CONNECTION COEFFICIENTS FOR ULTRASPHERICAL POLYNOMIALS WITH ARGUMENT DOUBLING AND GENERALIZED BISPECTRALITY

By

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Dedicated to the memory of Richard Askey

Abstract. We start by presenting a generalization of a discrete wave equation that is satisfied by the entries of the matrix coefficients of the refinement equation corresponding to the multiresolution analysis of Alpert. The entries are functions of two discrete variables and they can be expressed in terms of the Legendre polynomials. Next, we generalize these functions to the case of the ultraspherical polynomials and show that these new functions obey two generalized eigenvalue problems in each of the two discrete variables, which constitute a generalized bispectral problem.

1 Introduction

Let $\{P_n\}_{n=0}^{\infty}$ and $\{Q_n\}_{n=0}^{\infty}$ be two families of orthonormal polynomials whose orthogonality measures are $d\mu$ and $d\nu$, respectively. Then one can see that

$$P_i(t) = \sum_{j=0}^{i} c_{i,j} Q_j(t),$$

where the coefficients $c_{i,j}$ can be found in the following way:

$$c_{i,j} = \int P_i(t) Q_j(t)\, d\nu(t).$$

These coefficients are called connection coefficients and their nonnegativity for some particular cases of the ultraspherical polynomials is useful in the proof of the positivity of a certain ${}_3F_2$ function, which in turn, based on the work of Gasper

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and Askey and Gasper, played a significant role in the first proof of the Bieberbach conjecture [1]. Also there has been much work proving the nonnegativity of integrals of products of orthogonal polynomials times certain functions which was initiated by Askey in the late 1960's. These studies have been stimulated by the fact that some of those integrals have combinatorial interpretations (see [10]).

Another instance that we would like to mention is that in some early work leading to the theory of bispectral problems, a matrix S_1, whose entries are

$$(S_1)_{i,j} = \int_a^{\Omega} P_i(t)P_j(t)\, d\mu(t)$$

for some real a and Ω, was considered (for instance, see [8]). The question was to find the eigenvectors of S_1 which, since S_1 is a full matrix, is not an easy task. However, it was proposed to find a tridiagonal matrix commuting with S_1 in order to reduce the original problem to a problem of finding eigenvectors of the tridiagonal matrix, which is an easier and well-understood problem. It was shown to be possible to construct such tridiagonal matrices for some families of orthogonal polynomials and this is one of the fundamental ideas in the theory of bispectral problems.

The last instance we bring up here is that in [7] the Alpert multiresolution analysis was studied in detail and important in this study was the integral

$$f_{i,j} = \int_0^1 \hat{p}_i(t)\hat{p}_j(2t-1)dt,$$

where $\hat{p}_i$ is the orthonormal Legendre polynomial, i.e.,

$$\hat{p}_j(t) = k_j t^j + \text{lower degree terms}$$

with $k_j > 0$ and for any two nonnegative integers k and l we have

$$\int_{-1}^1 \hat{p}_k(t)\hat{p}_l(t)dt = \begin{cases} 0, & k \neq l; \\ 1, & k = l. \end{cases}$$

These coefficients are entries in the refinement equation associated with this multiresolution analysis. The fact that the Legendre polynomials are involved in the above integral allowed the authors in [7] to obtain many types of recurrence formulas in i and j including a generalized eigenvalue problem in each of the indices. These two equations together give rise to a bispectral generalized eigenvalue problem.

We begin by discussing a common property of the coefficients in all the above-mentioned cases: they satisfy a generalized 2D discrete wave equation. We observe numerically that a damped oscillatory behavior takes place in the case of the ultraspherical generalization of the coefficients $f_{i,j}$. In particular with

$$f_{i,j}^{(\lambda)} = \int_0^1 \hat{p}_i^{(\lambda)}(t)\hat{p}_j^{(\lambda)}(2t-1)(t(1-t))^{\lambda-1/2}dt,$$

where $\hat{p}_i^{(\lambda)}$ are the orthonormal ultraspherical polynomials and $\lambda > -1/2$, we find, for large i, the asymptotic formula

$$f_{i,j}^{(\lambda)} = k_j \frac{\cos(\pi(j+\frac{\lambda}{2}-\frac{i}{2}+\frac{1}{4}))}{\sqrt{\pi}i^{\lambda+1/2}} + O\Big(\frac{1}{i^{\lambda+3/2}}\Big),$$

where

$$k_j = \frac{1}{2^{j+1-2\lambda}}\sqrt{\frac{(2\lambda)_j}{j!(\lambda)_j(\lambda+1)_j\lambda\Gamma(2\lambda)}}\Gamma(2j+2\lambda+1)\Big(\lambda+\frac{1}{2}\Big)_j$$

which confirms the damped oscillatory behavior. We also derive some related properties and show that $f_{i,j}^{(\lambda)}$ satisfy a bispectral generalized eigenvalue problem of the form

$$\tilde{A}_i f_{i,j}^{(\lambda)} = \Big(j+\lambda-\frac{1}{2}\Big)\Big(j+\lambda+\frac{1}{2}\Big)B_i f_{i,j}^{(\lambda)},$$
$$\hat{A}_j f_{i,j}^{(\lambda)} = \Big(i+\lambda+\frac{1}{2}\Big)\Big(i+\lambda-\frac{1}{2}\Big)\hat{B}_j f_{i,j}^{(\lambda)},$$

where $\tilde{A}_i$, B_i are tridiagonal operators or second order linear difference operators acting on i and $\hat{A}_j$, $\hat{B}_j$ are tridiagonal operators acting on j. Each of the two above-given relations is a generalized eigenvalue problem and the theory of such problems is intimately related to biorthogonal rational functions (for instance, see [9], [11], [16]).

The paper is organized as follows. In Section 2 a vast generalization of the above integral is shown to give rise to a 2D wave equation and solutions to the special case of the above integral are plotted to show the oscillations. In Section 3 the Legendre case above is analyzed and various properties of the coefficients $f_{i,j}$ are derived. One point of this section is to derive the orthogonality property of these coefficients using that they come from special functions. In Section 4 the Legendre polynomials are replaced by the ultraspherical polynomials and their scaled weight. Here it is shown that the coefficients $f_{i,j}^{(\lambda)}$ satisfy a wave equation and also a bispectral generalized eigenvalue problem. Two proofs are given developing the generalized eigenvalue problem. One is based on the fact that the polynomials satisfy a differential equation and has the flavor of the proof given in [8] and the second follows from the formula for $f_{i,j}^{(\lambda)}$ in terms of a ${}_2F_1$ hypergeometric function. The two proofs emphasize different aspects of the problem that may be useful when viewing other orthogonal polynomial systems.

2 The 2D discrete wave equation

Let $\{P_n\}_{n=0}^{\infty}$ and $\{Q_n\}_{n=0}^{\infty}$ be two families of orthonormal polynomials with respect to two probability measures or, equivalently, two families that obey the three-term recurrence relations

$$a_{n+1}P_{n+1}(t) + b_nP_n(t) + a_nP_{n-1}(t) = tP_n(t), \quad n = 0, 1, 2, \ldots$$

and

$$c_{n+1}Q_{n+1}(t) + d_nQ_n(t) + c_nQ_{n-1}(t) = tQ_n(t), \quad n = 0, 1, 2, \ldots,$$

where the coefficients a_n and c_n are positive and the coefficients b_n and d_n are real. In particular, the first relations are

$$a_1P_1(t) + b_0P_0(t) = tP_0(t), \quad c_1Q_1(t) + d_0Q_0(t) = tQ_0(t).$$

Therefore we can set

$$a_0 = c_0 = 0$$

for the coefficients to be defined for $n = 0, 1, 2, \ldots$. Since the families are orthonormal with respect to probability measures we know that

$$P_0 = 1, \quad Q_0 = 1,$$

which are the initial conditions that allow to reconstruct each of the systems from the corresponding recurrence relation. It should be stressed here that by imposing these particular initial conditions we implicitly assume that the corresponding orthogonality measures are probability measures.

In addition, suppose we are given a measure σ on $\mathbb{R}$ with finite moments. Then, let us consider the coefficients

$$u_{i,j} = \int_{\mathbb{R}} P_i(t)Q_j(\alpha t + \beta)d\sigma(t), \tag{2.1}$$

where $\alpha \neq 0$ and β are complex numbers. It turns out that these coefficients constitute a solution of a generalized wave equation on the two-dimensional lattice.

Theorem 2.1 (cf. [10, Theorem 2.1]). *We have that*

$$a_{i+1}u_{i+1,j} + b_iu_{i,j} + a_iu_{i-1,j} = \frac{c_{j+1}}{\alpha}u_{i,j+1} + \frac{d_j - \beta}{\alpha}u_{i,j} + \frac{c_j}{\alpha}u_{i,j-1} \tag{2.2}$$

for $i, j = 0, 1, 2, \ldots$.

