Kitaev chain with a fractional twist

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The topological nontriviality of insulating phases of matter are by now well understood through topological K theory where the indices of the Dirac operators are assembled into topological classes. We consider in the context of the Kitaev chain a notion of a generalized Dirac operator where the associated Clifford algebra is centrally extended. We demonstrate that the central extension is achieved via taking rational operator powers of Pauli matrices that appear in the corresponding BdG Hamiltonian. Doing so introduces a pseudometallic component to the topological phase diagram within which the winding number is valued in $\mathbb{Q}$. We find that this phase hosts a mode that remains extended in the presence of weak disorder, motivating a topological interpretation of a nonintegral winding number. We remark that this is in correspondence with recent paper demonstrating that projective Dirac operators defined in the absence of spin$^{\mathrm{C}}$ structure have rational indices.

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I. INTRODUCTION

Dirac's proposal [1] of taking the square root of the Klein-Gordon equation has had a remarkable impact on theoretical physics. This simple manipulation doubled the number of particles and eventually led to the formalization of the quantum field theoretic concept of spin. With our modern understanding of the analytic-geometric aspects of spin, we know now that this proposal, while simple, is far from naive. In the context of topological materials, there has been a resurgence of Dirac's original intuition with the advent of square-root topological insulators and also square-root Weyl semimetals. Both systems emerge by taking the square root of either an appropriate tight-binding model [2], which can lead to a new class of topological insulator that allows robust edge states with codimension larger than one [3–10] or by stacking such two-dimensional (2D) square-root higher-order topological insulators with interlayer couplings in a double-helix pattern [11]. While both might seem artificial, the former has been observed in a photonic cage [12]. The key prediction of the square-root Weyl semimetal proposal [11] is the presence of Fermi arcs and hinge states that connect the projection of the Weyl points. Even the latter [11] has had an experimental realization. There have also been exciting developments in applying these ideas to Floquet systems [13,14] where the latter paper considers qth roots of the Floquet operator and makes contact with non-Hermitian physics, which is explored further in a similar context in Ref. [15].

From a field theoretic perspective, a QFT with second-order field equations defined on a manifold that admits spin structure will indeed have a meaningful Fermionic "square-root" theory. In fact, if the parent theory is not very exotic, the local square-root counterpart will not be either. Since taking a square-root is a topologically-nontrivial manipulation of the space of sections of the parent theory, the emergence of new/exotic topological features in these square root condensed matter systems is well motivated.

The special importance of the square root is clear. Alhough, one could ask if the (schematic) generalization $(\cdot)^{1/2} \to (\cdot)^{m/n}$ can be suitably made rigorous in a way that is applicable to topological condensed matter systems. We answer this question in the affirmative by a central extension of the Clifford algebra that closes for rational powers of the Dirac matrices. This new algebra is compatible not with the ordinary Dirac operator but with a pseudodifferential analog that is locally Dirac-like but only globalizes projectively. Such projective Dirac operators defined in the absence of a spin structure are shown to have rational topological index [16], albeit with theoretical machinery absent in the traditional treatment of topological Hamiltonian systems.

In this paper, we realize these formal ideas in a Kitaev chain [17] of spinless Fermions, chosen for its simplicity as well as its ubiquity, with the BdG doubled nearest-neighbour coupling Hamiltonians carry Pauli matrices raised to a rational power. The algebraic structures we need for our generalization live entirely in particle-hole space where we show that taking fractional powers gives rise to central extensions that lead to a fundamentally altered topological phase space. Most notably, we report the existence of a pseudometallic phase marked by a dense set of midspectrum modes with rational winding that resist localization in the presence of on-site disorder only when the untwisted theory is tuned to its topological phase. We conclude with a discussion of the field theoretic formulation of projective Dirac physics that motivates the interpretation of a rational winding number as a topological index and relate it to an extended classification scheme of topological phases.

II. RATIONAL POWERS OF BDG HAMILTONIANS

One of the standard models exhibiting topological superconductivity is a chain of spinless Fermions coupled by a nearest-neighbor p-wave pairing. This is the so-called Kitaev

