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Zeta statistics and Hadamard functions

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ABSTRACT

We introduce the Hadamard topology on the Witt ring of rational functions, giving a simultaneous refinement of the weight and point-counting topologies. Zeta functions of algebraic varieties over finite fields are elements of the rational Witt ring, and the Hadamard topology allows for a conjectural unification of results in arithmetic and motivic statistics: The completion of the Witt ring for the Hadamard topology can be identified with a space of meromorphic functions which we call Hadamard functions, and we make the meta-conjecture that any "natural" sequence of zeta functions which converges to a Hadamard function in both the weight and point-counting topologies converges also in the Hadamard topology. For statistics arising from Bertini problems, zero-cycles or the Batyrev-Manin conjecture, this yields an explicit conjectural unification of existing results in motivic and arithmetic statistics that were previously connected only by analogy. As evidence for our conjectures, we show that Hadamard convergence holds for many natural statistics arising from zero-cycles, as well as for the motivic height zeta function associated to the motivic Batyrev-Manin problem for split toric varieties.

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Contents

1. Introduction

1.1. Zeta functions

1.1.1. Zeta functions of varieties over finite fields

The zeta function of a variety $X/\mathbb{F}_q$ is the formal power series $Z_X(t) \in 1 + t\mathbb{Z}[[t]]$ defined by

$$Z_X(t) := \prod_{x \in X \text{ closed}} \frac{1}{1 - t^{\deg x}} = \sum_{j=0}^{\infty} |\mathrm{Sym}^j X(\mathbb{F}_q)| t^j.$$

It encodes the number of points of X over every finite extension of $\mathbb{F}_q$:

$$d \log Z_X(t) = \sum_{j=1}^{\infty} |X(\mathbb{F}_{q^j})| t^{j-1}. \tag{1.1.1.1}$$

The Grothendieck-Lefschetz fixed point formula implies that $Z_X(t)$ is the power-series expansion at 0 of a rational function, and that the zeroes and poles are determined by the eigenvalues of the Frobenius acting on the étale cohomology of X (up to cancellation between odd and even degree). We write $\mathcal{R}_1$ for the set of rational functions $f \in \mathbb{C}(t)$ such that $f(0) = 1$, and from now on consider $Z_X(t)$ as an element of $\mathcal{R}_1$.

1.1.2. Grothendieck rings of varieties

Let K be a field. We write $K_0(\mathrm{Var}/K)$ for the **modified** Grothendieck ring of varieties over K — it is the free abelian group on isomorphism classes $[X]$ of (not necessarily connected) varieties X/K, modulo the relations $[X_1 \sqcup X_2] = [X_1] + [X_2]$ and $[X] = [Y]$ if there is a map $X \to Y$ inducing a bijection on points over any algebraically closed field. This definition is equivalent to the classical definition of $K_0(\mathrm{Var}/K)$ via cut and paste relations in characteristic zero, but is better behaved in positive characteristic; we refer to [2, Section 2] for a detailed discussion. Write $\mathbb{L} := [\mathbb{A}^1] \in K_0(\mathrm{Var}/K)$, and

$$\mathcal{M}_K := K_0(\mathrm{Var}/K)[\mathbb{L}^{-1}].$$

There is a natural topology on $\mathcal{M}_K$ induced by the *dimensional filtration*: for every $d \in \mathbb{Z}$, we define $\mathrm{Fil}_d \mathcal{M}_K$ to be the subgroup of $\mathcal{M}_K$ generated by elements of the form $[X]\mathbb{L}^{-n}$ where X is a variety over K and $\dim X - n \leqslant d$. This gives us an increasing and exhaustive filtration on the ring $\mathcal{M}_K$. Denote the completion for this filtration by $\widehat{\mathcal{M}}_K$.

For a quasi-projective variety X over K, consider the *Kapranov zeta function*

$$Z_X^{\mathrm{Kap}}(t) := \sum_{j=0}^{\infty} [\mathrm{Sym}^j X] t^j \in 1 + t\mathcal{M}_K[[t]].$$

When the field K is finite, $Z_X^{\mathrm{Kap}}(t)$ specializes to $Z_X(t)$ via point counting, which replaces the class $[\mathrm{Sym}^j X]$ with the number $|\mathrm{Sym}^j X(K)|$ in the coefficients.

1.2. Arithmetic and motivic statistics

Many results in arithmetic statistics can be interpreted in terms of asymptotic properties of the number of $\mathbb{F}_q$-points on a sequence of varieties $X_n/\mathbb{F}_q$. Such results often have analogs in *motivic* statistics — these are asymptotic statements in $\widehat{\mathcal{M}}_K$. A beautiful example of such a correspondence between arithmetic and motivic results is given by the following:

Theorem. *Let K be a field and let $X \subset \mathbb{P}^n_K$ be a smooth projective variety. Denote by U_d the open subset of hypersurface sections in $\Gamma(\mathbb{P}^n_K, \mathcal{O}(d))$ which intersect X transversely. Then:*

(1) *(Poonen [20]) Assume $K = \mathbb{F}_q$ is finite. Then*

$$\lim_{d\to\infty} \frac{|U_d(\mathbb{F}_q)|}{|\Gamma(\mathbb{P}^n, \mathcal{O}(d))(\mathbb{F}_q)|} = Z_X(q^{-\dim X - 1})^{-1}.$$

(2) *(Vakil-Wood [22,23]) In $\widehat{\mathcal{M}}_K$, we have*

$$\lim_{d\to\infty} \frac{[U_d]}{[\Gamma(\mathbb{P}^n, \mathcal{O}(d))]} = Z_X^{\mathrm{Kap}}(\mathbb{L}^{-\dim X - 1})^{-1}.$$

Although the two results are compellingly similar for $K = \mathbb{F}_q$, neither of the two implies the other: the assignment sending a variety $X/\mathbb{F}_q$ to the number of points $|X(\mathbb{F}_q)|$ extends to the *point counting measure*, a map of rings $\mathcal{M}_{\mathbb{F}_q} \to \mathbb{R}$, but it is not continuous for the dimensional topology on $\mathcal{M}_{\mathbb{F}_q}$. The aim of this paper is to formulate a conjectural unification of such parallel statements in arithmetic and motivic statistics, and study some aspects of this conjecture.

The fundamental insight behind our conjecture is that, through the lens of the *zeta measure* (see (1.3.0.1) below), the aforementioned results of Poonen and Vakil-Wood may be viewed as convergence statements of the zeta functions of the varieties U_d, suitably renormalized, for two different, incompatible topologies on $\mathcal{R}_1$. Indeed, via (1.1.1.1) we can interpet Poonen's theorem, applied simultaneously over all finite extensions of $\mathbb{F}_q$, as a convergence result for the zeta functions $Z_{U_d}(t)$ in the *coefficient topology* on $\mathcal{R}_1$, induced by the product topology on the coefficients of the power series at zero. Because of this interpretation, we refer to this topology on $\mathcal{R}_1$ also as the *point-counting topology*. On the other hand, Vakil and Wood's result, via the Weil conjectures, implies a convergence statement about the functions $Z_{U_d}(t)$ in the *weight topology* on $\mathcal{R}_1$, where a function is considered small if all of its poles and zeroes are at large complex numbers. The point-counting and weight topologies are incompatible; below we introduce the *Hadamard topology* which refines both.

1.3. Rings of zeta functions

The set $\mathcal{R}_1$ has a ring structure given by Witt addition and multiplication: if we identify $\mathcal{R}_1$ with the group ring $\mathbb{Z}[\mathbb{C}^\times]$ via

$$f(t) \to -\mathrm{Div} f(t^{-1}), \text{ so, e.g., } (1-at)^{-1} \mapsto [a]$$

then Witt addition and multiplication are induced by addition and multiplication on the group ring $\mathbb{Z}[\mathbb{C}^\times]$. Alternatively, Witt addition $+_W$ is regular multiplication of rational functions: $f +_W g = fg$, and Witt multiplication $*_W$ is determined by

$$d\log f = \sum a_j t^j \text{ and } d\log g = \sum b_j t^j \implies d\log(f *_W g) = \sum a_j b_j t^j.$$

The assignment $X \mapsto Z_X(t)$ extends to the *zeta measure*, which is a map of rings

$$\begin{aligned} \mathcal{M}_{\mathbb{F}_q} &\to \mathcal{R}_1 \\ a &\mapsto Z_a(t). \end{aligned} \tag{1.3.0.1}$$

1.3.1. The Hadamard topology

Under the identification $\mathcal{R}_1 = \mathbb{Z}[\mathbb{C}^\times]$, the weight topology is induced by the norm

$$\left\| \sum a_n [z_n] \right\|_\infty = \sup |z_n|.$$

The point-counting topology is induced by the family of seminorms

$$\left\| \sum a_n [z_n] \right\|_j = \left| \sum a_n z_n^j \right| \text{ for all integers } j \geqslant 1.$$

We consider also the *Hadamard* topology, defined by the Hadamard norm

$$\left\| \sum a_n [z_n] \right\|_H = \sum |a_n||z_n|.$$

The Hadamard topology refines both the weight and point-counting topologies. Moreover, the completion of $\mathcal{R}_1$ for the Hadamard norm is naturally identified with a genuine space of meromorphic functions (as opposed to the completion for the weight topology, which is a space of formal divisors, or the completion for the point-counting topology, which is a space of formal power series):

Definition 1.3.2. A *Hadamard function* is a meromorphic function on $\mathbb{C}$ that can be written as a quotient $\frac{f}{g}$ where f and g are entire functions of genus zero.

We write $\mathcal{H}_1$ for the set of Hadamard functions f such that $f(0) = 1$. The Hadamard factorization theorem then yields

Theorem 1.3.3. *The completion of $\mathcal{R}_1$ for $|| \cdot ||_H$ is canonically identified with $\mathcal{H}_1$.*

1.4. The meta-conjecture

Because the Hadamard topology refines both the point-counting and weight topologies, asymptotics in the Hadamard topology give a common refinement of results in arithmetic and motivic statistics. Moreover, taking limits in the Hadamard topology retains the essential analytic characteristics of zeta functions, because these limits can be interpreted as meromorphic functions. For these reasons, it is natural to try to refine previous results in arithmetic and motivic statistics by studying them in the Hadamard topology. And in fact, we conjecture that any natural asymptotic which holds in both the weight and point-counting topologies should also hold in the Hadamard topology:

Meta-conjecture. *If $a_n \in \mathcal{M}_{\mathbb{F}_q}$ is a "natural" sequence of classes such that the sequence of zeta functions $Z_{a_n}(t)$ converges in both the point-counting and weight topology to some $f(t) \in \mathcal{H}_1$, then $Z_{a_n}(t) \to f(t)$ also in the Hadamard topology.*

The condition that $f \in \mathcal{H}_1$ is essential — without this condition there is no way to compare limits in the point-counting and weight topologies. Moreover, there are natural examples where limits exist in both topologies, but at least one of these limits is not expected to be a Hadamard function (cf. §1.5.6).

1.4.1. Hadamard convergence for Bertini problems

The theorems of Poonen and Vakil-Wood discussed above furnish an example where our meta-conjecture should apply. To see this, we must verify that the special value of the Kapranov zeta function appearing there is in fact a Hadamard function: We apply the zeta measure coefficientwise to $Z_X^{\mathrm{Kap}}(s)$ to obtain a series with coefficients in the ring $\mathcal{R}_1$, and then evaluate at $s = Z_{\mathbb{L}^{-m}}(t) = \frac{1}{1-q^{-m}t}$ for $m = \dim X + 1$. Indeed, for any $m > \dim X$, if we write

$$\zeta_X^{\mathrm{Kap}}(m) := 1 + Z_X(t)s + Z_{\mathrm{Sym}^2 X}(t)s^2 + \cdots |_{s=Z_{\mathbb{L}^{-m}}(t)} = \prod_{j \geqslant 1} Z_{\mathrm{Sym}^j X}(tq^{-mj}),$$

then the infinite product on the right (an infinite sum in the Witt ring structure) converges in the Hadamard topology to an invertible (for Witt multiplication) element of $\mathcal{H}_1$. Thus, in this case the meta-conjecture specializes to

Conjecture 1.4.2. *Let $X \subset \mathbb{P}^n_{\mathbb{F}_q}$ be a smooth projective variety and let $U_d \subset \Gamma(\mathbb{P}^n, \mathcal{O}(d))$ be the open subvariety of hypersurfaces intersecting X transversely. Then, in the Hadamard topology,*

$$\lim_{d\to\infty} Z_{U_d}(q^{-\dim U_d}t) = 1/_W \zeta_X^{\mathrm{Kap}}(\dim X + 1).$$

Here the notation $/_W$ denotes division in the Witt ring, and the inverse special value appearing on the right can be shown to live in $\mathcal{H}_1$ as a consequence of rationality of $Z_X^{\mathrm{Kap}}(t)$ after applying the zeta measure – see Example 4.2.3. In the statements of Theorem A and Theorem B below we will implicitly use the existence of inverses or quotients that can be justified in a similar way.