Proof. From (2.1) and the three-term recurrence relations we get that

$$
\begin{aligned}
a_{i+1}u_{i+1,j} &+ b_i u_{i,j} + a_i u_{i-1,j} \\
&= \int_{\mathbb{R}} (a_{i+1}P_{i+1}(t) + b_i P_i(t) + a_i P_{i-1}(t)) Q_j(\alpha t + \beta) d\sigma(t) \\
&= \int_{\mathbb{R}} t P_i(t) Q_j(\alpha t + \beta) d\sigma(t) \\
&= \frac{1}{\alpha} \int_{\mathbb{R}} P_i(t)(\alpha t + \beta) Q_j(\alpha t + \beta) d\sigma(t) - \frac{\beta}{\alpha} \int_{\mathbb{R}} P_i(t) Q_j(\alpha t + \beta) d\sigma(t) \\
&= \frac{1}{\alpha} \int_{\mathbb{R}} P_i(t)(c_{j+1}Q_{j+1}(\alpha t + \beta) + d_j Q_j(\alpha t + \beta) + c_j Q_{j-1}(\alpha t + \beta)) d\sigma(t) \\
&\quad - \frac{\beta}{\alpha} u_{i,j} \\
&= \frac{c_{j+1}}{\alpha} u_{i,j+1} + \frac{d_j}{\alpha} u_{i,j} + \frac{c_j}{\alpha} u_{i,j-1} - \frac{\beta}{\alpha} u_{i,j}
\end{aligned}
$$

and thus (2.2) holds. □

Remark 2.1. Given an equation of the form (2.2), then due to the Favard theorem the coefficients will uniquely determine the families $\{P_n\}_{n=0}^{\infty}$ and $\{Q_n\}_{n=0}^{\infty}$ of orthonormal polynomials. The measure σ is responsible for the initial state when $j = 0$ and j can be thought of as a discrete time. Namely, for a solution of the form (2.1) to exist they need to satisfy the initial condition

$$
u_{i,0} = \int_{\mathbb{R}} P_i(t) d\sigma(t),
$$

which means that given an initial function $u_{i,0}$ of the discrete space variable i, σ needs to be found. The latter problem is a generalized moment problem and in this particular case it is equivalent to a Hamburger moment problem.

It is also worth mentioning here that another type of cross-difference equations on $\mathbb{Z}_+^2$ was recently discussed in [2] and the construction was based on multiple orthogonal polynomials. Type I Legendre–Angelesco multiple orthogonal polynomials also arise in the wavelet construction proposed by Alpert [6].

Next, consider a particular case of the above scheme where P_n and Q_n are both orthonormal Legendre polynomials $\hat{p}_n$ and so verify the three-term recurrence relation

$$
\frac{(n+1)}{\sqrt{(2n+1)(2n+3)}} \hat{p}_{n+1}(t) + \frac{n}{\sqrt{(2n-1)(2n+1)}} \hat{p}_{n-1}(t) = t \hat{p}_n(t),
$$

for $n = 0, 1, 2, \ldots$. Set σ to be the Lebesgue measure on the interval $[0, 1]$. As a result, the coefficients (2.1) take the form

$$
f_{i,j} = \int_0^1 \hat{p}_i(t) \hat{p}_j(2t - 1) dt. \tag{2.3}
$$

Note that

$$f_{i,j} = 0, \quad j > i = 0, 1, 2, \dots, \tag{2.4}$$

which follows from the fact that the polynomials $\hat{p}_j(2t-1)$ are orthogonal on the interval [0, 1] with respect to the Lebesgue measure.

Since the coefficients of the three-term recurrence relation for the Legendre polynomials are explicitly known, the coefficients of equation (2.2) become explicit as well. The following Corollary can be found in [7].

Corollary 2.2. *The function $f_{i,j}$ satisfies*

$$\begin{aligned}\frac{j+1}{\sqrt{(2j+1)(2j+3)}}f_{i,j+1} + f_{i,j} + \frac{j}{\sqrt{(2j-1)(2j+1)}}f_{i,j-1} \\ = \frac{2(i+1)}{\sqrt{(2i+1)(2i+3)}}f_{i+1,j} + \frac{2i}{\sqrt{(2i-1)(2i+1)}}f_{i-1,j}\end{aligned} \tag{2.5}$$

for $i, j = 0, 1, 2, \dots$.

Figure 1 presents the MATLAB generated graphical representation of some behavior of the solution $f_{i,j}$ to equation (2.5), which is a generalization of the discretized wave equation.

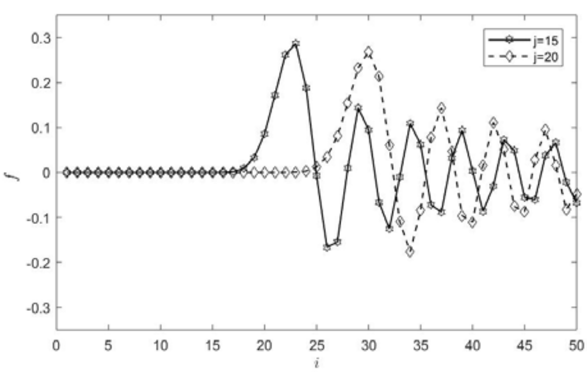

Figure 1. This picture demonstrates the moving wave. Here, one can see two graphs of the function $f = f(i) = f_{i,j}$ of the discrete space variable i at the two different discrete times $j = 15$ and $j = 20$.

Note that the form (2.1) of solutions of the discrete wave equations is useful for understanding the behavior of solutions because there are many asymptotic results for a variety of families of orthogonal polynomials; see, for example, [17].

3 Some further analysis of the coefficients $f_{i,j}$

In this section, we will obtain some properties of the coefficients $f_{i,j}$ based on the intuition and observations developed in [7]. In particular, we will rederive and expand upon some orthogonality properties of the coefficients $f_{i,j}$.

We begin with the following statement, which is based on formula (2.3) and some known properties of the Legendre polynomials.

Theorem 3.1. *Let k and l be two nonnegative integers. Then one has*

$$\sum_{j=0}^{\infty} f_{k,j} f_{l,j} = \begin{cases} 0, & \text{if } k \text{ and } l \text{ are of the same parity but not equal;} \\ 1, & \text{if } k = l; \\ (-1)^{\frac{k+l+1}{2}} \frac{k!l!\sqrt{2k+1}\sqrt{2l+1}}{2^{k+l-1}(k-l)(k+l+1)((\frac{k}{2})!)^2((\frac{l-1}{2})!)^2}, & \text{if } k \text{ and } l \text{ are of opposite parity.} \end{cases} \tag{3.1}$$

Proof. Without loss of generality, we can assume that $k \leq l$. Next observe that due to (2.4) the left-hand side of formula (3.1) is truncated to

$$\sum_{j=0}^{\infty} f_{k,j} f_{l,j} = \sum_{j=0}^{k} f_{k,j} f_{l,j},$$

which can be written as

$$\sum_{j=0}^{k} f_{k,j} f_{l,j} = \sum_{j=0}^{k} \int_0^1 \hat{p}_k(x)\hat{p}_j(2x-1)dx \int_0^1 \hat{p}_l(y)\hat{p}_j(2y-1)dy.$$

One can rewrite the expression in the following manner

$$\sum_{j=0}^{k} f_{k,j} f_{l,j} = \int_0^1 \hat{p}_l(y) \left(\int_0^1 \hat{p}_k(x) \sum_{j=0}^{k} \hat{p}_j(2x-1)\hat{p}_j(2y-1)dx \right) dy.$$

Since the Christoffel–Darboux kernel $2\sum_{j=0}^{k} \hat{p}_j(2x-1)\hat{p}_j(2y-1)$ is a reproducing kernel, we get

$$\sum_{j=0}^{k} f_{k,j} f_{l,j} = 2 \int_0^1 \hat{p}_k(y)\hat{p}_l(y)dy.$$

Next recall that one can explicitly compute the quantity

$$\int_0^1 \hat{p}_k(y)\hat{p}_l(y)dy$$

for any nonnegative integers k and l. If k and l have the same parity, the symmetry properties of the Legendre polynomials allow the above integral to be extended to the full orthonality interval $[-1, 1]$ which gives the first two parts of the Theorem. The third case of formula (3.1) is a consequence of [3, p. 173, Art. 91, Ex. 2]. □

One can also compute the inner product of vectors $f_{i,j}$ taken the other way.

