_Y(-D_i) \rightarrow \mathcal{O}_Y \rightarrow \mathcal{O}|_{D_i} \rightarrow 0$  and tensor it by  $\mathcal{L}(-D_J)$  where  $J \subset I \setminus \{i\}$ . For  $m > 0$  we have the following exact sequence:

$$R^m g_* \mathcal{L}(-D_J) \rightarrow R^m (g|_{D_i})_* (\mathcal{L}(-D_J)|_{D_i}) \rightarrow R^{m+1} g_* \mathcal{L}(-D_J - D_i).$$

We conclude by hypothesis that  $R^m (g|_{D_i})_* (\mathcal{L}(-D_J)|_{D_i}) = 0$ . ■

To prove that an irreducible component  $B_{ij} := B_i \cap B_j$  of  $\Gamma_i = \text{Diff}_{B_i}(\Delta - B_i)$  is CM, it is sufficient to show that  $\mathcal{O}_{B_i}(-B_{ij})$  is CM. By Claim 2.7 on  $X$ , naturality of the Fujita transforms and Claim 3.1.5 applied to  $\mathcal{L} = \mathcal{O}_Y$  and  $\{D_i = g^{\text{KF}}(-B_i)\}$  (resp.  $\mathcal{L} = \omega_Y$  and  $\{D_i = g^F(-B_i)\}$ ) the following higher direct images

$$R^m (g|_{B'_i})_* \mathcal{O}_{B'_i}((g|_{B'_i})^{\text{KF}}(-B'_{ij})) \text{ and } R^m (g|_{B'_i})_* \mathcal{O}_{B'_i}(K_{B'} - g^F(-B'_{ij})) \text{ for } m > 0$$

vanish, so we can apply Theorem 2.8 to conclude  $\mathcal{O}_{B_i}(-B_{ij})$  is CM. Once we know  $B_{ij}$  is CM, we deduce that  $B_{ij}$  is normal so  $(B_{ij}, \text{Diff}_{B_{ij}}(\Gamma_i - B_{ij}))$  is dlt. We can thus proceed with a straightforward induction process to show that all strata of  $\lfloor \Delta \rfloor$  are Cohen–Macaulay. Finally, the proof of [20, 4.16] now carries over to show that the lc centers of  $(X, \Delta)$  are exactly the strata of  $\lfloor \Delta \rfloor$ . We are only left to prove that the strata have rational singularities. We start by showing it for codimension one strata  $B_i$ . As above we can show  $R^m (g|_{B'_i})_* \mathcal{O}_{B'_i}$  and  $R^m (g|_{B'_i})_* \mathcal{O}_{B'_i}(K_{B'})$  vanish for  $m > 0$  and so we conclude  $B_i$  are rational. We continue by induction to conclude.

**Remark 3.2.** The natural inductive proof of (4) would be to show that G-R vanishing holds over  $(B_i, \text{Diff}_{B_i}(\Delta - B_i))$ . This is quite likely true, but it is not obvious how to prove it. To see what the difficulty is, fix a log resolution  $g: X' \rightarrow X$  and let  $B'_i \subset X'$  denote the birational transform of  $B_i$ . Given  $D' \sim_{g, \mathbb{R}} K_{X'} + B'_i + \Delta'$  where  $g_* \Delta' \leq \Delta$  and  $\lfloor \text{Ex}(\Delta') \rfloor = 0$ , we get that  $R^i(g|_{B'_i})_* \mathcal{O}_{B'}(D'_i|_{B'}) = 0$  for  $i > 0$ . Not every log resolution of  $(B_i, \text{Diff}_{B_i}(\Delta - B_i))$  is of this form, but, as we discussed in Section 2.3, this does not matter. The problem is that usually not every divisor  $D_{B_i} \sim_{g|_{B'}, \mathbb{R}} K_{B'_i} + \Delta'$  on  $B'_i$  is the restriction of a divisor from  $X'$  and we do not know how to overcome this. However, in many applications, we need G-R vanishing not for all divisors, but only for some that are constructed from  $(X, \Delta)$  in a “natural” way. If we are lucky then the construction of these divisors commutes with restriction, and the proof goes through. This is exactly the case with the proof of (4).

