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On Nonexistence of Splash Singularities for the α-SQG Patches

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Abstract
In this paper, we consider patch solutions to the α-SQG equation and derive new criteria for the absence of splash singularity where different patches or parts of the same patch collide in finite time. Our criterion refines a result due to Gancedo and Strain Gancedo and Strain (2014), providing a condition on the growth of curvature of the patch necessary for the splash and an exponential in time lower bound on the distance between patches with bounded curvature.

Keywords α-SQG equation · Patch Solutions · Slash Singularities

Mathematics Subject Classification 35Q86

1 Introduction

Recall that the family of α-SQG equations is given by

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = 0, \\ u = \nabla^{\perp}(-\Delta)^{-1+\alpha}\omega, \end{cases} \tag{1.1}$$

where $\nabla^{\perp} = (\partial_{x_2}, -\partial_{x_1})$ denotes the perpendicular gradient. The value $\alpha = 0$ in (1.1) corresponds to the Euler equation, and $\alpha = 1/2$ to the SQG equation. In general, models with α in the range $(0, 1)$ have been considered (Constantin et al. 2008; Chae et al. 2012). The α-SQG equations appear in atmospheric and ocean science (see

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Constantin et al. 1994; Held et al. 1995), and model evolution of temperature near the surface of a planet. Mathematically, the SQG equation has some similarities with the 3D Euler equation (Constantin et al. 1994) and has been a focus of much attention in recent years. The global regularity vs finite time blowup question for smooth initial data remains open for any $1 > \alpha > 0$. A singular scenario, closing front, has been presented in Constantin et al. (1994). However, later rigorous work (Cordoba 1998; Cordoba and Fefferman 2002; Cordoba et al. 2004) has shown that finite time blowup cannot happen in this scenario.

The SQG equation is in particular used to model frontogenesis: an interface with a sharp jump of temperature across it. In this context, patch solutions are natural. These are weak solutions of the equation that have form

$$\omega(x,t) = \sum_{j=1}^{n} \theta_j \chi_{\Omega_j(t)}(x),$$

where θ_j are constants, χ_S denotes the characteristic function of the set S, and $\Omega_j(t)$ are disjoint, regular regions evolving in time according to the Biot–Savart law (1.1) (this will be made more precise later). In the context of patches, the global regularity question is whether the patch solution conserves the initial regularity class of the boundaries $\partial\Omega(0)$, and whether different patches can collide or self-intersect. The existence and uniqueness of patch solutions for the 2D Euler equation is a consequence of Yudovich theory (see (Yudovich 1963; Marchioro and Pulvirenti 1994)), and global regularity has been proved by Chemin (Chemin 1993). For $\alpha > 0$, even the existence of patch solutions is not trivial. Local existence and uniqueness results of α-SQG patch solutions have been proved in Rodrigo (2005); Gancedo (2008); Cordoba et al. (2018); Chae et al. (2012). Numerical simulation in Cordoba et al. (2005) indicated a possible splash singularity where two patches touch each other with simultaneous formation of corners at the touch point, yet rigorous understanding of the phenomenon remained missing. For small $\alpha > 0$, finite time singularity formation has been proved for patches in the half-plane setting (Kiselev et al. 2016). This singularity formation happens near the hyperbolic point of the flow on the boundary, and in a scenario similar to very fast small-scale growth in solutions to 2D Euler equation (Kiselev and Sverak 2014; Kiselev and Li 2019) and conjectured singularity formation in the 3D Euler Hou-Luo scenario (Luo and Hou 2014). On the other hand, there are also recent numerical simulations by Scott and Dritschel (2014, 2019) which suggests a different pathway towards a singularity. In Scott and Dritschel (2014), an intricate self-similar cascade of filament instabilities is explored, where the picture roughly repeats in different locations at a geometric sequence of decreasing length scales and time intervals. In Scott and Dritschel (2019), it is suggested that filament pinching may happen in a simpler fashion, at one of the stages of the previous instability cascade. This filament pinching might be of the type of splash singularity, where different parts of the patch boundary touch each other. The formation of a splash singularity has been rigorously established for water waves (Castro et al. 2013; Coutand and Shkoller 2014), but the difference with α-patch case is that the wave interfaces can remain regular near the intersection. This cannot happen for the SQG patches, at least not in a

simple way, as the parts of the patch(es) with bounded curvature will never collide as shown by Gancedo and Strain (Gancedo and Strain 2014). Thus a simple α-SQG splash singularity can only happen along with the loss of regularity of the patch boundary, and rigorous examples of it remain missing possibly apart from (Kiselev et al. 2016) for $0 < \alpha < \frac{1}{24}$ and the extension (Gancedo and Patel 2021) to $0 < \alpha < \frac{1}{6}$ (where a fixed boundary is present and the precise picture of singularity is not established).

In this note, our goal is to sharpen the criterion of ruling out splash singularity for the α-SQG models, in order to understand what kind of phenomena—specifically, the rate of growth of certain norms associated with patch regularity—must appear for the splash to occur. We also prove a sharper separation result for patches with bounded curvature, improving the double exponential bound of Gancedo and Strain (2014) to just exponential in time. The key observation for these results is an additional folding odd symmetry in the Biot-Savart law.

1.1 Criteria of No Splash Singularities

In the context of this paper, we will interpret the splash singularity as in the following definition; we will recall the notion of patch solution more precisely in Sect. 2.1.

Definition 1.1 Let $k \in \mathbb{N}$ and $\gamma \in [0, 1]$. We say that a $C^{k,\gamma}$-patch solution $\Omega(t) = \cup_{j=1}^{N} \Omega_j(t)$ on $[0, T)$ develops a splash singularity as $t \to T^-$ if there exists $\varepsilon > 0$ and a fixed ball $B_\rho(x_0)$, $\rho > 0$ such that on $[T - \varepsilon, T)$ all of the following holds.