Theorem 3.2. *Let k and l be two nonnegative integers. Then one has*

$$\sum_{i=0}^{\infty} f_{i,k} f_{i,l} = \begin{cases} 0, & k \neq l; \\ 1/2, & \text{if } k = l. \end{cases} \tag{3.2}$$

Proof. Let n be a nonnegative integer. Then we can write

$$\sum_{i=0}^{n} f_{i,k} f_{i,l} = \sum_{i=0}^{n} \int_0^1 \hat{p}_i(x)\hat{p}_k(2x-1)dx \int_0^1 \hat{p}_i(y)\hat{p}_l(2y-1)dy,$$

which can be rewritten as follows

$$\sum_{i=0}^{n} f_{i,k} f_{i,l} = \int_{-1}^{1} \hat{p}_k(2x-1)\chi_{[0,1]}(x) \left(\sum_{i=0}^{n} \left(\int_{-1}^{1} \hat{p}_l(2y-1)\chi_{[0,1]}(y)\hat{p}_i(y)dy \right) \hat{p}_i(x) \right) dx.$$

Since the polynomials $\hat{p}_i$ form an orthonormal basis in $L_2([-1, 1], dt)$ we know that

$$\sum_{i=0}^{n} \left(\int_{-1}^{1} \hat{p}_l(2y-1)\chi_{[0,1]}(y)\hat{p}_i(y)dy \right) \hat{p}_i(x) \xrightarrow{L_2([-1,1],dt)} \hat{p}_l(2x-1)\chi_{[0,1]}(x)$$

as $n \to \infty$. As a result we arrive at the following relation

$$\begin{aligned} \sum_{i=0}^{\infty} f_{i,k} f_{i,l} &= \int_{-1}^{1} \hat{p}_k(2x-1)\chi_{[0,1]}(x)\hat{p}_l(2x-1)\chi_{[0,1]}(x)dx \\ &= \int_0^1 \hat{p}_k(2x-1)\hat{p}_l(2x-1)dx = \frac{1}{2}\int_{-1}^{1} \hat{p}_k(t)\hat{p}_l(t)dt, \end{aligned}$$

which finally gives (3.2). □

As a consequence we can say a bit more about the asymptotic behavior of the coefficients $f_{i,j}$.

Corollary 3.3. *Let k be a fixed nonnegative integer number. Then*

$$f_{i,k} \longrightarrow 0$$

as $i \to \infty$.

Proof. The statement immediately follows from the fact that the series

$$\sum_{i=0}^{\infty} f_{i,k}^2$$

converges. □

Remark 3.1. From (3.2) one gets that

$$\sum_{i=0}^{\infty} f_{i,j}^2 = 1/2$$

for any nonnegative j, which means that the energy of the wave represented by $f = f(i) = f_{i,j}$ is conserved over the discrete time j.

Remark 3.2. The fact that $f_{i,k}$ can be represented as a hypergeometric function allows a more precise asymptotic estimate; see formula (4.23).

4 The case of ultraspherical polynomials

In this section we will carry over our findings from the case of Legendre polynomials to the case of the family of ultraspherical polynomials which include the Legendre polynomials as a special case.

Recall that for $\lambda > -1/2$ an ultraspherical polynomial $\hat{p}_n^{(\lambda)}(t)$ is a polynomial of degree n that is the orthonormal polynomial with respect to the measure

$$(1 - t^2)^{\lambda - 1/2} dt.$$

In an analogous way to $f_{i,j}$, let us consider the function of the discrete variables i and j,

$$f_{i,j}^{(\lambda)} = \int_0^1 \hat{p}_i^{(\lambda)}(t) \hat{p}_j^{(\lambda)}(2t - 1)(t(1 - t))^{\lambda - 1/2} dt, \tag{4.1}$$

and notice that

$$f_{i,j} = f_{i,j}^{(1/2)}.$$

While this allows us to consider a more general case, the connection to multiresolution analysis seems to be lost due to the weight and there is no evident relation to multiresolution analysis for arbitrary $\lambda > -1/2$. Still, such a deformation of the coefficients $f_{i,j}$ gives an insight on how all these objects are connected to various problems some of which were mentioned in the introduction.

Note that the polynomials $\hat{p}_j^{(\lambda)}(2t-1)$ are orthogonal with respect to the measure

$$(t(1-t))^{\lambda-1/2}dt,$$

and since the orthonormal ultraspherical polynomials satisfy the three-term recurrence relation [17]

$$a_{n+1}\hat{p}_{n+1}^{(\lambda)}(t) + a_n\hat{p}_{n-1}^{(\lambda)}(t) = t\hat{p}_n^{(\lambda)}(t), \tag{4.2}$$

where $a_n = \frac{1}{2}\sqrt{\frac{n(n+2\lambda-1)}{(n+\lambda-1)(n+\lambda)}}$, the following corollary of Theorem 2.1 is immediate.

Corollary 4.1. *The function $f_{i,j}^{(\lambda)}$ satisfies*

$$a_{j+1}f_{i,j+1}^{(\lambda)} + f_{i,j}^{(\lambda)} + a_jf_{i,j-1}^{(\lambda)} = 2a_{i+1}f_{i+1,j}^{(\lambda)} + 2a_if_{i-1,j}^{(\lambda)} \tag{4.3}$$

for $i, j = 0, 1, 2, \ldots$.

Figure 2 demonstrates how the solution $f_{i,j}^{(\lambda)}$ of the discrete wave equation (4.3) changes with λ when j is fixed.

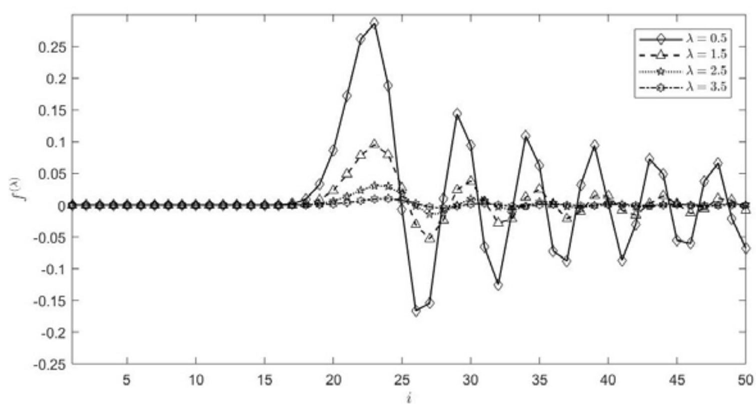

Figure 2. This picture shows the λ-evolution of the function $f^{(\lambda)} = f^{(\lambda)}(i) = f_{i,j}^{(\lambda)}$ of the discrete space variable i when the discrete time j is fixed and $j = 15$.

It is possible to generalize (3.1) and (3.2) to the case of the ultraspherical polynomials.

Theorem 4.2. *Let k and l be two nonnegative integer numbers. Then one has*

$$\sum_{j=0}^{\infty} f_{k,j}^{(\lambda)}f_{l,j}^{(\lambda)} = \frac{1}{2^{2\lambda}}\int_0^1 \hat{p}_k^{(\lambda)}(y)\hat{p}_l^{(\lambda)}(y)(y(1-y))^{\lambda-\frac{1}{2}}dy \tag{4.4}$$

for any $\lambda > -1/2$, and

$$\sum_{i=0}^{\infty} f_{i,k}^{(\lambda)}f_{i,l}^{(\lambda)} = \int_0^1 \hat{p}_k^{(\lambda)}(2x-1)\hat{p}_l^{(\lambda)}(2x-1)x^{2\lambda-1}\left(\frac{1-x}{1+x}\right)^{\lambda-\frac{1}{2}}dx \tag{4.5}$$

provided that $\lambda > 0$.

Proof. As before we can assume that $k \leq l$ therefore,

$$\sum_{j=0}^{\infty} f_{k,j}^{(\lambda)} f_{l,j}^{(\lambda)}$$

$$= \sum_{j=0}^{k} f_{k,j}^{(\lambda)} f_{l,j}^{(\lambda)} = \int_0^1 \hat{p}_l^{(\lambda)}(y)$$

$$\times \left(\int_0^1 \hat{p}_k^{(\lambda)}(x) \sum_{j=0}^{k} \hat{p}_j^{(\lambda)}(2x-1)\hat{p}_j^{(\lambda)}(2y-1)(x(1-x))^{\lambda-\frac{1}{2}} dx \right) (y(1-y))^{\lambda-\frac{1}{2}} dy.$$

Since the Christoffel–Darboux kernel

$$2^{2\lambda} \sum_{j=0}^{k} \hat{p}_j^{(\lambda)}(2x-1)\hat{p}_j^{(\lambda)}(2y-1)$$

is a reproducing kernel in the corresponding L_2-space, we get

$$\sum_{j=0}^{\infty} f_{k,j}^{(\lambda)} f_{l,j}^{(\lambda)} = \frac{1}{2^{2\lambda}} \int_0^1 \hat{p}_k^{(\lambda)}(y)\hat{p}_l^{(\lambda)}(y)(y(1-y))^{\lambda-\frac{1}{2}} dy.$$

To prove the second equality, consider the following representation of the finite sum:

$$\sum_{i=0}^{n} f_{i,k}^{(\lambda)} f_{i,l}^{(\lambda)} = \int_{-1}^1 \hat{p}_k^{(\lambda)}(2x-1)\chi_{[0,1]}(x)\frac{x^{\lambda-1/2}}{(1+x)^{\lambda-1/2}} P_n(x)(1-x^2)^{\lambda-1/2} dx,$$

where

$$P_n(x) = \sum_{i=0}^{n} \int_{-1}^1 \left((\hat{p}_l^{(\lambda)}(2y-1)\chi_{[0,1]}(y))\frac{y^{\lambda-1/2}}{(1+y)^{\lambda-1/2}} p_i^{(\lambda)}(y)(1-y^2)^{\lambda-1/2} dy \right) \hat{p}_i^{(\lambda)}(x).$$