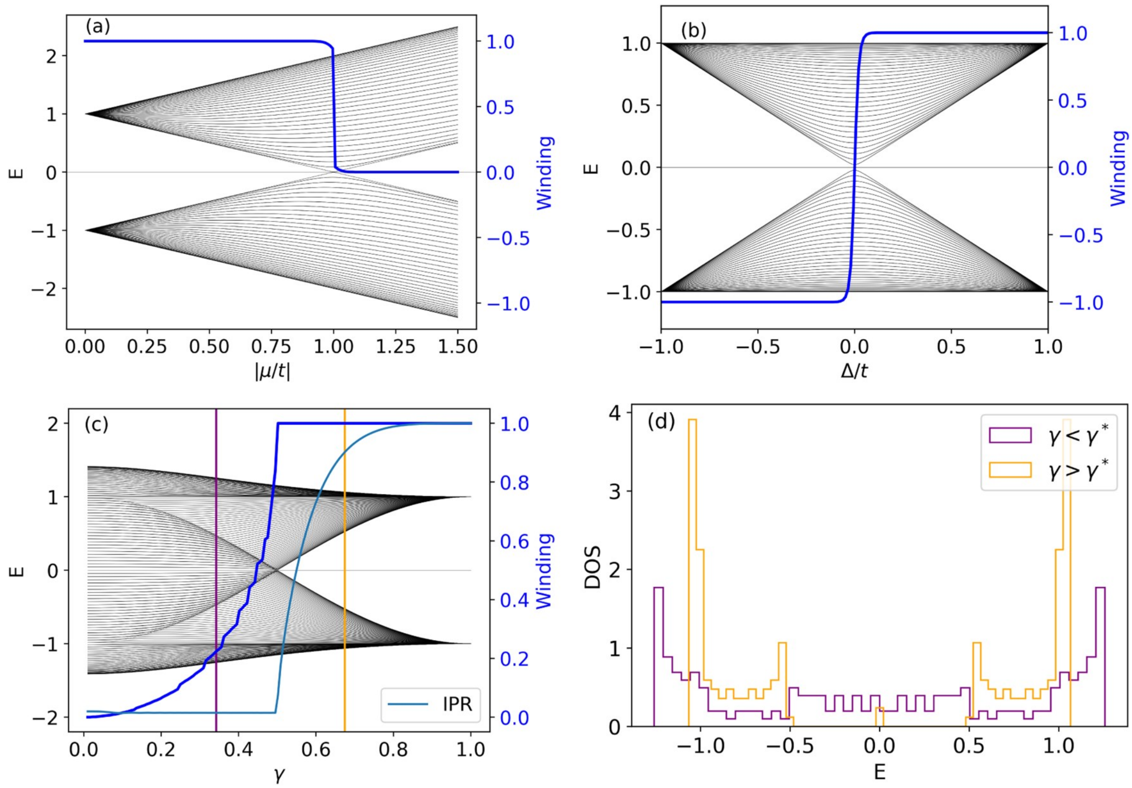

FIG. 1. Evolution of the spectrum of the ordinary Kitaev chain ($\gamma = 1$) as a function of (a) μ, for $\Delta = 1$ and (b) Δ, for $\mu = 0$. The standard critical point where the gap closes corresponds to $\mu = 1$ at which point the winding number transitions from 1 to zero. For any $\Delta = 0$, the winding number is sgn Δ and vanishes when $\Delta = 0$ and the fractional Kitaev chain with (c) γ for $\mu = 0$, $\Delta = 1$ and (d) its DOS along two energy slices. The winding number is approximated in real space based on the prescription given in [23]. Beyond the appearance of the delocalized midspectrum states, the μ and Δ spectral evolution for $\gamma \in (0, 1)$ does not provide any new information. We plot also the inverse participation ratio (IPR) of the zero mode in (c) to indicate (de)localization.

chain [17] with minimal Hamiltonian,

$$H = -\sum_i \left\{ \left(c^\dagger_{i+1} c_i - \frac{\Delta}{t} c^\dagger_{i+1} c^\dagger_i + \text{H.c.} \right) + \frac{\mu}{t} c^\dagger_i c_i \right\}. \quad (1)$$

This model, being a Fermionic theory, is constrained by Fermion parity. In general $\Delta \in \mathbb{C}$. Time reversal symmetry is present if the pairing is chosen in $\mathbb{R}$ and is broken otherwise. This puts the model in the symmetry classes [18] BDI or D respectively.

The standard way of dealing with the pairing term is to work in the doubled Bogoliubov-de Gennes (BdG) basis, $\psi_i := (c_i, c^\dagger_i)$, leading to the Hamiltonian

$$H = -\sum_i \left[\frac{\mu}{t} \psi^\dagger_i \sigma_z \psi_i + \left(\psi^\dagger_{i+1} \left(\sigma_z + \frac{\Delta}{t} i\sigma_y \right) \psi_i + \text{H.c.} \right) \right]. \quad (2)$$

With open boundary conditions, there exist localized Majorana modes on the boundaries of the chain in the topologically nontrivial phase. The familiar phase diagram of this model can be inferred from Figs. 1(a) and 1(b).