- There are only two disjoint branches $\mathcal{C}_1(t)$ and $\mathcal{C}_2(t)$ of the interface in the ball $B_\rho(x_0)$ that are simple curves, and $\mathcal{C}_1(t)$ and $\mathcal{C}_2(t)$ touch at a single point x_0 as $t \to T^-$.
- In the complement of the ball $B_\rho(x_0)$, the patch solution remains regular in the whole time interval $[0, T]$: the $C^{k,\gamma}$ norms of the patches remain bounded, different patches do not touch and individual patches do not self-intersect.
- The $C^{k,\gamma}$ regularity may be lost at time T as interfaces develop singular structures, but the singularity is localized at x_0.

For the sake of simplicity, we can think of all norms defined in terms of intrinsic arc length parametrization, with

$$\|\partial\Omega(t)\|_{C^{k,\gamma}} = \max_j \left(\sum_{l=0}^{k} \|\partial_s^l x_j(t)\|_\infty + \sup_{s,r} \frac{|\partial_s^k x_j(s) - \partial_s^k x_j(r)|}{|s - r|^\gamma} \right),$$

where x_i is the arc length parametrization of the ith patch $\Omega_i(t)$ and s, r are arc length parameters. The lack of self-intersection outside $B_\rho(x_0)$ can be rigorously defined as a positive lower bound for the arc chord ratio:

$$\min_{j, x_j(s), x_j(r) \notin B_\rho} \frac{|x_j(s) - x_j(r)|}{|s - r|} \geq c > 0.$$

Theorem 1.2 *Let* $0 < \alpha < 1$, $k := \lceil 2\alpha \rceil$ *and* $1 \geq \gamma \geq 2\alpha - \lceil 2\alpha \rceil + 1$. *Suppose that* ω *is a* $C^{k,\gamma}$*-patch solution to* (1.1) *on* $[0, T)$ *that forms a splash singularity at time* T. *Then we must have*

$$\int_0^T \|\partial\Omega(t)\|_{C^{k,\gamma}}^{\frac{k+2\alpha-1}{k+\gamma-1}} \, dt = \infty. \tag{1.2}$$

In particular, the curvature of $\partial\Omega$ controls splash singularity for all $\alpha \leq 1/2$ (as was shown in Gancedo and Strain (2014) for $\alpha = 1/2$).

1.2 Exponential Bound of Minimal Distance

In the case where we have a priori control on growth of the appropriate norms of patch solution, we can derive a lower bound on separation distance between different parts of the patch boundary provided that at any time the minimal distance is achieved at two points with certain properties. In particular, we need to make an assumption limiting the nature of how the patch boundary can approach itself.

Assumption 1.3 Assume that the $C^{k,\gamma}$ patch solution $\Omega(t)$ satisfies the following property: there exists $\eta > 0$ and $c > 0$ such that for all $t \in [0, T)$ and all $i = 1, \ldots, n$, $|x_i(s) - x_i(r)| \geq c|r - s|$ for all s, r with $|s - r| \leq \eta$. Here x_i is the arc length parametrization of the ith patch Ω_i.

We explain all the details later in the paper, but let us state here the main result (referring to forthcoming definitions).

Theorem 1.4 *Let* $0 < \alpha < 1$, $k := \lceil 2\alpha \rceil$, *and* $1 \geq \gamma \geq 2\alpha - \lceil 2\alpha \rceil + 1$. *Suppose that* ω *is a* $C^{k,\gamma}$*-patch solution to* (1.1) *on* $[0, T)$ *satisfying Assumption 1.3. Let* $d(t)$ *be the separation distance defined in* (3.14). *Assume in addition that for a.e. time* $t \in [0, T]$, *all points* p, q *such as* $|p - q| = \mathrm{d}(t)$ *are admissible in the sense of Definition 3.1. Then for all* $t \in [0, T]$, *we have*

$$\mathrm{d}(t) \geq \mathrm{d}(0) \exp\left(-C \int_0^t \|\partial\Omega(t')\|_{C^{k,\gamma}}^{\frac{k+2\alpha-1}{k+\gamma-1}} \, \mathrm{d}t'\right),$$

$C = C(\alpha, k, \gamma, \eta)$.

Note that if the curvature of $\partial\Omega(t)$ is bounded uniformly in time, we obtain an exponential in time lower bound on how quickly the patch boundaries can approach for $0 < \alpha \leq 1/2$.

We remark that simultaneously and independently of this paper, Jeon and Zlatoš (2021) were able to show the absence of patch singularities for α-patches with bounded $C^{1,2\alpha}$ norm without additional assumptions on the geometry of the solution for $0 \leq \alpha < 1/4$.

2 Preliminaries

2.1 Definition of a Patch Solution

The explicit form of the Biot-Savart law for the α-SQG equation (1.1) that is valid for smooth ω is given by

$$u(x,t) = P.V. \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^{2+2\alpha}} \omega(y,t)\, \mathrm{d}y \tag{2.1}$$

(we omit the constant $c(\alpha)$ in front of the integral). For patch solutions and $\alpha \geq 1/2$, the tangential to patch component of the velocity is infinite even for smooth patches, so we will only deal with normal component defining the patch evolution, i.e., (2.3). We adapt the definition of patch solution similar to Kiselev et al. (2016). Recall that the Hausdorff distance between any two sets A and B is given by

$$\mathrm{d}_H(A,B) = \max(\sup_{a\in A}\mathrm{dist}(a,B),\ \sup_{b\in B}\mathrm{dist}(A,b)).$$

Definition 2.1 Let $T > 0$, $N < \infty$ be an integer, and $\theta_i \in \mathbb{R}$ for $1 \leq i \leq N$. Suppose that $\Omega_i(t) \subset \mathbb{R}^2$ are bounded open sets whose boundaries $\partial\Omega_i(t)$ are pairwise disjoint closed C^1 curves for every $t \in [0,T)$. Let $\Omega = \bigcup_{1\leq i\leq N} \Omega_i$ and

$$\omega(\cdot,t) := \sum_{1\leq i\leq N} \theta_i \chi_{\Omega_i(t)}.$$

We say ω (or just Ω) is a patch solution for the α-SQG equation on $[0,T)$ if the followings are satisfied.