We state separately the case $X = \mathbb{P}^n$, which has a particularly simple form:

Conjecture 1.4.3. *Let $U_d \subset V_d := \Gamma(\mathbb{P}^n, \mathcal{O}(d))$ be the space of smooth hypersurfaces of degree d in $\mathbb{P}^n$. Then,*

$$\lim_{d\to\infty} Z_{U_d/\mathbb{F}_q}\left(q^{-\dim V_d}t\right) = Z_{\mathrm{GL}_{n+1}/\mathbb{F}_q}\left(q^{-(n+1)^2}t\right)$$

in the Hadamard topology. In particular, the sequence of rational functions

$$\frac{Z_{U_d/\mathbb{F}_q}(q^{-\dim V_d}t)}{Z_{\mathrm{GL}_{n+1}/\mathbb{F}_q}(q^{-(n+1)^2}t)}$$

converges uniformly on compact sets in $\mathbb{C}$ to the constant function 1.

For $n = 1$, Conjecture 1.4.3 is true, because for all $d \geqslant 2$

$$\frac{[U_d]}{[V_d]} = [\mathrm{GL}_2]\mathbb{L}^{-4} \in \mathcal{M}_{\mathbb{F}_q}.$$

For $n > 1$, however, already Conjecture 1.4.3 is completely open. As some partial evidence, we note that Tommasi [21] has established a cohomological stabilization result for moduli of smooth hypersurfaces in $\mathbb{P}^n_{\mathbb{C}}$ — cf. §1.7.1 below for more details on the relation between cohomological stabilization and Hadamard convergence.

Remark 1.4.4. Some of the material on Hadamard convergence developed in this work appeared already in the first version of [2] posted on arXiv. In particular, it was claimed there that Conjecture 1.4.2 could be proved in the case that $\dim X = 1$. No details were provided, and there was a mistake in the envisioned proof.

The point-counting [20] and motivic [2] Bertini theorems with Taylor coefficients furnish many more examples where we expect that the meta-conjecture should apply. The limits appearing in these theorems are special values of (motivic) Euler products, but, unfortunately we are currently unable to prove that these special values are Hadamard functions in any level of generality!

1.5. Results for zero-cycles

Our meta-conjecture was originally motivated by the Bertini examples discussed above, but for now these seem to be out of reach. On the other hand, there are a number of questions about zero-cycles that have been previously studied in both arithmetic and motivic statistics for which we can both formulate and prove concrete instances of the meta-conjecture. In particular, building on [13,22,6,12,19], we treat various problems involving colored effective zero-cycles with prescribed incidence relations. We also give an application to the motivic Batyrev-Manin conjecture as in [4]. These are the main results of this paper, and the main evidence that our meta-conjecture is reasonable.

1.5.1. Pattern-avoiding zero-cycles

In §4 we carry out a general study of convergence for densities of spaces of effective zero-cycles with prescribed allowable sets of labels. These generalize different densities considered previously in related contexts by Bourqui [4], Farb-Wolfson-Wood [13], and Vakil-Wood [22]. We establish fairly complete weight and point-counting convergence results, and find natural examples (and non-examples) of Hadamard convergence. For more details, we refer the reader to the beginning of §4; below we only highlight some examples.

1.5.2. Orthogonal pattern-avoiding zero-cycles

Let $X/\mathbb{F}_q$ be a geometrically irreducible quasi-projective variety and $k \geqslant 1$ an integer. For $\vec{d} = (d_1, \ldots, d_k) \in \mathbb{Z}_{\geqslant 0}^k$, we write

$$\mathrm{Sym}^{\vec{d}} X := \mathrm{Sym}^{d_1} X \times \mathrm{Sym}^{d_2} X \times \cdots \times \mathrm{Sym}^{d_k} X.$$

For $K/\mathbb{F}_q$ algebraically closed, we can view a point $s \in \mathrm{Sym}^{\vec{d}}(X)(K)$ as a tuple $(s_1, \ldots, s_k)$ of finitely supported functions on $X(K)$ with values in $\mathbb{Z}_{\geqslant 0}$. In particular, for each $x \in X(K)$, we obtain a label vector $\ell_s(x) := (s_1(x), \ldots, s_k(x)) \in \mathbb{Z}_{\geqslant 0}^k$. If we fix a finite subset $V \subset \mathbb{Z}_{\geqslant 0}^k$, then we can consider the locus

$$\mathcal{Z}_V^{\vec{d}}(X) \subset \mathrm{Sym}^{\vec{d}} X$$

whose K-points for algebraically closed K are exactly those s such that, for all $\vec{v} \in V$ and $x \in X(K)$, $\ell_s(x) \ngeqslant \vec{v}$ (i.e., ℓ_s avoids all of the patterns in V).

Example 1.5.3. If $V = \{(n, n, \ldots, n)\}$, then $\mathcal{Z}_V^{\vec{d}}(X)$ is the subvariety denoted $\mathcal{Z}_n^{\vec{d}}(X)$ in [13], which parameterizes tuples of effective zero cycles whose overlap has multiplicities bounded by n. In particular, $\mathcal{Z}_{\{(2)\}}^{(d)}(X) = C^d X$, the configuration space of d unordered distinct points on X.

A set of vectors V is orthogonal (for the standard inner product) if and only if for each $1 \leqslant i \leqslant k$, there is at most one vector $\vec{v} \in V$ with non-zero ith component. For $\vec{v} = (v_1, \ldots, v_k) \in \mathbb{Z}_{\geqslant 0}^k$, we write $|\vec{v}| = v_1 + \ldots + v_k$. We say that a set of vectors V is non-degenerate if it does not contain a $\vec{v}$ with $|\vec{v}| \leqslant 1$ (i.e. it does not contain the zero vector or the unit vector e_i for any i).

Theorem A. *If V is orthogonal and non-degenerate, then, in the Hadamard topology on $\mathcal{H}_1$,*

$$\lim_{d_1, d_2, \ldots, d_k \to \infty} Z_{\mathcal{Z}_V^{\vec{d}}(X)}(t) /_W Z_{\mathrm{Sym}^{\vec{d}}(X)}(t) = 1/_W \left(\prod_{\vec{v} \in V} \zeta_X^{\mathrm{Kap}}\left(|\vec{v}| \cdot \dim X\right) \right).$$

Note that all ring operations in this equation are taken in the Witt ring structure.

Remark 1.5.4. If V contains 0 then $\mathcal{Z}_V^{\vec{d}}(X) = \varnothing$ for any $\vec{d}$, and if V contains e_i then $\mathcal{Z}_V^{\vec{d}}(X) = \varnothing$ when $d_i \neq 0$. For V not orthogonal, see Section 4.5.1, particularly Theorem 4.5.8.

This theorem, and our other related results, provide a motivic lift of Theorem 1.9-2 of Farb-Wolfson-Wood [13], which describes the same phenomenon at the level of Hodge-Deligne polynomials in the special case of Example 1.5.3. This confirms the expectation of a motivic analog stated in [13, paragraph following Theorem 1.9]. Our proof is based on a simple identity of generating functions, generalizing the argument given by Vakil-Wood [22] for computing the density of $C^n X$ in $\mathrm{Sym}^n X$. In particular, this provides a shorter[1] proof of [13, Theorem 1.9-2].

Remark 1.5.5. Ho [18] has reinterpreted and extended the results of [13] using factorization cohomology. In particular, he constructs a natural rational homotopy type (a commutative dga computing the cohomology) attached to the density, then in [18, Proposition 7.7.7] obtains a simple explicit description from which one can deduce the connection with zeta values after taking the trace of Frobenius. Instead taking the characteristic power series of Frobenius, we recover the Hadamard function appearing above; thus this rational homotopy type has a meromorphic zeta function. It would be interesting to understand this phenomenon more generally!

1.5.6. A non-example of Hadamard convergence

We also study the density of the k-colored configuration spaces $C^{\vec{d}} X$ in $\mathrm{Sym}^{\vec{d}} X$ where $\vec{d} \in \mathbb{Z}_{\geqslant 0}^k$. This density converges as $\vec{d} \to \infty$ in the weight and point-counting topologies, and in Theorem 4.6.2 we show it converges in the Hadamard topology if $k < q^{\dim X}$. Some condition of this form appears to be necessary: for $k = 2$, $q = 2$, and $X = \mathbb{A}^1$, we have computed the limiting formal divisor to high precision, and the result strongly suggests that the limit is not a Hadamard function — cf. Remark 4.6.4.

1.5.7. Labeled configuration spaces

We also show Hadamard stabilization for labeled configuration spaces over unordered configuration spaces as studied in [19] in the motivic setting and [6] in the point-counting setting. This does not fit into the framework of allowable labels described above, but instead admits a natural interpretation as computing the moments of a motivic random variable over unordered configuration space. Concretely, we show (see §2.2 for the notation):

Theorem B. *Let λ be a partition and $X/\mathbb{F}_q$ a geometrically irreducible quasi-projective variety. Then, in the Hadamard topology on $\mathcal{H}_1$,*

[1] Of course, our technique does not say anything about the Leray spectral sequence analyzed in [13], and thus cannot establish any of the purely topological density results.

$$\lim_{d\to\infty} Z_{C^{\lambda\cdot\star^d}(X)}(t)/_W Z_{C^{|\lambda|+d}(X)}(t) = Z_{C_X^\lambda\left(\frac{1}{1+\mathbb{L}^{\dim X}}\right)}(t).$$

Here the right-hand-side is the zeta function of a very general notion of labeled configuration space, where the "space" of labels at each point is the class $\frac{1}{1+\mathbb{L}^{\dim X}}$.

Remark 1.5.8. The explicit computation of the limit in Theorem B is new (though closely related to [19, Corollary B]), and makes precise the statement that, after passing to the zeta measure, the universal family over $C^d X$ is asymptotically a motivic binomial random variable with parameters $N = X$ and $p = \frac{1}{1+\mathbb{L}^{\dim X}}$. Convergence in the Hadamard topology, without the explicit computation of the limit, can also be deduced (under a lifting hypothesis) from the étale homological stability results of Farb-Wolfson [12] (cf. also §1.7.1).

Remark 1.5.9. Following the strategy used in [19] to relate motivic stabilization of labeled configuration spaces and representation stability, one obtains the following consequence of Theorem B: given a Young diagram λ, the theory of representation stability attaches a natural sequence of locally constant ℓ-adic sheaves $\mathcal{V}_{\lambda,d}$ on $C^d X$ for d sufficiently large. Writing $L_{\lambda,d}(t)$ for the L-function of $\mathcal{V}_{\lambda,d}$, we find that the sequence $L_{\lambda,d}(tq^{-d\dim X})$ converges in the Hadamard topology.

1.6. Batyrev-Manin over function fields

Let K be a field and X a split toric variety over K, which is assumed smooth and projective. Let U be its open orbit. For every integer $d \geqslant 0$, we denote by $[U_{0,d}]$ the quasi-projective variety parameterizing K-morphisms $\mathbb{P}^1_K \to X$ with image intersecting U, and of anticanonical degree d. Let ρ be the rank of the Picard group of X. We are interested in the *motivic height zeta function*

$$Z(T) = \sum_{d\geqslant 0} [U_{0,d}] T^d.$$

In the finite field case, the specialization via point counting of $Z(T)$ has been extensively studied by Bourqui [3,5], in a much more general setting (for morphisms from a curve of arbitrary genus to not necessarily split toric varieties). In [4], Bourqui also addressed the motivic problem over an arbitrary K. Combining his method therein with our results, we show:

Theorem C.

(1) *There exists an integer $a \geqslant 1$ and a real number $\delta > 0$ such that the series*

$$(1-(\mathbb{L}T)^a)^\rho \left(\sum_{d\geqslant 0} [U_{0,d}] T^d\right) \tag{1.6.0.1}$$

converges for $||T|| < ||\mathbb{L}||^{-1+\delta}$ *in the dimensional topology (see §4.2.1). Its value at* $\mathbb{L}^{-1}$ *is non-zero and can be described explicitly by the special value of a motivic Euler product.*

(2) *Assume now* $K = \mathbb{F}_q$ *finite. Then the specialization of (1.6.0.1) via the zeta measure converges in the point counting topology. If* q *is larger than some explicit bound, it converges in the Hadamard topology.*

We refer to §5 and in particular to Theorem 5.3.1 for a more precise version with explicit bounds and values. The result in the dimensional topology is obtained simply by substituting the more versatile notion of motivic Euler product from [1] for the one used by Bourqui in [4]. The point counting convergence was already known in greater generality (for curves of any genus) by [3]. The Hadamard convergence is an application of the results of the section on zero-cycles.

This problem is an instance of the function field Batyrev-Manin conjecture (classically, the Batyrev-Manin conjecture deals with counting points of bounded height on algebraic varieties defined over number fields). As far as the authors are aware, Theorem C is the first result in the literature giving a unified treatment of a case of the function-field Batyrev-Manin problem in the point counting and motivic setting outside of situations where the motivic height zeta function is rational.

1.7. Obstacles and strategies

Our results for zero-cycles are all, in the end, obtained by explicit computations and estimates with generating functions. By contrast, in the Bertini setting which first motivated this work, similar manipulations with generating functions do not appear useful — instead, to prove point-counting and weight stabilization results, one uses inclusion-exclusion to compare values at a finite step to truncated Euler products.

The versions of inclusion-exclusion that come into play are quite different in the motivic and arithmetic settings, and, in particular, there does not seem to be an obvious way to merge the point-counting argument with the motivic argument in order to control the error term in the Hadamard topology. It would, however, be quite interesting if such an argument could be made!