We view the standard BdG Hamiltonian in dimension one as arising from the representation theory of spin_{2n}. We will focus on the case of BdG doubled spinless Fermions, where the algebraic object of interest is the $su(2)$ particle-hole algebra, of which the blocks of the real-space Hamiltonians furnish representations. One possible modification of standard BdG Hamiltonians describing linearized superconducting pairing is to enlarge this algebra.

To this end, one commonly defines a general operator power by the integral

$$A^\gamma = \frac{1}{\Gamma(-\gamma)} \int_0^\infty \frac{dt}{t^{1+\gamma}} (e^{-tA} - \mathrm{I}), \quad (3)$$

for $\gamma \in [0, 1]$, though this requires a strictly positive spectrum. Of course, when the operator in question is Hermitian, we have the more intuitive definition,

$$A^\gamma = U \text{diag}(\text{spec} A)^\gamma U^\dagger. \quad (4)$$

We now imagine taking rational powers of the BdG Hamiltonian sub-blocks in Eq. (2), valued in 2×2 Pauli matrices.

Indeed, σ_i^γ is not in su(2) for $\gamma = 1$. Recall that for (ijk), any permutation of $\{1, 2, 3\}$,

$$[\sigma_i, \sigma_j] = 2i\epsilon^{ijk}\sigma_k. \quad (5)$$

To close the algebra for σ_i^γ, we proceed as follows. We first define $z_\gamma := (-1)^\gamma$. Using Eq. (4), we obtain the relations

$$\sigma_k^\gamma = \frac{1 + z_\gamma}{2}\mathrm{I} + \frac{1 - z_\gamma}{2}\sigma_k,$$
$$\sigma_k = -\frac{2\sigma_k^\gamma}{z_\gamma - 1} + \frac{z_\gamma + 1}{z_\gamma - 1}\mathrm{I}. \quad (6)$$

These relations can be used to bring the resultant commutator

$$\left[\sigma_i^\gamma, \sigma_j^\gamma\right] = \frac{i}{2}\varepsilon^{ijk}(z_\gamma - 1)^2\sigma_k \tag{7}$$

into the form

$$\left[\sigma_i^\gamma, \sigma_j^\gamma\right] = i\varepsilon^{ijk}\sigma_k^\gamma(1 - z_\gamma) + \frac{1}{2}\frac{z_\gamma + 1}{z_\gamma - 1}\mathbb{I}, \tag{8}$$

where it is evident that the algebra has a central extension with $w\mathbb{I}$ in the center of the algebra generated by the σ_i^γ's. This importantly gives rise to a projective representation (due to Bargmann's theorem [19]), i.e., a homomorphism of $SU(2)$ into $PGL(2, \mathbb{C})$. As we will see later, this is the main feature of our paper. The topological nature—specifically the rationality of the winding number—is a fundamental property of the projectivity of the representations involved in the construction of the Hamiltonian.

Particle-hole-space Hamiltonians that are valued in this new algebra via

$$\sum_i \sigma_i^\gamma f^i(k) \tag{9}$$

are normal but non-Hermitian in general. While breaking from Hermiticity is interesting on its own right, we opt to restore it by defining a self-adjoint rational power,

$$\tilde{\sigma}_k^\gamma = \frac{\sigma_k^\gamma}{2} + \frac{\left(\sigma_k^\gamma\right)^\dagger}{2}. \tag{10}$$

The Hermitian central extension results in commutators of the form

$$\tilde{\sigma}_k^\gamma = \frac{1 - \cos\pi\gamma}{2}\tilde{\sigma}_k + \frac{1 + \cos\pi\gamma}{2}\mathbb{I}, \tag{11}$$

$$\left[\tilde{\sigma}_i^\gamma, \tilde{\sigma}_j^\gamma\right] = i\varepsilon^{ijk}\tilde{\sigma}_k^\gamma(1 - \cos\pi\gamma) + \frac{\mathbb{I}}{2}\frac{1 + \cos\pi\gamma}{1 - \cos\pi\gamma}. \tag{12}$$

We note that when $\gamma \to 1$, $\tilde{\sigma}_k^\gamma$ goes smoothly over to $\tilde{\sigma}_k$.

We can now define the fractionally twisted BdG Hamiltonian for the Kitaev chain,

$$H = \sum_i \left(\frac{\mu}{t}\psi_i^\dagger\sigma_z\psi_i + \left(\psi_{i+1}^\dagger\left(\tilde{\sigma}_z^{\gamma_h} + i\frac{1}{t}\tilde{\sigma}_y^{\gamma_p}\right)\psi_i + \text{H.c.}\right)\right), \tag{13}$$

where $\gamma_{h,p} \in \mathbb{Q}$ is the matrix power of the hopping and pairing Hamiltonians respectively. The couplings are not included in the power to ease comparisons between the fractional and integral models. Further, we will initially develop the case of $\gamma \equiv \gamma_p$, $\gamma_h = 1$. The cases where $\gamma_h \in (0, 1]$, $\gamma_p = 1$ and $\gamma_{p,h} \in (0, 1]$ lead to qualitatively similar results, although the modified hopping term furnishes a more interesting phase diagram, as we discuss in Sec. III C.