If $\lambda > 0$ then

$$P_n(x) \xrightarrow{L_2([-1,1],(1-x^2)^{\lambda-1/2}dx)} \hat{p}_l^{(\lambda)}(2x-1)\chi_{[0,1]}(x)\frac{x^{\lambda-1/2}}{(1+x)^{\lambda-1/2}}$$

as $n \to \infty$. Next, since the functional

$$F(g) = \int_{-1}^1 \hat{p}_k^{(\lambda)}(2x-1)\chi_{[0,1]}(x)\frac{x^{\lambda-1/2}}{(1+x)^{\lambda-1/2}} g(x)(1-x^2)^{\lambda-1/2} dx,$$

is continuous for $\lambda > 0$ we arrive at the following:

$$\sum_{i=0}^{\infty} f_{i,k}^{(\lambda)} f_{i,l}^{(\lambda)} = \int_0^1 \hat{p}_k^{(\lambda)}(2x-1))\hat{p}_l^{(\lambda)}(2x-1)x^{2\lambda-1}\left(\frac{1-x}{1+x}\right)^{\lambda-\frac{1}{2}} dx$$

which completes the proof. □

Remark 4.1. The first integral in the above Theorem can be evaluated with the use of the equations (4.7.30) in [17], namely

$$I^1_{k,l} = \frac{1}{2^\lambda} k_k k_l I^2_{k,l},$$

where

$$k_l = 2^l \sqrt{\frac{(\lambda)_l(\lambda+1)_l}{l!(2\lambda)_l}},$$

and

$$I^2_{k,l} = \int_0^1 p^\lambda_k(y) p^\lambda_i(y)(y(1-y))^{\lambda-1/2} dy.$$

With the use of the formulas alluded to above in [17] we find

$$\begin{aligned} &I^2_{2k,2l} \\ &= (-1)^{k+l} \frac{(1/2)_k (1/2)_l \Gamma(\lambda+\frac{1}{2})^2}{(k+\lambda)_k (l+\lambda)_l \Gamma(2\lambda+1)} \\ &\quad \times \sum_{j=0}^k \frac{(-k)_j (k+\lambda)_j (\lambda+1/2)_{2j}}{(1)_j (1/2)_j (2\lambda+1)_{2j}} {}_4F_3\left(\begin{matrix} -l, l+\lambda, j+\frac{\lambda}{2}+\frac{1}{4}, j+\frac{\lambda}{2}+\frac{3}{4} \\ \frac{1}{2}, j+\lambda+1, j+\lambda+\frac{1}{2} \end{matrix}; 1\right), \end{aligned}$$

$$\begin{aligned} &I^2_{2k,2l+1} \\ &= (-1)^{k+l} \frac{(1/2)_k (3/2)_l \Gamma(\lambda+\frac{1}{2}) \Gamma(\lambda+\frac{3}{2})}{(k+\lambda)_k (l+\lambda+1)_l \Gamma(2\lambda+2)} \\ &\quad \times \sum_{j=0}^k \frac{(-k)_j (k+\lambda)_j (\lambda+3/2)_{2j}}{(1)_j (1/2)_j (2\lambda+2)_{2j}} {}_4F_3\left(\begin{matrix} -l, l+\lambda+1, j+\frac{\lambda}{2}+\frac{3}{4}, j+\frac{\lambda}{2}+\frac{5}{4} \\ \frac{3}{2}, j+\lambda+1, j+\lambda+\frac{3}{2} \end{matrix}; 1\right), \end{aligned}$$

and

$$\begin{aligned} &I^2_{2k+1,2l+1} \\ &= (-1)^{k+l} \frac{(3/2)_k (3/2)_l \Gamma(\lambda+\frac{1}{2}) \Gamma(\lambda+\frac{5}{2})}{(k+\lambda+1)_k (l+\lambda+1)_l \Gamma(2\lambda+3)} \\ &\quad \times \sum_{j=0}^k \frac{(-k)_j (k+\lambda+1)_j (\lambda+5/2)_{2j}}{(1)_j (3/2)_j (2\lambda+3)_{2j}} {}_4F_3\left(\begin{matrix} -l, l+\lambda+1, j+\frac{\lambda}{2}+\frac{5}{4}, j+\frac{\lambda}{2}+\frac{7}{4} \\ \frac{3}{2}, j+\lambda+2, j+\lambda+\frac{3}{2} \end{matrix}; 1\right). \end{aligned}$$

Note that all of the above hypergeometric functions are balanced. Furthermore, for $\lambda = 1/2$ one of the terms in the numerator cancels a denoninator term so they all become balanced ${}_3F_2$'s and can be summed using the Pfaff–Saalschiitz formula. The remaining sums in turn reduce to the Legendre case discussed earlier.

At this point we are unable to determine whether for certain values of λ the above sums simplify or if there is any orthogonality as in the Legendre case. Another interesting problem is the asymptotics of the above sums.

A formula for the second integral in the above Theorem may be obtained using equation (4.7.6) (first formula) in [17] and is

$$\begin{aligned}
&\int_0^1 \hat{p}_k^{(\lambda)}(2x-1)\hat{p}_l^{(\lambda)}(2x-1)x^{2\lambda-1}\left(\frac{1-x}{1+x}\right)^{\lambda-\frac{1}{2}}dx \\
&= (-1)^{k+l}k_k^\lambda k_l^\lambda \frac{(\lambda+\frac{1}{2})_k(\lambda+\frac{1}{2})_l}{(k+2\lambda)_k(l+2\lambda)_l}\Gamma(\lambda+1/2) \\
&\times \sum_{j=0}^{k}\sum_{n=0}^{l}\frac{(-k)_j(k+2\lambda)_j(-l)_n(l+2\lambda)_n\Gamma(j+n+2\lambda)}{(1)_j(\lambda+1/2)_j(1)_n(\lambda+1/2)_n\Gamma(j+n+3\lambda+1/2)} \\
&\times {}_2F_1\left(\begin{matrix}\lambda-1/2, j+n+2\lambda \\ j+n+3\lambda+1/2\end{matrix}; -1\right).
\end{aligned}$$

The next step is to obtain a generalized eigenvalue problem which will be a 1D-relation for the function $f_{i,j}^{(\lambda)}$, unlike (4.3). Our first approach uses the fact that the ultraspherical polynomials satisfy second order differential equations and apparently the approach can be generalized to the case of polynomials satisfying differential equations such as Krall polynomials, Koornwinder's generalized Jacobi polynomials and some Sobolev orthogonal polynomials.

Theorem 4.3. *Let j be a fixed nonnegative integer number. Then the function* $f = f(i) = f_{i,j}^{(\lambda)}$ *of the discrete variable i satisfies the generalized eigenvalue problem*

$$\begin{aligned}
&2((i+\lambda)^2-1/4)\left(i+\lambda+\frac{3}{2}\right)\sqrt{\frac{i+2\lambda}{(i+1)(i+\lambda+1)(\lambda+i)}}f_{i+1,j}^{(\lambda)} \\
&+2((i+\lambda)^2-1/4)\left(i+\lambda-\frac{3}{2}\right)\sqrt{\frac{i}{(i-1+\lambda)(i-1+2\lambda)(\lambda+i)}}f_{i-1,j}^{(\lambda)} \\
&= \left(j+\lambda-\frac{1}{2}\right)\left(j+\lambda+\frac{1}{2}\right) \\
&\times\left[2(i+\lambda-1/2)\sqrt{\frac{i+2\lambda}{(i+1)(i+\lambda+1)(\lambda+i)}}f_{i+1,j}^{(\lambda)}+4f_{i,j}^{(\lambda)}\right. \\
&\left.+2(i+\lambda+1/2)\sqrt{\frac{i}{(i-1+\lambda)(i-1+2\lambda)(\lambda+i)}}f_{i-1,j}^{(\lambda)}\right],
\end{aligned} \tag{4.6}$$

for $i = 0, 1, 2, \ldots$ *and, here, the number* $(j+\lambda-\frac{1}{2})(j+\lambda+\frac{1}{2})$ *is the corresponding generalized eigenvalue.*

Remark 4.2. For the case $\lambda = 1/2$, formula (4.6) was obtained in [7].