(1) Each $\partial\Omega_i$ is continuous in t with respect to the Hausdorff distance d_H.
(2) Denote $\partial\Omega(t) := \cup_i \partial\Omega_i(t)$. Then for every t while patch solution exists

$$\lim_{h\to 0+} \frac{d_H\left(\partial\Omega(t+h), X^h_{u_n(\cdot,t)}[\partial\Omega(t)]\right)}{h} = 0 \tag{2.2}$$

where $X^h_v[E] := \{x + hv(x) : x \in E\}$ and $u_n = (u\cdot n)n$ is the normal to the boundary $\partial\Omega(\cdot,t)$ component of the velocity field given by

$$u_n(x,t) = n(x) P.V. \int_{\mathbb{R}^2} \frac{n(x)\cdot(x-y)^\perp}{|x-y|^{2+2\alpha}} \omega(y,t)\, \mathrm{d}y \quad \text{for} x \in \partial\Omega, \tag{2.3}$$

where $n(x)$ is the unit normal vector of $\partial\Omega$.

The tangential component of the velocity (2.1) is infinite for any regularity of the boundary starting from $\alpha = 1/2$, and this explains our difference with the definition of Kiselev et al. (2016) where only small values of α were considered.

The following elementary lemma shows that $u_n = (u\cdot n)n$ is well defined for patches with even less regularity than our case; in the above formula (2.3), we first

take the inner product with the normal at $x \in \partial\Omega$ and then integrate according to (2.1). In the next lemma, we abuse notation and denote by Ω a single fixed $C^{1,\gamma}$ domain (not necessarily a patch solution).

Lemma 2.2 *Let $0 < \alpha < 1$, and suppose that Ω is a compact, connected $C^{1,\gamma}$ domain with $\gamma > \max(0, 2\alpha - 1)$. Then $u_n = (u \cdot n)n$, the normal to $\partial\Omega$ component of the velocity u, computed according to (2.3) with $\omega(x) = \chi_\Omega(x)$ is well defined and finite at all points on $\partial\Omega$, and is continuous on $\partial\Omega$ with*

$$\|u_n\|_{C(\partial\Omega)} \leq C(\Omega). \tag{2.4}$$

Proof Fix a point on $\partial\Omega$, and choose a system of coordinates with this point at its origin such that x_1 axis is along the tangent to $\partial\Omega$. Then

$$u_n = -P.V. \int_\Omega \frac{y_1}{|y|^{2+2\alpha}}\, dy. \tag{2.5}$$

Consider a square $S_R = [-R, R]^2$ with $R = 0.1\|\partial\Omega\|^{-1}_{C^{1,\gamma}}$ centered at the origin, with one of its sides parallel to y_1 axis. Let $f(y_1)$ be a function such that the graph $y_2 = f(y_1)$ within S_R coincides with the part of $\partial\Omega$ containing the origin. If S_R contains more than the curve of $\partial\Omega$ passing through the origin, make R smaller so that there is only one curve of $\partial\Omega$ in it. The part in (2.5) coming from integration over S^c_R can be estimated from above by

$$\int_{|y|\geq R} |y|^{-1-2\alpha} \chi_\Omega(y)\, \mathrm{d}y \leq C(\Omega)\max(1, R^{1-2\alpha}) \tag{2.6}$$

due to the compactness of Ω. To exploit the odd in y_1 symmetry, for any $y_1 \geq 0$ we introduce

$$\overline{f}(y) := \max\{f(y), f(-y))\} \quad \text{and} \quad \underline{f}(y) := \min\{f(y), f(-y))\}. \tag{2.7}$$

Then, with (2.7) we have

$$\left| \int_{\Omega\cap S_R} \frac{y_1}{|y|^{2+2\alpha}} d\, y \right| \leq \int_0^R \int_{\underline{f}(y_1)}^{\bar{f}(y_1)} \frac{y_1}{|y|^{2+2\alpha}}\, \mathrm{d}y_2 \mathrm{d}y_1. \tag{2.8}$$

To facilitate the estimates let us note that since f is $C^{1,\gamma}$ and $f(0) = f'(0) = 0$, we have

$$|\overline{f}(y_1) - \underline{f}(y_1)| \leq \|\partial\Omega\|_{C^{1,\gamma}} |y_1|^{1+\gamma} \quad \text{for all} |y_1| \leq R.$$

Also, in the region of integration on the right of (2.8) $y_2 \leq y_1$ due to the regularity of $\partial\Omega$ and choice of R. Then we can continue the estimate (2.8) and obtain upper bound

by

$$C(\Omega)\int_0^R y_1^{2+\gamma} y_1^{-2-2\alpha} \mathrm{d}y_1 \leq C(\Omega) R^{1+\gamma-2\alpha}. \tag{2.9}$$

Given that R only depends on Ω, the bounds (2.6) and (2.9) imply (2.4) for the L^∞ norm of u_n. The continuity of u_n on $\partial\Omega$ follows by Vitali convergence theorem. Indeed, let us perform an odd reflection like above in the integrals defining $u_n(x)$ and $u_n(x')$. Then the regions of integration and integrands converge pointwise as $x' \to x$ along $\partial\Omega$. Also, the integrands are uniformly integrable due to elementary estimates using the structure of integration regions similarly to (2.8), (2.9). □

In this note, we do not discuss questions of the existence and uniqueness of patch solutions. Existence of patch solutions with C^∞ or Sobolev regularity follows from the contour equation analysis in Gancedo (2008); Rodrigo (2005) for $0 < \alpha \leq 1/2$, and in Chae et al. (2012) for $\alpha > 1/2$. The uniqueness of patch solutions in the whole space setting is known for $\alpha \leq 1/2$ (Gancedo 2008; Cordoba et al. 2018) and is open to the best of our knowledge in the case $\alpha > 1/2$.