The off-diagonal Hamiltonian block can be expressed in terms of $\sigma_{y,z}$ and the identity matrix using Eq. (11). In that regard, our model with the rational power of σ_y can be reinterpreted as the Kitaev chain with an additional nearest-neighbour-coupling term that mixes the hopping and pairing strengths,

$$H = \sum_i \left(\frac{\mu}{t}\psi_i^\dagger\sigma_z\psi_i + \left(\psi_{i+1}^\dagger\left(\sigma_z + i\frac{1}{t}\left(\frac{1 - \cos\pi\gamma}{2}\tilde{\sigma}_k + \frac{1 + \cos\pi\gamma}{2}\mathbb{I}\right)\right)\psi_i + \text{H.c.}\right)\right). \tag{14}$$

We opt for the former representation of the model as the latter obfuscates the underlying algebraic structure from which the model emerges. However, Eq. (14), appears to be more experimentally transparent.

The choice of which Hamiltonian block to raise to a rational power is somewhat arbitrary. In discussing the phase diagram, we consider also taking powers of the hopping Hamiltonian.

III. TOPOLOGICAL CHARACTERIZATION OF THE FRACTIONALLY TWISTED KITAEV CHAIN

In momentum space, we have the generic effective Dirac Hamiltonian, $H_D = \sum_i f_i(k)\sigma^i$, where, roughly speaking, for a 1 dimensional Brillouin zone (BZ), the degree of the map between spheres, $f_i : S^1 \to \mathbb{C}P^1$, characterizes the integral winding number of the Hamiltonian. More precisely, one is interested in casting the topological classification of such Dirac-like Hamiltonians [20] as the classification of Dirac operators acting on sections of Dirac bundles. This objective is formalized using (twisted) topological K theory to develop a periodic table of topological insulators [21,22].

A. Winding number

In the case of the BDI class in dimension 1 the Dirac fibers are particle-hole spaces and, index $\not{D} \in \mathbb{Z}$, corresponding to a specific topological number like the $\hat{A}$ genus. Practically, this topological invariant can be captured by computing the winding number of the ground state.

In fact, more generally one can consider a family of such theories where the Dirac operators $\not{D}_\gamma$, are parametrized by some manifold, which, for concreteness, we take to be S^1. Then, there still holds a version of the index theorem. Namely, the geometric incarnation, which is the Atiyah-Singer index theorem and it comes from globalizing representations of the Clifford algebras by means of spin$^{\mathbb{C}}$ structures [whose existence depends only on a topological invariant, the third Stiefel-Whitney class, $w_3(M) \in H_3(M; \mathbb{Z})$], of the underlying manifold M. What is more remarkable is that even in the case in which such spin$^{\mathbb{C}}$ structures exist only projectively, a version of the index theorem where the index is valued in twisted K theory and is now a rational number [16] still holds.

In our context, we take $\gamma \in S^1 = \mathbb{R}/\mathbb{Z}$, by periodicity, and we observe that in the geometric realization of the model in the Brillouin zone, the effective base space of the projective Dirac operator with parameters is $M = \mathbb{C}P^1 \times S^1$. The work of Melrose *et al.* [16] requires the twisted K theory be done by means of torsion classes in $H_3(M;\mathbb{Z})$ (which are absent in our case). We therefore cannot infer that the analytic and topological indices are equal here, since we would be twisting with nontorsion classes. Instead, we probe the topological nature of the fractional analytical index numerically in the proceeding section. Further, we contend that the main property that ensures their version of the index theorem holds requires only projective representations. We further elaborate on this in Sec. IV.

B. Computation of the real space winding number

In the conventional case of $\gamma = 1$, one has that

$$\nu = \frac{1}{2\pi i}\int_{S^1} \mathrm{tr}[h^{-1}(k)dh(k)] \in \mathbb{Z} \tag{15}$$

where $h(k)$ is one of the pairs of the BdG doubled Hamiltonian blocks. In addition to being able to define a winding number in the presence of disorder, we have the additional complication of this integral expression holding only for some $\gamma \in \mathbb{Q}$ in general. In reality, the distinction/separation of Bloch subbundles over the Brillouin zone depends on γ so we are in need of a formulation of the winding number that can be extended beyond the integers.