Proof. To make all the formulas shorter and, more importantly transparent, let us introduce the following operators:

$$\begin{aligned} A_i &= 2\Big(i+\lambda+\frac{3}{2}\Big)\sqrt{\frac{i+2\lambda}{(i+1)(i+\lambda+1)(\lambda+i)}}E_+ \\ &\quad + 2\Big(i+\lambda-\frac{3}{2}\Big)\sqrt{\frac{i}{(i-1+\lambda)(i-1+2\lambda)(\lambda+i)}}E_- \\ &= a_i^1E_+ + a_i^2E_- \end{aligned} \tag{4.7}$$

and

$$\begin{aligned} B_i &= 4I + 2(i+\lambda-1/2)\sqrt{\frac{i+2\lambda}{(i+1)(i+\lambda+1)(\lambda+i)}}E_+ \\ &\quad + 2(i+\lambda+1/2)\sqrt{\frac{i}{(i-1+\lambda)(i-1+2\lambda)(\lambda+i)}}E_- \\ &= 4I + b_i^1E_+ + b_i^2E_-, \end{aligned} \tag{4.8}$$

where I is the identity operator and E_+, E_- are the forward and backward shift operators on i, respectively. With these notations, equation (4.6) can be rewritten as

$$(i(i+2\lambda)+\lambda^2-1/4)A_if_{i,j}^{(\lambda)} = (j(j+2\lambda)+\lambda^2-1/4)B_if_{i,j}^{(\lambda)} \tag{4.9}$$

or

$$i(i+2\lambda)A_if_{i,j}^{(\lambda)} + (\lambda^2-1/4)(A_i-B_i)f_{i,j}^{(\lambda)} = j(j+2\lambda)B_if_{i,j}^{(\lambda)}, \tag{4.10}$$

since

$$\begin{aligned} A_i - B_i &= -4I + 4\sqrt{\frac{i+2\lambda}{(i+1)(i+\lambda+1)(\lambda+i)}}E_+ \\ &\quad - 4\sqrt{\frac{i}{(i-1+\lambda)(i-1+2\lambda)(\lambda+i)}}E_-. \end{aligned} \tag{4.11}$$

As is known [17], the orthonormal ultraspherical polynomials satisfy the differential equation

$$\frac{d}{dt}\Big((t(1-t))^{\lambda+1/2}\frac{d}{dt}\hat{p}_j^{(\lambda)}(2t-1)\Big) + j(j+2\lambda)(t(1-t))^{\lambda-1/2}\hat{p}_j^{(\lambda)}(2t-1) = 0. \tag{4.12}$$

Thus after two integration by parts we have

$$\begin{aligned}
&j(j+2\lambda)B_i f_{i,j}^{(\lambda)} \\
&= -\int_0^1 \frac{d}{dt}((t(1-t))^{\lambda+1/2}\frac{d}{dt}B_i\hat{p}_i^{(\lambda)}(t))\hat{p}_j^{(\lambda)}(2t-1)dt \\
&= -\int_0^1 \Big(\Big(t(1-t)\frac{d^2}{dt^2}+(\lambda+1/2)(1-2t)\frac{d}{dt}\Big)B_i\hat{p}_i^{(\lambda)}(t)\Big)\hat{p}_j^{(\lambda)}(2t-1)(t(1-t))^{\lambda-1/2}dt \\
&= -\int_0^1 \Big((1-t^2)\frac{d^2}{dt^2}-(2\lambda+1)t\frac{d}{dt}\Big)B_i\hat{p}_i^{(\lambda)}(t))\hat{p}_j^{\lambda}(2t-1)(t(1-t))^{\lambda-1/2}dt \\
&\quad -\int_0^1 \Big((t-1)\frac{d^2}{dt^2}+(\lambda+1/2)\frac{d}{dt}\Big)B_i\hat{p}_i^{(\lambda)}(t))\hat{p}_j^{\lambda}(2t-1)(t(1-t))^{\lambda-1/2}dt.
\end{aligned}$$

Now

$$\begin{aligned}
&-\Big((1-t^2)\frac{d^2}{dt^2}-(2\lambda+1)t\frac{d}{dt}\Big)B_i\hat{p}_i^{(\lambda)}(t)) \\
&\quad = (i+1)(i+1+2\lambda)b_i^1\hat{p}_{i+1}^{(\lambda)}(t)+i(i+2\lambda)4\hat{p}_i^{(\lambda)}(t)+(i-1)(i-1+2\lambda)b_i^2\hat{p}_{i-1}^{(\lambda)}(t).
\end{aligned}$$

Since

$$(i\pm 1)(i\pm 1+2\lambda)2\Big(i+\lambda\mp\frac{1}{2}\Big)-i(i+2\lambda)2\Big(i+\lambda\pm\frac{3}{2}\Big)\mp 4\Big(\lambda^2-\frac{1}{4}\Big)=0,$$

it follows that

$$\begin{aligned}
&(j(j+2\lambda)B_i-i(i+2\lambda)A_i-(\lambda^2-1/4)(A_i-B_i))f_{i,j}^{(\lambda)} \\
&= -\int_0^1 \Big((t-1)\frac{d^2}{dt^2}+(\lambda+1/2)\frac{d}{dt}\Big)B_i\hat{p}_i^{(\lambda)}(t))\hat{p}_j^{(\lambda)}(2t-1)(t(1-t))^{\lambda-1/2}dt \\
&\quad +4(i(i+2\lambda)+\lambda^2-1/4)\int_0^1 \hat{p}_i^{(\lambda)}(t)\hat{p}_j^{(\lambda)}(2t-1)(t(1-t))^{\lambda-1/2}dt.
\end{aligned} \tag{4.13}$$

From equations (4.2) and (4.8) we find

$$b_i^1 = 4\frac{i+\lambda-1/2}{i+1}a_{i+1} = 4(1+\frac{\lambda-3/2}{i+1})a_{i+1}$$

and

$$b_i^2 = 4\frac{i+\lambda+1/2}{i+2\lambda-1}a_i = 4(1-\frac{\lambda-3/2}{i+2\lambda-1})a_i.$$

The substitution of these relations in (4.13) leads to the following:

$$\begin{aligned}
B_i\hat{p}_i^{(\lambda)}(t) &= 4\Big(1+\frac{\lambda-3/2}{i+1}\Big)a_{i+1}\hat{p}_{i+1}^{(\lambda)}(t)+4\Big(1-\frac{\lambda-3/2}{i+2\lambda-1}\Big)a_i\hat{p}_{i-1}^{(\lambda)}(t)+4\hat{p}_i^{(\lambda)}(t) \\
&= 4\Big(1+t+\frac{\lambda-3/2}{i+1}t\Big)\hat{p}_i^{(\lambda)}(t)-8a_i\frac{(\lambda-3/2)(\lambda+i)}{(i+1)(i+2\lambda-1)}\hat{p}_{i-1}^{(\lambda)}(t)),
\end{aligned}$$

where the recurrence formula, (4.2), has been used to obtain the last equation. Using the first equation in [17, equation (4.7.28)] gives

$$\frac{d}{dt}\hat{p}_{i-1}^{(\lambda)}(t) = 2\frac{(i+\lambda-1)a_i}{i}\Big(t\frac{d}{dt}\hat{p}_i^{(\lambda)}(t) - i\hat{p}_i^{(\lambda)}(t)\Big)$$

so we find that

$$\begin{aligned}\frac{d}{dt}B_i\hat{p}_i^{(\lambda)}(t) &= 4\frac{d}{dt}\Big(1+t+\frac{\lambda-3/2}{i+1}t\Big)\hat{p}_i^{(\lambda)}(t) - 8a_i\frac{(\lambda-3/2)(\lambda+i)}{(i+1)(i+2\lambda-1)}\frac{d}{dt}\hat{p}_{i-1}^{(\lambda)}(t))\\ &= 4(\lambda-1/2)\hat{p}_i^{(\lambda)}(t) + 4(1+t)\frac{d}{dt}\hat{p}_i^{(\lambda)}(t).\end{aligned}$$

Thus we have

$$\begin{aligned}&\Big((1-t)\frac{d}{dt} - (\lambda+1/2)\Big)\frac{d}{dt}B_i\hat{p}_i^{(\lambda)}(t)\\ &\qquad = 4\Big((1-t^2)\frac{d^2}{dt^2} - (2\lambda+1)t\frac{d}{dt} - (\lambda^2-1/4)\Big)\hat{p}_i^{(\lambda)}(t)\end{aligned}$$

and the result follows. □

Remark 4.3. We can see that equation (4.6) has the form

$$\tilde{A}_i f_{i,j}^{(\lambda)} = \Big(j+\lambda-\frac{1}{2}\Big)\Big(j+\lambda+\frac{1}{2}\Big)B_i f_{i,j}^{(\lambda)},$$

where

$$\tilde{A}_i = (i+\lambda-1/2)(i+\lambda+1/2)A_i,$$

with the operators A_i and B_i given by (4.7) and (4.8), respectively. The above-given proof shows that the generalized eigenvalue problem (4.6) is a consequence of the fact that ultraspherical polynomials are eigenfunctions of a second order differential operator of a specific form. Difference equations for connection coefficients have also been investigated in [14] and [20] using differential equations.

There is another way to see the validity of equation (4.6). We first prove the following statement.

