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The L_p chord Minkowski problem

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Abstract: Chord measures are newly discovered translation-invariant geometric measures of convex bodies in $\mathbb{R}^n$, in addition to Aleksandrov-Fenchel-Jessen's area measures. They are constructed from chord integrals of convex bodies and random lines. Prescribing the L_p chord measures is called the L_p chord Minkowski problem in the L_p Brunn-Minkowski theory, which includes the L_p Minkowski problem as a special case. This article solves the L_p chord Minkowski problem when $p > 1$ and the symmetric case of $0 < p < 1$.

Keywords: chord integral, chord measure, L_p surface area measure, L_p chord measure, L_p Minkowski problem, L_p chord Minkowski problem

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1 Introduction

The classical Minkowski problem asks for the existence, uniqueness, and regularity of a convex body whose surface area measure is equal to a pre-given spherical Borel measure. When the given measure has a positive continuous density, the Minkowski problem is also known as the problem of prescribing the Gauss curvature in differential geometry. For other important geometric measures of convex bodies, similar Minkowski problems have been studied in convex geometry, differential geometry, and partial differential equations. The study of Minkowski problems has motivated the study of fully nonlinear partial differential equations, geometric curvature flows, and geometric inequalities.

The surface area measure of a convex body in Euclidean space is a Borel measure on the unit sphere, which was introduced by Aleksandrov-Fenchel-Jessen in the 1930s. It is the differential of the volume functional over convex bodies. Similar concepts for surface area and other quermassintegrals, called area measures, were also introduced by them. Area measures are translation invariant. Another family of geometric measures associated with quermassintegrals is the curvature measures of Federer. Area measures and curvature measures are fundamental concepts in the classical Brunn-Minkowski theory.

In the 1970s, Lutwak introduced the dual Brunn-Minkowski theory [34]. The duality between projections and intersections of convex bodies and their connections with harmonic analysis were the focus in the 1990s. Significant breakthroughs were made, see, for example, [7,10,15,29,35,54], and the books of Gardner [11] and Koldobsky [30]. However, what acts as the dual counterpart of the geometric measures in the

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Brunn-Minkowski theory was not clear until the work of Huang et al. [23] in 2016. Their discovery of dual curvature measures gives dual concepts to Federer's curvature measures. The dual Minkowski problem posed by them has led to a number of articles in a short period of time, see, for example, [6,13,14,25,26,31,45,57,58]. Dual curvature measures are differentials of the dual quermassintegrals. Dual quermassintegrals are origin dependent and thus not translation invariant. Their translation invariant analogues are *chord integrals*, which are basic geometric invariants in integral geometry [46,47,53].

Very recently, Lutwak et al. [38] constructed the geometric measures, which are the differentials of chord integrals and then called *chord measures*. Chord measures are translation invariant-like area measures. Minkowski problems associated with chord measures were posed in [38]. These geometric problems give new Monge-Ampère-type partial differential equations. The *chord Minkowski problem* includes the classical Minkowski problem as a special case and an unsolved Christoffel-Minkowski problem as a critical case, while the *chord log-Minkowski problem* includes the logarithmic Minkowski problem as an important case. Thus, the new chord Minkowski problems greatly enrich the area of Minkowski problems. Xi et al. [38] solved the *chord Minkowski problem*, except for the critical case of the Christoffel-Minkowski problem and partially solved the symmetric case of the *chord log-Minkowski problem*. They also posed the more general *L_p chord Minkowski problem*, which includes the L_p Minkowski problem as a special case. This article solves the L_p chord Minkowski problem when $p > 1$ and the symmetric case of $0 < p < 1$. These results generalize known results for the corresponding cases of the L_p Minkowski problem.

Let $\mathcal{K}^n$ be the collection of convex bodies (compact convex sets with nonempty interior) in $\mathbb{R}^n$. For $K \in \mathcal{K}^n$, the *chord integral* $I_q(K)$ of K is defined as follows:

$$I_q(K) = \int_{\mathscr{L}^n} |K \cap \ell|^q \mathrm{d}\ell, \quad q \geq 0,$$

where $|K \cap \ell|$ denotes the length of the chord $K \cap \ell$, and the integration is with respect to the (appropriately normalized) Haar measure on the affine Grassmannian $\mathscr{L}^n$ of lines in $\mathbb{R}^n$. Chord integrals contain volume $V(K)$ and surface area $S(K)$ as two important special cases:

$$I_1(K) = V(K), \quad I_0(K) = \frac{\omega_{n-1}}{n\omega_n} S(K), \quad I_{n+1}(K) = \frac{n+1}{\omega_n} V(K)^2,$$

where ω_n is the volume enclosed by the unit sphere S^{n-1}.

It was shown in [38] that the differential of $I_q(K)$ defines a finite Borel measure $F_q(K, \cdot)$ on S^{n-1}. Precisely, for convex bodies K and L in $\mathbb{R}^n$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0^+} I_q(K + tL) = \int_{S^{n-1}} h_L(v) \mathrm{d}F_q(K, v), \quad q \geq 0, \tag{1.1}$$

where $F_q(K, \cdot)$ is called the *qth chord measure* of K and h_L is the support function of L. The cases of $q = 0, 1$ of this formula are classical, which are the variational formulas of surface area and volume. There are

$$F_0(K, \cdot) = \frac{(n-1)\omega_{n-1}}{n\omega_n} S_{n-2}(K, \cdot), \quad F_1(K, \cdot) = S_{n-1}(K, \cdot),$$

where $S_{n-2}(K, \cdot)$ is the $(n-2)$th order area measure of K, and $S_{n-1}(K, \cdot)$ is the $(n-1)$th order area measure of K (i.e., the classical surface area measure of K).

The **chord Minkowski problem** states:

If μ is a finite Borel measure on S^{n-1}, what are the necessary and sufficient conditions for the existence of a convex body K that solves the equation,

$$F_q(K, \cdot) = \mu \ ?$$

This is a new Minkowski problem except $q = 0, 1$. The case of $q = 1$ is the classical Minkowski problem for surface area measure, and the case of $q = 0$ is the unsolved Christoffel-Minkowski problem for the $(n-2)$th area measure. When $q > 0$, the solution to the chord Minkowski problem was given in [38].

The L_p Minkowski problem was posed by Lutwak [36] in the early 1990s. He defined the fundamental concept of L_p surface area measure in the L_p Brunn-Minkowski theory, which has led to fruitful studies in several areas, including affine isoperimetric and Sobolev inequalities [8,19–22,39,41,55], affine surface areas and valuations [27,32,33,44,50,51,56], and Minkowski problems [9,17,24,26,28,42]. The Minkowski problem of prescribing L_p surface area measures is the L_p Minkowski problem. Many cases of the L_p Minkowski problem have been solved. However, critical cases of the problem and uniqueness for $p < 1$ remain open. In particular, the centro-affine Minkowski problem [9] and the logarithmic Brunn-Minkowski conjecture [2] are highly interesting.

Denote by $\mathcal{K}_o^n$ the sub-collection of $\mathcal{K}^n$ of convex bodies that contain the origin in their interiors, and by $\mathcal{K}_e^n$ the sub-collection of $\mathcal{K}^n$ of convex bodies that are symmetric about the origin. The L_p surface area measure can be extended to a two-parameter family of geometric measures, called L_p *chord measures*. The *(p,q)th chord measure*, $F_{p,q}(K,\cdot)$ of $K \in \mathcal{K}_o^n$ is defined as follows:

$$\mathrm{d}F_{p,q}(K,\cdot) = h_K^{1-p}\mathrm{d}F_q(K,\cdot), \quad p \in \mathbb{R},\ q \geq 0,$$

where h_K is the support function of K and $F_q(K,\cdot)$ is the qth chord measure of K. When $q = 1$, $F_{p,1}(K,\cdot)$ is the L_p surface area measure. When $q = 0$, $F_{p,0}(K,\cdot)$ is the L_p $(n-2)$th area measure. When $p = 1$, $F_{1,q}(K,\cdot)$ is just the qth chord measure $F_q(K,\cdot)$.

The L_p **chord Minkowski problem** asks:

Let μ be a finite Borel measure on S^{n-1}, $p \in \mathbb{R}$, and $q \geq 0$. What are the necessary and sufficient conditions for the existence of a convex body $K \in \mathcal{K}_o^n$ that solves the equation:

$$F_{p,q}(K,\cdot) = \mu\ ? \tag{1.2}$$

When $p = 1$, it is the chord Minkowski problem, and the $q = 1$ case is the L_p Minkowski problem.

When the given measure μ has a density f that is an integrable nonnegative function on S^{n-1}, equation (1.2) becomes a new Monge-Ampère-type partial differential equation:

$$\det(\nabla_{ij}h + h\delta_{ij}) = \frac{h^{p-1}f}{\widetilde{V}_{q-1}([h],\nabla h)}, \quad \text{on } S^{n-1},$$

where h is the unknown function on S^{n-1}, which is extended via homogeneity to $\mathbb{R}^n$, while ∇h is the Euclidean gradient of h in $\mathbb{R}^n$, the spherical Hessian of h with respect to an orthonormal frame on S^{n-1} is $(\nabla_{ij}h)$, δ_{ij} is the Kronecker delta, and $\widetilde{V}_{q-1}([h],\nabla h)$ is the $(q-1)$th dual quermassintegral of the Wulff-shape $[h]$ of h with respect to the point ∇h (see next section for the precise definition).