3 Absence of Splash Singularities

3.1 Geometric Configuration

Given our definition of splash singularity, starting from time $t = T - \varepsilon$ there exists a fixed ball $B_\rho(x_0)$ and a pair of points $p(t), q(t) \in B_\rho$ such that $|p - q|$ is the minimal distance between any two patches (or the maximum of the arc-chord condition if it is different parts of the same patch that form splash singularity). Furthermore, for each $t \in [T-\varepsilon, T)$, $\partial\Omega(t) \cap B_\rho$ consists of two disjoint simple $C^{k,\gamma}$ curves one of which contains p and the other q. Since there is only one touching point x_0 at time T and the motion of $\partial\Omega_j$ is continuous in time, we can freely assume that any such pair $p, q \in B_{\rho/4}(x_0)$: This will be true starting from some time $t \geq T - \varepsilon$.

3.2 Estimates for the Velocity Difference

We now prove a result on relative velocity of the points p and q inspired by the above discussion. Given k and γ, define

$$r(t) = \frac{1}{4}\min(\rho, (0.01\|\partial\Omega(t)\|_{C^{k,\gamma}})^{-\frac{1}{k+\gamma-1}}). \tag{3.1}$$

Without loss of generality, we will assume that $\rho \leq 1$ and so $r < 1$.

Definition 3.1 Let $0 < \alpha < 1$, $k := \lceil 2\alpha \rceil$, and $1 \geq \gamma \geq 2\alpha - \lceil 2\alpha \rceil + 1$. Suppose that ω is a $C^{k,\gamma}$-patch solution of (1.1) and $\Omega(t)$ is the union of the patches.

We say $p, q \in \partial\Omega(t)$ is a pair of admissible points at time t if

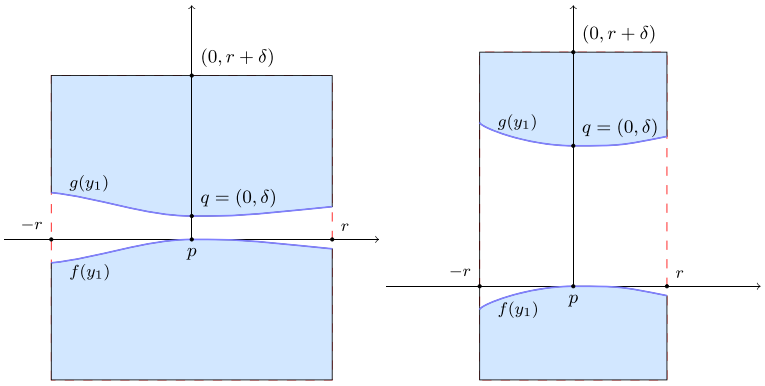

Fig. 1 Illustration of a pair of admissible points p, q. The left has $|p-q| \ll r$ while the right has $|p-q| \gg r$

- $\partial\Omega(t) \cap (B_{2r}(p) \cup B_{2r}(q))$ consists of two disjoint curves $\mathcal{C}_1$ and $\mathcal{C}_2$, one containing p and the other q; here r is given by (3.1).
- the distance between p and q forms a local minimum distance, i.e.

$$|p - q| = \operatorname{dist}(\mathcal{C}_1, \mathcal{C}_2).$$

Note that in general, there could be other patches between $\mathcal{C}_1$ and $\mathcal{C}_2$ for example if $|p - q| \gg r$ - the definition of admissible points does not preclude that.

Proposition 3.2 *Let $0 < \alpha < 1$, $k := \lceil 2\alpha \rceil$, and $1 \geq \gamma \geq 2\alpha - \lceil 2\alpha \rceil + 1$. Suppose that ω is a $C^{k,\gamma}$-patch solution and there is a pair of admissible points $p, q \in \partial\Omega$ at some time t.*

Then the difference of the normal to patch components u_n of the velocity u defined by (2.3) satisfies

$$\left|u_n(p,t) - u_n(q,t)\right| \leq C \|\partial\Omega(t)\|_{C^{k,\gamma}}^{\frac{k+2\alpha-1}{k+\gamma-1}} |p - q|. \tag{3.2}$$

where the constant C depends on α, γ, ρ, and the couplings θ_j of different patches.

Proof For simplicity, we drop the time variable t in the proof. Let us set up the coordinate system center at the point p such that the segment of minimum distance is on the x_2-axis and $q = (0, \delta)$, $\delta := |p - q|$. Parametrize the two patch interfaces by $(x_1, f(x_1))$ for the bottom piece and $(x_1, g(x_1))$ for the top piece. It is not difficult to check that with our regularity assumptions on $\partial\Omega$ and the choice of r (3.1), such a representation is valid for $x_1 \in [-r, r]$. See Fig. 1 for an illustration. Note that when $|q - p| \gg r$, there might be other branches of the patches between $p - q$, but this does not affect our argument.

Denote the coupling constants of the top and bottom patches by θ_1 and θ_2, respectively. The vertical velocity at the points p and q coincides with the normal component

of u and is given by

$$u_n(p) = \theta_1 \int_{E_p} \frac{y_1}{|y|^{2+2\alpha}}\,\mathrm{d}y + \theta_2 \int_{E_q} \frac{y_1}{|y|^{2+2\alpha}}\,\mathrm{d}y + \int_{\mathbb{R}^2\setminus E} \frac{y_1}{|y|^{2+2\alpha}}\omega(y)\,\mathrm{d}y$$

and

$$\begin{aligned} u_n(q) = \theta_1 \int_{E_p} \frac{y_1}{|y-(0,\delta(t))|^{2+2\alpha}}\,\mathrm{d}y + \theta_2 \int_{E_q} \frac{y_1}{|y-(0,\delta(t))|^{2+2\alpha}}\,\mathrm{d}y \\ + \int_{\mathbb{R}^2\setminus E} \frac{y_1}{|y-(0,\delta(t))|^{2+2\alpha}}\omega(y)\,\mathrm{d}y \end{aligned}$$

where $E = E_p \cup E_q$ and

$$E_p := \{x \in \mathbb{R}^2 : -r \le x_1 \le r, \quad -r \le x_2 \le f(x_1)\}$$

while

$$E_q := \{x \in \mathbb{R}^2 : -r \le x_1 \le r, \quad g(x_1) \le x_2 \le r + \delta\}$$