Proposition 4.4. *The following representation holds*

$$f_{i,j}^{(\lambda)} = \begin{cases} 0, & i<j;\\ \frac{1}{2^{3j+1}}\sqrt{\frac{i!(\lambda+1)_i(2\lambda)_i(2\lambda)_j}{j!(\lambda)_i(\lambda)_j(\lambda+1)_j}}\frac{(i+2\lambda)_j}{(\lambda+\frac{1}{2})_j(i-j)!}{}_2F_1\left({-i+j,\ i+j+2\lambda \atop 2j+2\lambda+1};\ \frac{1}{2}\right), & i\geq j.\end{cases} \tag{4.14}$$

Proof. Write

$$f_{i,j}^{(\lambda)} = k_{i,j,\lambda}\int_0^1 p_i^{(\lambda)}(t)p_j^{(\lambda)}(2t-1)(t(1-t))^{\lambda-1/2}dt, \tag{4.15}$$

where $p_n^{(\lambda)}$ is the monic orthogonal polynomial and

$$k_{i,j,\lambda} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\frac{1}{2})\sqrt{\pi}} 2^{i+j+2\lambda+1} \sqrt{\frac{(\lambda)_i(\lambda+1)_i}{i!(2\lambda)_i}} \sqrt{\frac{(\lambda)_j(\lambda+1)_j}{j!(2\lambda)_j}}. \tag{4.16}$$

If we denote the integral in equation (4.15) as $I^{(1)}$, we find using the representation

$$p_i^{(\lambda)}(t) = 2^i \frac{(\lambda+\frac{1}{2})_i}{(i+2\lambda)_i} {}_2F_1\left(\begin{matrix} -i,\ i+2\lambda \\ \lambda+\frac{1}{2} \end{matrix};\ \frac{1-t}{2}\right)$$

that

$$I^{(1)} = 2^{i+j}(-1)^j \frac{(\lambda+\frac{1}{2})_i}{((i+2\lambda)_i)} \frac{(\lambda+\frac{1}{2})_j}{((j+2\lambda)_j)} I^{(2)}, \tag{4.17}$$

with

$$\begin{aligned} I^{(2)} &= \int_0^1 {}_2F_1\left(\begin{matrix} -i,\ i+2\lambda \\ \lambda+\frac{1}{2} \end{matrix};\ \frac{1-t}{2}\right) {}_2F_1\left(\begin{matrix} -j,\ j+2\lambda \\ \lambda+\frac{1}{2} \end{matrix};\ t\right) (t(1-t))^{\lambda-1/2} dt \\ &= \sum_{k=0}^{i} \frac{(-i)_k(i+2\lambda)_k}{(1)_k(\lambda+\frac{1}{2})_k 2^k} \sum_{n=0}^{j} \frac{(-j)_n(j+2\lambda)_n}{(1)_n(\lambda+\frac{1}{2})_n} \int_0^1 (1-t)^{k+\lambda-1/2} t^{n+\lambda-1/2} dt. \end{aligned}$$

The integral can be evaluated as

$$\frac{\Gamma(k+\lambda+\frac{1}{2})\Gamma(n+\lambda+\frac{1}{2})}{\Gamma(k+n+2\lambda+1)} = \frac{(\lambda+\frac{1}{2})_k(\lambda+\frac{1}{2})_n\Gamma(\lambda+\frac{1}{2})^2}{(2\lambda+1)_k(k+2\lambda+1)_n\Gamma(2\lambda+1)}.$$

From the Chu–Vandermonde formula the sum on n yields

$$\sum_{n=0}^{j} \frac{(-j)_n(j+2\lambda)_n}{(1)_n(k+2\lambda+1)_n} = \frac{(k-j+1)_j}{(k+2\lambda+1)_j},$$

and the sum on k now becomes

$$\sum_{k=j}^{i} \frac{(-i)_k(i+2\lambda)_k(k-j+1)_j}{(1)_k(2\lambda+1)_k(k+2\lambda+1)_j 2^k} = \sum_{k=0}^{i-j} \frac{(-i)_{k+j}(i+2\lambda)_{k+j}(k+1)_j}{(1)_{k+j}(2\lambda+1)_{k+j}(k+j+2\lambda+1)_j 2^{k+j}}.$$

With the identities

$$(k+b)_j = \frac{(j+b)_k(b)_j}{(b)_k}, \quad (a)_{k+j} = (a+j)_k(a)_j,$$

the above sum becomes

$$\begin{aligned} &\sum_{k=0}^{i-j} \frac{(-i)_{k+j}(i+2\lambda)_{k+j}(k+1)_j}{(1)_{k+j}(2\lambda+1)_{k+j}(k+j+2\lambda+1)_j 2^{k+j}} \\ &= \frac{(-i)_j(i+2\lambda)_j(1)_j}{(2\lambda+1)_j(j+2\lambda+1)_j} \sum_{k=0}^{i-j} \frac{(-i+j)_k(i+j+2\lambda)_j}{(1)_j(2j+2\lambda+1)_k} \frac{1}{2^k} \\ &= \frac{(-i)_j(i+2\lambda)_j(1)_j}{(2\lambda+1)_j(j+2\lambda+1)_j} {}_2F_1\left(\begin{matrix} -i+j,\ i+j+2\lambda \\ 2j+2\lambda+1 \end{matrix};\ \frac{1}{2}\right). \end{aligned}$$

Combining all this together gives the result. □

The above hypergeometric representation (4.14) for $f_{i,j}^{(\lambda)}$ gives a recurrence relation among them.

Another Proof of Theorem 4.3. To see this use the contiguous relation (see [1, equation (2.5.15)])

$$\begin{aligned} &2b(c-b)(b-a-1)\,{}_2F_1\left(\begin{matrix} a-1,\ b+1 \\ c \end{matrix};\ \frac{1}{2}\right) \\ &\qquad - (b-a)(b+a-1)(2c-b-a-1)\,{}_2F_1\left(\begin{matrix} a,\ b \\ c \end{matrix};\ \frac{1}{2}\right) \\ &\qquad - 2a(b-c)(b-a+1)\,{}_2F_1\left(\begin{matrix} a+1,\ b-1 \\ c \end{matrix};\ \frac{1}{2}\right) = 0, \end{aligned}$$

which, with $a = -i + j$, $b = i + j + 2\lambda$, and $c = 2j + 2\lambda + 1$, yields the equation

$$\begin{aligned} &(2i+2\lambda-1)(j+2\lambda+i+1)(i+1-j)\sqrt{\frac{i+2\lambda}{(i+1)(i+\lambda+1)(\lambda+i)}}f_{i+1,j}^{(\lambda)} \\ &\qquad -(2j+2\lambda-1)(2j+2\lambda+1)f_{i,j}^{(\lambda)} \\ &\qquad +(2i+2\lambda+1)(i-j-1)(i+j+2\lambda-1) \\ &\qquad \times\sqrt{\frac{i}{(i-1+\lambda)(i-1+2\lambda)(\lambda+i)}}f_{i-1,j}^{(\lambda)} = 0. \end{aligned} \tag{4.18}$$

The latter relation leads to (4.6). □

A generalized eigenvalue problem can also be found for i fixed. To this end we need to use the relation

$${}_2F_1\left(\begin{matrix} -n,\ b \\ c \end{matrix};\ x\right) = \frac{(b)_n}{(c)_n}(-x)^n\,{}_2F_1\left(\begin{matrix} -n,\ -c-n+1 \\ -b-n+1 \end{matrix};\ 1/x\right).$$

Therefore we find that

$$\begin{aligned} &{}_2F_1\left(\begin{matrix} -i+j,\ i+j+2\lambda \\ 2j+2\lambda+1 \end{matrix};\ 1/2\right) \\ &= \frac{(i+j+2\lambda)_{i-j}}{(2j+2\lambda+1)_{i-j}}(-2)^{j-i}\,{}_2F_1\left(\begin{matrix} -i+j,\ -i-j-2\lambda \\ -2i-2\lambda+1 \end{matrix};\ 2\right). \end{aligned} \tag{4.19}$$

Following the steps used to obtain the recurrence formula for j fixed in the second proof we find that

$$c_j f_{i,j-1}^{(\lambda)} + d_j f_{i,j+1}^{(\lambda)} + e_j f_{i,j}^{(\lambda)} = 0,$$

where

$$c_j = -2(i+j+2\lambda-1)(i-j+1)(2j+2\lambda+1)(j+\lambda+1),$$

$$d_j = -4(i-j-1)(i+j+2\lambda+1)\left(j+\lambda-\frac{1}{2}\right)\sqrt{\frac{j(j+1)(j+\lambda-1)(j+\lambda+1)}{(j+2\lambda-1)(j+2\lambda)}},$$

and

$$e_j = -2(2i+2\lambda+1)(2i+2\lambda-1)(j+\lambda+1)\sqrt{\frac{j(j+\lambda)(j+\lambda-1)}{(j+2\lambda-1)}}$$
$$+6(2j+2\lambda-1)(2j+2\lambda+1)(j+\lambda+1)\sqrt{\frac{j(j+\lambda-1)(j+\lambda)}{(j+2\lambda-1)}}.$$

Since

$$(i+j+2\lambda\mp 1)(i-j\pm 1) = \left(i+\lambda+\frac{1}{2}\right)\left(i+\lambda-\frac{1}{2}\right) - \left(j+\lambda\mp\frac{1}{2}\right)\left(j+\lambda\mp\frac{3}{2}\right)$$

the above recurrence can be recast as the generalized eigenvalue equation

$$\hat{A}_j f_{i,j}^{(\lambda)} = \left(i+\lambda+\frac{1}{2}\right)\left(i+\lambda-\frac{1}{2}\right)\hat{B}_j f_{i,j}^{(\lambda)},$$

where the operator $\hat{A}_j$ is the second order difference operator

$$\begin{aligned}\hat{A}_j &= (j+\lambda+1/2)(j+\lambda-1/2))\\ &\times\left(12I+(2j+2\lambda+3)\sqrt{\frac{j+1}{(j+\lambda)(j+2\lambda)(j+\lambda+1)}}\hat{E}_+\right.\\ &\left.+(2j+2\lambda-3)\sqrt{\frac{(j+2\lambda-1)}{j(j+\lambda)(j+\lambda-1)}}\hat{E}_-\right),\end{aligned} \tag{4.20}$$

and the operator $\hat{B}_j$ is another second order difference operator given by the formula

$$\begin{aligned}\hat{B}_j = &\left(4I+(2j+2\lambda-1)\sqrt{\frac{j+1}{(j+\lambda)(j+2\lambda)(j+\lambda+1)}}\hat{E}_+\right.\\ &\left.+(2j+2\lambda+1)\sqrt{\frac{(j+2\lambda-1)}{j(j+\lambda)(j+\lambda-1)}}\hat{E}_-\right),\end{aligned} \tag{4.21}$$

the operator I is the identity operator, and $\hat{E}_+$, $\hat{E}_-$ are the forward and backward shift operators on j, respectively. Thus we have just proved the following statement.