We first solve the symmetric case of the L_p chord Minkowski problem when $p, q > 0$.

Theorem 1.1. *Let $p, q > 0$. If μ is an even finite Borel measure on S^{n-1} that is not concentrated on a great subsphere, then there exists a symmetric, convex body $K \in \mathcal{K}_e^n$ such that*

$$F_{p,q}(K,\cdot) = \mu, \quad \text{when } p \neq n+q-1,$$
$$\frac{F_{p,q}(K,\cdot)}{V(K)} = \mu, \quad \text{when } p = n+q-1.$$

When $q = 1$, this result is a solution to the symmetric L_p Minkowski problem, see [18,36,42]. When $p > 1$, the symmetric condition can be dropped. We have the following solution:

Theorem 1.2. *Let $p > 1$ and $q > 0$. If μ is a finite Borel measure on S^{n-1} that is not concentrated in any closed hemisphere, then there exists a convex body K with nonnegative support function $h_K \geq 0$ so that*

$$\mathrm{d}F_q(K,\cdot) = h_K^{p-1}\mathrm{d}\mu, \quad \text{when } p \neq n+q-1,$$
$$\frac{\mathrm{d}F_q(K,\cdot)}{V(K)} = h_K^{p-1}\mathrm{d}\mu, \quad \text{when } p = n+q-1.$$

Moreover, $h_K > 0$ if μ is discrete or if $p \geq n$.

Again, when $q = 1$, this result is a solution to the L_p Minkowski problem, see [9,28].

2 Preliminaries

2.1 Wulff shape

Let $\Omega \subset S^{n-1}$ be a closed subset that is not contained in any hemisphere. For $h \in C(\Omega)$, the *Wulff-shape* $[h]$ is a compact convex set defined by

$$[h] = \{x \in \mathbb{R}^n : x \cdot v \le h(v), \quad \forall v \in \Omega\}.$$

Clearly, $h_{[h]}(v) \le h(v)$. A useful fact is that, when $[h] \in \mathcal{K}^n$, the support of $S_{n-1}([h], \cdot)$ must be contained in Ω.

In particular, let $v_1, \dots, v_N \quad (N > n+1)$ be unit vectors that are not contained in any closed hemisphere, and let $\Omega = \{v_1, \dots, v_N\}$. For $z = (z_1, \dots, z_N) \in \mathbb{R}^N$, we write

$$[z] = P(z) = \bigcap_{k=1}^{N} \{x \in \mathbb{R}^n : x \cdot v_i \le z_i\}.$$

Define $\mathcal{P}(v_1, \dots, v_N)$ by

$$\mathcal{P}(v_1, \dots, v_N) = \{P(z) : z \in \mathbb{R}^N \quad \text{such that } P(z) \in \mathcal{K}^n\}.$$

Denote $\mathbb{R}_+ = (0, \infty)$. Clearly, if $z \in \mathbb{R}_+^N$, then $P(z) \in \mathcal{K}_o^n$.

2.2 Chord integrals and chord measures

Let $K \in \mathcal{K}^n$. For $z \in \operatorname{int} K$ and $q \in \mathbb{R}$, the qth dual quermassintegral $\widetilde{V}_q(K, z)$ of K with respect to z is

$$\widetilde{V}_q(K, z) = \frac{1}{n} \int_{S^{n-1}} \rho_{K,z}(u)^q \mathrm{d}u,$$

where $\rho_{K,z}(u) = \max\{\lambda > 0 : z + \lambda u \in K\}$ is the radial function of K with respect to z. When $z \in \partial K$, $\widetilde{V}_q(K, z)$ is defined in the way that the integral is only over those $u \in S^{n-1}$ such that $\rho_{K,z}(u) > 0$. In another word,

$$\widetilde{V}_q(K, z) = \frac{1}{n} \int_{\rho_{K,z}(u)>0} \rho_{K,z}(u)^q \mathrm{d}u, \quad \text{whenever } z \in \partial K.$$

In this case, for $\mathcal{H}^{n-1}$-almost all $z \in \partial K$, we have

$$\widetilde{V}_q(K, z) = \frac{1}{2n} \int_{S^{n-1}} X_K(z, u)^q \mathrm{d}u,$$

where the *parallel X-ray* of K is the nonnegative function on $\mathbb{R}^n \times S^{n-1}$ defined by

$$X_K(z, u) = |K \cap (z + \mathbb{R}u)|, \quad z \in \mathbb{R}^n, \quad u \in S^{n-1}.$$

When restricting to $q > 0$, the dual quermassintegral is the Riesz potential of the characteristic function, that is,

$$\widetilde{V}_q(K, z) = \frac{q}{n} \int_K |x - z|^{q-n} \mathrm{d}x.$$

Note that this immediately allows an extension of $\widetilde{V}_q(K, \cdot)$ to $\mathbb{R}^n$. See [38] for an equivalent definition via radial function. By a change-of-variable, we have

$$\widetilde{V}_q(K, z) = \frac{q}{n} \int_{K-z} |y|^{q-n} \mathrm{d}y,$$

and since, for $q > 0$, the integrand $|y|^{q-n}$ is locally integrable, we immediately conclude that in this case, the dual quermassintegral $\widetilde{V}_q(K, z)$ is continuous in z.

Let $K \in \mathcal{K}^n$. The X-ray $X_K(x, u)$ and the radial function $\rho_{K,z}(u)$ have the following relation:

$$X_K(x, u) = \rho_{K,z}(u) + \rho_{K,z}(-u), \quad \text{when} \quad K \cap (x + \mathbb{R}u) = K \cap (z + \mathbb{R}u) \neq \varnothing. \tag{2.1}$$

When $z \in \partial K$, then either $\rho_{K,z}(u) = 0$ or $\rho_{K,z}(-u) = 0$ for almost all $u \in S^{n-1}$, and thus

$$X_K(z, u) = \rho_{K,z}(u), \quad \text{or } X_K(z, u) = \rho_{K,z}(-u), \quad z \in \partial K,$$

for almost all $u \in S^{n-1}$. Then, the chord integral $I_q(K)$ can be represented as follows:

$$I_q(K) = \frac{1}{n\omega_n} \int_{S^{n-1}} \int_{u^\perp} X_K(x, u)^q \mathrm{d}x \mathrm{d}u, \quad q \geq 0.$$

An elementary property of the functional I_q is its homogeneity. If $K \in \mathcal{K}^n$ and $q \geq 0$, then

$$I_q(tK) = t^{n+q-1} I_q(K),$$

for $t > 0$. By compactness of K, it is simple to see that the chord integral $I_q(K)$ is finite whenever $q \geq 0$.

Let $K \in \mathcal{K}^n$ and $q > 0$. The chord measure $F_q(K, \cdot)$ is a finite Borel measure on S^{n-1} given by

$$F_q(K, \eta) = \frac{2q}{\omega_n} \int_{\nu_K^{-1}(\eta)} \widetilde{V}_{q-1}(K, z) \mathrm{d}\mathcal{H}^{n-1}(z), \quad \text{for each Borel } \eta \subset S^{n-1},$$

where $\nu_K : \partial K \to S^{n-1}$ is the Gauss map that takes boundary points of K to their corresponding outer unit normals. Note that by convexity of K, its Gauss map ν_K is almost everywhere defined on ∂K with respect to the $(n-1)$-dimensional Hausdorff measure. The significance of the chord measure $F_q(K, \cdot)$ is that it comes from differentiating, in a certain sense, the chord integral I_q, see (1.1). It is simple to see that the chord measure $F_q(K, \cdot)$ is absolutely continuous with respect to the surface area measure $S_{n-1}(K, \cdot)$. In particular, for each $P \in \mathcal{P}(v_1, \ldots, v_N)$, we have that the chord measure $F_q(P, \cdot)$ is supported entirely on $\{v_1, \ldots, v_N\}$. It was shown in [38, Theorem 4.3] that

$$I_q(K) = \frac{1}{n + q - 1} \int_{S^{n-1}} h_K(v) \mathrm{d}F_q(K, v). \tag{2.2}$$

When $q > 0$, a useful integral formula demonstrated in [38, Lemma 5.3] is

$$2n \int_{\partial K} \widetilde{V}_{q-1}(K, z) g(\nu_K(z)) \mathrm{d}\mathcal{H}^{n-1}(z) = \int_{S^{n-1}} \int_{\partial K} X_K(z, u)^{q-1} g(\nu_K(z)) \mathrm{d}\mathcal{H}^{n-1}(z) \mathrm{d}u,$$

for any $g \in C(S^{n-1})$. Therefore, for each $K \in \mathcal{K}^n$, we have

$$\begin{aligned} \int_{S^{n-1}} g(v) \mathrm{d}F_q(K, v) &= \frac{q}{n\omega_n} \int_{S^{n-1}} \int_{\partial K} X_K(z, u)^{q-1} g(\nu_K(z)) \mathrm{d}\mathcal{H}^{n-1}(z) \mathrm{d}u \\ &= \frac{q}{n\omega_n} \int_{S^{n-1}} \int_{S^{n-1}} X_K(\rho_K(w)w, u)^{q-1} h_K(\alpha_K(w))^{-1} \rho_K(w)^n g(\alpha_K(w)) \mathrm{d}w \mathrm{d}u. \end{aligned} \tag{2.3}$$

Here, $\alpha_K(w) = \nu_K(w\rho_K(w))$ is the radial Gauss map, and we have used the short-hand $\rho_K = \rho_{K,o}$.