**Remark 3.3.** A general result on rationality of dlt singularities in every dimension similar to Theorem 3.1(2) has been proven in [25, Corollary 10.18], where it is proved that  $X$  has rational singularities if it is Cohen–Macaulay and  $X$  is potentially dlt.

#### 4 Grauert–Riemenschneider Vanishing Theorem

The aim of this section is to prove G-R vanishing for dlt three-folds.

**Proof of Theorem 1.3.** Since the statement of the theorem is local, we can assume  $X$  to be affine. Let  $g: Y \rightarrow (X, \Delta)$  be a projective log resolution such that the exceptional divisor  $E := E_1 + \cdots + E_n$  supports a  $g$ -ample  $\mathbb{Q}$ -divisor whose existence is guaranteed by [26, Corollary 3]. By [22, Theorem 9], (see also [4]), we can run a relative MMP starting with  $(X^0, \Theta^0) := (Y, \Theta := g_*^{-1} \Delta_Y + \sum_{i=1}^n E_i)$  over  $X$  as in [22, Theorem 2]. Let

$$\phi_i: (X^{i-1}, \Theta^{i-1}) \dashrightarrow (X^i, \Theta^i)$$

be the  $i$ -th step of the MMP and  $g_i: X^i \rightarrow Y$  the corresponding map to  $Y$ .

As we proved in Section 2.3, G-R vanishing holds over the snc pair  $(X^0, \Theta^0)$ . We show the following:

**Claim 4.1.** If G-R vanishing holds over  $(X^{i-1}, \Theta^{i-1})$  then it also holds over  $(X^i, \Theta^i)$ .

At the end of the MMP now we conclude that G-R vanishing holds over  $(X^m, \Theta^m)$ . Therefore it remains to show Claim 4.1 for each MMP step. If  $\phi_i: (X^{i-1}, \Theta^{i-1}) \dashrightarrow (X^i, \Theta^i)$  is a divisorial contraction that contracts a divisor  $S_{i-1} \subset X^{i-1}$ , then we use Proposition 4.1. For this we need to check that assumption (4.1.3) is satisfied. If  $S_{i-1}$  is contracted to a curve, then K-V vanishing holds by [30, Theorem 3.3]. If  $S_{i-1}$  is mapped to a closed

point, then we use [1, Theorem 1.1]. This is where we use the assumption that the residue fields of closed points are perfect of characteristic  $p > 5$ .

If  $\phi_i: (X^{i-1}, \Theta^{i-1}) \dashrightarrow (X^i, \Theta^i)$  is a flip, then we apply Proposition 4.2, which works for any excellent 3-dimensional scheme. ■

For divisorial contractions we have the following general result.

**Proposition 4.1.** Let  $(X, S + \Delta)$  be an excellent dlt pair where  $S$  is a prime divisor,  $g: X \rightarrow Z$  a proper birational morphism contracting  $S$ . Assume that

1. G-R vanishing holds over  $(X, S + \Delta)$ ,
2.  $-S$  is  $g$ -ample, and
3. K-V vanishing holds for  $g|_S: (S, \text{Diff}_S \Delta) \rightarrow g(S)$  (as in Definition 2.2).

Then G-R vanishing holds over  $(Z, g_* \Delta)$ .

**Proof.** We use Proposition 2.3. Choose any log resolution  $\pi: Y \rightarrow X$ . Assume that we have  $D_Y \sim_{\mathbb{R}} K_Y + \Delta'_Y$ . Set  $D_X := \pi_*(D_Y)$ . Then  $\pi_* \mathcal{O}_Y(D_Y) = \mathcal{O}_X(D_X)$  by Claim 2.5. Since  $(X, S + \Delta)$  satisfies G-R vanishing, Theorem 3.1 shows that  $S$  is normal and  $\mathcal{O}_X(D_X - mS)$  is  $S_3$  along  $S$  since  $(D_X - mS + (S + \Delta - \Delta'_X))$  is  $\mathbb{R}$ -Cartier.