The difference of normal velocities reads

$$\begin{aligned} u_n(q) - u_n(p) &= \theta_1 \int_{E_p} y_1 \left(\frac{1}{|y|^{2+2\alpha}} - \frac{1}{|y-(0,\delta)|^{2+2\alpha}}\right)\mathrm{d}y \\ &\quad + \theta_2 \int_{E_q} y_1 \left(\frac{1}{|y|^{2+2\alpha}} - \frac{1}{|y-(0,\delta)|^{2+2\alpha}}\right) \\ &\quad + \int_{\mathbb{R}^2\setminus E} y_1 \left(\frac{1}{|y|^{2+2\alpha}} - \frac{1}{|y-(0,\delta)|^{2+2\alpha}}\right)\omega(y)\,\mathrm{d}y \\ &:= I_1 + I_2 + I_3. \end{aligned}$$

Estimates of I_1

We first split the integral by y_2 axis

$$\begin{aligned} I_1 = \theta_1 \int_{0\le y_1\le r} \int_{-r\le y_2\le f(y_1)} y_1 \left(\frac{1}{|y|^{2+2\alpha}} - \frac{1}{|y-(0,\delta)|^{2+2\alpha}}\right)\mathrm{d}y \\ - \theta_1 \int_{0\le y_1\le r} \int_{-r\le y_2\le f(-y_1)} y_1 \left(\frac{1}{|y|^{2+2\alpha}} - \frac{1}{|y-(0,\delta)|^{2+2\alpha}}\right)\mathrm{d}y \end{aligned} \tag{3.3}$$

where in the second integral we have applied a change of variable $y_1 \mapsto -y_1$.

To exploit the odd-in-y_1 symmetry, for any $y_1 \ge 0$ we introduce

$$\overline{f}(y) := \max\{f(y), f(-y))\} \quad \text{and} \quad \underline{f}(y) := \min\{f(y), f(-y))\}.$$

With these, due to the opposite signs in (3.3), we have

$$|I_1| \le \theta_1 \int_{0\le y_1\le r} \int_{\underline{f}(y_1)\le y_2\le \overline{f}(y_1)} y_1 \left| \frac{1}{|y|^{2+2\alpha}} - \frac{1}{|y-(0,\delta)|^{2+2\alpha}} \right| \mathrm{d}y. \tag{3.4}$$

To facilitate the estimates let us note that since $f(0) = f'(0) = 0$, when f is $C^{1,\gamma}$, we have

$$|\overline{f}(y_1) - \underline{f}(y_1)| \lesssim \|\partial\Omega\|_{C^{1,\gamma}} |y_1|^{1+\gamma}$$

and when f is $C^{2,\gamma}$, by the Fundamental Theorem of Calculus we have

$$\begin{aligned} |\overline{f}(y_1) - \underline{f}(y_1)| &= \left| \int_0^{y_1} (f'(x) + f'(-x)) dx \right| \\ &= \left| \int_0^{y_1} \int_0^x (f''(z) - f''(-z))\, dz dx \right| \lesssim \|\partial\Omega\|_{C^{2,\gamma}} |y_1|^{2+\gamma}. \end{aligned}$$

We may write these two as one estimate with $k = 1, 2$:

$$|\overline{f}(y_1) - \underline{f}(y_1)| \lesssim \|\partial\Omega\|_{C^{k,\gamma}} |y_1|^{k+\gamma}. \tag{3.5}$$

By the Mean Value theorem, for some z_2 lying between y_2 and $y_2 - \delta$ we have

$$\left| \frac{1}{|y|^{2+2\alpha}} - \frac{1}{|y-(0,\delta)|^{2+2\alpha}} \right| = c_\alpha \delta \frac{|z_2|}{(y_1^2 + z_2^2)^{2+\alpha}} \le \frac{c_\alpha \delta(\delta + |y_2|)}{|y_1|^{4+2\alpha}}. \tag{3.6}$$

To bound I_1, we consider two separate integral regions in (3.4) where $\delta \le y_1 \le r$ and where $y_1 \le \min\{\delta, r\}$. A direct computation using (3.4) and (3.6) shows that

$$I_1 \lesssim \int_{y_1\le \min\{\delta,r\}} \int_{\underline{f}(y_1)\le y_2\le \overline{f}(y_1)} \frac{\mathrm{d}y}{y_1^{1+2\alpha}} + \delta \int_{\delta\le y_1\le r} \int_{\underline{f}(y_1)\le y_2\le \overline{f}(y_1)} \frac{(\delta + |y_2|)\mathrm{d}y}{y_1^{3+2\alpha}}.$$

Here in the first integral we use a simple direct estimate on the integrand in (3.4). Note that the set $\delta \le y_1 \le r$ may be empty, in which case we do not need to bound the latter integral. Then, applying (3.5), we have

$$\begin{aligned} I_1 \lesssim \|\partial\Omega\|_{C^{k,\gamma}} \int_{0\le y_1\le \min\{\delta,r\}} y_1^{k+\gamma-2\alpha-1}\, \mathrm{d}y_1 + \|\partial\Omega\|_{C^{k,\gamma}} \delta^2 \int_{\delta\le y_1\le r} y_1^{k+\gamma-3-2\alpha}\, \mathrm{d}y_1 \\ + \delta \int_{\delta\le y_1\le r} \int_{\underline{f}(y_1)\le y_2\le \overline{f}(y_1)} |y_2| y_1^{-3-2\alpha}\, \mathrm{d}y. \end{aligned} \tag{3.7}$$