For each $p \in \mathbb{R}$ and $K \in \mathcal{K}_o^n$, the L_p chord measure $F_{p,q}(K, \cdot)$ is defined as follows:

$$\mathrm{d}F_{p,q}(K, v) = h_K(v)^{1-p} \mathrm{d}F_q(K, v).$$

It was shown in [38] that the differential of the chord integral I_q with respect to the L_p Minkowski combinations leads to the L_p chord measure: for $p \neq 0$,

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} I_q(K +_p t \cdot L) = \frac{1}{p} \int_{S^{n-1}} h_L^p(v) \mathrm{d}F_{p,q}(K, v),$$

where $K +_p t \cdot L$ is the L_p Minkowski combination between K and L defined via Wulff shape as follows:

$$K +_p t \cdot L = \left[(h_K^p + t h_L^p)^{\frac{1}{p}} \right].$$

It is worth pointing out that there is a similar formula for the $p = 0$ case, which leads to the cone-chord measure studied in [38]. Since the cone-chord measure and its Minkowski problem are not considered in the current work, we omit this formulation.

2.3 Weak continuity of L_p chord measures

We prove in this subsection the weak continuity of the L_p chord measure $F_{p,q}(K, \cdot)$ in K with respect to the Hausdorff metric.

For each $x \in \mathbb{R}^n$ and $u \in S^{n-1}$, we will write $x|u^\perp$ as the image point of the orthogonal projection of x onto $u^\perp$. Similarly, for each subset $E \subset \mathbb{R}^n$, we write

$$E|u^\perp = \{x|u^\perp : x \in E\}.$$

We will need the following lemma obtained in [23].

Lemma 2.1. [23, Lemma 2.2] *Let $K_i \in \mathcal{K}_o^n$ be such that $K_i \to K \in \mathcal{K}_o^n$ in the Hausdorff metric as $i \to \infty$. Then, for $\mathcal{H}^{n-1}$-a.e. $w \in S^{n-1}$,*

$$\alpha_{K_i}(w) \to \alpha_K(w), \quad \text{as } i \to \infty.$$

A generalized dominated convergence theorem will be needed to establish the weak continuity of chord measures: Suppose f_k, ϕ_k, f, and ϕ are integrable functions in a measure space with $f_k \to f$ and $\phi_k \to \phi$, while $|f_k| \le \phi_k$, almost everywhere. If $\int \phi_k \to \int \phi$, then $\int f_k \to \int f$.

We first show that chord measures are weakly continuous.

Theorem 2.2. *Let $q > 0$ and $K_i \in \mathcal{K}^n$. If $K_i \to K \in \mathcal{K}^n$, then the chord measure $F_q(K_i, \cdot)$ converges to $F_q(K, \cdot)$ weakly.*

Proof. Since the chord measure is translation invariant, we can assume without loss of generality that $K_i, K \in \mathcal{K}_o^n$. Then, by (2.3), for any $g \in C(S^{n-1})$,

$$\int_{S^{n-1}} g(v) \mathrm{d}F_q(K_i, v) = \frac{q}{n\omega_n} \int_{S^{n-1}} \int_{S^{n-1}} X_{K_i}\big(\rho_{K_i}(w)w, u\big)^{q-1} h_{K_i}\big(\alpha_{K_i}(w)\big)^{-1} \rho_{K_i}(w)^n g\big(\alpha_{K_i}(w)\big) \mathrm{d}w \mathrm{d}u.$$

Let

$$f_i(w, u) = X_{K_i}\big(\rho_{K_i}(w)w, u\big)^{q-1} h_{K_i}\big(\alpha_{K_i}(w)\big)^{-1} \rho_{K_i}(w)^n g\big(\alpha_{K_i}(w)\big)$$

and

$$f(w, u) = X_K(\rho_K(w)w, u)^{q-1} h_K(\alpha_K(w))^{-1} \rho_K(w)^n g(\alpha_K(w)).$$

On the one hand, since $K_i \to K \in \mathcal{K}_o^n$, we have $\rho_{K_i}(w) \to \rho_K(w)$. This and Lemma 2.1 further show that $h_{K_i}(\alpha_{K_i}(w)) \to h_K(\alpha_K(w))$ a.e. $w \in S^{n-1}$. Moreover, $g \in C(S^{n-1})$ and Lemma 2.1 imply $g(\alpha_{K_i}(w)) \to g(\alpha_K(w))$ a.e. $w \in S^{n-1}$. Note that ∂K is line-free in direction u for almost all $u \in S^{n-1}$. For such a u, the projection point $(\rho_K(w)w)|u^\perp$ belongs to $\mathrm{int}(K|u^\perp)$ (relative interior) for almost all $w \in S^{n-1}$. Then,

$$X_{K_i}\big(\rho_{K_i}(w)w, u\big) \to X_K(\rho_K(w)w, u), \quad \text{a.e. } w \in S^{n-1}.$$

Overall, we have

$$f_i(w, u) \to f(w, u), \quad \text{as } i \to \infty,$$

for $\mathcal{H}^{n-1} \times \mathcal{H}^{n-1}$-a.e. $(w, u) \in S^{n-1} \times S^{n-1}$.

On the other hand, since $g \in C(S^{n-1})$ and $K_i \to K \in \mathcal{K}_o^n$, there exists a constant $c > 0$, such that

$$|g(v)| \leq c h_{K_i}(v)^{-1} \quad \text{for all } i \in \mathbb{N}, v \in S^{n-1}.$$

Let

$$\phi_i(w, u) = c X_{K_i}\big(\rho_{K_i}(w)w, u\big)^{q-1}\rho_{K_i}(w)^n$$

and

$$\phi(w, u) = c X_K(\rho_K(w)w, u)^{q-1}\rho_K(w)^n.$$

Then,

$$|f_i| \leq \phi_i \quad \text{and} \quad |f| \leq \phi.$$

By (2.2), we have

$$\int_{S^{n-1}}\int_{S^{n-1}} \phi_i(w, u)\mathrm{d}w\mathrm{d}u = \frac{cn(n+q-1)\omega_n}{q} I_q(K_i)$$

and

$$\int_{S^{n-1}}\int_{S^{n-1}} \phi(w, u)\mathrm{d}w\mathrm{d}u = \frac{cn(n+q-1)\omega_n}{q} I_q(K).$$

By the fact that $I_q(K_i) \to I_q(K)$, applying the generalized dominated convergence theorem, we obtain

$$\lim_{i\to\infty}\int_{S^{n-1}}\int_{S^{n-1}} f_i(w, u)\mathrm{d}w\mathrm{d}u = \int_{S^{n-1}}\int_{S^{n-1}} f(w, u)\mathrm{d}w\mathrm{d}u,$$

which completes the proof. □

As a corollary, one immediately obtains the following weak continuity property for L_p chord measures.

Corollary 2.3. *Let $q > 0$ and $K_i \in \mathcal{K}^n$. If $K_i \to K \in \mathcal{K}^n$,*
(1) *when $p < 1$, with the additional assumption that $o \in K_i \cap K$, then $F_{p,q}(K_i, \cdot)$ converges to $F_{p,q}(K, \cdot)$ weakly;*
(2) *when $p > 1$, with the additional assumption that $o \in \operatorname{int}(K_i) \cap \operatorname{int}(K)$, then $F_{p,q}(K_i, \cdot)$ converges to $F_{p,q}(K, \cdot)$ weakly.*

Proof. Note that

$$\mathrm{d}F_{p,q}(K_i, \cdot) = h_{K_i}^{1-p}\mathrm{d}F_q(K_i, \cdot).$$

In the case $p < 1$, note that since both h_{K_i} and h_K are nonnegative, we obtain from the uniform convergence of h_{K_i} to h_K the fact that $h_{K_i}^{1-p}$ converges to h_K^{1-p} uniformly. In the case $p > 1$, with the additional assumption that both K_i and K contain the origin in their respective interiors, along with the fact that $K_i \to K$, we conclude that h_{K_i} and h_K are uniformly bounded away from 0. As a consequence, we also have the uniform convergence of $h_{K_i}^{1-p}$ to h_K^{1-p}.

The desired weak convergence now readily follows from Theorem 2.2. □

3 Variation problems for chord measures

3.1 Variation formula

The following variational formula was shown in [38].