1.7.1. Betti bounds and Hadamard convergence

Another angle of attack for Conjecture 1.4.2 is by proving étale cohomological stability and sub-exponential growth for the cohomology of U_d, as in the alternative proof of Hadamard convergence for labeled configuration spaces mentioned in Remark 1.5.8. This approach is particularly appealing in the specific case of Conjecture 1.4.3, in light of Tommasi's [21] results on cohomological stability in characteristic zero.

In fact, it turns out that one does not need the full strength of cohomological stability for this kind of argument: in §7, we show that weight convergence combined with suitable bounds on Betti numbers implies Hadamard convergence. This seems like a promising strategy for proving new instances of our meta-conjecture.

1.8. Organization

In §2 we recall some basic notation and results on (very) generalized configuration spaces, motivic Euler products, pre-λ rings, and power structures. In §3, we introduce the Witt ring, its various topologies, and the zeta measure. The heart of the paper is §4, where we prove a general convergence result on spaces of pattern-avoiding effective zero-cycles and deduce Theorem A. We also discuss the case where the vectors in the set V are non-orthogonal: using a Möbius function formalism, we show Hadamard convergence over $\mathbb{F}_q$ for q larger than some explicit bound, and study some interesting boundary cases. In §5, we apply our results from the previous section to prove Theorem C. In §6, we prove Theorem B, and in §7 we explain the link with cohomological stability. Finally, in Appendix A we give some computations related to the boundary cases for Hadamard convergence discussed in §4.

1.9. Acknowledgments

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2. Recollections

In this section we recall some basic definitions and results on generalized configuration spaces, motivic Euler products, and power structures on pre-λ rings.

2.1. Grothendieck rings

In this paper $K_0(\mathrm{Var}/K)$, $\mathcal{M}_K$, $\widehat{\mathcal{M}_K}$, etc. are all built starting with the *modified* Grothendieck ring of varieties (see 1.1.2 above); this is equivalent to the standard definition via cut and paste relations in characteristic zero but in characteristic p gives a better-behaved quotient.

We also consider, for X/K a variety, the relative Grothendieck rings $K_0(\mathrm{Var}/X)$, $\mathcal{M}_X$, and $\widehat{\mathcal{M}_X}$, defined completely analogously but starting with varieties over X instead of $\operatorname{Spec} K$.

We refer the reader to [2, Section 2] for more details on these points.

2.2. Generalized configuration spaces and motivic Euler products

We will briefly cover the basic definitions for the reader's convenience; for further discussion and properties of generalized configuration spaces and motivic Euler products beyond what is included here, we refer the reader to [2, Sections 3.2, 6.1 and 6.2].

2.2.1. Generalized configuration spaces

Suppose given a *label set* S and a finite multiset λ supported on S, i.e. an element of $\mathbb{Z}^S_{\geqslant 0}$ that is zero on all but finitely many $s \in S$. We write $|\lambda| = \sum_{s\in S} \lambda(s)$. We denote by $\lambda \cdot *^d$ any multiset λ' that adds a new element of multiplicity d to λ, i.e. when $\lambda' = \lambda + d\cdot s$ and $\lambda(s) = 0$ for some $s \in S$.

For a quasi-projective variety X/K we define the λ-labeled configuration space of X to be

$$C^\lambda X := \left(\left(\prod_{s\in S} X^{\lambda(s)}\right) \setminus \Delta\right) / \prod_s \Sigma_{\lambda(s)},$$

where Δ is the big diagonal and Σ_k denotes the permutation group on k elements so that the product group acts in the obvious way. The points of $C^\lambda X$ in an algebraically closed field K are given by labellings of $|\lambda|$ distinct points in $X(K)$ by elements of S such that the total multiset of labels is equal to λ. For example, if $\lambda = (a_1, a_2, \dots, a_k) \in \mathbb{Z}^k_{\geqslant 0}\setminus\{\vec{0}\}$, then $C^\lambda X$ is the configuration space of $a_1 + a_2 + \ldots + a_k$ distinct points on X with a_i of the points labeled by i for each $1 \leqslant i \leqslant k$; in other words, a colored configuration space of X.

The construction generalizes to allow, for each $s \in S$, a *space* of labels, here interpreted to be a variety Y_s/X. One obtains a variety $C^\lambda\left((Y_s/X)_{s\in S}\right)$, given by

$$C^\lambda\left((Y_s/X)_{s\in S}\right) := \left(\left(\prod_{s\in S} Y_s^{\lambda(s)}\right) \setminus \Delta\right) / \prod_s \Sigma_{\lambda(s)},$$

with a natural map to $C^\lambda X$ (here Δ is the inverse image of the big diagonal in the definition of $C^\lambda X$).

2.2.2. Motivic Euler products

The results of [1] allow one to extend the construction of generalized configuration spaces to allow the spaces of labels Y_s to be replaced with classes of labels a_s in a relative Grothendieck ring $K_0(\mathrm{Var}/X), \mathcal{M}_X$, or $\widehat{\mathcal{M}_X}$. The result is a class $C^\lambda_X\left((a_s)_{s\in S}\right)$ in the corresponding relative Grothendieck ring over $C^\lambda X$ (the actual definition is explained in 2.2.4 below); when $a_s = [Y_s/X]$ (i.e. the class of Y_s in a relative Grothendieck ring over X) we have the natural identity

$$[C^\lambda\left((Y_s/X)_{s\in S}\right)/C^\lambda X] = C^\lambda_X\left((a_s)_{s\in S}\right).$$

Remark 2.2.3. We write $C^\lambda_X(a)$ if all a_s are taken to be equal to the same class a.

In the above, if the label set S is taken to be the non-zero elements of an abelian monoid M, then a multiset λ as above is called a *generalized partition*. In this case, it makes sense to consider the sum of its elements $\sum\lambda \in M$, and for $m \in M$ we say $\lambda \vdash m$ or λ *partitions* m if $\sum\lambda = m$. This setup applies in particular when M is a free abelian monoid, e.g. $M = \mathbb{Z}^k_{\geqslant 0}$.

This extension is carried out so as to give a reasonable notion of an "infinite product over X", or, a *motivic Euler product*, satisfying the natural properties one would expect for manipulating products. Indeed, for $a_i \in K_0(\mathrm{Var}/X)$, one defines

$$\prod_{x\in X}\left(1 + a_{1,x}t + a_{2,x}t^2 + \dots\right) := 1 + \sum_{m\geqslant 1}\left(\sum_{\lambda\vdash m} C^\lambda_X\left((a_s)_{s\in\mathbb{N}}\right)\right)t^m \in 1 + tK_0(\mathrm{Var}/K)[[t]]$$

where the sums for each coefficient are obtained by first applying the forgetful maps

$$K_0(\mathrm{Var}/C^\lambda X) \to K_0(\mathrm{Var}/K).$$

One can replace $K_0(\mathrm{Var}/K)$ here with $\mathcal{M}_K$ or $\widehat{\mathcal{M}_K}$. We can also make a similar construction for an abelian monoid M as above by using the ring of power series in the variables t_m, $m \in M$, with $t_{m_1}t_{m_2} = t_{m_1+m_2}$. In this setting, the definition of the motivic Euler product becomes

$$\prod_{x\in X}\left(1 + \sum_{s\in M\backslash\{0\}} a_{s,x}t_s\right) := 1 + \sum_{m\in M\backslash\{0\}}\left(\sum_{\lambda\vdash m} C^\lambda_X\left((a_s)_{s\in M\backslash\{0\}}\right)\right)t_m.$$

Standard power series in one variable are obtained using $M = \mathbb{Z}_{\geqslant 0}$ via the identification $t^m = t_1^m = t_m$. More generally, for $M = \bigoplus_{i\in I}\mathbb{Z}_{\geqslant 0}$, we obtain power series on the variables $t_i, i \in I$. In particular, $M = \mathbb{Z}^k_{\geqslant 0}$ gives power series on variables $t_1, \dots, t_k$, and because of this for $m \in \mathbb{Z}^k_{\geqslant 0}$ we frequently write the product $\mathbf{t}^m = \prod_{1\leqslant i\leqslant k} t_i^{m(i)}$ in place of t_m in the above.

2.2.4. Definition

We briefly recall the definition of the classes $C^{\lambda}_X((a_s)_{s\in S})$ used above: The first step is to define for an element $a \in K_0(\mathrm{Var}/X)$ (or $\mathcal{M}_X$ or $\widehat{\mathcal{M}_X}$) its symmetric powers $(\mathrm{Sym}^n_X(a))_{n\geqslant 1}$, in such a way that

$$\mathrm{Sym}^n_X(a) \in K_0(\mathrm{Var}/\mathrm{Sym}^n(X)),$$

and so that for all $a, b \in K_0(\mathrm{Var}/X)$,

$$\mathrm{Sym}^n_X(a+b) = \sum_{k=0}^{n} \mathrm{Sym}^k_X(a) \boxtimes \mathrm{Sym}^{n-k}_X(b). \tag{2.2.4.1}$$

In other words, one lifts the relative Kapranov zeta function on $K_0(\mathrm{Var}/X)$ so that the coefficient of t^i lives in $K_0(\mathrm{Var}/\mathrm{Sym}^i X)$ rather than $K_0(\mathrm{Var}/X)$.

Then, for a partition λ and classes a_s, we consider

$$\mathrm{Sym}^{\lambda}_X((a_s)_{s\in S}) := \prod_s \mathrm{Sym}^{\lambda(s)}_X(a_s) \in K_0\left(\mathrm{Var}/\prod_s \mathrm{Sym}^{\lambda(s)} X\right).$$

Pulling back via the inclusion $C^{\lambda}X \to \prod_s \mathrm{Sym}^{\lambda(s)} X$ gives $C^{\lambda}_X((a_s)_{s\in S})$. We often write this restriction with the subscript "$*$", or even "$*, X$" if we want to emphasize that the diagonal was removed at the level of points of X. If $a_s = a$ for all s we also just write a instead of (a_s). So, e.g.,

$$C^{\lambda}_X(a) = \left(\mathrm{Sym}^{\lambda}_X(a)\right)_* = \left(\prod_s \mathrm{Sym}^{\lambda(s)}_X(a)\right)_{*,X}. \tag{2.2.4.2}$$

In particular, the variety $\prod_s \mathrm{Sym}^{\lambda(s)} X$ will be denoted $\mathrm{Sym}^{\lambda} X$.

We note that, by the definitions and [1, Proposition 3.7.0.4], for any $k : S \to \mathbb{Z}$,

$$C^{\lambda}_X((\mathbb{L}^{k(s)} a_s)_{s\in S}) = \mathbb{L}^{\sum_{s\in S} k(s)\lambda(s)} C^{\lambda}_X((a_s)_{s\in S}). \tag{2.2.4.3}$$

2.3. Pre-λ rings and power structures

Recall (e.g., from [19]) that a pre-λ ring is a ring R equipped with a group homomorphism

$$\begin{aligned} \lambda_t : (R, +) &\to (1 + tR[[t]], \times) \\ r &\mapsto 1 + \lambda_1(r)t + \lambda_2(r)t^2 + \dots \end{aligned}$$

such that $\lambda_1(r) = r$. We require always the further condition that $\lambda_t(1) = 1 + t$.

It is equivalent, and for us usually more convenient, to give the homomorphism

$$\sigma_t : r \mapsto \lambda_{-t}(-r) = 1 + \sigma_1(r)t + \sigma_2(r)t^2 + \dots,$$

obtained by making the substitution $t \to -t$ in $\lambda_t(-r)$. The condition $\lambda_1(r) = r$ is equivalent to $\sigma_1(r) = r$ and the condition $\lambda_t(1) = 1 + t$ is equivalent to $\sigma_t(1) = \frac{1}{1-t}$.

The operations λ_i and σ_i on R are conveniently packaged and extended as a pairing

$$\Lambda \times R \to R$$

for Λ the ring of symmetric functions – for e_k the elementary symmetric functions and h_k the complete symmetric functions we have

$$(e_k, r) = \lambda_k(r),\ (h_k, r) = \sigma_k(r),$$

and for any fixed $r \in R$ the induced map $(\bullet, r) : \Lambda \to R$ is a ring homomorphism.

Example 2.3.1. If G is a finite group and R is the complex representation ring of G, then R is equipped with a natural pre-λ ring structure such that, for any representation in V with corresponding class $[V] \in R$ (which is also identified with the trace of V, viewed as a conjugation invariant function on G),

$$\lambda_k([V]) = [\textstyle\bigwedge^k V],\ \sigma_k([V]) = [\mathrm{Sym}^k V].$$

For any $f \in \Lambda$, $(f, [V])$, viewed as a conjugation-invariant function on G, is the function whose value on $g \in G$ is obtained by applying f to the eigenvalues of g acting on V.