Since $3 \ge k + \gamma \ge 1 + 2\alpha$, the first two integrals in (3.7) are bounded by $\|\partial\Omega\|_{C^{k,\gamma}} \delta r^{k+\gamma-1-2\alpha}$. For the last integral in (3.7), we use (3.5) together with a non-optimal bound (only optimal when $k = 1$)

$$|\overline{f}(y_1) + \underline{f}(y_1)| \le \|\partial\Omega\|_{C^{k,\gamma}} |y_1|^{1+\gamma}. \tag{3.8}$$

to obtain that

$$\int_{\delta\le y_1\le r}\int_{\underline{f}(y_1)\le y_2\le \overline{f}(y_1)} |y_2| y_1^{-3-2\alpha}\,\mathrm{d}y \lesssim \int_{\delta\le y_1\le r} \Big|\underline{f}(y_1)-\overline{f}(y_1)\Big|\Big|\underline{f}(y_1)+\overline{f}(y_1)\Big| y_1^{-3-2\alpha}\,\mathrm{d}y_1$$
$$\lesssim \|\partial\Omega\|^2_{C^{k,\gamma}}\int_{\delta\le y_1\le r} y_1^{k+2\gamma-2-2\alpha}\,\mathrm{d}y.$$

Since $k+2\gamma-2-2\alpha\ge\gamma-1>-1$, combining the terms in (3.7) we obtain

$$I_1\lesssim \|\partial\Omega\|_{C^{k,\gamma}}\delta r^{k+\gamma-2\alpha-1}\big(1+\|\partial\Omega\|_{C^{k,\gamma}} r^{\gamma}\big). \tag{3.9}$$

Estimates of I_2

Similarly to the estimation of I_1, for any $y\ge 0$ we introduce

$$\overline{g}(y):=\max\{g(y),g(-y))\}\quad\text{and}\quad \underline{g}(y):=\min\{g(y),g(-y))\}.$$

to obtain

$$|I_2|\le\int_{0\le y_1\le r}\int_{\underline{g}(y_1)\le y_2\le\overline{g}(y_1)|} y_1\left|\frac{1}{|y|^{2+2\alpha}}-\frac{1}{|y-(0,\delta)|^{2+2\alpha}}\right|\mathrm{d}y.$$

Since $g(0)=\delta$ and $g'(0)=0$, by the same reasoning for (3.5), we have for $k=1,2$,

$$|\overline{g}(y_1)-\underline{g}(y_1)|\le\|\partial\Omega\|_{C^{k,\gamma}}|y_1|^{k+\gamma},$$

and

$$|\overline{g}(y_1)+\underline{g}(y_1)|\le 2\delta+\|\partial\Omega\|_{C^{k,\gamma}}|y_1|^{1+\gamma}.$$

A similar argument shows that I_2 shares the same estimate as I_1:

$$I_2\lesssim\|\partial\Omega\|_{C^{k,\gamma}}\delta r^{k+\gamma-2\alpha-1}\big(1+\|\partial\Omega\|_{C^{k,\gamma}} r^{\gamma}\big). \tag{3.10}$$

Estimates of I_3:

For the last term I_3, we consider two cases: $2\delta<r$ and $2\delta\ge r$ and will show that in both cases

$$I_3\lesssim\delta r^{-2\alpha}.$$

Case 1: $2\delta\le r$:

By the Mean Value theorem,

$$\left|\frac{1}{|y|^{2+2\alpha}}-\frac{1}{|y-(0,\delta)|^{2+2\alpha}}\right|\leq c_\alpha\delta\max_{z_2\in[y_2-\delta,y_2]}\frac{|z_2|}{(y_1^2+z_2^2)^{2+\alpha}}. \tag{3.11}$$

For $y\in(\mathbb{R}^2\setminus E)\cap\mathrm{supp}(\omega)$ we have, due to the definition of admissible points, that $|y|\geq r$. Since $\delta\leq r/2$, it follows that

$$\max_{z_2\in[y_2-\delta,y_2]}\frac{|z_2|}{(y_1^2+z_2^2)^{2+\alpha}}\leq\max_{z_2\in[y_2-\delta,y_2]}\frac{1}{(y_1^2+z_2^2)^{\frac{3}{2}+\alpha}}\leq 2^{3+2\alpha}|y|^{-3-2\alpha}. \tag{3.12}$$

It follows from (3.11), (3.12) and the definition of I_3 that

$$I_3\lesssim\delta\int_{|y|\geq r}\frac{|\omega(y)|}{|y|^{2+2\alpha}}\,\mathrm{d}y\lesssim\delta r^{-2\alpha}. \tag{3.13}$$

Case 2: $2\delta\geq r$:

In this case, we split the integral:

$$\begin{aligned}I_3=&\int_{\widetilde{E}^c}y_1\left(\frac{1}{|y|^{2+2\alpha}}-\frac{1}{|y-(0,\delta)|^{2+2\alpha}}\right)\omega(y)\,\mathrm{d}y\\&+\int_{|y-(0,\delta/2)|>4\delta}y_1\left(\frac{1}{|y|^{2+2\alpha}}-\frac{1}{|y-(0,\delta)|^{2+2\alpha}}\right)\omega(y)\,\mathrm{d}y:=I_{31}+I_{32}\end{aligned}$$

where $\widetilde{E}^c$ is defined by

$$\widetilde{E}^c:=\{y\in\Omega(t):|y-(0,\delta/2)|\leq 4\delta\}\cap E^c,$$

with $(0,\frac{\delta}{2})$ being the mid-point between p and q.