Theorem 3.1. [38, Theorem 5.5] *Let $q > 0$, and Ω be a compact subset of S^{n-1} that is not contained in any closed hemisphere. Suppose that $g : \Omega \to \mathbb{R}$ is continuous and $h_t : \Omega \to (0, \infty)$ is a family of continuous functions given as follows:*

$$h_t = h_0 + tg + o(t, \cdot),$$

for each $t \in (-\delta, \delta)$ for some $\delta > 0$. Here, $o(t, \cdot) \in C(\Omega)$ and $o(t, \cdot)/t$ tends to 0 uniformly on Ω as $t \to 0$. Let K_t be the Wulff shape generated by h_t and K be the Wulff shape generated by h_0. Then,

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} I_q(K_t) = \int_{\Omega} g(v)\mathrm{d}F_q(K, v). \tag{3.1}$$

Remark 3.2. Note that the aforementioned theorem is slightly different from Theorem 5.5 in [38]. Indeed, the domain of g in Theorem 5.5 in [38] is S^{n-1} and is changed to Ω here. Despite the change, the proof, however, works for any Ω without any essential changes once we realize the fact that, for $h : \Omega \to (0, \infty)$ and for almost all $x \in \partial[h]$, we have $v_{[h]}(x) \in \Omega$. For completeness, we include a detailed proof in the Appendix.

Note that the special case of $q = 1$ of the variational formula (3.1) is the volume variational formula of Aleksandrov. Taking Ω to be a finite set, we immediately obtain the following corollary for the discrete case.

Corollary 3.3. *Let $p, q > 0$, $z = (z_1, \ldots, z_N) \in \mathbb{R}_+^N$, $\beta = (\beta_1, \ldots, \beta_N) \in \mathbb{R}^N$, and $v_1, \ldots, v_N$ be N unit vectors that are not contained in any closed hemisphere. For sufficiently small $|t|$, consider $z_i(t)^p = z_i^p + t\beta_i > 0$ and*

$$P_t = [z(t)] = \bigcap_{i=1}^{N}\{x \in \mathbb{R}^n : x \cdot v_i \le z_i(t) = (z_i^p + t\beta_i)^{\frac{1}{p}}\}.$$

Then, for $q > 0$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} I_q(P_t) = \frac{1}{p}\sum_{i=1}^{N}\beta_i F_{p,q}(P_0, v_i). \tag{3.2}$$

Here, in proving (3.2), we used the fact that $F_q(P_0, \cdot)$ is supported entirely on $\{v_1, \ldots, v_N\}$.

3.2 Maximization problems

The goal of this subsection is to convert the existence of the solution to the L_p chord Minkowski problem to the existence of the solution to maximization problem.

For each $\Omega \subset S^{n-1}$, we will write $C(\Omega)$ for the set of continuous functions on Ω. The set $C^+(\Omega) \subset C(\Omega)$ will denote the subset consisting only of positive functions. Similarly, the set $C_e^+(\Omega)$ consists only of positive, symmetric continuous functions on Ω. We will write supp μ for the support of a measure μ.

Let $p, q \neq 0$ and $\Omega \subset S^{n-1}$ be a compact subset that is not contained in any closed hemisphere. For each non-zero finite Borel measure μ on S^{n-1}, define the functional $\Phi_{p,q} : C^+(\Omega) \to \mathbb{R}^n$ by

$$\Phi_{p,q}(h) = \frac{1}{n+q-1}\log I_q([h]) - \frac{1}{p}\log\int_{\Omega} h(v)^p \mathrm{d}\mu(v).$$

Theorem 3.4. *Let $p > 1$, $q > 0$, and μ be a nonzero finite Borel measure on S^{n-1} that is not concentrated in any closed hemisphere. Suppose $\Omega \subset S^{n-1}$ is a compact subset such that* supp $\mu \subset \Omega$. *If the maximization problem*

$$\sup\{\Phi_{p,q}(h) : h \in C^+(\Omega)\}$$

has a solution $h_0 \in C^+(\Omega)$, then there exists $K_0 \in \mathcal{K}_o^n$ such that

$$F_{p,q}(K_0, \cdot) = \mu, \quad \text{if } p \neq n + q - 1,$$
$$\frac{F_{p,q}(K_0, \cdot)}{V(K_0)} = \mu, \quad \text{if } p = n + q - 1.$$

Proof. Let $g \in C(\Omega)$. Define $h_t = h_0 + tg$. For sufficiently small $|t|$, the family $h_t \in C^+(\Omega)$. Using the fact that h_0 is a maximizer and Theorem 3.1, we have

$$\begin{aligned} 0 &= \frac{1}{n+q-1}\frac{d}{dt}\bigg|_{t=0}(\log I_q([h_t])) - \frac{1}{p}\frac{d}{dt}\bigg|_{t=0}\left(\log \int_\Omega h_t(v)^p d\mu(v)\right) \\ &= \frac{1}{(n+q-1)I_q([h_0])}\int_\Omega g(v)dF_q([h_0], v) - \int_\Omega g(v)h_0^{p-1}(v)d\mu(v) \Big/ \int_\Omega h_0^p d\mu. \end{aligned}$$

Since $g \in C(\Omega)$ is arbitrary and using the fact that $h_0 = h_{[h_0]}$, $F_q([h_0], \cdot)$-almost everywhere, we have

$$\frac{F_{p,q}([h_0], \cdot)}{I_q([h_0])} = \frac{(n+q-1)}{\int_\Omega h_0^p d\mu}\mu(\cdot) \quad \text{on } \Omega. \tag{3.3}$$

Note that the measure $F_{p,q}(K, \cdot)$ is homogeneous of degree $n + q - p - 1$ in K. Therefore, we may rescale $[h_0]$ and obtain $K_0 \in \mathcal{K}_o^n$ such that

$$F_{p,q}(K_0, \cdot) = \mu \quad \text{on } \Omega$$

if $p \neq n + q - 1$, and

$$\frac{F_{p,q}(K_0, \cdot)}{V(K_0)} = \mu \quad \text{on } \Omega$$

if $p = n + q - 1$. Noting that both μ and $F_{p,q}(K_0, \cdot)$ are concentrated on Ω, we reach the desired conclusion. □

Remark 3.5. In fact, it is clear from the proof that the convex body K_0 obtained in Theorem 3.4 is a rescaling of $[h_0]$, that is $K_0 = c[h_0]$, where

$$c = \left(\frac{\int_\Omega h_0^p d\mu}{(n+q-1)I_q([h_0])}\right)^{\frac{1}{n+q-p-1}},$$

if $p \neq n + q - 1$. If $p = n + q - 1$,

$$c = \left(\frac{I_1([h_0])\int_\Omega h_0^p d\mu}{(n+q-1)I_q([h_0])}\right)^{-\frac{1}{n}}.$$

Taking $\Omega = \{v_1, \dots, v_N\}$, where the $v_i \in S^{n-1}$ are not contained entirely in any closed hemisphere, from the fact that $\Phi_{p,q}$ is homogeneous of degree 0, we immediately obtain the following discrete version of the maximization problem.

Theorem 3.6. *Let $p > 1$, $q > 0$, and*

$$\mu = \sum_{i=1}^N \alpha_i \delta_{\{v_i\}}$$

be a finite discrete measure on S^{n-1}, where $\alpha_i > 0$ and $v_i \in S^{n-1}$. Suppose $v_1, \dots, v_N$ are not contained entirely in any closed hemisphere. If the maximization problem

$$\sup\left\{I_q(P(z)) : \sum_{i=1}^{N}\alpha_i z_i^p \leq 1, z=(z_1,\ldots,z_N)\in\mathbb{R}_+^N\right\} \tag{3.4}$$

has a solution $z^0 \in \mathbb{R}_+^N$, *then there exists a polytope* P_0 *containing the origin in its interior such that*

$$F_{p,q}(P_0,\cdot)=\mu, \quad \text{if } p\neq n+q-1,$$
$$\frac{F_{p,q}(P_0,\cdot)}{V(P_0)}=\mu, \quad \text{if } p=n+q-1.$$

Using virtually the same argument, we may obtain the symmetric version of Theorem 3.4.

Theorem 3.7. *Let* $p, q > 0$ *and* μ *be a nonzero even finite Borel measure on* S^{n-1} *that is not concentrated in any closed hemisphere. If the maximization problem*

$$\sup\{\Phi_{p,q}(h) : h\in C_e^+(S^{n-1})\} \tag{3.5}$$

has a solution $h_0 \in C_e^+(S^{n-1})$, *then there exists* $K_0 \in \mathcal{K}_e^n$ *such that*

$$F_{p,q}(K_0,\cdot)=\mu, \quad \text{if } p\neq n+q-1,$$
$$\frac{F_{p,q}(K_0,\cdot)}{V(K_0)}=\mu, \quad \text{if } p=n+q-1.$$

4 The even L_p chord Minkowski problem when $p, q > 0$

In this section, we solve the even L_p chord Minkowski problem when $p, q > 0$.

We will use B to denote the centered unit ball in $\mathbb{R}^n$ and we will write $(\cdot)_+ : \mathbb{R} \to [0,\infty)$ for the function given as

$$(t)_+ = \begin{cases} t, & \text{if } t\geq 0,\\ 0, & \text{otherwise}, \end{cases}$$

for each $t \in \mathbb{R}$.