To this end, we follow the algorithm developed in Ref. [23,24]. Their noncommutative geometric construction of the real-space winding number remains well defined for chiral systems in odd dimensions in the presence of arbitrarily strong disorder. The latter feature enables the algorithm to produce winding numbers valued in $\mathbb{Q}$ (though approximating quantized integral winding numbers in their case) and hence makes it suitable for computing the fractional winding number as a proxy for the rational analytic index where there is an analogous challenge of the fractional twist introducing midspectrum states (see Ref. [25] for a comparison with other common algorithms).

We begin by expressing the $\gamma = 1$ Hamiltonian in block-off-diagonal form

$$H_1 \simeq \begin{pmatrix} 0 & Q_1 \\ Q_1^\dagger & 0 \end{pmatrix} \tag{16}$$

where Q_1 is taken to be the unitary that results in flat bands. In this limit, Eq. (15) can be translated into real-space with the noncommutative dictionary,

$$\begin{aligned} q(k) &\longleftrightarrow Q_1, \\ \int \frac{dk}{2\pi i}\mathrm{tr}(A_k) &\longleftrightarrow \mathrm{Tr}(\cdot), \\ dq(k) &\longleftrightarrow -i[X, Q_1], \end{aligned}$$

where Tr is the real space trace per unit volume and $q(k)$. Then, the winding number has the real-space representation [24],

$$\nu = \mathrm{Tr}\left(Q_1^{-1}[X, Q_1]\right), \tag{17}$$

with X the position operator. The real-space trace, while not integral in general (and hence not a topological index), remains well defined when the spectral gap closes. We can therefore generalize immediately to $1 \to \gamma$ where the commutator in question is written [26]

$$[X, Q_\gamma] = \sum_{l=1} c_l l^X Q_\gamma l^{-X}, \quad c_l = \frac{l^{N+1}}{1-l}. \tag{18}$$

Of course, these expressions hold only approximately in finite volume.

C. Phase structure

We first concentrate on flat-band Hamiltonians, which occur at vanishing chemical potential. We will orient ourselves with respect to the $\gamma = 1$ case where $\mu = 0$ and $\Delta = 0$ correspond to a topologically nontrivial phase of the Majorana chain. We compare in Fig. 1 the spectral evolution under $\gamma \in (0, 1]$ to the familiar parametrization of the ordinary Ki-taev chain phase space. For completeness, we provide also the (γ, μ) and (γ, Δ) cuts of the topological phase space probed by ν in Figs. 2(a) and 2(b). For $\gamma > \gamma^*$, with γ^* separating the gapped and gapless phases, the phase diagrams of the ordinary and fractionally twisted Kitaev chain are identical.

The most notable feature of the spectral evolution as a function of γ is the transition to a metallic region with rational winding number for $\gamma < \gamma^* \equiv 1/2$. Although this metallic state is surprising in Dirac systems with two independent moduli (which do not afford any additional tunable parameters that allow the bands to cross generically), this metallic state is in accordance with the von Neumann-Wigner theorem: A Hamiltonian describing dynamics in an m dimensional phase space will have level crossing on an $m-2$ dimensional manifold. In our case, the Hamiltonian has blocks valued in the modified algebra of Eq. (8), which introduces a third independent parameter γ, controlling the central element. Hence, levels can cross on curves (rather than points) in the moduli space of the Kitaev chain spectra.

For $\gamma > \gamma^*$, the topological phase remains robust, as indicated by the winding number plateau at $\nu = +1$. A stable pair of $E = 0$ boundary modes indicates that the bulk-boundary correspondence is intact in this regime. The boundary Majorana modes delocalize as $\gamma \to \gamma^*$ from above. This is evidenced by the vanishing inverse-participation (IPR),

$$I_q := \sum_x |\psi(x)|^{2q},$$

of the midspectrum mode $\psi(x)$. In the present context, with $q = 2$, the IPR is a direct measure of the inverse localization length. A perfectly localized state has IPR 1 while a perfectly delocalized state has IPR 0. This is not, however, accompanied by a vanishing winding number. One normally expects $\nu \to 0^-$ and $\nu \to 1^+$ at a topological phase transition as we see in Figs. 1(a) and 1(b). Curiously, the winding number interpolates between $+1$ and 0 in the delocalized regime, independent of finite size effects. If we can interpret the rational winding number as a form of topological obstruction, the delocalized phase breaks the bulk-boundary correspondence, as the presence of the metallic modes are insensitive to the boundary.