For I_{31}, we use the triangle inequality and the bound

$$\left|\frac{y_1}{|y|^{2+2\alpha}}\right|+\left|\frac{y_1}{|y-(0,\delta)|^{2+2\alpha}}\right|\leq\frac{1}{|y|^{1+2\alpha}}+\frac{1}{|y-(0,\delta)|^{1+2\alpha}}$$

to obtain

$$\begin{aligned}I_{31}&\lesssim\int_{\widetilde{E}^c}\frac{1}{|y|^{1+2\alpha}}\,\mathrm{d}y+\int_{\widetilde{E}^c}\frac{1}{|y-(0,\delta)|^{1+2\alpha}}\,\mathrm{d}y\\&\lesssim\int_{r\leq|y|\leq 5\delta}\frac{1}{|y|^{1+2\alpha}}\,\mathrm{d}y\lesssim\delta^{1-2\alpha}\lesssim\delta r^{-2\alpha}.\end{aligned}$$

For I_{32}, similarly to (3.11) and (3.12), for $y\in\mathbb{R}^2\setminus\widetilde{E}^c$ we have

$$\left|\frac{1}{|y|^{2+2\alpha}}-\frac{1}{|y-(0,\delta)|^{2+2\alpha}}\right|\leq c_\alpha\delta\max_{z_2}\frac{z_2}{(y_1^2+z_2^2)^{2+\alpha}}\lesssim\delta\frac{1}{|y|^{3+2\alpha}}.$$

Inserting this bound into the integral I_{32} gives

$$I_{32} \lesssim \delta \int_{|y|\geq 2\delta} \frac{1}{|y|^{2+2\alpha}}\,\mathrm{d}y \lesssim \delta^{1-2\alpha} \lesssim \delta r^{-2\alpha}.$$

Putting together I_{31} and I_{32} yields

$$I_3 \lesssim \delta r^{-2\alpha}.$$

Combined estimates:

Now observe that with our choice of r (3.1), $r^{k+\gamma-1} \leq \|\partial\Omega\|^{-1}_{C^{k,\gamma}}$, and combining the estimates of I_1, I_2, and I_3, we arrive at (3.2):

$$\begin{aligned}\big|u_n(q)-u_n(p)\big| &\lesssim \delta\Big(r^{-2\alpha} + r^{k+\gamma-2\alpha-1}\|\partial\Omega\|_{C^{k,\gamma}}\big(1+\|\partial\Omega\|_{C^{k,\gamma}}r^{\gamma}\big)\Big)\\ &\lesssim \delta\Big(r^{-2\alpha} + r^{-2\alpha}\big(1+\|\partial\Omega\|_{C^{k,\gamma}}r^{\gamma}\big)\Big)\\ &\lesssim \delta\|\partial\Omega\|_{C^{k,\gamma}}^{\frac{2\alpha}{k+\gamma-1}}\big(1+\|\partial\Omega\|_{C^{k,\gamma}}r^{\gamma}\big) \lesssim \|\partial\Omega\|_{C^{k,\gamma}}^{\frac{k+2\alpha-1}{k+\gamma-1}}|p-q|.\end{aligned}$$

Here the constants in the last two inequalities may depend on ρ. □

3.3 Estimates on the Distance and Proof of Main Results

Now fix $1 > \alpha > 0$ and suppose that we have a $C^{k,\gamma}$ patch solution $\Omega(t) = \cup_{j=1}^{N}\Omega_j(t)$ with $k = \lceil 2\alpha\rceil$, and $1 \geq \gamma \geq 2\alpha - \lceil 2\alpha\rceil + 1$ that forms a splash singularity described in Definition 1.1 at some point x_0 at time T (it is not hard to generalize the argument to the case where splash happens simultaneously at a finite number of different points).

For any $i \neq j$, let us define $d_{ij}(t) = \mathrm{dist}(\Omega_i(t), \Omega_j(t))$. To control patch self-intersections, fix a small parameter $\eta > 0$, and define

$$d_{ii}(t) = \min_{s,r:|s-r|\geq\eta}|x_i(s) - x_i(r)|,$$

where $x_i(s)$ is the arc length parametrization of $\Omega_i(t)$. Given any splash singularity, one can choose sufficiently small $\eta > 0$ so that we must have $d_{ii}(t) \to 0$ as $t \to T$; indeed, it suffices to choose $\eta = \rho$, the radius of the ball from Definition 1.1.

Finally, define the minimal distance $d(t)$ between different patches (or different branches of the same patch if itself-intersects) by

$$d(t) = \min_{1\leq i,j\leq n} d_{ij}(t). \tag{3.14}$$

Due to our definition of splash singularity, there exists $\varepsilon > 0$ such that for every $T - \varepsilon \leq t < T$, all points such that $d(t) = |p-q|$ must lie in $B_{\rho/4}(x_0)$. Note that due to our definition of patch solution and boundedness of the normal component of the

velocity u_n at $\partial\Omega$ for our patch regularity assured by Lemma 2.2, all functions $d_{ik}(t)$ are Lipschitz in time, and therefore $d(t)$ retains this property.

Now we are ready to prove

Proposition 3.3 *Let $0 < \alpha < 1$, $k := \lceil 2\alpha \rceil$, and $1 \geq \gamma \geq 2\alpha - \lceil 2\alpha \rceil + 1$. Suppose that ω is a $C^{k,\gamma}$-patch solution that forms a splash singularity as in Definition 1.1 at time T. Then the minimal distance $d(t)$ defined in (3.14) is a Lipschitz function of time and there exists $\varepsilon > 0$ such that for almost every time $t \in [T - \varepsilon, T)$*

$$d'(t) \geq -Cd(t)\|\partial\Omega\|_{C^{k,\gamma}}^{\frac{k+2\alpha-1}{k+\gamma-1}}. \tag{3.15}$$

Proof Since $d(t)$ is Lipschitz, it is almost everywhere differentiable by the Rademacher theorem, and moreover $d(t_2) - d(t_1) = \int_{t_1}^{t_2} d'(t)\,dt$ (see e.g. (Evans and Gariepy 1992)). Fix a time $t \in [T - \varepsilon, T)$; according to our definition of splash singularity, $\partial\Omega \cap B_\rho(x_0)$ consists of two disjoint simple curves that we will denote $\mathcal{C}_1(t)$ and $\mathcal{C}_2(t)$ and we can assume that $d(t) = \text{dist}(\mathcal{C}_1(t) \cap B_{\rho/2}(x_0), \mathcal{C}_2(t) \cap B_{\rho/2}(x_0))$.