Lemma 4.1. *Let* $p > 0$ *and* μ *be a finite Borel measure that is not concentrated in any closed hemisphere. If* $K \in \mathcal{K}_o^n$ *and*

$$\int_{S^{n-1}} h_K(v)^p \mathrm{d}\mu(v) \leq 1,$$

then there exists a positive constant $c_p(\mu)$ *depending only on* μ *and* p, *such that*

$$K \subset c_p(\mu)B,$$

In particular,

$$c_p(\mu)^{-p} = \min_{u\in S^{n-1}} \int_{S^{n-1}} (u\cdot v)_+^p \mathrm{d}\mu(v) > 0. \tag{4.1}$$

Proof. Since μ is not concentrated in any closed hemisphere, the function

$$u \mapsto \int_{S^{n-1}} (u\cdot v)_+^p \mathrm{d}\mu(v)$$

is strictly positive on S^{n-1} and continuous. By the compactness of S^{n-1}, the constant $c_p(\mu)^{-p}$ defined in (4.1) is positive.

Denote $R_0 = \max\{|x| : x \in K\}$. Then, there exists $u_0 \in S^{n-1}$ such that $R_0 u_0 \in K$. Note that $o \in K$. Thus, we obtain

$$R_0^p \cdot c_p(\mu)^{-p} \leq \int_{S^{n-1}} (R_0 u_0 \cdot v)_+^p \mathrm{d}\mu(v) \leq \int_{S^{n-1}} h_K(v)^p \mathrm{d}\mu(v) \leq 1,$$

and hence $R_0 \leq c_p(\mu)$, which implies what we wanted by the choice of R_0. □

The following theorem is Theorem 1.1.

Theorem 4.2. *Let $p, q > 0$. If μ is an even finite Borel measure on S^{n-1} that is not concentrated on a great subsphere, then there exists a symmetric, convex body $K \in \mathcal{K}_e^n$ such that*

$$F_{p,q}(K, \cdot) = \mu, \quad \text{when } p \neq n + q - 1, \tag{4.2}$$

$$\frac{F_{p,q}(K, \cdot)}{V(K)} = \mu, \quad \text{when } p = n + q - 1. \tag{4.3}$$

Proof. Let h_i be a sequence of functions in $C_e^+(S^{n-1})$ such that

$$\Phi_{p,q}(h_i) \to \sup\{\Phi_{p,q}(h) : h \in C_e^+(S^{n-1})\}.$$

Denote $K_i = [h_i]$. Since $\Phi_{p,q}(\cdot)$ is 0-homogeneous, we may assume that

$$\int_{S^{n-1}} h_i^p \mathrm{d}\mu = 1.$$

It follows from Lemma 4.1 and the fact $h_{K_i} \leq h_i$ that K_i are uniformly bounded. By Blaschke's selection theorem, there exists a subsequence, which will still be denoted as K_i (since there is no confusion), such that

$$K_i \to K_0,$$

where K_0 is origin-symmetric, convex, and compact.

If int K_0 is empty, then $I_q(K_i) \to 0$, and hence $\Phi_{p,q}(h_i) \to -\infty$, which contradicts the fact that h_i is a maximizing sequence. Thus, $K_0 \in \mathcal{K}_e^n$, and as a result, $h_0 = h_{K_0} \in C_e^+(S^{n-1})$. Moreover,

$$\begin{aligned}
\Phi_{p,q}(h_0) &= \frac{1}{n+q-1} \log I_q([h_0]) - \frac{1}{p} \log \int_{S^{n-1}} h_0(v)^p \mathrm{d}\mu(v) \\
&= \lim_{i\to\infty} \left(\frac{1}{n+q-1} \log I_q([h_i]) - \frac{1}{p} \log \int_{S^{n-1}} h_{K_i}(v)^p \mathrm{d}\mu(v) \right) \\
&\geq \lim_{i\to\infty} \left(\frac{1}{n+q-1} \log I_q([h_i]) - \frac{1}{p} \log \int_{S^{n-1}} h_i(v)^p \mathrm{d}\mu(v) \right) \\
&= \lim_{i\to\infty} \Phi_{p,q}(h_i).
\end{aligned}$$

Therefore, h_0 is a maximizer for the maximization problem (3.5). By Theorem 3.7, we obtain (4.2) and (4.3). □

5 The discrete L_p chord Minkowski problem when $p > 1$ and $q > 0$

In this section, we solve the discrete L_p chord Minkowski problem when $p > 1$ and $q > 0$ without the assumption that the given measure μ is even.

The following lemma shows the non-degeneracy of the maximizer to the maximization problem (3.4).

Lemma 5.1. *Let $p > 1$, $q > 0$, and*

$$\mu = \sum_{i=1}^{N} \alpha_i \delta_{\{v_i\}}$$

be a finite discrete measure on S^{n-1}, where $\alpha_i > 0$ and $v_i \in S^{n-1}$. Suppose $v_1, \dots, v_N$ are not contained entirely in any closed hemisphere. If $z^0 = (z_1^0, \dots z_N^0) \in \mathbb{R}^N$ is such that $z_i^0 \geq 0$ and satisfies

$$\sum_{i=1}^{N} \alpha_i (z_i^0)^p \leq 1$$

and

$$I_q(P(z^0)) = \sup\left\{ I_q(P(z)) : \sum_{i=1}^{N} \alpha_i z_i^p \leq 1, z = (z_1, \dots, z_N) \in \mathbb{R}_+^N \right\}, \tag{5.1}$$

then $z^0 \in \mathbb{R}_+^N$.

Proof. We argue by contradiction and assume that at least one $z_i^0 = 0$. We write $P_0 = P(z^0)$. Therefore, $o \in \partial P_0$. For simplicity, we write $h_i = h_{P_0}(v_i) \geq 0$. Let

$$J = \{i : h_i = 0\}.$$

Since $o \in \partial P_0$, we have that J is non-empty. It is also simple to see that $J \neq \{1, \dots, N\}$. Indeed, if that is not the case, then $P_0 = \{o\}$, which implies that $I_q(P(z^0)) = 0$ and this is a contradiction to (5.1). (Clearly, there is some $z \in \mathbb{R}_+^N$ in the domain of the maximization problem (5.1) and $I_q(P(z)) > 0$.) By the same argument, P_0 must have a nonempty interior.

For each $t > 0$, let

$$z_i(t) = \begin{cases} t^{\frac{1}{p}}, & \text{if } i \in J, \\ (h_i^p - at)^{\frac{1}{p}}, & \text{if } i \notin J, \end{cases}$$

where

$$a = \frac{\sum_{i \in J} \alpha_i}{\sum_{i \notin J} \alpha_j} > 0.$$

Clearly, for sufficiently small $t > 0$, we have $z(t) = (z_1(t), \dots, z_N(t)) \in \mathbb{R}_+^N$. Note that

$$\sum_{i=1}^{N} \alpha_i z_i(t)^p = \sum_{i \in J} \alpha_i t + \sum_{i \notin J} \alpha_i (h_i^p - at) = \sum_{i \notin J} \alpha_i h_i^p = \sum_{i=1}^{N} \alpha_i h_i^p \leq \sum_{i=1}^{N} \alpha_i (z_i^0)^p \leq 1.$$

For simplicity, write $P_t = P(z(t))$. Note that according to Corollary 3.3, the functional $I_q(P_t)$ is differentiable in t for sufficiently small $t > 0$. Note also that $I_q(P_t)$ is continuous at $t = 0$. Therefore, by the mean value theorem and the fact that $p > 1$,

$$\begin{aligned} p \cdot \left(\frac{I_q(P_t) - I_q(P_0)}{t} \right) &= p \cdot \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=\theta} I_q(P_t) \\ &= \sum_{i \in J} h_{P_\theta}(v_i)^{1-p} F_q(P_\theta, v_i) - \sum_{i \notin J} a h_{P_\theta}(v_i)^{1-p} F_q(P_\theta, v_i) \\ &\geq \sum_{i \in J} t^{\frac{1-p}{p}} F_q(P_\theta, v_i) - \sum_{i \notin J} a h_{P_\theta}(v_i)^{1-p} F_q(P_\theta, v_i) \end{aligned}$$

for some $\theta \in (0, t)$ that depends on t.

Since $o \in \partial P_0$, there must exist $i_0 \in J$ such that P_0 has a facet (with positive $\mathcal{H}^{n-1}$-area) with v_{i_0} as its normal. Therefore, by the definition of F_q, we have $F_q(P_0, \{v_{i_0}\}) > 0$. By Theorem 2.2, when $t > 0$ is sufficiently small, we have

$$F_q\big(P_\theta, v_{i_0}\big) \geq \frac{1}{2}F_q\big(P_0, v_{i_0}\big) > 0.$$

By the fact that $P_t \to P_0$ and the choice of J, we know that for $t > 0$ sufficiently small, there exists $c_0 > 0$ such that $h_{P_\theta}(v_i) \geq c_0$ for each $i \notin J$. Combining these facts together with (2.2), we have

$$\begin{aligned} p \cdot \left(\frac{I_q(P_t) - I_q(P_0)}{t}\right) &\geq t^{\frac{1-p}{p}}\frac{1}{2}F_q\big(P_0, v_{i_0}\big) - ac_0^{-p}\sum_{i\notin J} h_{P_\theta}(v_i)F_q(P_\theta, v_i) \\ &\geq t^{\frac{1-p}{p}}\frac{1}{2}F_q\big(P_0, v_{i_0}\big) - ac_0^{-p}\sum_{i=1}^{N} h_{P_\theta}(v_i)F_q(P_\theta, v_i) \\ &= t^{\frac{1-p}{p}}\frac{1}{2}F_q\big(P_0, v_{i_0}\big) - ac_0^{-p}(n+q-1)I_q(P_\theta) \\ &> 0, \end{aligned}$$

when $t > 0$ is sufficiently small. This implies the existence of $t_0 > 0$ such that $I_q(P_{t_0}) > I_q(P_0)$, or, equivalently, $I_q(z(t_0)) > I_q(P(z^0))$. This is in contradiction to (5.1). □

With the above lemma, we obtain the following solution to the discrete L_p chord Minkowski problem when $p > 1$, $q > 0$.