Example 2.3.2. The Kapranov zeta function gives a pre-λ ring structure on $K_0(\mathrm{Var}/K)$, $\mathcal{M}_K$, and $\widehat{\mathcal{M}_K}$ via

$$\sigma_t([X]) = Z_X^{\mathrm{Kap}}(t).$$

Remark 2.3.3. In categories of a combinatorial nature such as varieties or sets, one has symmetric powers but no exterior powers. However, the original formulation of (pre-)λ-rings takes places in categories of vector bundles, where exterior powers are natural. This explains why we put the emphasis on σ-operations instead of λ-operations as in classical presentations of this topic. In the literature, there is typically no restriction on $\lambda_t(1)$ and $\sigma_t(1)$ – without this condition, one can define a new pre-λ ring by swapping the σ and λ-operations, so our requirements on $\lambda_t(1)$ and $\sigma_t(1)$ serve to eliminate this confusion. Our choice is the "right one" in the sense that the operations enforced by this convention on Grothendieck rings of combinatorial and linear categories are compatible with natural functors like passing from a group action on a set to the induced permutation representation or from a variety to its compactly supported cohomology.

Remark 2.3.4. A pre-λ ring R is a λ-ring if the map λ_t is a pre-λ ring homomorphism (for the Witt ring and pre-λ structure on $1+tR[[t]]$), i.e. if it is also multiplicative and if it identifies the pre-λ ring structures. If R is torsion free over $\mathbb{Z}$, then in terms of Adams operations (see 2.3.6) this is equivalent to asking that $(p_m, \bullet)$ be a ring homomorphism and $(p_{m_1}, (p_{m_2}, \bullet)) = (p_{m_1 m_2}, \bullet)$. The natural pre-$\lambda$ ring structure on the Grothendieck ring of a symmetric monoidal category is in fact a λ-ring structure, but it is not known whether the pre-λ ring structure on $K_0(\mathrm{Var}/K)$ and its variants is a λ-ring structure.

2.3.5. Power structures on pre-λ rings

In [15,16] (see also [19]) it is explained how a pre-λ structure on a ring R extends naturally to a *power structure*, which gives a systematic way to make sense of expressions like

$$\left(1+\sum a_{\vec{d}}\,\mathbf{t}^{\vec{d}}\right)^b$$

for $a_{\vec{d}}, b \in R$; the result is a new power series with coefficients in R and constant term 1, and we have

$$(1+t)^r = \lambda_t(r), \quad \left(\frac{1}{1-t}\right)^r = \sigma_t(r).$$

In the case of $K_0(\mathrm{Var}/K)$ and its variants, the power structure attached to the Kapranov zeta function (viewed as a pre-λ structure as in Example 2.3.2) is closely related to motivic Euler products: in fact, it exactly captures the motivic Euler products with constant coefficients. Indeed, for classes $a_{\vec{d}} \in K_0(\mathrm{Var}/K)$, we have

$$\left(1+\sum_{\vec{d}\in\mathbb{Z}^k_{\geqslant 0}\setminus\{\vec{0}\}} a_{\vec{d}}\,\mathbf{t}^{\vec{d}}\right)^{[X]} = \prod_{x\in X}\left(1+\sum_{\vec{d}\in\mathbb{Z}^k_{\geqslant 0}\setminus\{\vec{0}\}} a_{\vec{d}}\,\mathbf{t}^{\vec{d}}\right)$$

where in the right to interpret the motivic Euler product we pull back the elements $a_{\vec{d}}$ to $K_0(\mathrm{Var}/X)$ as constant classes. In particular, we obtain

$$Z_X^{\mathrm{Kap}}(t) = \left(\frac{1}{1-t}\right)^{[X]} = \prod_{x\in X}\frac{1}{1-t}.$$

2.3.6. Computing simple powers

In a useful special case, we now explain a direct explicit formula for computing these powers (or equivalently, the motivic Euler products). This formula will be used to establish several estimates in §4.

To that end, we consider the power sum symmetric functions

$$p_m = x_1^m + x_2^m + \ldots \in \Lambda$$

as well as their Möbius-inverted counterparts

$$p'_m = \frac{1}{m} \sum_{d|m} \mu(m/d) p_d \in \Lambda[1/m]$$

studied in [19]. We note that for R a pre-λ ring,

$$(p_m, \bullet) : R \to R$$

is an additive homomorphism (these are the Adams operations). Indeed, this follows from the identity

$$\sum_{i \geqslant 0} p_{i+1} t^i = d \log \sum_{i \geqslant 0} h_i t^i, \tag{2.3.6.1}$$

and the fact that σ_t is a homomorphism. As a consequence, we also have that

$$(p'_m, \bullet) : R[1/m] \to R[1/m]$$

is a homomorphism of additive groups.

As shown in [19, Lemma 2.8], we have

Lemma 2.3.7. *Suppose $f(t) \in \mathbb{Z}[[t_1, \ldots, t_n]]$ with constant coefficient 1 and $r \in R$ for a pre-λ ring R. Then, in $(R \otimes_{\mathbb{Z}} \mathbb{Q})[[t_1, \ldots, t_n]]$,*

$$\log\left(f(t_1, t_2, \ldots, t_k)^r\right) = \sum_{m \geqslant 1} p'_m(r) \log f(t_1^m, t_2^m, \ldots, t_k^m),$$

where the exponentiation on the left-hand side is for the power structure determined by the pre-λ structure on R and $\log(1 + \ldots)$ is evaluated via the formal series

$$\log(1+s) = s - \frac{s^2}{2} + \frac{s^3}{3} - \ldots .$$

3. The ring of Hadamard functions

3.1. The Witt ring structure on rational functions

We start by explaining the ring structure on the set $\mathcal{R}_1$ of complex rational functions f with $f(0) = 1$. One can identify $\mathcal{R}_1$ with a Grothendieck ring: let $\mathbf{Rep}_{\mathbb{Z}}$ be the category of pairs (V, ρ) where V is a finite dimensional complex vector space and ρ is a representation of $\mathbb{Z}$ on V (to give ρ is equivalent to giving the automorphism $\rho(1)$ of V). The characteristic power series of a linear map induces an injective map

$$
\begin{aligned}
K_0(\mathbf{Rep}_{\mathbb{Z}}) &\to 1 + t\mathbb{C}[[t]] \\
[(V,\rho)] &\mapsto \frac{1}{\det(1 - t\rho(1))}
\end{aligned}
$$

with image $\mathcal{R}_1$. The induced addition on $\mathcal{R}_1$ is multiplication of power series, called Witt addition; the induced multiplication is called Witt multiplication, and the set $\mathcal{R}_1$ equipped with this ring structure is also known as the rational Witt ring of $\mathbb{C}$.

We note that $K_0(\mathbf{Rep}_{\mathbb{Z}})$ is also naturally isomorphic to the group ring $\mathbb{Z}[\mathbb{C}^\times]$, where the class $[a]$ in the group ring is matched with the class of the 1-dimensional representation with $\rho(1)$ given by multiplication by a. The induced identification of $\mathcal{R}_1$ with $\mathbb{Z}[\mathbb{C}^\times]$ sends $f \in \mathcal{R}_1$ to the divisor of $\frac{1}{f(1/t)}$.

3.1.1. The big Witt ring

These ring structures extend naturally (e.g., by continuity in the coefficients) to $1 + t\mathbb{C}[[t]]$; the result is the big Witt ring $W(\mathbb{C})$. The subring

$$W(\mathbb{Z}) = 1 + t\mathbb{Z}[[t]] \subset W(\mathbb{C})$$

also admits a natural interpretation as the Grothendieck ring $K_0(\mathbf{AlFin}\ \mathbb{Z}\textbf{-set})$ of the *almost finite cyclic sets* of [9]: Here an almost finite cyclic set is a set S with an action of $\mathbb{Z}$ such that the fixed points $X^{n\mathbb{Z}}$ are finite for each n and $X = \bigcup_n X^{n\mathbb{Z}}$. If we denote by $a_n(S)$ the (finite) number of orbits of length n in S, then the identification is induced by

$$[S] \mapsto Z_S(t) = \prod_{n=1}^{\infty} \left(\frac{1}{1-t^n}\right)^{a_n(S)}.$$

Note that there is a commutative diagram

$$
\begin{tikzcd}
K_0(\mathbf{Fin}\ \mathbb{Z}\textbf{-set}) \arrow[r] \arrow[d] & K_0(\mathbf{AlFin}\ \mathbb{Z}\textbf{-set}) = W(\mathbb{Z}) = 1 + t\mathbb{Z}[[t]] \arrow[d] \\
K_0(\mathbf{Rep}_{\mathbb{Z}}) = \mathcal{R}_1 \arrow[r] & W(\mathbb{C}) = 1 + t\mathbb{C}[[t]]
\end{tikzcd}
\tag{3.1.1.1}
$$

Here the left vertical arrow sends a $\mathbb{Z}$-set S to the permutation representation $\mathbb{C}[S]$, and its image consists of the functions in $\mathcal{R}_1$ with zero and pole sets both given by unions of Galois-orbits of roots of unity, or equivalently the functions

$$\prod_{n=1}^{\infty} \left(\frac{1}{1-t^n}\right)^{a_n}$$

with $a_n \in \mathbb{Z}$ and equal to zero for n sufficiently large. In particular, this can be used to show that the two ring structures on $\mathcal{R}_1 \cap W(\mathbb{Z})$ agree.

3.2. λ-ring structure and Adams operations

The symmetric monoidal structure on $\mathbf{Rep}_{\mathbb{Z}}$ equips the Grothendieck ring $K_0(\mathbf{Rep}_{\mathbb{Z}})$ with the structure of a λ-ring with σ-operations induced by symmetric powers and λ-operations induced by exterior powers. Under the isomorphism with $\mathbb{Z}[\mathbb{C}^\times]$, the σ- and λ-operations are determined by

$$\sigma_s([a]) = 1 + \sigma_1([a])s + \sigma_2([a])s^2 + \cdots = 1 + [a]s + [a^2]s^2 + \cdots$$

and

$$\lambda_s([a]) = 1 + \lambda_1([a])s + \lambda_2([a])s^2 + \cdots = 1 + [a]s.$$

Thus, in $K_0(\mathbf{Rep}_{\mathbb{Z}})$ with the λ-ring-structure described above, (2.3.6.1) gives

$$\sum_{i\geqslant 0} p_{i+1}([a])s^i = \frac{\sigma_s([a])'}{\sigma_s([a])} = \frac{[a]}{1-[a]s},$$

and in particular, the Adams operations are given by $p_i([a]) = (p_i, [a]) = [a^i]$.

Finally, we note that symmetric powers of sets define σ-operations for a λ-ring structure on $W(\mathbb{Z})$, and using (3.1.1.1) we find the two λ-ring structures agree on

$$W(\mathbb{Z}) \cap \mathcal{R}_1 \subset W(\mathbb{C}).$$

3.3. Topologies on rational functions

We describe three topologies on $\mathcal{R}_1$.

3.3.1. The point counting topology

There is a natural injective map

$$\mathcal{R}_1 \hookrightarrow 1 + t\mathbb{C}[[t]]$$

given by taking the power series expansion at zero. The *point counting topology* on $\mathcal{R}_1$ is induced by the product topology on the coefficients of

$$1 + t\mathbb{C}[[t]] = \mathbb{C}^{\mathbb{N}}.$$

Note that $\mathcal{R}_1$ is dense when viewed as a subset of $1+t\mathbb{C}[[t]]$ so that the completion of $\mathcal{R}_1$ for the point counting topology is identified with $1+t\mathbb{C}[[t]]$. The addition, multiplication, and λ-ring structure are continuous for the point counting topology, so that they extend to $1+t\mathbb{C}[[t]]$ which is thus a complete topological λ-ring; this is the big Witt ring $W(\mathbb{C})$ discussed already in 3.1.1 above.

Instead of taking the power-series expansion of a rational function f, one could instead take the power-series expansion of $d\log f$, and we would obtain the same topology. In fact, $d\log f$ gives a bijection

$$1 + t\mathbb{C}[[t]] \to t\mathbb{C}[[t]] = \mathbb{C}^{\mathbb{N}}$$

that is an isomorphism of topological rings when $\mathbb{C}^{\mathbb{N}}$ on the right is equipped with the product topology and ring structure. The coefficients of $d\log f$ are also called the *ghost coordinates* on the big Witt ring.

Using this observation, we can also describe the point counting topology in terms of $\mathbb{Z}[\mathbb{C}^\times] = \mathcal{R}_1$. It is induced by the family of semi-norms $||\cdot||_j$, $j = 1, 2, \ldots$

$$\left|\left|\sum_a k_a[a]\right|\right|_j = \left|\sum_a k_a a^j\right|.$$

Indeed, this follows from the above discussion and the computation

$$d\log\left(\prod_a (1-ta)^{-k_a}\right) = \sum_{j=1}^{\infty}\left(\sum_a k_a a^j\right) t^{j-1}.$$

We note that there is no natural description of the completion of $\mathcal{R}_1 = \mathbb{Z}[\mathbb{C}^\times]$ for the point counting topology in terms of divisors on $\mathbb{C}^\times$.

3.3.2. The weight topology

In the weight topology, a basis of open neighborhoods of $f \in \mathcal{R}_1$ is given by, for each $r > 0$, the set of all rational functions g with the same zeroes and poles as f on the ball $|t| \leqslant r$. In particular, a sequence converges if and only if on every bounded set the zeroes and poles eventually stabilize.