Due to our definition of patch solution, we have that for small $h > 0$

$$\begin{aligned} d(t+h) &= \text{dist}(\mathcal{C}_1(t+h), \mathcal{C}_2(t+h)) \\ &= \text{dist}(X^h_{u_n(\cdot,t)}(\mathcal{C}_1(t)), X^h_{u_n(\cdot,t)}(\mathcal{C}_2(t))) + o(h), \end{aligned} \tag{3.16}$$

where $o(h)$ means a quantity that goes to zero as $h \to 0^+$.

Our aim will be to derive a lower bound on $d(t+h)$ using (3.16). Let $S \in \mathcal{C}_1(t) \times \mathcal{C}_2(t)$ be the set of all pairs of points $(p, q) \in \mathcal{C}_1(t) \times \mathcal{C}_2(t)$ such that $d(t) = |p - q|$. Due to our definition of admissible points and (3.1), any pair p, q is admissible and so the estimates of Proposition 3.2 apply.

To bound the first term on the right-hand side of (3.16), let us fix a small number $\varepsilon_1 > 0$. Define a distance ζ on $\mathcal{C}_1(t) \times \mathcal{C}_2(t)$ by

$$\zeta((x, y), (p, q)) = |x - p| + |y - q| \quad \text{for} (x, y), (q, p) \in \mathcal{C}_1 \times \mathcal{C}_2.$$

Let us denote S_{ε_1} the set of pairs of points (x, y) such that $\zeta((x, y), S) < \varepsilon_1$. This admits a decomposition of $\mathcal{C}_1(t) \times \mathcal{C}_2(t)$:

$$\mathcal{C}_1(t) \times \mathcal{C}_2(t) = S_{\varepsilon_1} \cup S^c_{\varepsilon_1} \tag{3.17}$$

where the compact set $S^c_{\varepsilon_1} := \mathcal{C}_1(t) \times \mathcal{C}_2(t) \backslash S_{\varepsilon_1}$ consists of pairs (x, y) that are away from the admissible ones.

Suppose first that $(x, y) \in S^c_{\varepsilon_1}$, that is, $\zeta((x, y), S) \geq \varepsilon_1 > 0$. Then there exists $\eta(\varepsilon_1) > 0$ such that $|x - y| \geq d(t) + \eta(\varepsilon_1)$. Indeed, $|x - y| - d(t)$ is a continuous function on the compact set $S^c_{\varepsilon_1}$, and so it has a minimum that is clearly positive. Therefore, for all $(x, y) \in S^c_{\varepsilon_1}$, we have

$$\begin{aligned} &|x + u_n(x)h - y - u_n(y)h| \geq |x - y| - 2\|u_n\|_{C(\partial\Omega)}h \\ &\geq d(t) + \eta(\varepsilon_1) - C(\Omega)h \geq d(t) \end{aligned} \tag{3.18}$$

for all $h > 0$ sufficiently smaller than $\eta(\varepsilon_1)$. Thus, the points that are not in S_{ε_1} are not important in derivation of the lower bound on $d'(t)$.

Now consider $(x, y) \in S_{\varepsilon_1}$. Find $(p, q) \in S$ such that $\zeta((x, y), (p, q)) \leq \varepsilon_1$. Note that by Lemma 2.2 and compactness of $\partial\Omega$, u_n is uniformly continuous on $\partial\Omega$. Therefore, since $|x - p| \leq \varepsilon_1$ and $|y - q| \leq \varepsilon_1$, there exists $\sigma(\varepsilon_1)$ such that $|u_n(x) - u_n(p)| \leq \sigma$ and $|u_n(y) - u_n(q)| \leq \sigma$, and $\sigma(\varepsilon_1) \to 0$ as $\varepsilon_1 \to 0$. Applying these considerations along with the bound (3.2), we find that for $(x, y) \in S_{\varepsilon_1}$, the following bound holds:

$$
\begin{aligned}
&.|x + u_n(x)h - y - u_n(y)h| \geq |x - y| - h|u_n(x) - u_n(y)| \\
&\geq |p - q| - \zeta((x, y), (p, q)) \\
&\quad - h|u_n(x) - u_n(p)| - h|u_n(p) - u_n(q)| - h|u_n(q) - u_n(y)| \\
&\geq d(t) - \varepsilon_1 - 2h\sigma(\varepsilon_1) - Chd(t)\|\partial\Omega\|_{C^{k,\gamma}}^{\frac{k+2\alpha-1}{k+\gamma-1}} \qquad (3.19)
\end{aligned}
$$

It follows from (3.16), (3.18), and (3.19) that given any $\varepsilon_1 > 0$, for any sufficiently small $h > 0$, we have

$$
d(t + h) \geq d(t) - \varepsilon_1 - 2h\sigma(\varepsilon_1) - Chd(t)\|\partial\Omega\|_{C^{k,\gamma}}^{\frac{k+2\alpha-1}{k+\gamma-1}} - E(h),
$$

where $E(h) = o(h)$. Since this inequality holds for every $\varepsilon_1 > 0$, (3.15) follows for a.e. $t \in [T - \varepsilon, T)$. □

Proof of Theorems 1.2 and 1.4 We only proof Theorem 1.2, since given the Assumption 1.3, the proof of Theorem 1.4 follows along the same lines.

According to Proposition 3.3, for a.e. $t \in [T - \varepsilon, T)$ before the splash singularity (3.15) holds. By Gronwall lemma, we must have that

$$
\int_0^T \|\partial\Omega(t)\|_{C^{k,\gamma}}^{\frac{k+2\alpha-1}{k+\gamma-1}} \, \mathrm{d}t = \infty,
$$

completing the proof of Theorem 1.2.

□

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