Integer winding numbers are conventionally tied to the existence of distinct static bands. Standard theory suggests that if bands cross or touch, the winding of the ground state is no longer unambiguously connected to some topological

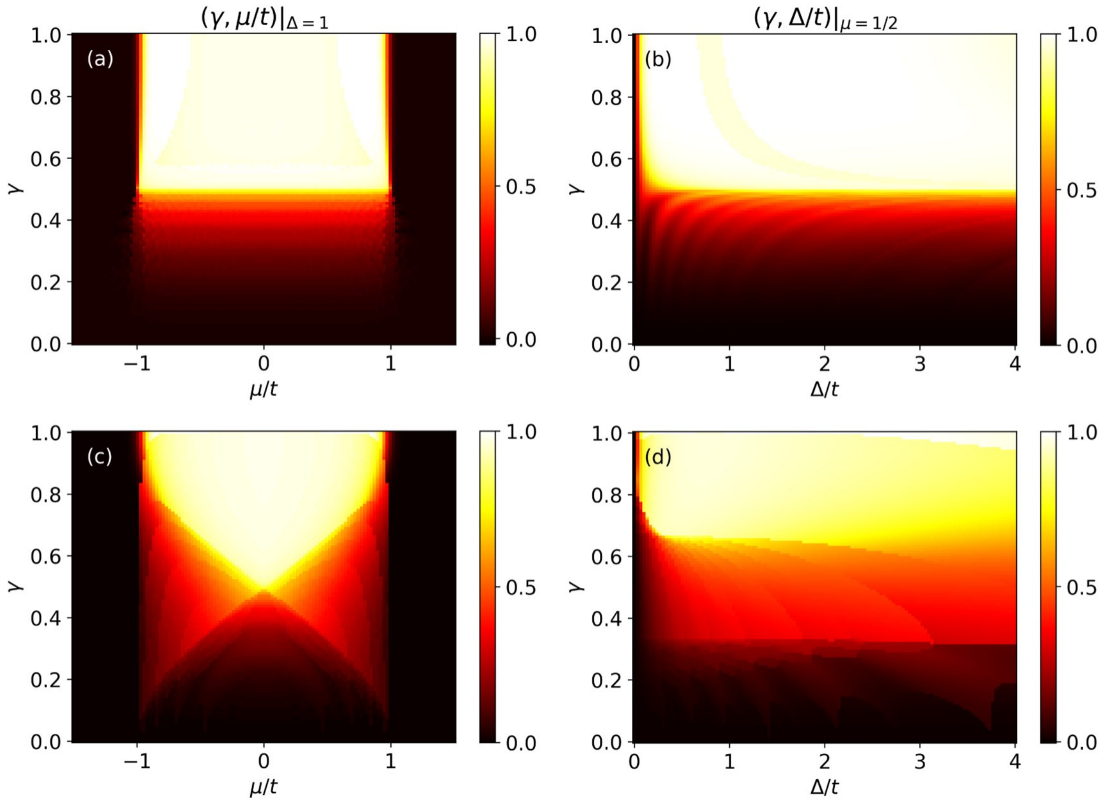

FIG. 2. Heatmap of winding number in the $(\gamma, \mu/t)|_{1=1}$ and $(\gamma, 1/t)|_{\mu=1/2}$ planes of phase space for (a), (b) fractional pairing, $\gamma \equiv \gamma_p$ and (c), (d) fractional hopping, $\gamma \equiv \gamma_h$.

invariant of a Hilbert sub-bundle (or an index). In the present case, however, there do exist well defined band edges [see Fig. 1(d)], albeit with a continuum of midspectrum modes. We claim that the rational winding numbers obtained reflect the rational (analytic) index of $\not{D}_\gamma$ as in the discussion above.

Figures 2(c) and 2(d) depicts the phase diagram of the case where $\gamma_h \equiv \gamma$, $\gamma_p = 1$ in Eq. (13). Common to both configurations of the model is a robust regime of integral winding number that transition to domains of rational winding number. Further, the case where the hopping is fractionalized indicates the existence of a locus of γ^*'s that separate two regimes of rational winding number with a kink [bright-red curves in Figs. 2(c) and 2(d)]. We defer a detailed analysis of the implications of this feature to later work. Our more pressing goal now is to interpret the rational analytic index (the fractional winding number) as a rational topological index for the pseudometallic phase.

The nontriviality of the pseudometallic phase

Returning to the case of fractional pairing, we further probe the spectrum in the delocalized regime by partitioning the chain into two regions with different values of γ that straddle γ^*. With open boundary conditions, this corresponds to a Kitaev chain where the right half-chain is in the well understood insulating $\nu = 1$ phase and the left half-ring hosts the delocalized midspectrum modes. Tracking the two modes closest to $E = 0$ along the chain in Fig. 3, we find boundary localization on the right and delocalization across the left half-chain. When we tune the pseudometallic left half-chain to be topologically trivial with $\mu > |\mu^*| \equiv 1$ while maintaining the right half-chain at $\nu = 1$, we notice that there exists a midspectrum delocalized mode in the spectrum that coalesces into a Majorana mode localized at the domain boundary.