Theorem 4.5. *Let i be a fixed nonnegative integer number. Then the function $f = f(j) = f_{i,j}^{(\lambda)}$ of the discrete variable j satisfies the generalized eigenvalue problem*

$$\hat{A}_j f_{i,j}^{(\lambda)} = \left(i+\lambda+\frac{1}{2}\right)\left(i+\lambda-\frac{1}{2}\right)\hat{B}_j f_{i,j}^{(\lambda)}$$

for $i = 0, 1, 2, \ldots$ and where the operators $\hat{A}_j$ and $\hat{B}_j$ are given by (4.20) *and* (4.21)*, respectively. Also, here, $(i+\lambda+\frac{1}{2})(i+\lambda-\frac{1}{2})$ is the corresponding generalized eigenvalue.*

Remark 4.4. For the case $\lambda = 1/2$, Theorem 4.5 was obtained in [7].

Recall that it is said that a function $\Psi(x, y)$ is a solution of a bispectral problem if it satisfies the following:

$$A\Psi(x, y) = g(y)\Psi(x, y),$$
$$B\Psi(x, y) = f(x)\Psi(x, y),$$

where A, B are some operators, with A acting only on x and B acting only on y, and f, g are some functions [5]. It is shown in [13] that if A and B are tridiagonal operators, then the solutions of the corresponding discrete bispectral problem are related to the Askey–Wilson polynomials.

The problem we are dealing with in this section is the following generalization of a bispectral problem:

$$\begin{aligned} A\Psi(i, j) &= g(j)B\Psi(i, j),\\ C\Psi(i, j) &= f(i)D\Psi(i, j), \end{aligned} \tag{4.22}$$

where i, j are discrete variables, the operators A and B are tridiagonal operators acting on the index i, and C, D are tridiagonal operators acting on the index j. Note that each equation in (4.22) is a generalized eigenvalue problem and, hence, the problem (4.22) includes a bispectral problem as a particular case (for instance, when B and D are the identity operators).

Setting $\Psi(i, j) = f_{i,j}^{(\lambda)}$ we see that Theorems 4.3 and 4.5 tell us that $f_{i,j}^{(\lambda)}$ is a solution of a generalized bispectral problem of the form (4.22). It would be interesting to find a characterization of such generalized bispectral problems as was done in [13] for discrete bispectral problems.

Using the asymptotic results for the Gauss hypergeometric function from [15] (see also [19], [12], [18]) one can get asymptotic behavior of the solution $f_{i,j}^{(\lambda)}$ for j fixed and when i tends to infinity.

Theorem 4.6. *For sufficiently large i the following formula holds:*

$$f_{i,j}^{(\lambda)} = k_j \frac{\cos(\pi(j + \frac{\lambda}{2} - \frac{i}{2} + \frac{1}{4}))}{\sqrt{\pi} i^{\lambda+1/2}} + O\left(\frac{1}{i^{\lambda+3/2}}\right) : \tag{4.23}$$

where

$$k_j = \frac{1}{2^{j+1-\lambda}} \sqrt{\frac{(2\lambda)_j}{j!(\lambda)_j(\lambda+1)_j\lambda\Gamma(2\lambda)} \frac{\Gamma(2j+2\lambda+1)}{(\lambda+\frac{1}{2})_j}}. \tag{4.24}$$

Proof. According to Proposition 4.4 for $i \geq j$ we have

$$f_{i,j}^{(\lambda)} = \frac{1}{2^{3j+1}} \sqrt{\frac{i!(\lambda+1)_i(2\lambda)_i(2\lambda)_j}{j!(\lambda)_i(\lambda)_j(\lambda+1)_j}} \frac{(i+2\lambda)_j}{(\lambda+\frac{1}{2})_j(i-j)!} {}_2F_1\left(\begin{matrix} -i+j,\ i+j+2\lambda \\ 2j+2\lambda+1 \end{matrix} ;\ \frac{1}{2}\right).$$

The hypergeometric function is of type B in the terminology of [15] and since its argument is real the relevant formula is (4.7) with saddle points $t_{s1} = 1 + i$ and $t_{s2} = 1 - i$, which gives

$$
\begin{aligned}
{}_2F_1&\left(\begin{matrix}-i+j,\ i+j+2\lambda\\ 2j+2\lambda+1\end{matrix};\frac{1}{2}\right)\\
&= \frac{2^{j+\lambda}\Gamma(2j+2\lambda+1)}{\sqrt{\pi}\, i^{2j+2\lambda+1/2}}\Big(\cos(\pi(j+\lambda/2-i/2+1/4))+O\Big(\frac{1}{i}\Big)\Big).
\end{aligned}
$$

Then, since

$$\sqrt{\frac{i!(\lambda+1)_i(2\lambda)_i}{(\lambda)_i}}\frac{(i+2\lambda)_j}{(i-j)!} = \sqrt{\frac{1}{\lambda\Gamma(2\lambda)}}\, i^{(2j+\lambda)}(1+O(1/i)),$$

the result follows. □

Remark 4.5. Formula (4.23) along with the fact that $f_{i,j}^{(\lambda)} = 0$ for $i < j$ show that the moving wave behavior of the solution demonstrated in Figure 1 is also characteristic for the solution $f_{i,j}^{(\lambda)}$ of the discrete wave equation (4.3) for any $\lambda > -1/2$.

Another useful asymptotic is when $i = k_1 t$ and $j = k_2 t$, where $k_1 > k_2$ are fixed and t is large.

Theorem 4.7. *For $k_1 t$ and $k_2 t$ integers with $k_1 > k_2 > 0$, and $\frac{\sqrt{2}k_2}{k_1} > 1$,*

$$
\begin{aligned}
f_{k_1 t, k_2 t}^{(\lambda)} &= \frac{c(\epsilon,\lambda)}{2^{k_1 t+1}(k_1 t)^{\frac{1}{2}}}\left(\frac{1+\hat{b}(\epsilon)}{\epsilon-\hat{b}(\epsilon)}\right)^{(k_1-k_2)t}\\
&\quad\times\left(\frac{1+2\epsilon-\hat{b}(\epsilon)}{1+\epsilon}\right)^{(k_1+k_2)t+2\lambda}(1+O(1/t)),
\end{aligned}
\tag{4.25}
$$

where

$$c(\epsilon,\lambda) = \epsilon^{\lambda}\frac{1}{\sqrt{\pi(1-\epsilon^2)(2\epsilon^2-1)^{\frac{1}{2}}}},\tag{4.26}$$

$\epsilon = \frac{k_2}{k_1}$, and $\hat{b}(\epsilon) = \sqrt{2\epsilon^2-1}$.

Proof. In this case the representation given by equation (4.19) is most convenient. An application of Pfaff's transformation yields

$${}_2F_1\left(\begin{matrix}-i+j,\ -i-j-2\lambda\\ -2i-2\lambda+1\end{matrix};2\right) = (-1)^{i-j}{}_2F_1\left(\begin{matrix}-i+j,\ -i+j+1\\ -2i-2\lambda+1\end{matrix};2\right).$$

Now the use of transformation T3 in [15] on the hypergeometric function on the right-hand side of the above equation yields, with $a = -i + j$ and $b = -i + j + 1$,

$$\begin{aligned} &{}_2F_1\left(\begin{matrix} -i+j,\ -i+j+1 \\ -2i-2\lambda+1 \end{matrix};\ 2\right) \\ &= -\frac{2^{i+j+2\lambda-1}(i-j)!}{(i+j+2\lambda+1)_{i-j-1}}\,{}_2F_1\left(\begin{matrix} i-j+1,\ -i-j-2\lambda+1 \\ 2 \end{matrix};\ 1/2\right), \end{aligned}$$

since the first term in T3 is equal to zero. Thus

$$f_{i,j}^{(\lambda)} = d_{i,j}\,{}_2F_1\left(\begin{matrix} i-j+1,\ -i-j-2\lambda+1 \\ 2 \end{matrix};\ 1/2\right), \tag{4.27}$$

where

$$d_{i,j} = (-1)^{i-j+1}2^{j+2\lambda-1}\sqrt{\frac{(i+\lambda)(j+\lambda)i!\Gamma(2\lambda+j)}{j!\Gamma(2\lambda+i)}}.$$