Viewed as the group ring $\mathbb{Z}[\mathbb{C}^\times]$, a basis of open neighborhoods of zero is given by the set of all finite sums $\sum_{a\in\mathbb{C}^\times} k_a[a]$ supported on the closed ball of radius r around $0 \in \mathbb{C}$ (here we are using that $[a]$, as a rational function, has a pole at a^{-1}), and a basis of open neighborhoods at any other point is given by translation.

The completion $\widehat{\mathbb{Z}[\mathbb{C}^\times]}^w$ of $\mathbb{Z}[\mathbb{C}^\times]$ for the weight topology can be described as the set of formal sums

$$\sum_{a\in\mathbb{C}^\times} k_a[a]$$

whose support is a discrete subset of $\mathbb{C}$ and whose set of accumulation points in $\mathbb{C} \sqcup \infty$ is contained in $\{0\}$. The addition, multiplication, and λ-ring structure are all continuous for the weight topology, and extend to these formal sums.

We note that there is no natural description of the completion of $\mathcal{R}_1$ for the weight topology in terms of the power series expansion at 0.

3.3.3. The Hadamard topology

The Hadamard topology is most simply described under the isomorphism $\mathcal{R}_1 \to \mathbb{Z}[\mathbb{C}^\times]$, where it is the topology induced by the sub-multiplicative norm

$$\left|\left|\sum k_a[a]\right|\right|_H = \sum |k_a||a|.$$

It is easy to see the point counting and weight topologies on $\mathcal{R}_1$ are not comparable (i.e., neither is finer than the other). However:

Lemma 3.3.4. *The Hadamard topology refines both the point counting and weight topologies on $\mathcal{R}_1$.*

Proof. Each of the semi-norms $||\cdot||_j$ defining the point counting topology is continuous for the norm $||\cdot||_H$, and thus the Hadamard topology refines the point counting topology.

To compare with the weight topology, it suffices to observe that if $f = \sum k_a[a]$ is supported inside the closed ball of radius r, then so is any g with $||f - g||_H < r$. □

3.4. The ring of Hadamard functions

We define the Hadamard-Witt ring $\mathcal{W}$ to be the completion of $\mathbb{Z}[\mathbb{C}^\times]$ for the norm $||\cdot||_H$. It can be identified with the set of discretely supported divisors

$$\sum_{a\in\mathbb{C}^\times} k_a[a]$$

such that $\sum_{a\in\mathbb{C}^\times} |k_a||a| < \infty$. It is an elementary computation to check that the multiplication and σ (or λ) operations are continuous, so that they extend to $\mathcal{W}$ which is thus a complete topological λ-ring.

A *Hadamard* function is a meromorphic function f on $\mathbb{C}$ such that f can be written as a quotient $f = \frac{g}{h}$ where g and h are both entire functions of genus zero. In the next lemma, we extend the identification of $\mathbb{Z}[\mathbb{C}^\times]$ with $\mathcal{R}_1$ to an identification of $\mathcal{W}$ with the set $\mathcal{H}_1$ of Hadamard functions f such that $f(0) = 1$.

Lemma 3.4.1. *If $\sum_{a\in\mathbb{C}^\times} k_a[a] \in \mathcal{W}$ and*

$$\sum_{a\in\mathbb{C}^\times} k_a[a] = \sum_{a\in\mathbb{C}^\times} k_a^+[a] + \sum_{a\in\mathbb{C}^\times} k_a^-[a]$$

is the unique decomposition with $k_a^+ \geqslant 0$ and $k_a^- \leqslant 0$, then the infinite products

$$\prod_a \left(\frac{1}{1-ta}\right)^{k_a^-} \text{ and } \prod_a \left(\frac{1}{1-ta}\right)^{-k_a^+}$$

converge uniformly on compact sets to entire functions of genus zero f^- and f^+. Furthermore, the map

$$\sum_{a\in\mathbb{C}^\times} k_a[a] \mapsto \frac{f^-}{f^+}$$

induces a bijection $\mathcal{W} \to \mathcal{H}_1$ extending the bijection $\mathbb{Z}[\mathbb{C}^\times] \to \mathcal{R}_1$.

Proof. This is an immediate consequence of the Hadamard factorization theorem in the case of genus zero entire functions. □

Because the Hadamard topology refines the point counting and weight topologies, there are natural maps between the completions, and these maps have natural function-theoretic interpretations:

(1) The map from the Hadamard completion to the point counting completion is given by taking $f(t) \in \mathcal{H}_1$ to its power series at 0.
(2) The map from the Hadamard completion to the weight completion is given by taking $f(t) \in \mathcal{H}_1$ to the divisor of $\frac{1}{f(1/t)}$.

The constructions in this section are summarized by the following diagram:

$$\begin{tikzcd}
 & \mathcal{R}_1 = \mathbb{Z}[\mathbb{C}^\times] \arrow[dl, "\substack{\text{point}\\ \text{counting}\\ \text{topology}}"'] \arrow[d, "\substack{\text{Hadamard}\\ \text{topology}}"'] \arrow[dr, "\substack{\text{weight}\\ \text{topology}}"] & \\
W(\mathbb{C}) & \mathcal{H}_1 = \mathcal{W} \arrow[l, hook', "\text{Taylor expansion at zero}"] \arrow[r, hook, "f\mapsto -\mathrm{Div}(f(t^{-1}))"'] & \widehat{\mathbb{Z}[\mathbb{C}^\times]}^w
\end{tikzcd}$$

3.5. The zeta measure

Zeta functions of varieties give an interesting source of elements of the ring $\mathcal{R}_1$. In fact, we have the following:

Proposition 3.5.1. *The assignment $X \mapsto Z_X(t)$ induces a map of pre-λ-rings*

$$K_0(\mathrm{Var}/\mathbb{F}_q) \to \mathcal{R}_1,$$

where $\mathcal{R}_1$ is equipped with the Witt ring structure.

Proof. It suffices to prove the same with $\mathcal{R}_1$ replaced by $1 + t\mathbb{Z}[[t]] = W(\mathbb{Z})$, because the zeta function of any variety is contained in $\mathcal{R}_1 \cap 1 + t\mathbb{Z}[[t]] \subset 1 + t\mathbb{C}[[t]] = W(\mathbb{C})$. Then, we claim the zeta measure is induced by the functor

$$\mathrm{Var}/\mathbb{F}_q \to \mathbf{AlFin}\ \mathbb{Z}\textbf{-set}$$

sending $X/\mathbb{F}_q$ to the set $X(\overline{\mathbb{F}_q})$ with the action of $\mathrm{Frob}_q^{\mathbb{Z}}$.

It is clear that this induces a map of abelian groups on K_0: indeed, the functor factors through the localization of $\mathrm{Var}/\mathbb{F}_q$ at radicial surjective maps (those maps which induce bijections on points over algebraically closed fields), and then applying the naive K_0 to both sides gives the desired map — here the naive K_0 is the free group on isomorphism classes modulo turning finite disjoint unions into sums, and it is shown in [2] that the localization combined with naive K_0 on the left recovers $K_0(\mathrm{Var}/\mathbb{F}_q)$. Since products and symmetric powers are preserved by the functor, we find that this is furthermore a map of pre-λ rings.

The orbits of length n in $X(\overline{\mathbb{F}_q})$ correspond to closed points of degree n in X, and thus by definition we have

$$Z_X(t) = Z_{X(\overline{\mathbb{F}_q})}(t)$$

where the right-hand side is the assignment defined in 3.1.1 inducing

$$K_0(\mathbf{AlFin}\ \mathbb{Z}\textbf{-set}) = 1 + t\mathbb{Z}[[t]],$$

and we conclude. $\square$

Remark 3.5.2. The element $\mathbb{L} \in K_0(\mathrm{Var}/\mathbb{F}_q)$ gets sent to $\frac{1}{1-qt}$, which thanks to the way we chose our normalizations has associated element $[q] \in \mathbb{Z}[\mathbb{C}^\times]$ and therefore Hadamard norm q. Since $[q]$ is invertible for Witt multiplication in $\mathcal{R}_1$ with inverse $[q^{-1}]$, we also see that the zeta measure induces a ring morphism $\mathcal{M}_{\mathbb{F}_q} \to \mathcal{R}_1$.

Remark 3.5.3. Let $X/\mathbb{F}_q$ be a variety. Then the different (semi-)norms introduced in Section 3.3 are expressed in the following way when applied to $Z_X(q^{-N}t)$ for some $N \geqslant 1$: For every $j \geqslant 1$, $||Z_X(q^{-N}t)||_j = q^{-Nj}|X(\mathbb{F}_{q^j})|$. Moreover, by Deligne's results on weights, we have

$$||Z_X(q^{-N}t)||_\infty = q^{\dim X - N},$$

and

$$||Z_X(q^{-N}t)||_H \leqslant q^{-N} \sum_{i=0}^{2\dim X} \dim_{\mathbb{Q}_\ell} H^i_c(X_{\bar{\mathbb{F}}_q}, \mathbb{Q}_\ell)|q|^{\frac{i}{2}},\ [\gcd(\ell, q) = 1]$$

with equality if X is smooth projective, in which case compactly supported ℓ-adic cohomology can also be replaced with regular ℓ-adic cohomology.

Example 3.5.4. It is straightforward to cook up sequences of $a_n \in \mathcal{M}_{\mathbb{F}_q}$ that seemingly violate our meta-conjecture, i.e. whose zeta measures converge in the point counting and weight topologies to the same $f \in \mathcal{H}_1$ but not in the Hadamard topology. For example, take

$$a_n = (n!)\mathbb{L}^{-2n} \prod_{i=1}^{n} (\mathbb{L} - q^i) \in \mathcal{M}_{\mathbb{F}_q}.$$

Then, $||Z_{a_n}||_\infty \to 0$ and $||Z_{a_n}||_j \to 0$ for all $j \geqslant 1$. Thus, in both the weight and point-counting topologies, $Z_{a_n} \to 1$, which is certainly a Hadamard function! On the other hand,

$$||Z_{a_n}||_H \geqslant \frac{n!}{q^n}$$

so the sequence does not converge in the Hadamard topology.

However we maintain that this is not a *natural* sequence to consider. We refer the reader to Remark 4.6.4 for an example of a natural sequence where a different issue occurs — there we find a sequence that converges in both the point-counting and weight topologies, but *not* to a Hadamard function, so that the two limits cannot even be compared and our meta-conjecture does not apply.

3.6. The Kapranov zeta function and its special values

As explained in the introduction, special values of Kapranov's zeta function often appear as limits in natural motivic statistics questions.

Let X be a quasi-projective variety over $\mathbb{F}_q$, and consider the series

$$Z^{\mathrm{Kap}}_{X,\mathrm{zeta}}(s) = 1 + Z_X(t)s + Z_{\mathrm{Sym}^2 X}(t)s^2 + \cdots \in 1 + s\mathcal{R}_1[[s]], \tag{3.6.0.1}$$

obtained from the usual Kapranov zeta function by applying the zeta measure.

Remark 3.6.1. Recall that a rational function $f \in \mathcal{R}_1$ has for every $i \geqslant 1$ a ghost coordinate $g_i(f)$ given by the coefficient of t^{i-1} in $d\log f$. As can be verified either from the series or rational function expansion of $Z^{\mathrm{Kap}}_{X,\mathrm{zeta}}(s)$, applying the ith ghost coordinate map g_i to each coefficient, we obtain

$$g_i(Z^{\mathrm{Kap}}_{X,\mathrm{zeta}}(s)) = Z_{X_{\mathbb{F}_{q^i}}}(s).$$

Thus, one way to think of the Kapranov zeta function of $X/\mathbb{F}_q$ is as working simultaneously with the Hasse-Weil zeta functions of the base changes of X to all finite extensions of $\mathbb{F}_q$.

By Proposition 3.5.1, the zeta measure is a map of pre-λ rings, and therefore

$$Z^{\mathrm{Kap}}_{X,\mathrm{zeta}}(s) = \sigma_s(Z_X(t)).$$

It follows from the formula for the σ-operations on $\mathcal{R}_1$ that for *any* rational function $f \in \mathcal{R}_1$, if we factorize

$$f = \frac{\prod(1 - ta_i)}{\prod(1 - tb_i)}$$

then

$$\sigma_s(f) = \frac{\prod(1 - s[a_i])}{\prod(1 - s[b_i])}. \tag{3.6.1.1}$$

In particular, combined with Deligne's results on weights we obtain the following proposition that will be crucial in §4:

Proposition 3.6.2. *If $X/\mathbb{F}_q$ is geometrically irreducible, then*

$$Z^{\mathrm{Kap}}_{X,\mathrm{zeta}}(s) \in 1 + s\mathcal{R}_1[[s]]$$

is "a rational function with smallest pole or zero given by a simple pole at $[q^{-\dim X}]$", that is,

$$(1 - [q^{\dim X}]s)Z^{\mathrm{Kap}}_{X,\mathrm{zeta}}(s) = \frac{\prod(1 - s[a_i])}{\prod(1 - s[b_i])}$$

where the products are finite and $|a_i|, |b_i| \leqslant q^{\dim X - 1/2}$.

In particular, this gives a way to make sense of special value in $\mathcal{H}_1$ by a simple evaluation for any values of s such that the denominator is invertible in $\mathcal{H}_1$. The significance of knowing the locations of the smallest pole or zero is that it allows us to control the convergence of the corresponding *series* expansion, which is what will come up naturally in our applications.