Theorem 5.2. *Let $p > 1$, $q > 0$, and μ be a discrete measure on S^{n-1} that is not concentrated in any closed hemisphere. Then, there is a polytope $P \in \mathcal{K}_o^n$ such that*

$$F_{p,q}(P, \cdot) = \mu, \quad \text{when } p \neq n+q-1,$$

$$\frac{F_{p,q}(P, \cdot)}{V(P)} = \mu, \quad \text{when } p = n+q-1.$$

Proof. Suppose

$$\mu = \sum_{i=1}^{N}\alpha_i\delta_{\{v_i\}},$$

where $v_1, \ldots, v_N$ are N unit vectors not contained in any closed hemisphere, and $\alpha_1, \ldots, \alpha_N > 0$.

Let $z(k) = (z_1(k), \ldots, z_N(k)) \in \mathbb{R}_+^N$ be a maximizing sequence to (3.4), that is,

$$\sum_{i=1}^{N}\alpha_i z_i(k)^p \leq 1 \tag{5.2}$$

and

$$I_q(P(z(k))) \to \sup\left\{I_q(P(z)) : \sum_{i=1}^{N}\alpha_i z_i^p \leq 1, z = (z_1, \ldots, z_n) \in \mathbb{R}_+^N\right\}.$$

By (5.2) and the fact that $\alpha_i > 0$, it is clear that $z_i(k)$ are uniformly bounded in k and i. Therefore, we may pick a subsequence, which we still denote by $z(k)$ such that $z(k) \to z^0 \in \mathbb{R}^N$. Since $z_i(k) > 0$, we have $z_i^0 \geq 0$. Moreover, we have

$$\sum_{i=1}^{N}\alpha_i(z_i^0)^p = \lim_{k\to\infty}\sum_{i=1}^{N}\alpha_i z_i(k)^p \leq 1$$

and

$$I_q(P(z^0)) = \lim_{k\to\infty} I_q(P(z(k))) = \sup\left\{I_q(P(z)) : \sum_{i=1}^{N}\alpha_i z_i^p \le 1, z = (z_1, \ldots, z_n) \in \mathbb{R}_+^N\right\}.$$

Therefore, by Lemma 5.1, we conclude that $z^0 \in \mathbb{R}_+^N$. In fact, since z^0 maximizes (3.4), by the monotonicity of I_q, it must be the case that

$$\sum_{i=1}^{N}\alpha_i(z_i^0)^p = 1.$$

Therefore, z^0 is a maximizer to the maximization problem (3.4). Now, we may use Theorem 3.6 to find the desired polytope $P \in \mathcal{K}_o^n$. □

A normalized version of Theorem 5.2 together with a boundness estimate will be useful in the next section.

Corollary 5.3. *Let p, q, and μ be the same as in* Theorem 5.2. *Then, there is a polytope P_0 such that*

$$\frac{F_{p,q}(P_0, \cdot)}{I_q(P_0)} = \mu(\cdot), \tag{5.3}$$

and P_0 satisfies

$$P_0 \subset (n + q - 1)^{\frac{1}{p}} c_p(\mu) B, \tag{5.4}$$

where $c_p(\mu)$ is given in Lemma 4.1.

Proof. Following the proof of Theorem 5.2, we may find a maximizer z to the maximization problem (3.4), and z satisfies

$$\sum_{i=1}^{N}\alpha_i z_i^p = 1.$$

From (3.3) in the proof of Theorem 3.4, we have that $P_0 = (n + q - 1)^{\frac{1}{p}} \cdot [z]$ satisfies (5.3). The desired bound (5.4) for P_0 follows immediately from Lemma 4.1. □

6 The L_p chord Minkowski problem for general measures when $p > 1$ and $q > 0$

This section is focused on solving the existence of a solution to the L_p chord Minkowski problem when $p > 1$, $q > 0$, and the given measure μ is not necessarily discrete or even.

Lemma 6.1. *Let $p \ge 1$ and $(\mu_i)_{i\in\mathbb{N}}$ be a sequence of nonzero finite Borel measures that converges weakly to a nonzero finite Borel measure μ. Suppose μ is not concentrated entirely in any closed hemisphere. Then, for sufficiently large i, we have*

$$c_p(\mu_i) \le 2c_p(\mu).$$

Here, $c_p(\mu)$ is defined by (4.1) *in* Lemma 4.1.

Proof. Define

$$f_\mu(u) = \int_{S^{n-1}} (u \cdot v)_+^p \mathrm{d}\mu(v).$$

Since $\mu_i \rightharpoonup \mu$ weakly,

$$f_{\mu_i}(u) \to f_\mu(u), \quad \forall u \in S^{n-1}.$$

Since $f_{\mu_i}^{1/p}$ and $f_\mu^{1/p}$ are support functions, point-wise convergence implies uniform convergence. Note that f_μ is always positive following the fact that μ is not concentrated in any closed hemisphere. Therefore, when i is sufficiently large,

$$c_p(\mu_i) = (\min_{u\in S^{n-1}} f_{\mu_i}(u))^{-1/p} \leq 2(\min_{u\in S^{n-1}} f_\mu(u))^{-1/p} = 2c_p(\mu). \qquad \square$$

Theorem 6.2. *Let $p > 1$ and $q > 0$. If μ is a finite Borel measure on S^{n-1} that is not concentrated in any closed hemisphere, then there exists a convex body K with nonnegative support function $h_K \geq 0$ so that*

$$\mathrm{d}F_q(K,\cdot) = h_K^{p-1}\mathrm{d}\mu, \quad \text{when } p \neq n+q-1,$$
$$\frac{\mathrm{d}F_q(K,\cdot)}{V(K)} = h_K^{p-1}\mathrm{d}\mu, \quad \text{when } p = n+q-1.$$

Proof. Choose a sequence of discrete measure $(\mu_i)_{i\in\mathbb{N}}$ such that μ_i converges to μ weakly. Since μ is not concentrated in any closed hemisphere, we may choose μ_i so that μ_i is not concentrated in any closed hemisphere either.

By Corollary 5.3, for each i, there is a polytope $P_i \in \mathcal{K}_o^n$ such that

$$I_q(P_i)h_{P_i}(v)^{p-1}\mathrm{d}\mu_i(v) = \mathrm{d}F_q(P_i, v)$$

and

$$P_i \subset (n+q-1)^{\frac{1}{p}}c_p(\mu_i)B.$$

By Lemma 6.1, $(P_i)_{i\in\mathbb{N}}$ is uniformly bounded. Thus, there is a subsequence of (P_i) converging to a compact, convex set K_0 that contains the origin (not necessarily as an interior point). We claim that K_0 has nonempty interior. If this is not the case, then $I(P_i) \to 0$. This is in contradiction to P_i being (rescaled versions of) the maximizer to (5.1) (with μ replaced by μ_i) and the fact that μ_i converges to μ weakly.

By Theorem 2.2, the uniform convergence of support functions, and the continuity of the chord integral, we have

$$I_q(K_0)h_{K_0}(v)^{p-1}\mathrm{d}\mu(v) = \mathrm{d}F_q(K_0, v).$$

By the homogeneities of $F_q(K,\cdot)$, $I_q(K)$, $V(K)$, and h_K in K, we may rescale K_0 and obtain $K \in \mathcal{K}^n$ with nonnegative support function such that

$$\mathrm{d}F_q(K,\cdot) = h_K^{p-1}\mathrm{d}\mu$$

if $p \neq n+q-1$, and

$$\frac{\mathrm{d}F_q(K,\cdot)}{V(K)} = h_K^{p-1}\mathrm{d}\mu$$

if $p = n+q-1$. $\square$

When $p \geq n$, we may further show that the solution obtained in Theorem 6.2 contains the origin as an interior point. We require the following lemma.