Set  $c := \text{coeff}_S \Delta'_Y$ . Since  $S$  is  $g$ -exceptional, we have  $0 \leq c < 1$ , hence

$$D_X \sim_{\mathbb{R}} K_X + \Delta'_X = K_X + S + (\Delta'_X - cS) + (1 - c)(-S)$$

shows that assumption (2.3.1) and (2.3.4) are also satisfied. Thus  $R^i g_* \mathcal{O}_X(D_X) = 0$  for  $i > 0$  near  $g(S)$  and the Leray spectral sequence gives that  $R^i (g \circ \pi)_* \mathcal{O}_Y(D_Y) = 0$  for  $i > 0$  near  $g(S)$ . ■

Next we show that G-R vanishing is preserved by many 3-dimensional flips.

**Proposition 4.2.** Let  $(X, S + \Delta)$  be a 3-dimensional excellent dlt pair and

$$(C \subset X, S + \Delta) \xrightarrow{g} Z \xleftarrow{g^+} (C^+ \subset X^+, S^+ + \Delta^+)$$

a flip. Assume that

1. G-R vanishing holds over  $(X, S + \Delta)$ , and
2.  $-S$  is  $g$ -ample.

Then G-R vanishing holds over  $(X^+, S^+ + \Delta^+)$ .

**Proof.** Choose a common log resolution  $\pi: Y \rightarrow X$  and  $\pi^+: Y \rightarrow X^+$  and assume we have  $D_Y \sim_{\mathbb{R}} K_Y + \Delta'_Y$ . Following the proof of Proposition 4.1, we show that G-R vanishing holds over  $(Z, g_*(S + \Delta))$ . Now we show how to deduce G-R vanishing on  $(X^+, S^+ + \Delta^+)$ .

By G-R vanishing for excellent surfaces,  $R^i \pi_*^+ \mathcal{O}_Y(D_Y)$  is supported on closed points for  $i \geq 1$ . Since the dimension of the exceptional locus of  $g^+$  is one, we conclude also that  $R^i g_*^+(\pi_*^+ \mathcal{O}_Y(D_Y)) = 0$  for  $i \geq 2$ . As G-R vanishing holds on  $(X^+, S^+ + \Delta^+)$  we can apply Lemma 4.3 to  $F := \mathcal{O}_Y(D_Y)$  and  $W := X^+$  to conclude that  $R^i \pi_*^+ \mathcal{O}_Y(D_Y) = 0$  for  $i > 0$ . ■

**Lemma 4.3.** Let  $\tau: Y \rightarrow W$  and  $\pi: W \rightarrow Z$  be proper morphisms and  $F$  a coherent sheaf on  $Y$ . Assume that

1.  $\dim \text{Supp } R^i \tau_* F \leq 0$  for  $i \geq 1$  and
2.  $R^i \pi_*(\tau_* F) = 0$  for  $i \geq 2$ .

Then the following are equivalent.

1.  $R^i(\pi \circ \tau)_* F = 0$  for  $i \geq 1$ .
2.  $R^i \tau_* F = 0$  for  $i \geq 1$  and  $R^1 \pi_*(\tau_* F) = 0$ .

**Proof.** The only nonzero terms in the Leray spectral sequence are  $\pi_* R^i \tau_* F$  and  $R^j \pi_*(\tau_* F)$  for  $j = 0, 1$ . This gives that

$$R^i(\pi \circ \tau)_* F = \pi_* R^i \tau_* F \quad \text{for } i \geq 2$$

and we have an exact sequence

$$0 \rightarrow R^1 \pi_*(\tau_* F) \rightarrow R^1(\pi \circ \tau)_* F \rightarrow \pi_* R^1 \tau_* F \rightarrow 0.$$

■

## 5 Applications to Mori Fiber Spaces

In this section we describe some applications of the vanishing theorems proven in Section 4 to the birational geometry of three-folds in positive characteristic.

### 5.1 Vanishing for log Fano contractions

We prove the vanishing of higher direct images of the structure sheaf for log Fano contractions in characteristic  $p \geq 7$ .

**Theorem 5.1.** Let  $k$  be a perfect field of characteristic  $p \geq 7$ . Let  $f: X \rightarrow Z$  be a proper contraction morphism between quasi-projective normal varieties over  $k$ . Suppose that there exists an effective  $\mathbb{Q}$ -divisor  $\Delta \geq 0$  such that

1.  $(X, \Delta)$  is a klt three-fold pair;
2.  $-(K_X + \Delta)$  is  $f$ -big and  $f$ -nef;
3.  $\dim(Z) \geq 1$ .