4. Hadamard stabilization for effective zero-cycles

In this section we investigate Hadamard convergence for sequences of motivic densities arising from prescribing a set of allowable labels for effective zero cycles. After setting up the notation and problem, in Theorem 4.3.5 we give a general condition that guarantees these densities exist. As immediate corollaries we obtain stabilization for the Hodge measure over $\mathbb{C}$ (Corollary 4.3.7) and for the weight topology on zeta functions (Corollary 4.3.8). For X stably rational, these results hold already in the dimension topology on the Grothendieck ring. The proof of Theorem 4.3.5 is based on the generating function argument used by Vakil-Wood [22] in their study of motivic densities of configuration spaces.

We then apply our convergence criterion to obtain Hadamard convergence in various more specific settings. In §4.4-4.5, we consider *pattern-avoiding* zero-cycles, generalizing spaces considered by Farb-Wolfson-Wood [13] and by Bourqui [4]. In particular, we obtain Hadamard convergence for the spaces considered in [13] as well as simpler proofs of some results of [13] (cf. §4.4.2), and we also clarify some results of [4] (cf. Remark 4.5.7).

In §4.6, we consider densities with a finite set of allowable labels. The main case of interest is the universal one, corresponding to configuration spaces with a fixed set of k labels. Here the behavior is more delicate: if $q^{\dim X} > k$, then we obtain Hadamard convergence, but otherwise we obtain natural examples that converge in both the weight and point counting topology, but *not* to a Hadamard function.

4.1. Notation, examples, and a general density problem

We consider the monoid $\mathbb{Z}^k_{\geqslant 0}$. For a subset $A \subset \mathbb{Z}^k_{\geqslant 0}\backslash\{0\}$ (our set of *allowable labels*) and $\vec{d} \in \mathbb{Z}^k_{\geqslant 0}$, we write $X^{\vec{d}}_A$ for the configuration space of points in X with labels in A summing to $\vec{d}$. In other words, points in $X^{\vec{d}}_A$ can be thought of as finite formal sums $\sum_{x\in X} \vec{a}_x x$ such that each $\vec{a_x} \in A$ and $\sum_x \vec{a}_x = \vec{d}$. Each point of $X^{\vec{d}}_A$ thus determines a partition of $\vec{d}$ into elements of A, and there is a natural decomposition of $X^{\vec{d}}_A$ as a disjoint union of configuration spaces over such partitions:

$$X^{\vec{d}}_A = \bigsqcup_{\lambda \vdash_A \vec{d}} C^\lambda X. \tag{4.1.0.1}$$

Here the subscript in $\vdash_A$ signifies that each part of λ is an element of A and the notation C^λ is as in §2.2.1. In terms of generating functions, by the definition of motivic Euler products and the decomposition (4.1.0.1), in $\mathcal{M}_K$ we have the identity

$$\sum_{\vec{d}\in\mathbb{Z}^k_{\geqslant 0}} [X^{\vec{d}}_A]\mathbf{t}^{\vec{d}} = \prod_{x\in X}\left(1+\sum_{\vec{a}\in A}\mathbf{t}^{\vec{a}}\right).$$

Example 4.1.1.

(1) Taking $A = \mathbb{Z}^k_{\geqslant 0}\backslash\{0\}$, we obtain

$$[X^{\vec{d}}_{\mathbb{Z}^k_{\geqslant 0}\backslash\{0\}}] = [\mathrm{Sym}^{\vec{d}}X] = [\mathrm{Sym}^{d_1}X \times \mathrm{Sym}^{d_2}X \times \cdots \times \mathrm{Sym}^{d_k}X]$$

and

$$\begin{aligned}\sum_{\vec{d}\in\mathbb{Z}^k_{\geqslant 0}} [X^{\vec{d}}_A]\mathbf{t}^{\vec{d}} &= \prod_{x\in X}\left(1+\sum_{\vec{a}\in A}\mathbf{t}^{\vec{a}}\right)\\ &= \prod_{x\in X}\frac{1}{(1-t_1)(1-t_2)\dots(1-t_k)}\end{aligned}$$

$$= Z_X^{\mathrm{Kap}}(t_1)Z_X^{\mathrm{Kap}}(t_2)\cdots Z_X^{\mathrm{Kap}}(t_k).$$

In general, we may think of $X_A^{\vec{d}}$ as a constructible subset of $\mathrm{Sym}^{\vec{d}}X$, i.e. as a parameter space of k-tuples of effective zero cycles on X.

(2) If A is the collection of standard basis vectors $\vec{e}_i, 1 \leqslant i \leqslant k$, then

$$X_A^{\vec{d}} = C^{\vec{d}}(X),$$

the colored configuration space of $\sum d_i$ points in X with k colors and d_i points of color i. The generating function is then

$$\sum_{\vec{d}\in\mathbb{Z}_{\geqslant 0}^k} [X_A^{\vec{d}}]\mathbf{t}^{\vec{d}} = \prod_{x\in X}(1 + t_1 + t_2 + \cdots + t_k).$$

(3) Taking A to be the complement of $\mathbb{Z}_{\geqslant 0}^k + (m, m, \dots, m)$ in $\mathbb{Z}_{\geqslant 0}^k\backslash\{0\}$, we find that $[X_A^{\vec{d}}]$ equals the class of k-tuples of effective zero cycles on X that overlap in a zero cycle with multiplicities less than m. We then have

$$\begin{aligned}\sum_{\vec{d}\in\mathbb{Z}_{\geqslant 0}^k} [X_A^{\vec{d}}]\mathbf{t}^{\vec{d}} &= \prod_{x\in X}\left(1 + \sum_{\vec{a}\in A}\mathbf{t}^{\vec{a}}\right)\\ &= \prod_{x\in X}\frac{(1-(t_1t_2\cdots t_k)^m)}{(1-t_1)(1-t_2)\cdots(1-t_k)}.\\ &= \frac{Z_X^{\mathrm{Kap}}(t_1)Z_X^{\mathrm{Kap}}(t_2)\cdots Z_X^{\mathrm{Kap}}(t_k)}{Z_X^{\mathrm{Kap}}\left((t_1t_2\cdots t_k)^m\right)}.\end{aligned}$$

4.1.2. A density problem

Recall that a *motivic measure* is a ring morphism $\phi : \mathcal{M}_K \to R$ valued in some ring R. We will often write a_ϕ for $\phi(a)$.

If $A \subset B$, then $X_A^{\vec{d}} \subset X_B^{\vec{d}}$, and for suitable motivic measures ϕ, it is natural to consider the asymptotic density

$$\lim_{d\to\infty}\frac{[X_A^{\vec{d}}]_\phi}{[X_B^{\vec{d}}]_\phi} \tag{4.1.2.1}$$

where here $\vec{d} \to \infty$ means each entry is going to ∞.

We restrict here to the case where A contains the standard basis vectors $\vec{e}_i$, $1 \leqslant i \leqslant k$; this ensures that $X_A^{\vec{d}}$ has dimension $\dim X \cdot \sum \vec{d}$, and includes the cases of Example 4.1.1. It is then natural to take $B = \mathbb{Z}_{\geqslant 0}^k\backslash\{0\}$, so that we are studying densities in the full motivic probability space of k-tuples of effective zero cycles. In this setup, we give in Theorem 4.3.5 below a condition on X, A, and ϕ that guarantees the limit (4.1.2.1) exists, and, moreover identifies the limit as a special value of a power series expressed as a motivic Euler product. Before establishing this criterion, we give some basic notation for discussing convergence of power series with coefficients in $\mathcal{M}_K$.

4.2. Normed motivic measures and convergence of series

4.2.1. Normed motivic measures

We call a motivic measure $\phi : \mathcal{M}_K \to R$ normed if R is complete for a sub-multiplicative norm $||\cdot||$ and $||\mathbb{L}_\phi^{-1}|| < 1$. We have in mind especially the cases:

(1) $K = \mathbb{F}_q$ and ϕ the zeta measure to the completion of $\mathcal{R}_1$ for either the weight or Hadamard topologies.
(2) K arbitrary and ϕ the map $\mathcal{M}_K \to \widehat{\mathcal{M}}_K$ to the completion for the dimension topology. In this case, we fix the norm to be $||a|| = 2^{\dim a}$ where we define $\dim a := \inf\{d \in \mathbb{Z}, a \in \mathrm{Fil}_d \widehat{\mathcal{M}}_K\}$.
(3) $K = \mathbb{C}$ and ϕ the Hodge measure, with a norm defined similarly using the weight filtration on $\widehat{K_0(\mathrm{HS})}$.

Note that on $\mathcal{M}_{\mathbb{F}_q}$ we can also access the point-counting topology for the zeta measure through this setup by treating each ghost coordinate individually, i.e. by considering for each k the measure induced by $X \mapsto |X(\mathbb{F}_{q^k})|$ as a $\mathbb{C}$-valued measure.

4.2.2. Absolute convergence and radius of convergence

The following is completely elementary, but it will be useful to spell it out clearly before we start manipulating values of convergent power series for normed motivic measures.

If R is complete for a norm $||\cdot||$, then we say a series

$$\sum_{\vec{d} \in \mathbb{Z}^k_{\geqslant 0}} r_{\vec{d}}$$

for $r_{\vec{d}} \in R$ *converges absolutely* if the series of real numbers $\sum ||r_{\vec{d}}||$ converges, i.e. if the limit of partial sums converges. An absolutely convergent series converges and its limit is independent of reordering the terms; moreover, the Cauchy product of two absolutely convergent series is absolutely convergent and its limit is the product of the limits of the two series.

Given a power series

$$f(\mathbf{t}) = \sum_{\vec{d} \in \mathbb{Z}^k_{\geqslant 0}} a_{\vec{d}} \mathbf{t}^{\vec{d}} \in R[[t_1, \ldots, t_k]],$$

the *radius of convergence* of $f(\mathbf{t})$ is

$$\rho(f) = \frac{1}{\limsup_{\vec{d}} ||a_{\vec{d}}||^{1/\sum \vec{d}}} \in [0, +\infty].$$

If $s_1, \dots, s_k \in R$ are such that $||s_i|| < \rho(f)$ for all i, then the series obtained by formally substituting s_i for t_i converges absolutely, and we write $f(s_1, \dots, s_k)$ for its value in R.

If $f(\mathbf{t})$ and $g(\mathbf{t})$ both have radius of convergence $\geqslant \rho_0$, then so does the formal power series $h(\mathbf{t}) = f(\mathbf{t})g(\mathbf{t})$, and $h(s_1, \dots, s_k) = f(s_1, \dots, s_k)g(s_1, \dots, s_k)$ for $||s_i|| < \rho_0$. In particular, we highlight the following point: if $f(\mathbf{t})$ has invertible constant coefficient then it admits a formal inverse $\frac{1}{f}(\mathbf{t}) \in R[[t_1, \dots, t_k]]$. If both f and $1/f$ have radius of convergence $\geqslant \rho_0$ and $||s_i|| < \rho_0$, then we find

$$f(s_1, \dots, s_k) \cdot \frac{1}{f}(s_1, \dots, s_k) = 1, \text{ so } \frac{1}{f}(s_1, \dots, s_k) = \frac{1}{f(s_1, \dots, s_k)}.$$

Example 4.2.3. Let $X/\mathbb{F}_q$ be a geometrically irreducible variety. From Proposition 3.6.2 and the above discussion, we deduce that, as power series in $\mathcal{H}_1[[t]]$,

(1) $Z^{\mathrm{Kap}}_{X,\mathrm{zeta}}(t)$ and $1/Z^{\mathrm{Kap}}_{X,\mathrm{zeta}}(t)$ have radius of convergence $\geqslant q^{-\dim X}$ and converge to mutually inverse values for $|t| < q^{-\dim X}$.
(2) Let $f(x) = (1 - t[q^{\dim X}])Z^{\mathrm{Kap}}_{X,\mathrm{zeta}}(t)$. Then $f(x)$ and $1/f(x)$ have radius of convergence $\geqslant q^{-\dim X + 1/2}$, and converge to mutually inverse values for $|t| < q^{-\dim X + 1/2}$.

Indeed, given the formula for $Z^{\mathrm{Kap}}_{X,\mathrm{zeta}}(t)$ as a rational function in Proposition 3.6.2 and the behavior of radius of convergence of products, it suffices to observe that, for $a \in \mathbb{C}^\times$, the formal power series with coefficients in $\mathcal{H}_1$

$$\frac{1}{1 - [a]t} = 1 + [a]t + [a]^2t^2 + \dots$$

has radius of convergence $\frac{1}{|a|}$.

4.2.4. A useful lemma

We establish a useful convergence lemma for the dimensional topology. Before stating it, recall that a motivic Euler product is *by definition* a power series with coefficients in $\mathcal{M}_K$ – that is, the product symbol is only a notationally convenient way of defining a series. In particular, when discussing convergence of a motivic Euler product, one is always discussing convergence of a series – there is indeed no other way that the convergence can be interpreted.