This result is necessary but insufficient as an argument for the topological nontriviality of a rational winding number. Next, we introduce on-site disorder in the form of a random Gaussian vector with mean μ (the chemical potential) and variance σ^2. That is, we have

$$H_{\text{onsite}} = \tilde{\mu}\sigma_z; \quad \tilde{\mu} \sim N(\mu, \sigma^2) \tag{19}$$

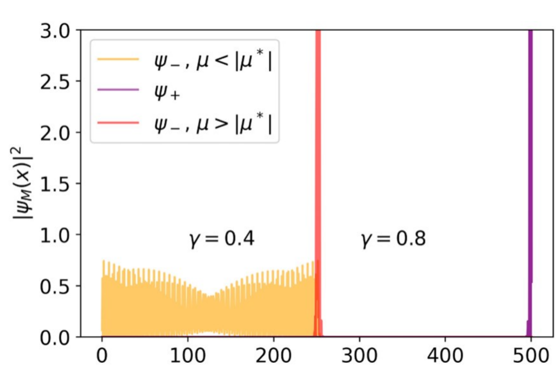

FIG. 3. $|\psi_M(x)|^2$ corresponding to the two states closest to $E = 0$, denoted $\psi_\pm$. One of these states is delocalized over the region with $\gamma < \gamma^*$ while the other is localized at the boundary of the region with $\gamma > \gamma^*$. The left domain has a modulated on-site potential that can switch between topological and trivial phases.

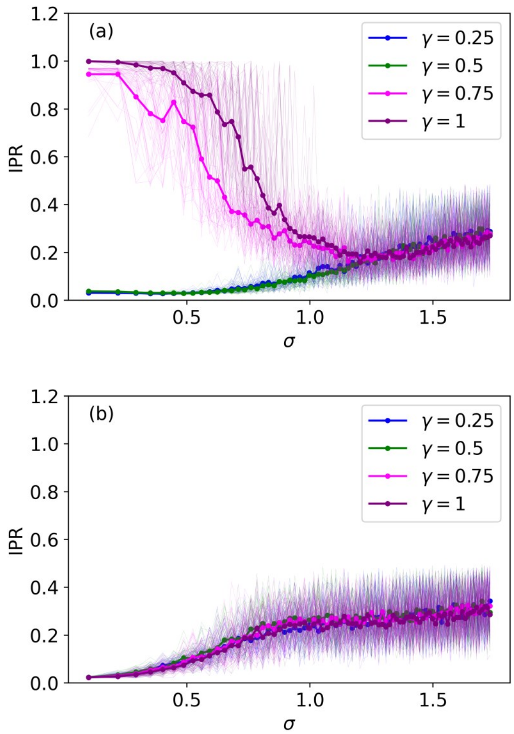

FIG. 4. Inverse participation ratio for increasing normal disorder variance for characteristic choices of γ for (a) the topological phase with $\mu = 0$ and (b) the trivial phase with $\mu = 2$. The light lines indicate the evolution of ensembles of the disorder realizations and the bold lines indicate ensemble averages.

for each diagonal block of the Hamiltonian. We compute the inverse participation ratio of the midspectrum state for different values of γ. Notably, when $\mu < |\mu^*|$, the delocalized mode that is in correspondence with a Majorana mode in the $\gamma > \gamma^*$ phase initially resists localization. Around $\sigma \approx 1$ for which Anderson localization [27] obtains is precisely where the localized boundary modes present when $\gamma > \gamma^*$ take the form of generic randomly localized states. When the chemical potential is tuned to the trivial phase where a Majorana modes is absent for $\gamma > \gamma^*$ but midspectrum states exist for $\gamma < \gamma^*$, the onset of Anderson physics is immediate at $\sigma > 0$ with no sensitivity to the choice of γ. These facts are illustrated in Fig. 4. Note that the two choices of mean chemical potential, $\mu = 0$ and $\mu = 2$ have the same bandgap in the clean limit.

We conclude, therefore, that while the bulk-boundary correspondence is lost for $\gamma < \gamma^*$, there exist a particular pair of extended states that are in correspondence with the topological boundary modes present when $\gamma > \gamma^*$. This is a strong indication that the $\mathbb{Q}$ valued winding number should be taken seriously as an indicator of nontrivial bulk topology in the pseudometallic phase.