Then the natural map  $\mathcal{O}_Z \rightarrow \mathbf{R}f_*\mathcal{O}_X$  is an isomorphism.

**Proof.** It is sufficient to prove  $R^if_*\mathcal{O}_X = 0$  for  $i > 0$ . By [30, Theorem 3.3] and [5, Corollary 1.8], there exists an open subset  $U \subset Z$  such that  $(R^if_*\mathcal{O}_X)|_U = 0$  and  $Z \setminus U$  is a finite set of points. Let  $z$  be a closed point in  $Z \setminus U$ . By [12, Proposition 2.15], there exists a contraction  $\pi: Y \rightarrow Z$  and an effective  $\mathbb{Q}$ -divisor  $\Delta_Y$  on  $Y$  such that

- (i)  $(Y, \Delta_Y)$  is a  $\mathbb{Q}$ -factorial plt pair (in particular,  $Y$  is klt),
- (ii)  $S := (\pi^{-1}(z))_{\text{red}}$  is an irreducible component of  $\lfloor \Delta_Y \rfloor$ ,
- (iii)  $-(K_Y + \Delta_Y)$  is  $\pi$ -ample and  $-S$  is  $\pi$ -nef.

Let us consider the following diagram

$$\begin{array}{ccc} W & \xrightarrow{\varphi} & Y \\ \psi \downarrow & & \downarrow \pi \\ X & \xrightarrow{f} & Z, \end{array}$$

where  $\varphi$  and  $\psi$  are log resolutions. Since  $X$  and  $Y$  have rational singularities by [1, Corollary 1.3], to prove  $R^if_*\mathcal{O}_X = 0$  in a neighbourhood of  $z$ , it is sufficient to prove  $R^i\pi_*\mathcal{O}_Y = 0$  for  $i > 0$ . This follows from Proposition 2.3 and [1, Theorem 1.1]. ■

The lifting of birational contractions between klt pairs was treated in [14, Corollary 5.1]). Combining Theorem 5.1 with [9, Theorem 3.1] we get the following analog for log Fano contractions.

**Corollary 5.2.** Let  $k$  be a perfect field of characteristic  $p \geq 7$ . Let  $(X, \Delta)$  be a klt three-fold pair and let  $f: X \rightarrow Z$  be a proper contraction morphism over  $k$  between quasi-projective varieties. Assume that

1.  $X$  lifts to  $W_m(k)$  (resp. formally lifts to  $W(k)$ ),
2.  $-(K_X + \Delta)$  is  $f$ -big and  $f$ -nef,

Then the morphism  $f: X \rightarrow Z$  lifts to  $W_m(k)$  (resp. formally lifts to  $W(k)$ ).

## 5.2 Three-fold conic bundles

By [29, Theorem 3.8], the base scheme of a three-fold conic bundle has  $W\mathcal{O}$ -rational singularities over perfect fields of characteristic  $p \geq 7$ . The same holds for  $p = 5$  using [15]. We show that the base of a three-fold conic bundle has rational singularities for  $p \geq 7$ .

**Theorem 5.3.** Let  $k$  be a perfect field of characteristic  $p \geq 7$ . Let  $(X, \Delta)$  be a klt three-fold log pair and let  $\pi: X \rightarrow S$  be a proper contraction morphism onto a surface  $S$  over  $k$ . If  $-(K_X + \Delta)$  is  $\pi$ -big and  $\pi$ -nef, then  $S$  has rational singularities.

**Proof.** By [1, Corollary 1.3],  $X$  has rational singularities and  $\mathcal{O}_S \rightarrow \mathbf{R}\pi_*\mathcal{O}_X$  is a quasi-isomorphism by Theorem 5.1. Since  $S$  is a normal surface, it is Cohen–Macaulay and thus we conclude by Proposition 2.14. ■

It would be natural to expect that  $S$  has klt singularities, at least for sufficiently large  $p > 0$ . It is easy to see that  $S$  is smooth if the total space is smooth, see [18, 4.11.2].