Lemma 4.2.5. *Let $1 + \sum_{i \geqslant 2} a_i T^i \in \mathbb{Z}[[T]]$ be a power series with no term of degree 1. Then the motivic Euler product*

$$f(t) = \prod_{x \in X} \left(1 + \sum_{i \geqslant 2} a_i t^i\right),$$

viewed as a power series with coefficients in $\widehat{\mathcal{M}}_K$, has radius of convergence $\geqslant ||\mathbb{L}||^{-\frac{\dim X}{2}}$. Moreover, for $||s|| < ||\mathbb{L}||^{-\frac{\dim X}{2}}$, $f(s)$ is an invertible element of $\widehat{\mathcal{M}}_K$.

Proof. Because the formal inverse

$$\frac{1}{1+\sum_{i\geqslant 2} a_i T^i} \in \mathbb{Z}[[T]]$$

also satisfies the hypotheses of theorem and

$$1/f(t) = \prod_{x\in X} \frac{1}{1+\sum_{i\geqslant 2} a_i T^i},$$

the statement about invertibility of the value $f(s)$ will follow once we have established the claim about the radius of convergence.

To compute the radius of convergence, we note that the term of degree n of the expansion of this motivic Euler product is a sum over partitions of n. Because $a_1 = 0$, the contribution of each partition (n_i) such that $\sum in_i = n$ is bounded in dimension by

$$\sum_{i\geqslant 2} n_i \dim X \leqslant \frac{1}{2}\sum_{i\geqslant 1} in_i \dim X = \frac{n\dim X}{2}.$$

The result then follows immediately from the formula for the radius of convergence. □

4.3. Weak rationality and a convergence criterion

Given a power series $f(\mathbf{t})$ with coefficients in $\mathcal{M}_K$ and a normed motivic measure ϕ, the ϕ-radius of convergence of f is the radius of convergence of the power series obtained by applying ϕ to the coefficients of f.

Definition 4.3.1. For X a geometrically irreducible K-variety, we say $Z_X^{\mathrm{Kap}}(t)$ is *weakly rational* for a normed motivic measure ϕ if the power series $(1-\mathbb{L}^{\dim X}t)Z_X^{\mathrm{Kap}}(t)$ and its inverse both have ϕ-radius of convergence strictly larger than $||\mathbb{L}_\phi^{-\dim X}||$.

For $K = \mathbb{F}_q$ and ϕ the point-counting measure to $\mathbb{C}$, weak rationality follows from Deligne's results on weights – indeed, the smallest zero or pole of the rational function $Z_X(t)$ is a pole of multiplicity one at $q^{-\dim X}$. For our applications, what we will need is precisely the convergence of the series expressions obtained after removing this pole, and this motivates our definition of weak rationality.

We note that closely related conditions have previously been considered in the literature. In particular, weak rationality is stronger than the motivic stabilization of symmetric powers (MSSP) of [22] and the extension MSSP* of [19]. In practice, however, weak rationality holds whenever MSSP is known, in particular:

Proposition 4.3.2. *Let X be a geometrically irreducible K-variety.*

(1) *If X is stably rational then $Z_X^{\mathrm{Kap}}(t)$ is weakly rational for the map to the completed Grothendieck ring $\widehat{\mathcal{M}}_K$.*

(2) *If $K = \mathbb{F}_q$ then $Z_X^{\mathrm{Kap}}(t)$ is weakly rational for the zeta measure to $\mathcal{H}_1$.*
(3) *If $K = \mathbb{C}$ then $Z_X^{\mathrm{Kap}}(t)$ is weakly rational for the Hodge measure to $\widehat{K_0(\mathrm{HS})}$.*

Proof. Case (2) follows from Example 4.2.3, and case (3) follows by a similar argument using rationality of the Kapranov zeta function under the Hodge measure.

Case (1) is very similar to the argument for MSSP in [22, §4]: X stably rational means that for some n, $X \times \mathbb{A}^n$ is birational to $\mathbb{A}^{\dim X+n}$, and since

$$(1 - \mathbb{L}^{\dim X\times\mathbb{A}^n} t) Z_{X\times\mathbb{A}^n}^{\mathrm{Kap}}(t) = (1 - \mathbb{L}^{\dim X}(\mathbb{L}^n t)) Z_X^{\mathrm{Kap}}(\mathbb{L}^n t),$$

it suffices to assume that X is rational. If X is rational, then there are varieties Y_1 and Y_2 with $\dim Y_i < \dim X$ such that $[X] - [Y_1] = [\mathbb{A}^{\dim X}] - [Y_2]$. Then

$$Z_X^{\mathrm{Kap}}(t) = \frac{Z_{Y_1}^{\mathrm{Kap}}(t)}{Z_{Y_2}^{\mathrm{Kap}}(t)} Z_{\mathbb{A}^{\dim X}}^{\mathrm{Kap}}(t),$$

and since $Z_{\mathbb{A}^{\dim X}}^{\mathrm{Kap}}(t) = \frac{1}{1-\mathbb{L}^{\dim X}t}$ is cleared by multiplying by $1 - \mathbb{L}^{\dim X}t$, we conclude weak rationality; indeed, $Z_{Y_i}^{\mathrm{Kap}}(t)^{\pm 1}$ has radius of convergence $\geqslant ||\mathbb{L}^{-\dim Y_i}||$. $\square$

Remark 4.3.3. Essentially the same argument as in the proof of Proposition 4.3.2-(3) shows more generally that having a weakly rational Kapranov zeta function is invariant under stable birational equivalence (as is MSSP; see [22, 1.26-(i)]).

4.3.4. A convergence criterion

Adapting the strategy used to study the motivic densities of configuration spaces in [22], we find

Theorem 4.3.5. *If $Z_X^{\mathrm{Kap}}(t)$ is weakly rational for ϕ and the power series*

$$\frac{\sum_{\vec{d}\in\mathbb{Z}_{\geqslant 0}^k} [X_A^{\vec{d}}]\mathbf{t}^{\vec{d}}}{Z_X^{\mathrm{Kap}}(t_1) Z_X^{\mathrm{Kap}}(t_2)\cdots Z_X^{\mathrm{Kap}}(t_k)} = \prod_{x\in X} (1-t_1)(1-t_2)\cdots(1-t_k)\left(1 + \sum_{\vec{a}\in A} \mathbf{t}^{\vec{a}}\right) \quad (4.3.5.1)$$

converges absolutely at $t_1 = t_2 = \cdots = t_k = \mathbb{L}_\phi^{-\dim X}$ to a value ζ, then

$$\lim_{\vec{d}\to\infty} \frac{[X_A^{\vec{d}}]_\phi}{[\mathrm{Sym}^{\vec{d}} X]_\phi} = \zeta.$$

Proof. In what follows, we write $n = \dim X$. By weak rationality,

$$(1 - \mathbb{L}^n t_1)\cdots(1 - \mathbb{L}^n t_k) Z_X^{\mathrm{Kap}}(t_1)\cdots Z_X^{\mathrm{Kap}}(t_k)$$

converges absolutely at $\mathbf{t} = (\mathbb{L}_\phi^{-n}, \dots, \mathbb{L}_\phi^{-n})$ to an invertible element. In particular, the sequence of partial sums $\frac{[\mathrm{Sym}^{\vec{d}} X]_\phi}{\mathbb{L}_\phi^{n \sum \vec{d}}}$ converges to an invertible element as $\vec{d} \to \infty$.

If the quotient power series

$$\frac{\prod_{x \in X} (1 + \sum_{\vec{a} \in A} \mathbf{t}^{\vec{a}})}{Z_X^{\mathrm{Kap}}(t_1) Z_X^{\mathrm{Kap}}(t_2) \cdots Z_X^{\mathrm{Kap}}(t_k)} = \prod_{x \in X} (1 - t_1) \cdots (1 - t_k) \left(1 + \sum_{\vec{a} \in A} \mathbf{t}^{\vec{a}}\right)$$

also converges absolutely at $\mathbf{t} = (\mathbb{L}_\phi^{-n}, \dots, \mathbb{L}_\phi^{-n})$, then multiplying we find that

$$(1 - \mathbb{L}^n t_1) \cdots (1 - \mathbb{L}^n t_k) \prod_{x \in X} \left(1 + \sum_{\vec{a} \in A} \mathbf{t}^{\vec{a}}\right)$$

converges absolutely at $\mathbf{t} = (\mathbb{L}_\phi^{-n}, \dots, \mathbb{L}_\phi^{-n})$. In particular, the sequence of partial sums $\frac{[X_A^{\vec{d}}]_\phi}{\mathbb{L}_\phi^{n|\vec{d}|}}$ converges, and the quotient $\frac{[X_A^{\vec{d}}]_\phi}{[\mathrm{Sym}^{\vec{d}} X]_\phi}$ converges to the value of

$$\prod_{x \in X} (1 - t_1)(1 - t_2) \cdots (1 - t_k) \left(1 + \sum_{\vec{a} \in A} \mathbf{t}^{\vec{a}}\right)$$

at $\mathbf{t} = (\mathbb{L}_\phi^{-n}, \dots, \mathbb{L}_\phi^{-n})$. $\square$

Remark 4.3.6. The "local factor" at a geometric point x of X,

$$(1 - t_1)(1 - t_2) \cdots (1 - t_k) \left(1 + \sum_{\vec{a} \in A} \mathbf{t}^{\vec{a}}\right)\Bigg|_{(\mathbb{L}^{-\dim X}, \dots, \mathbb{L}^{-\dim X})},$$

is the asymptotic density as $\vec{d} \to \infty$ of the subset of $\mathrm{Sym}^{\vec{d}} X$ where x has an allowable label. Indeed, this can be verified essentially as in the proof above, using

$$\left(\prod_{\{x\}} \left(1 + \sum_{\vec{a} \in A} \mathbf{t}^{\vec{a}}\right) \prod_{X \setminus \{x\}} \frac{1}{(1 - t_1)} \cdots \frac{1}{(1 - t_k)}\right) \Bigg/ \prod_X \frac{1}{(1 - t_1)} \cdots \frac{1}{(1 - t_k)}$$
$$= \prod_{\{x\}} (1 + t_1) \dots (1 + t_k) \left(1 + \sum_{\vec{a} \in A} \mathbf{t}^{\vec{a}}\right).$$

Here on the first line the numerator is the generating function for the subsets of $\mathrm{Sym}^{\vec{d}} X$ where x has an allowable label and the denominator is the generating function for $\mathrm{Sym}^{\vec{d}} X$. Thus the theorem can be thought of as establishing the asymptotic independence of these local conditions at each point x.

Corollary 4.3.7. *For $X/\mathbb{C}$ irreducible,*

$$\lim_{\vec{d}\to\infty} \frac{[X_A^{\vec{d}}]_{\mathrm{HS}}}{[\mathrm{Sym}^{\vec{d}} X]_{\mathrm{HS}}} = \left(\prod_{x\in X} (1-t_1)(1-t_2)\cdots(1-t_k) \left(1 + \sum_{\vec{a}\in A} \mathbf{t}^{\vec{a}}\right)\right)\Bigg|_{(\mathbb{L}_{\mathrm{HS}}^{-\dim X},\ldots,\mathbb{L}_{\mathrm{HS}}^{-\dim X})}.$$

If X is stably rational, then this holds already in $\widehat{\mathcal{M}_{\mathbb{C}}}$.

Proof. By Proposition 4.3.2, the Kapranov zeta function $Z_X^{\mathrm{Kap}}(t)$ is always weakly rational for the Hodge measure, and is weakly rational in $\widehat{\mathcal{M}_{\mathbb{C}}}$ if X is stably rational. On the other hand, since there are no degree 1 terms in

$$(1-t_1)(1-t_2)\cdots(1-t_k)\left(1 + \sum_{\vec{a}\in A} \mathbf{t}^{\vec{a}}\right),$$

we can use Lemma 4.2.5 to check that the motivic Euler product converges absolutely in the dimension topology on $\widehat{\mathcal{M}_{\mathbb{C}}}$ and thus also for the weight topology after passing to the Hodge measure. □

Arguing similarly, we also obtain

Corollary 4.3.8. *For $X/\mathbb{F}_q$ geometrically irreducible,*

$$\lim_{\vec{d}\to\infty} Z_{X_A^{\vec{d}}}/_W Z_{\mathrm{Sym}^{\vec{d}}(X)} = \left(\prod_{x\in X} (1-t_1)(1-t_2)\cdots(1-t_k) \left(1 + \sum_{\vec{a}\in A} \mathbf{t}^{\vec{a}}\right)\right)\Bigg|_{([q^{-\dim X}],[q^{-\dim X}],\ldots)}$$

in the weight topology. If X is stably rational, this holds already in $\widehat{\mathcal{M}_{\mathbb{F}_q}}$.

4.4. Pattern-avoiding zero-cycles I

A natural generalization of the setup in Example 4.1.1-(3) is given by the notion of pattern-avoiding zero cycles: we fix a subset V of labels, and then take $A = A(V)$ to consist of every label not lying above V (where a label $\vec{a}$ lies above V if there is some $\vec{v} \in V$ such that $\vec{a} - \vec{v} \in \mathbb{Z}^k_{\geqslant 0}$). Thus, in this case $X_A^{\vec{d}}$ is the space $\mathcal{Z}_V^{\vec{d}}(X)$ considered in section 1.5.2. In order for A to contain all basis vectors, we assume here that all the vectors in V have norm at least 2.

Example 4.4.1.

(1) $V = (m, m, \ldots, m)$ recovers Example 4.1.1-(3).