This becomes

$$\begin{aligned} d_{k_1t,k_2t} &= (-1)^{(i-j+1}2^{j+2\lambda-1}\left(\frac{j^{2\lambda}}{i^{2\lambda-2}}\right)^{1/2}(1+O(1/i)) \\ &= (-1)^{(k_1-k_2)t+1}2^{k_2t+2\lambda-1}\left(\frac{k_2}{k_1}\right)^{\lambda}(k_1t)(1+O(1/t)). \end{aligned} \tag{4.28}$$

The hypergeometric function on the right hand side of equation (4.27) is in the form to use the type B formulas in [15] and leads to considering the hypergeometric function

$${}_2F_1\left(\begin{matrix} \epsilon_1 w+1,\ -w-2\lambda+1 \\ 2 \end{matrix};\ 1/2\right)$$

where $\epsilon_1 w$ is an integer. Equation (4.4) in [15] shows that the saddle points occur at $\frac{1+\epsilon_1}{2} \pm \sqrt{(\frac{1+\epsilon_1}{2})^2 - 2\epsilon_1}$. If the discriminant is positive both saddles are real and equation (4.9) in [15] yields

$$\begin{aligned} &{}_2F_1\left(\begin{matrix} \epsilon_1 w+1,\ -w-2\lambda+1 \\ 2 \end{matrix};\ 1/2\right) \\ &= \frac{(-1)^{\epsilon_1 w+1}}{w^{\frac{3}{2}}\sqrt{\pi\epsilon_1 b(\epsilon_1)}}\left(\frac{1+\epsilon_1+b(\epsilon_1)}{1-\epsilon_1-b(\epsilon_1)}\right)^{\epsilon_1 w}\frac{(3-\epsilon_1-b(\epsilon_1))^{w+2\lambda}}{2^{2w+4\lambda-\frac{1}{2}}}(1+O(1/w)), \end{aligned}$$

where

$$b(\epsilon_1) = \sqrt{(1+\epsilon_1)^2 - 8\epsilon_1}. \tag{4.29}$$

With $\epsilon_1 = \frac{k_1-k_2}{k_1+k_2}$ and $w = (k_1+k_2)t$ the above equations yield (4.25). □

Remark 4.6. When the discriminant is negative, the two saddle points are conjugates of each other and so in this case equation (4.7) in [15] is used to obtain the asymptotics for ${}_2F_1\left(\begin{smallmatrix}\epsilon_1 t+1, & -t-2\lambda+1 \\ & 2\end{smallmatrix}; 1/2\right)$, which then are used to obtain the asymptotics of $f^{(\lambda)}_{k_1t,k_2t}$.

We finish this section with a couple of statements starting with the recurrence formulas. Write the recurrence formula in equation (4.6) as

$$a_{i,j}f^{(\lambda)}_{i+1,j} + b_{i,j}f^{(\lambda)}_{i,j} + c_{i,j}f^{(\lambda)}_{i-1,j} = 0, \tag{4.30}$$

and the recurrence formula in j as

$$\hat{a}_{i,j}f^{(\lambda)}_{i,j+1} + \hat{b}_{i,j}f^{(\lambda)}_{i,j} + \hat{c}_{i,j}f^{(\lambda)}_{i,j-1} = 0, \tag{4.31}$$

with $i \geq j \geq 0$.

We can now prove the following simple statement.

Proposition 4.8. *Given $a_{i,j}$, $b_{i,j}$, $c_{i,j}$ and $\lambda > -1/2$, for each $j > 0$ the unique solution of equation* (4.30) *with initial conditions*

$$f_{j-1,j} = 0, \quad f_{j,j} = \int_0^1 \hat{p}^{(\lambda)}_j(t)\hat{p}^{(\lambda)}_j(2t-1)(t(1-t))^{\lambda-1/2}dt$$

is the function

$$f_{i,j} = I^{(\lambda)}_{i,j} := \int_0^1 \hat{p}^{(\lambda)}_i(t)\hat{p}^{(\lambda)}_j(2t-1)(t(1-t))^{\lambda-1/2}dt.$$

If $j = 0$, $\lambda > -1/2$, and $\lambda \neq 1/2$, then $f_{0,0} = I^{(\lambda)}_{0,0}$ gives the unique solution $f_{i,0} = I^{(\lambda)}_{i,0}$. If $\lambda = 1/2$, then the initial conditions $f_{0,0} = I^{(1/2)}_{0,0}$ and $f_{1,0} = I^{(1/2)}_{1,0}$ are needed to give $f_{i,j} = I^{(1/2)}_{i,j}$.

Proof. For $j > 0$, $a_{i,j} \neq 0$ for $i \geq j$ so the result follows from equation (4.30). For $j = 0$ and $\lambda \neq 1/2$, $c_{0,0} = 0 \neq a_{0,0}$ so that only $f_{0,0}$ is needed to compute $f_{1,0}$. The remaining $f_{i,j}$ are computed in the standard fashion from equation (4.30). For the last case when $\lambda = 1/2$, $a_{0,0} = 0 = b_{i,0}$ so $f_{2,0} = \frac{c_{1,0}}{a_{1,0}}f_{0,0}$ and $f_{3,0} = \frac{c_{2,0}}{a_{2,0}}f_{1,0}$. The remaining $f_{i,0}$ are computed in the same way using the fact that $a_{i,0} \neq 0$ for $i > 0$.□

Similarly, for the recurrence in j we have the following.

Proposition 4.9. *Given $a_{i,j}$, $b_{i,j}$, $c_{i,j}$ and $\lambda > -1/2$, for each $i > 0$ the unique solution of equation* (4.31) *with initial conditions $f_{j,j+1} = 0$ and $f_{j,j} = I^{(\lambda)}_{j,j}$ is*

$$f_{i,j} = I^{(\lambda)}_{i,j}.$$

Since $\hat{c}_{i,j}$, $\hat{b}_{i,j}$, and $\hat{a}_{i,j}$ are not equal to zero for $i \geq j$ the result follows from equation (4.31).

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REFERENCES

[1] G. E. Andrews, R. Askey and R. Roy, *Special Functions*, Cambridge University Press, Cambridge, 1999.

[2] A.I. Aptekarev, M. Derevyagin and W. Van Assche, *On 2D discrete Schrödinger operators associated with multiple orthogonal polynomials*, J. Phys. A **48** (2015), 065201.

[3] W. E. Byerly, *An Elementary Treatise on Fourier's Series and Spherical, Cylindrical, and Ellipsoidal Harmonics, with Applications to Problems in Mathematical Physics*, Dover, New York, 1959.

[4] M. de Sainte-Catherine and G. Viennot, *Combinatorial interpretation of integrals of products of Hermite, Laguerre and Tchebycheff polynomials*, in *Orthogonal Polynomials and Applications (Bar-le-Duc, 1984)*, Springer, Berlin, 1985, pp. 120–128.

[5] J. J. Duistermaat and F. A. Grünbaum, *Differential equations in the spectral parameter*, Comm. Math. Phys. **103** (1986), 177–240.

[6] J. S. Geronimo, P. Iliev and W. Van Assche, *Alpert Multiwavelets and Legendre-Angelesco multiple orthogonal polynomials*, SIAM. J. Math. Anal. **49** (2017), 626–645.

[7] J.S. Geronimo and F. Marcellán, *On Alpert multiwavelets*, Proc. Amer. Math. Soc. **143** (2015), 2479–2494.

[8] F. A. Grünbaum, *A new property of reproducing kernels for classical orthogonal polynomials*, J. Math. Anal. Appl. **95** (1983), 491–500.

[9] D. Gupta and D. Masson, *Contiguous relations, continued fractions and orthogonality*, Trans. Amer. Math. Soc. **350** (1998), 769–808.

[10] M. Ismail, A. Kasraoui and J. Zeng, *Separation of variables and combinatorics of linearization coefficients of orthogonal polynomials*, J. Combin. Theory Ser. A **120** (2013), 561–599.

[11] M. Ismail and D. Masson, *Generalized orthogonality and continued fractions*, J. Approx. Theory **83** (1995), 1–40.

[12] D. S. Jones, *Asymptotics of the hypergeometric function*, Math. Methods Appl. Sci. **24** (2001), 369–389.

[13] F. Levstein and L. F. Matusevich, *The discrete version of the bispectral problem*, in *The Bispectral Problem*, American Mathematical Society, Providence, RI, 1998, pp. 93–104.

[14] S. Lewanowicz, *Construction of a recurrence relation of the lowest order for coefficients of the Gegenbauer series*, Zastos. Mat. **15** (1976), 345–396.

[15] R. B. Paris, *Asymptotics of the Gauss hypergeometric function with large parameters, I*, J. Class. Anal. **2** (2013), 183–203.

[16] V. Spiridonov and A. Zhedanov, *Spectral transformation chains and some new biorthogonal rational functions*, Comm. Math. Phys. **210** (2000), 49–83.

[17] G. Szegő, *Orthogonal Polynomials*, American Mathematical Society, Providence, RI, 1975.

[18] N. M. Temme, *Large parameter cases of the Gauss hypergeometric function*, J. Comput. Appl. Math. **153** (2003), 441–462.

[19] G. N. Watson, *Asymptotic expansions of hypergeometric functions*, Trans. Cambridge Philos. Soc. **22** (1918) 277–308.

[20] J. Wimp, *Computation with Recurrence Relations*, Pitman, Boston, MA, 1984.

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