## 5.3 Pathological examples in higher dimension

The aim of this section is to show new examples of Mori fibre spaces whose bases have non-lc singularities in characteristic  $p > 0$ . The first examples where the singularities of the bases were non-klt (but still log canonical) were constructed for  $p = 2, 3$  in [31, Theorem 1.1]. Our construction is based on the work of Yasuda on wild quotient singularities [32, 33].

Fix an algebraically closed field  $k$  of characteristic  $p > 0$  and the cyclic group  $C_p := \mathbb{Z}/p\mathbb{Z}$ . The Jordan normal form theorem says that every  $C_p$ -representation decomposes as a direct sum of indecomposable representations  $V = \bigoplus_i V_i$  where  $\dim V_i \leq p$ . Yasuda introduces the invariant

$$D_V := \sum_i \binom{\dim V_i}{2},$$

and proves that if  $D_V \geq 2$ , then the quotient variety  $X := V/C_p$  is terminal (resp. canonical, log canonical) if and only if  $D_V > p$  (resp.  $D_V \geq p$ ,  $D_V \geq p - 1$ ).

**Example 5.4.** Assume that  $p \geq 5$  and let  $V_3$  denote the indecomposable  $C_p$ -representation of dimension 3. Then  $Y := V_3/C_p$  is not log canonical.

Next consider  $(\mathbb{P}_k^1)^p$  with the cyclic permutation  $C_p$ -action. Set  $X := ((\mathbb{P}^1)^p \times V_3)/C_p$  where the action of  $C_p$  on  $((\mathbb{P}^1)^p \times V_3)$  is the diagonal one. At the fixed points, in local affine charts, the action is the sum of two irreducible representations  $V_p \oplus V_3$ . Thus  $X$  has terminal singularities by Yasuda's theorem.

Note finally that  $\mathrm{Pic}((\mathbb{P}^1)^p)^{C_p} \simeq \mathbb{Z}$ . We thus conclude that the coordinate projection

$$\pi : X = ((\mathbb{P}^1)^p \times V_3)/C_p \rightarrow V_3/C_p = Y$$

is a Mori fiber space,  $X$  has terminal singularities, and  $Y$  is not even log canonical.

Note that  $Y$  is a hypersurface singularity by [32, Example 6.23] while  $X$  is not Cohen–Macaulay by [10].

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## References

- [1] Arvidsson, E., F. Bernasconi, and J. Lacini. "On the Kawamata–Viehweg vanishing for log del Pezzo surfaces in positive characteristic." (2020): arXiv e-prints arXiv:2006.03571.
- [2] Bernasconi, F. "Kawamata–Viehweg vanishing fails for log del Pezzo surfaces in characteristic 3." *J. Pure Appl. Algebra* 225, no. 11 (2021): 106727, 16.
- [3] Bernasconi, F. "Non-normal purely log terminal centres in characteristic  $\geq 3$ ." *Eur. J. Math.* 5, no. 4 (2019): 1242–51.
- [4] Bhatt, B., L. Ma, Z. Patakfalvi, K. Schwede, K. Tucker, J. Waldron, and J. Witaszek. "Globally  $\mathbb{A}^1$ -regular varieties and the minimal model program for threefolds in mixed characteristic." (2020): arXiv e-prints arXiv:2012.15801.
- [5] Bernasconi, F. and H. Tanaka. "On del Pezzo fibrations in positive characteristic." *J. Inst. Math. Jussieu* (2019): arXiv e-prints arXiv:1903.10116v3.
- [6] Cossart, V., U. Jannsen, and S. Saito. "Desingularization: Invariants and Strategy." *Lecture Notes in Mathematics*, vol. 2270. Springer, 2020.
- [7] Chatzistamatiou, A. and K. Rülling. "Vanishing of the higher direct images of the structure sheaf." *Compos. Math.* 151, no. 11 (2015): 2131–44.