Lemma 6.3. *Let $q > -1$. If $K_i \in \mathcal{K}^n$ converges to $K \in \mathcal{K}^n$ as $i \to \infty$, then there exists a positive constant $c(n,q,K)$ such that for $\mathcal{H}^{n-1}$-almost all $z \in \partial K_i$ and every i,*

$$\widetilde{V}_q(K_i, z) \geq c(n,q,K) > 0.$$

Proof. Recall that by convexity, for $\mathcal{H}^{n-1}$-almost all $z \in \partial K_i$, there is a unique tangent plane to K_i at z. As a consequence, the set

$$\mathcal{H}^{n-1}(\{u \in S^{n-1} : \rho_{K_i,z}(u) > 0\}) = \frac{\omega_n}{2},$$

for $\mathcal{H}^{n-1}$-almost all $z \in \partial K_i$.

If $q = 0$, based on the earlier observation, there is nothing to prove as $\widetilde{V}_q(K_i, z)$ is a constant $\mathcal{H}^{n-1}$-almost everywhere.

If $-1 < q < 0$, then for $\mathcal{H}^{n-1}$-almost all $z \in \partial K_i$,

$$\widetilde{V}_q(K_i, z) = \frac{1}{n} \int_{S^{n-1}} \rho_{K_i,z}(u)^q \mathrm{d}u \geq \frac{\omega_n}{2n} D(K_i)^q > 0,$$

where $D(K_i)$ denotes the diameter of K_i. Since $K_i \to K$, one has $D(K_i) \to D(K)$, and one can easily obtain the positive constant $c(n, q, K)$ in this case.

Let us now deal with the $q > 0$ case. Recall that in this case $\widetilde{V}_q(K, z)$ is continuous in z. Moreover, if L is a convex body contained in K,

$$\widetilde{V}_q(L, z) \leq \widetilde{V}_q(K, z).$$

Since $K_i \to K \in \mathcal{K}^n$, there exist two balls B_1 and B_2 (not necessarily centered at the origin) such that

$$B_1 \subset K_i \subset B_2, \quad \forall i \in \mathbb{N}.$$

Note that

$$\widetilde{V}_q(K_i, z) \geq \widetilde{V}_q(B_1, z), \quad \forall z \in \partial K_i.$$

Since $\partial K_i \subset B_2$, we have

$$\widetilde{V}_q(K_i, z) \geq \min_{z \in B_2} \widetilde{V}_q(B_1, z) > 0.$$

□

The following lemma is extracted from Section 4 of [28].

Lemma 6.4. [28] *Let $p \geq n$. Suppose $P_i \in \mathcal{K}_o^n$ are polytopes, and $P_i \to K \in \mathcal{K}^n$ as $i \to \infty$. If there exists a constant $c > 0$ independent of i such that*

$$\int_{S^{n-1}} h_{P_i}^{1-p}(v) \mathrm{d}S_{P_i}(v) < c,$$

then K contains the origin in its interior.

Theorem 6.5. *If we further assume $p \geq n$, the convex body K obtained in* Theorem 6.2 *must be in $\mathcal{K}_o^n$. In particular, we have*

$$F_{p,q}(K, \cdot) = \mu, \quad \text{when } p \neq n + q - 1, \tag{6.1}$$

$$\frac{F_{p,q}(K, \cdot)}{V(K)} = \mu, \quad \text{when } p = n + q - 1. \tag{6.2}$$

Proof. Let P_i be the convergent subsequence with limit K_0 that was obtained in the proof of Theorem 6.2.

By the definition of L_p chord measure,

$$F_{p,q}(P_i, S^{n-1}) = \frac{2q}{\omega_n} \int_{\partial P_i} \widetilde{V}_{q-1}(P_i, z)(z \cdot \nu_K(z))^{1-p} \mathrm{d}\mathcal{H}^{n-1}(z).$$

By Lemma 6.3, there exists a positive uniform lower bound $c(n, q-1, K_0)$ of $\widetilde{V}_{q-1}(P_i, z)$. Thus,

$$\int_{S^{n-1}} h_{P_i}^{1-p}(v)dS_{P_i}(v) \le \frac{\omega_n}{2qc(n, q-1, K_0)} F_{p,q}(P_i, S^{n-1}).$$

Since

$$\frac{F_{p,q}(P_i, \cdot)}{I_q(P_i)} = \mu_i \rightharpoonup \mu \quad \text{weakly},$$

and $I_q(P_i) \to I_q(K_0) > 0$, we infer that

$$\int_{S^{n-1}} h_{P_i}^{1-p}(v)\mathrm{d}S_{P_i}(v)$$

has a uniform upper bound. It follows from Lemma 6.4 that $K_0 \in \mathcal{K}_o^n$. Now (6.1) and (6.2) follow from this and Theorem 6.2. □

Theorem 1.2 follows from Theorems 5.2, 6.2, and 6.5.

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Appendix

The aim of this appendix is to give detailed proof of Theorem 3.1, which is a general version of Theorem 5.5 in [38]. Its proof is actually based on a careful examination of the approach in [38].

Let $\Omega \subset S^{n-1}$ be a compact set that is not contained in any closed hemisphere, $g \in C(\Omega)$, and $\delta > 0$. Let $h_t : \Omega \to (0, \infty)$ be a family of continuous functions defined by

$$h_t(v) = h_0(v) + tg(v) + o(t, v), \qquad t \in (-\delta, \delta) \text{ and } v \in \Omega,$$

where $o(t, \cdot) \in C(\Omega)$, and $o(t, \cdot)/t \to 0$ uniformly on Ω, as $t \to 0$. Recall that the Wulff-shape $[h_t]$ of h_t is

$$[h_t] = \{x \in \mathbb{R}^n : x \cdot v \leq h_t(v) \quad \text{for all } v \in \Omega\}. \tag{A1}$$

We require that $[h_0]$ has a nonempty interior throughout this section and as a consequence, the set $[h_t]$ also has nonempty interior for sufficiently small $|t|$.

The following differential formula was established in [23], for almost all $u \in S^{n-1}$,

$$\left.\frac{\mathrm{d}\rho_{[h_t]}(u)}{\mathrm{d}t}\right|_{t=0} = \frac{g(v_{[h_0]}(y))}{u \cdot v_{[h_0]}(y)}, \tag{A2}$$

where $y = \rho_{[h_0]}(u)u$. We remark that even though g is only defined on Ω, the right side of (A2) makes sense for almost all $u \in S^{n-1}$. This is because the normal vector $v_{[h_0]}(x) \in \Omega$ for $\mathcal{H}^{n-1}$-almost all $x \in \partial[h_0]$.

The following is the differential formula for the extended radial function, which is a slight extension of (A2). For a point $z \in \mathbb{R}^n$, we denote $h_{t,z}$ to be the translation of h_t,

$$h_{t,z}(v) = h_t(v) - z \cdot v.$$

Lemma A.1. *Let $K_t = [h_t]$ be the Wulff shape defined by* (A1) *and $K = K_0$. If z is an interior point of K, then for almost all $u \in S^{n-1}$,*

$$\left.\frac{\mathrm{d}\rho_{K_t,z}(u)}{\mathrm{d}t}\right|_{t=0} = \frac{g(v_K(z + \rho_{K,z}(u)u))}{u \cdot v_K(z + \rho_{K,z}(u)u)}. \tag{A3}$$

Proof. Since

$$\rho_{K_t,z}(u) = \rho_{K_t - z}(u),$$

we obtain

$$\begin{aligned} K_t - z &= \{x - z : x \in \mathbb{R}^n \quad \text{and} \quad x \cdot v \leq h_t(v), \quad \text{for all } v \in \Omega\} \\ &= \{y \in \mathbb{R}^n : y \cdot v \leq h_0(v) - z \cdot v + tg(v) + o(t, v), \quad \text{for all } v \in \Omega\} \\ &= [h_{t,z}]. \end{aligned}$$

Thus,

$$\left.\frac{\mathrm{d}\rho_{K_t,z}(u)}{\mathrm{d}t}\right|_{t=0} = \left.\frac{\mathrm{d}\rho_{h_{t,z}}(u)}{\mathrm{d}t}\right|_{t=0}.$$

Since z is an interior point of $[h_0]$, the body $[h_{0,z}] = K - z$ contains the origin in its interior. By (A2),

$$\left.\frac{\mathrm{d}\rho_{[h_{t,z}]_t}(u)}{\mathrm{d}t}\right|_{t=0} = \frac{g(v_K(y))}{u \cdot v_K(y)},$$

where $y = z + \rho_{K,z}(u)u \in \partial K$. The desired formula (A3) follows. □

By using (A3), we now derive the differential formula for the X-ray function.

Lemma A.2. *Let $K_t = [h_t]$ be the Wulff shape defined by* (A1) *and $K = K_0$. If $u \in S^{n-1}$, then for almost all x in the interior of $K|u^\perp$,*

$$\frac{\mathrm{d}X_{K_t}(x,u)}{\mathrm{d}t}\bigg|_{t=0} = \frac{g(\nu_K(y))}{u\cdot\nu_K(y)} - \frac{g(\nu_K(y^-))}{u\cdot\nu_K(y^-)}, \tag{A4}$$

where y and y^- are the upper and lower points of $\partial K \cap (x + \mathbb{R}u)$.