IV. FIELD THEORETIC PERSPECTIVE

Recall here that a Dirac bundle over a Riemannian manifold is a bundle of Clifford modules equipped with a Clifford connection. One can define the Dirac operator, D, as a first-order differential operator acting on the space of sections of such bundles. Further, by equipping the Dirac bundle with the compatible action of the Clifford algebra Cl, one defines on its sections the operator $\not{D} := s \cdot D$, $s \in$ Cl. It is the index of this operator, which also encodes the constraints of symmetries S, T, and C, that can be related to the topological classification of Dirac-like lattice systems realizing those symmetries. It is therefore to be expected that nontrivially altering the Clifford algebra action (say, by a central extension) will change the index of $\not{D}$.

Let us begin by defining a theory of Majorana fermions on, for definiteness, $\mathbb{C}P^1$, equipped with the spin structure σ,

$$S = i\int_{\mathbb{C}P^1} \bar{\psi}\not{D}_\sigma\psi. \tag{20}$$

Here, the Dirac operator is chosen to be

$$\not{D}_\sigma = \gamma^1 D_1 + \gamma^2 D_2 + \gamma^3 m, \quad \gamma^3 = \gamma^1\gamma^2. \tag{21}$$

The free n-Majorana path integral is given by the Pffafian,

$$Z = \mathrm{Pf}(\not{D}_{\mathbb{C}P^1})^n. \tag{22}$$

This theory has two distinct phases, parameterized by mod 2 Index$(\not{D}_\sigma) \in \{1, 0\}$. Depending on this (integral) index,

$$Z \to (-1)^{\mathrm{Index}(\not{D}_\sigma)}Z$$

under $m \to -m$. The theory under consideration in our note is one where the operator $\not{D}_\sigma$ is replaced by

$$\not{D}_{\rho,\sigma} = \tilde{\gamma}^1 D_1 + \tilde{\gamma}^2 D_2 + \tilde{\gamma}^3 m, \tag{23}$$

where ρ : Cl $\to PGL(n)$ is a projective representation arising from a central extension of Cl and $\tilde{\gamma}^i = \rho(\gamma^i)$. In Ref. [16] the authors show that when the global manifold M (representing the parameter space of the family of generalized Dirac operators) does not admit a spin$^{\mathbb{C}}$ structure but via a torsion class in $H^3(M, \mathbb{Z})$, it admits one up to projective representations (or up to central extension). This makes the $\hat{A}$ genus in the familiar index theorem not an integer but a rational number.

In turn, under $m \to -m$, the Pfaffian picks up a phase $(-1)^{2s}$, with $s \in \mathbb{Q}$. Taking $s \equiv p/q$ with $p, q \in \mathbb{Z}$, one finds that the $q-$Majorana theory with the projective Dirac operator recovers the topological phase structure of the standard 2−Majorana theory, where the sign change of the Pfaffian is controlled by the number of zero-mode pairs present in the spectrum. This prompts an analogy between the fractional analytic index and a Majorana zero-mode carrying a *fraction* of the topological index.

The intuitive connection to ordinary Majorana physics makes use of the fact that a spin$^{\mathbb{C}}$ structure exists whereas one might think the introduction of the projective Dirac operator demands the absence of spin structure. We argue that the salient feature is not that the spin$^{\mathbb{C}}$ structure does not exist, but that the (generalized) Dirac operator comes about from a central extension of the Clifford algebra. Hence the rationality of the winding number. We remark that the operator $\not{D}_\gamma$ of equation (23) (by virtue of the periodicity in $\gamma \in [0, 2]$) defines an operator on $M = \mathbb{C}P^1 \times S^1$. This manifold has no torsion classes in $H^3(M, \mathbb{Z})$ and indeed does admit a spin$^{\mathbb{C}}$ structure, but the arguments of [16] carry through to the case where the Dirac operator does not arise from a spin$^{\mathbb{C}}$ but from a projective representation.

V. CONCLUDING REMARKS

The introduction of a twist in the Kitaev Hamiltonian in the form of an operator power of a Pauli matrix centrally extends the Clifford algebra and gives rise to Hamiltonian blocks that are projective representations of the particle-hole symmetry. This turns out to be intimately related to the notion of twisted K theory, which now replaces K theory in the classification of topological materials. The notable physical consequences of this extension is the appearance of a pseudometallic phase within which the winding number is valued in $\mathbb{Q}$, with the bulk gap closing precisely at $\gamma = 1/2$, the square-root case. In analogy with the rational analytic index of the projec-tive Dirac operators remaining homotopy invariants [16], we presented numerical evidence in the form of localization resisting metallic modes for the topological nontriviality this new metallic phase. Finally, we postulated that along with class BDI topological insulators in dimension 1 (chosen for their simplicity), the entire periodic table is interspersed with nonintegral topological indices when the Dirac operator is realized only projectively. Hence, the scheme posed here, based on Eq. (14) opens up a potentially new route to engineering topological materials.

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