- [8] Cascini, P. and H. Tanaka. "Purely log terminal threefolds with non-normal centres in characteristic two." *Am. J. Math.* 141, no. 4 (2019): 941–79.
- [9] Cynk, S. and D. van Straten. "Small resolutions and non-liftable Calabi–Yau threefolds." *Manuscripta Math.* 130, no. 2 (2009): 233–49.
- [10] Ellingsrud, G. and T. Skjelbred. "Profondeur d’anneaux d’invariants en caractéristique  $p$ ." *Compos. Math.* 41, no. 2 (1980): 233–44.
- [11] Fujita, T. "A relative version of Kawamata–Viehweg vanishing theorem." Unpublished manuscript. Univ. of Tokyo.
- [12] Gongyo, Y., Y. Nakamura, and H. Tanaka. "Rational points on log Fano threefolds over a finite field." *J. Eur. Math. Soc. (JEMS)* 21, no. 12 (2019): 3759–95.
- [13] Hironaka, H. "Resolution of singularities of an algebraic variety over a field of characteristic zero. I, II." *Ann. Math.* 79, no. 2 (1964): 109–203; *ibid.* (2) 79 (1964): 205–326.
- [14] Hacon, C. D. and J. Witaszek. "On the rationality of Kawamata log terminal singularities in positive characteristic." *Algebraic Geometry* 6, no. 5 (2019): 516–29.
- [15] Hacon, C. D. and J. Witaszek. "The Minimal Model Program for threefolds in characteristic five." (2019): arXiv e-prints arXiv:1911.12895.
- [16] Kawasaki, T. "On Macaulayfication of Noetherian schemes." *Trans. Amer. Math. Soc.* 352, no. 6 (2000): 2517–52.
- [17] Kollár, J. and S. Mori. "Birational Geometry of Algebraic Varieties." *Cambridge Tracts in Mathematics*, vol. 134. Cambridge: Cambridge University Press, 1998. With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original.
- [18] Kollár, J. "Extremal rays on smooth threefolds." *Ann. Sci. École Norm. Sup. (4)* 24, no. 3 (1991): 339–61.
- [19] Kollár, J. "A local version of the Kawamata–Viehweg vanishing theorem." *Pure Appl. Math. Q.* 7, no. 4 (2011): 1477–94. Special Issue: in memory of Eckart Viehweg.
- [20] Kollár, J. "Singularities of the Minimal Model Program." *Cambridge Tracts in Mathematics*, vol. 200. Cambridge: Cambridge University Press, 2013. With a collaboration of Sándor Kovács.
- [21] Kollár, J. "Coherence of local and global hulls." *Methods Appl. Anal.* 24, no. 1 (2017): 63–70.
- [22] Kollár, J. "Relative MMP without  $\mathbb{Q}$ -factoriality." (2020): arXiv e-prints arXiv:2012.05327v2.
- [23] Kovács, S. J. "A characterization of rational singularities." *Duke Math. J.* 102, no. 2 (2000): 187–91.
- [24] Kovács, S. J. "The Splitting Principle and Singularities." *Compact Moduli Spaces and Vector Bundles*. Contemp. Math., vol. 564. 195–204. Providence, RI: Amer. Math. Soc, 2012.
- [25] Kovács, S. J. "Rational singularities." (2017): arXiv e-prints arXiv:1703.02269v9.
- [26] Kollár, J. and J. Witaszek. "Resolution and alteration with ample exceptional divisor." (2021): arXiv e-prints arXiv:2102.03162.
- [27] Lipman, J. "Rational singularities, with applications to algebraic surfaces and unique factorization." *Inst. Hautes Études Sci. Publ. Math.* 36 (1969): 195–279.
- [28] Murayama, T. "Relative vanishing theorems for  $\mathbb{Q}$ -schemes." (2021): arXiv e-prints arXiv:2101.10397.

- [29] Nakamura, Y. and H. Tanaka. "Witt Nadel vanishing theorem for threefolds." *Compos. Math.* 156, no. 3 (2020): 435–75.
- [30] Tanaka, H. "Minimal model program for excellent surfaces." *Univ. Grenoble. Ann. Institut Fourier. Univ. Grenoble I* 68, no. 1 (2018): 345–76.
- [31] Tanaka, H. "Pathologies on Mori fibre spaces in positive characteristic." *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* 20, no. 3 (2020): 1113–34.
- [32] Yasuda, T. "The  $p$ -cyclic McKay correspondence via motivic integration." *Compos. Math.* 150, no. 7 (2014): 1125–68.
- [33] Yasuda, T. "Discrepancies of  $p$ -cyclic quotient varieties." *J. Math. Sci. Univ. Tokyo* 26, no. 1 (2019): 1–14.