Proof. Since x is an interior point of $K|u^\perp$, we can pick an interior point z in K so that

$$K \cap (x + \mathbb{R}u) = K \cap (z + \mathbb{R}u).$$

By (2.1) and (A3), we have

$$\frac{\mathrm{d}X_{K_t}(x,u)}{\mathrm{d}t}\bigg|_{t=0} = \frac{\mathrm{d}\rho_{K_t,z}(u)}{\mathrm{d}t}\bigg|_{t=0} + \frac{\mathrm{d}\rho_{K_t,z}(-u)}{\mathrm{d}t}\bigg|_{t=0} = \frac{g(\nu_K(y))}{u\cdot\nu_K(y)} - \frac{g(\nu_K(y^-))}{u\cdot\nu_K(y^-)}.$$ □

The following two lemmas from [38] are required.

Lemma A.3. [38, Lemma 5.3] *Suppose $K \in \mathcal{K}^n$, $q > -1$, and g is a continuous function on S^{n-1}. Then,*

$$2n\int_{\partial K} \widetilde{V}_q(K,z)g(\nu_K(z))\mathrm{d}\mathcal{H}^{n-1}(z) = \int_{S^{n-1}}\int_{\partial K} X_K(z,u)^q g(\nu_K(z))\mathrm{d}\mathcal{H}^{n-1}(z)\mathrm{d}u.$$

Lemma A.4. [38, Lemma 4.8] *If $K \in \mathcal{K}^n_o$, then for all $q > 0$,*

$$\begin{aligned} I_q(K) &= \frac{2q}{(n+q-1)n\omega_n}\int_{\partial K}\int_{S_z^+}(z\cdot\nu_K(z))\rho_{K,z}(u)^{q-1}\mathrm{d}u\mathrm{d}\mathcal{H}^{n-1}(z) \\ &= \frac{2q}{(n+q-1)\omega_n}\int_{\partial K}(z\cdot\nu_K(z))\widetilde{V}_{q-1}(K,z)\mathrm{d}\mathcal{H}^{n-1}(z). \end{aligned}$$

A generalized dominated convergence theorem will be needed to establish the lemma to follow: Suppose f_k, ϕ_k, f, and ϕ are integrable functions in a measure space with $f_k \to f$ and $\phi_k \to \phi$, while $|f_k| \le \phi_k$, almost everywhere. If $\int\phi_k \to \int\phi$, then $\int f_k \to \int f$. The following lemma is the crucial technical lemma needed in order to establish the differential formula for chord integrals.

Lemma A.5. *Suppose $q > 0$. Let $K_t = [h_t]$ be the Wulff shape defined by* (A1) *and $K = K_0$. Then, there is a class of nonnegative integrable functions $\phi_t(x,u)$ defined for $u \in S^{n-1}$ and $x \in u^\perp$ such that*

$$\left|\frac{1}{t}(X_{K_t}(x,u)^q - X_K(x,u)^q)\right| \le \phi_t(x,u). \tag{A5}$$

Moreover, the limit function $\lim_{t\to 0}\phi_t(x,u)$ *is integrable and*

$$\lim_{t\to 0}\int_{S^{n-1}}\int_{u^\perp}\phi_t(x,u)\mathrm{d}x\mathrm{d}u = \int_{S^{n-1}}\int_{u^\perp}\lim_{t\to 0}\phi_t(x,u)\mathrm{d}x\mathrm{d}u. \tag{A6}$$

Proof. Since h_0 is positive and continuous, the origin is inside the interior of $K = [h_0]$. Since g in (A1) is continuous, and since $o(t,\cdot)/t \to 0$ uniformly on S^{n-1}, there exist constants $c, \delta' > 0$ so that

$$\left|g(v) + \frac{o(t,v)}{t}\right| \le ch_0(v), \quad \text{for all} \quad v \in \Omega,\ t \in (-\delta',\delta').$$

Then,

$$(1 - c|t|)K \subset K_t \subset (1 + c|t|)K.$$

Thus,

$$\left| \frac{1}{t}(X_{K_t}(x, u)^q - X_K(x, u)^q) \right| \le \phi_t(x, u), \tag{A7}$$

where

$$\phi_t(x, u) = \frac{1}{|t|}(X_{(1+c|t|)K}(x, u)^q - X_{(1-c|t|)K}(x, u)^q).$$

Thus, (A5) holds. The homogeneity of $I_q(K, u)$, gives us

$$\begin{aligned} \int_{u^\perp} \phi_t(x, u)\mathrm{d}x &= \frac{1}{|t|}(I_q((1 + c|t|)K, u) - I_q((1 - c|t|)K, u)) \\ &= \frac{1}{|t|}((1 + c|t|)^{n+q-1} - (1 - c|t|)^{n+q-1})I_q(K, u). \end{aligned}$$

Therefore,

$$\int_{S^{n-1}} \int_{u^\perp} \phi_t(x, u)\mathrm{d}x\mathrm{d}u = \frac{1}{|t|}((1 + c|t|)^{n+q-1} - (1 - c|t|)^{n+q-1})n\omega_n I_q(K),$$

and thus,

$$\lim_{t\to 0} \int_{S^{n-1}} \int_{u^\perp} \phi_t(x, u)\mathrm{d}x\mathrm{d}u = 2c(n + q - 1)n\omega_n I_q(K).$$

On the other hand, by (A4), when x is an interior point of $K|u^\perp$, we have

$$\lim_{t\to 0} \phi_t(x, u) = 2qcX_K(x, u)^{q-1}\left(\frac{h_K(\nu_K(y))}{u \cdot \nu_K(y)} - \frac{h_K(\nu_K(y^-))}{u \cdot \nu_K(y^-)} \right),$$

where y and y^- are the two boundary points of $\partial K \cap (x + \mathbb{R}u)$. Since, for almost all $u \in S^{n-1}$, $\partial K \cap (x + \mathbb{R}u)$ consists of at most two points, we obtain

$$\begin{aligned} \int_{S^{n-1}} \int_{u^\perp} \lim_{t\to 0} \phi_t(x, u)\mathrm{d}x\mathrm{d}u &= 2qc \int_{S^{n-1}} \int_{K|u^\perp} X_K(x, u)^{q-1}\left(\frac{h_K(\nu_K(y))}{u \cdot \nu_K(y)} - \frac{h_K(\nu_K(y^-))}{u \cdot \nu_K(y^-)} \right)\mathrm{d}x\mathrm{d}u \\ &= 2qc \int_{S^{n-1}} \int_{\partial K} X_K(y, u)^{q-1} h_K(\nu_K(y))\mathrm{d}\mathcal{H}^{n-1}(y)\mathrm{d}u. \end{aligned}$$

By Lemmas A.4 and A.3, we obtain

$$I_q(K) = \frac{q}{(n + q - 1)n\omega_n} \int_{S^{n-1}} \int_{\partial K} X_K(y, u)^{q-1} h_K(\nu_K(y))\mathrm{d}\mathcal{H}^{n-1}(y)\mathrm{d}u.$$

Therefore, we obtain

$$\int_{S^{n-1}} \int_{u^\perp} \lim_{t\to 0} \phi_t(x, u)\mathrm{d}x\mathrm{d}u = 2c(n + q - 1)n\omega_n I_q(K).$$

Thus, both sides of equation (A6) are equal to $2c(n + q - 1)n\omega_n I_q(K)$. □

We are now ready to prove Theorem 3.1.

Proof of Theorem 3.1. By Lemma A.5, there is a family of nonnegative integrable functions $\phi_t(x, u)$ satisfying (A5) and (A6). Then, by the generalized dominated convergence theorem and Lemmas A.5, A.2, and A.3, we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} I_q(K_t) &= \frac{1}{n\omega_n}\lim_{t\to 0}\int_{S^{n-1}}\int_{u^\perp}\frac{1}{t}(X_{K_t}(x,u)^q - X_K(x,u)^q)\mathrm{d}x\mathrm{d}u \\
&= \frac{1}{n\omega_n}\int_{S^{n-1}}\int_{u^\perp}\lim_{t\to 0}\frac{1}{t}(X_{K_t}(x,u)^q - X_K(x,u)^q)\,\mathrm{d}x\mathrm{d}u \\
&= \frac{q}{n\omega_n}\int_{S^{n-1}}\int_{K|u^\perp} X_K(x,u)^{q-1}\left(\frac{g(\nu_K(y))}{u\cdot\nu_K(y)} - \frac{g(\nu_K(y^-))}{u\cdot\nu_K(y^-)}\right)\mathrm{d}x\mathrm{d}u \\
&= \frac{q}{n\omega_n}\int_{S^{n-1}}\int_{\partial K} X_K(y,u)^{q-1}g(\nu_K(y))\mathrm{d}\mathcal{H}^{n-1}(y)\mathrm{d}u \\
&= \frac{2q}{\omega_n}\int_{\partial K}\widetilde{V}_{q-1}(K,z)g(\nu_K(z))\mathrm{d}\mathcal{H}^{n-1}(z) \\
&= \int_{S^{n-1}} g(v)\mathrm{d}F_q(K,v) \\
&= \int_{\Omega} g(v)\mathrm{d}F_q(K,v),
\end{aligned}$$

where y and y^- are the two boundary points of $\partial K \cap (x + \mathbb{R}u)$. Here, in the last equality, we used the fact that $F_q([h_0], \cdot)$ is concentrated on Ω. □