(2) Let $V = \{(1,2),(2,1)\}$. Then $X^{\vec{d}}_{A(V)}$ parameterizes pairs of effective zero-cycles $(C_1, C_2) \in \mathrm{Sym}^{d_1}(X) \times \mathrm{Sym}^{d_2}(X)$ such that we can write $C_1 = C_1' + D$ and $C_2 = C_2' + D$ where C_1', C_2' and D have disjoint supports and D is reduced (i.e. all points have multiplicity at most one).

(3) Bourqui [4, Section 3] studied spaces of "intersection-avoiding" zero-cycles, which corresponds to requiring each vector in V to have all of its coordinates equal to 0 or 1.

4.4.2. Orthogonal patterns

Suppose we take V to be a collection of orthogonal vectors of norm larger than one (generalizing Example 4.4.1-(1)). Then, a straightforward computation shows

$$\left(1 + \sum_{\vec{a}\in A} \mathbf{t}^{\vec{a}}\right) = \left(\prod_{i=1}^{k} \frac{1}{1-t_i}\right) \prod_{\vec{v}\in V} (1 - \mathbf{t}^{\vec{v}}).$$

So, in this case (4.3.5.1) simplifies to

$$\prod_{\vec{v}\in V} Z_X^{\mathrm{Kap}}(\mathbf{t}^{\vec{v}})^{-1}.$$

In any of the normed motivic measures we consider each term in this finite product converges absolutely at $\mathbf{t} = (\mathbb{L}^{-\dim X}, \mathbb{L}^{-\dim X}, \ldots)$ because $|\vec{v}| > 1$, thus the product converges absolutely. Theorem 4.3.5 then gives Theorem A, and we also obtain convergence in the Hodge measure for $X/\mathbb{C}$, and in the dimension topology if X is stably rational. In particular, if $V = \{(m, m, \ldots, m)\}$ as in Example 4.1.1-(3), then we obtain a short proof of Theorem 1.9-2 of [13] and provide the motivic lift predicted there.

Remark 4.4.3. Note that while the generating function for any A can be written as an infinite product of zeta functions, it is quite special that we obtain a finite product in this case.

4.5. Pattern-avoiding zero-cycles II

We now consider the more general case where the vectors in V are not necessarily orthogonal.

4.5.1. Möbius functions

It will be helpful to rewrite the power series in Theorem 4.3.5 using the notion of Möbius function appearing in [4, Section 3].

Definition 4.5.2. The local Möbius function $\mu_V : \mathbb{Z}^k_{\geqslant 0} \to \mathbb{Z}$ is defined recursively by the relation

$$\mathbf{1}_{A(V)\cup\{0\}}(\vec{n}) = \sum_{0\leqslant \vec{n}'\leqslant \vec{n}} \mu_V(\vec{n}'),$$

where the left-hand side is the characteristic function of the set $A(V) \cup \{0\}$.

Remark 4.5.3. It is immediate from the definition that

- $\mu_V(0) = 1$;
- for any $\vec{n} \in A(V)$, $\mu_V(\vec{n}) = 0$;
- for any minimal $\vec{v} \in V$, $\mu_V(\vec{v}) = -1$.

Lemma 4.5.4. *We have*

$$(1-t_1)\cdots(1-t_k)\left(1+\sum_{\vec{a}\in A(V)} \mathbf{t}^{\vec{a}}\right) = \sum_{\vec{n}\in\mathbb{Z}^k_{\geqslant 0}} \mu_V(\vec{n})\mathbf{t}^{\vec{n}}.$$

Proof. This follows by dividing both sides by $(1-t_1)\cdots(1-t_k)$, expanding the right-hand side and using the definition of μ_V. □

Example 4.5.5. In the case where $V = \{(m,\dots,m)\}$, we get $\mu_V(m,\dots,m) = -1$, and $\mu_V(\vec{n}) = 0$ for any other non-zero vector $\vec{n}$, so we recover Example 4.1.1-(3). More generally, if the vectors in V are orthogonal, one can check that we recover the expression in Section 4.4.2.

It is worth showing how this notion is related to the notion of Möbius function of a poset. For every $\vec{v} \in \mathbb{Z}^k_{\geqslant 0}$, we denote by $\vec{v}(i)$ its i-th coordinate. Define $\vec{v}_{\max}$ to be the vector given by

$$\vec{v}_{\max}(i) = \max_{\vec{v}\in V} \vec{v}(i).$$

Define $P_V := \{0\} \cup \bigcup_{\vec{v}\in V} \{\vec{n},\ \vec{v} \leqslant \vec{n} \leqslant \vec{v}_{\max}\}$. Then we have

Proposition 4.5.6.

(1) *The restriction of the function μ_V to P_V is equal to the Möbius function of the poset P_V.*
(2) *The function μ_V is zero outside of the finite set P_V.*

Proof.

(1) Since for any $\vec{n} \in A(V)$, $\mu_V(\vec{n}) = 0$, the two functions satisfy the same recurrence relation for all $\vec{n} \leqslant \vec{v}_{\max}$, so coincide on all these elements.

(2) Let $\vec{m} \in \mathbb{Z}_{\geqslant 0}^k \backslash P_V$. If $\vec{m} \in A(V)$, then we have already observed that $\mu_V(\vec{m}) = 0$. Otherwise, since $\vec{m} \notin P_V$, we have $\vec{m} \nleqslant \vec{v}_{\max}$ so there exists an index i such that $\vec{m}(i) > \max_{\vec{v} \in V} \vec{v}(i)$. Note that, identifying coefficients in Lemma 4.5.4, we get

$$\mu_V(\vec{m}) = \sum_{\vec{m}' \text{s.t.} (\vec{m}-\vec{m}') \in \{0,1\}^k} (-1)^{\Sigma(\vec{m}-\vec{m}')} \mathbf{1}_{A(V) \cup \{\vec{0}\}}(\vec{m}').$$

Now, the $\vec{m}'$ in the sum break up naturally into pairs with the same coordinates outside of the ith index, and because for all terms we have $\vec{m}'(i) \geqslant \vec{m}(i) - 1 \geqslant \max_{\vec{v} \in V} v(i)$, in each pair either both vectors or neither will lie in $A(V)$. Thus, the contributions of the two vectors in each pair cancel, and we obtain zero for the sum. □

Remark 4.5.7. We comment further on Example 4.4.1-(3). In this case, Bourqui [4, Section 3] writes $B_{\min} \subset \{0,1\}^k$ where we write V, and the label generating function $1 + \sum_{\vec{a} \in A(V)} \mathbf{t}^{\vec{a}}$ is what Bourqui denotes as Q_B; the product with the polynomial $(1-t_1)\cdots(1-t_k)$ is P_B, and the analogue of our Lemma 4.5.4 is Bourqui's Lemme 3.1. The coefficients of (4.3.5.1) are then the values of the motivic Möbius function of [4, Section 3.3], and the formula after specializing to Chow motives in characteristic zero in [4, Theorem 3.3] follows from the identification of the motivic Euler product with a pre-λ power and [19, Lemma 2.8].

Because of the definition of motivic Euler products used there, Bourqui's results are valid after tensoring the Grothendieck ring with $\mathbb{Q}$ and specializing to Chow motives. Our setting does not require these procedures, and can be thought of as a strengthening and generalization of the results of [4, Section 3], answering in particular Bourqui's Question 3.5: it boils down to verifying the identity

$$\frac{\sum_{\vec{d} \in \mathbb{Z}_{\geqslant 0}^k} [X^{\vec{d}}_{A(V)}] \mathbf{t}^{\vec{d}}}{Z_X^{\mathrm{Kap}}(t_1) Z_X^{\mathrm{Kap}}(t_2) \cdots Z_X^{\mathrm{Kap}}(t_k)} = \prod_{x \in X} \left(\sum_{\vec{n}} \mu_V(\vec{n}) \mathbf{t}^{\vec{n}} \right),$$

which follows from Lemma 4.5.4 after taking motivic Euler products. As explained by Bourqui, a positive answer to his Question 3.5 ensures that Corollary 3.4 in [4] is valid at the level of the Grothendieck ring of varieties, which in turn gives a lift of his main theorem to the Grothendieck ring of varieties. We give more details about this in Section 5, where we also address Hadamard convergence.

In the remainder of this subsection we prove the following convergence theorem for spaces of pattern-avoiding zero-cycles:

Theorem 4.5.8. *Suppose $X/\mathbb{F}_q$ is irreducible. Let $A = A(V)$ for some finite set of vectors $V \subset \mathbb{Z}_{\geqslant 0}^k$ of norms at least 2, and denote by e the minimum of the sums of the coordinates of the vectors in V. Then,*

$$Z_{X_A^{\vec{d}}}/_W Z_{\mathrm{Sym}^{\vec{d}}(X)}$$

converges as $\vec{d} \to \infty$ in the weight and point-counting topologies on $\mathbb{Z}[\mathbb{C}^\times]$. If $\sum_{\vec{n}\in\mathbb{Z}^k_{\geqslant 0}-\{0\}} |\mu_V(\vec{n})| < q^{e\dim X}$, then it converges in the Hadamard topology.

Example 4.5.9. Let $V = \{(2,1),(1,2)\}$ as in Example 4.4.1-(2). Then Theorem 4.5.8 tells us that

$$Z_{X_{A(V)}^{\vec{d}}}/_W Z_{\mathrm{Sym}^{\vec{d}}(X)}$$

converges in the Hadamard topology for any value of q. Indeed, in this case $e = 3$ and the only non-zero values of μ_V are

$$\mu_V(2,1) = \mu_V(1,2) = -1 \text{ and } \mu_V(2,2) = 1,$$

so the inequality becomes $3 < q^3$, which is satisfied for any prime power q.

Remark 4.5.10. In general, we do not necessarily expect Hadamard convergence to hold for all values of q. See Remark 4.6.4 for a discussion of this phenomenon, and an example of Hadamard non-convergence.

To prove Theorem 4.5.8, we will apply Theorem 4.3.5, and we start by establishing some bounds which will be needed to check convergence of the motivic Euler product appearing in its hypotheses. In the following we use the notation of 2.3.6.

Lemma 4.5.11. *For R a pre-λ ring, $r \in R$ and $a_1,\dots,a_k \in \mathbb{Z}$, the coefficient of $\mathbf{u}^{\vec{d}}$ in*

$$\log\left((1 + a_1u_1 + \cdots + a_ku_k)^r\right),$$

an element in $R\otimes_{\mathbb{Z}}\mathbb{Q}$, is

$$-\sum_{m\vec{d'}=\vec{d}} \binom{\sum \vec{d'}}{\vec{d'}} \frac{(-1)^{\sum \vec{d'}}}{\sum \vec{d'}} a_1^{d'_1}\cdots a_k^{d'_k} p'_m(r). \tag{4.5.11.1}$$

In particular, if $R\otimes_{\mathbb{Z}}\mathbb{Q}$ is normed, we find that the sum of the norms of the coefficients of $\mathbf{u}^{\vec{d}}$ for a fixed total degree $d = \sum \vec{d}$ is bounded by

$$\sum_{m|d} \left(\sum_i |a_i|\right)^{d/m} ||p'_m(r)||. \tag{4.5.11.2}$$

Proof. The formula (4.5.11.1) is obtained by expanding the formula given in Lemma 2.3.7 in this case. We then obtain the estimate (4.5.11.2) by summing norms for fixed m and all $\vec{d}$ in (4.5.11.1). $\square$

Lemma 4.5.12. *Let X be an irreducible variety over $\mathbb{F}_q$. Let*

$$P(t_1,\dots,t_k) = 1 + \sum_{\vec{\imath}} b_{\vec{\imath}} \mathbf{t}^{\vec{\imath}} \in \mathbb{Z}[t_1,\dots,t_k]$$

be a polynomial such that $P(t_1,\dots,t_k) - 1$ has only terms of degree at least $e \geqslant 2$. The motivic Euler product

$$\prod_{x \in X} P(t_1,\dots,t_k)$$

converges absolutely at $t_1,\dots,t_k = [q^{-\dim X}]$ in the point counting topology. If $\sum_{\vec{\imath}} |b_{\vec{\imath}}| < q^{e \dim X}$, then it also converges absolutely in the Hadamard topology.

Proof. Since motivic Euler products commute with monomial substitutions (see [2, Section 6.5]), and since each non-constant monomial is of degree at least e, we can reduce to verifying the convergence of

$$\prod_{x \in X} (1 + a_1 u_1 + \cdots + a_n u_n)$$

for $|u_i| \leqslant q^{-e \dim X}$, where n is the number of non-constant monomials appearing in P, and $a_1,\dots,a_n$ are the coefficients $b_{\vec{\imath}}$, arbitrarily relabeled. Because the coefficients are constant, the motivic Euler product is equivalent to the power

$$(1 + a_1 u_1 + \cdots + a_n u_n)^{[X]}$$

Moreover, by Proposition 3.5.1 the zeta measure is a map of pre-λ rings, so we can compute the image as

$$(1 + a_1 u_1 + \cdots + a_n u_n)^{Z_X(s)}$$

and apply Lemma 4.5.11.

We first treat the Hadamard case. We extend the Hadamard norm on $\mathbb{Z}[\mathbb{C}^\times]$ to $\mathbb{Q}[\mathbb{C}^\times]$ in the obvious way; it then suffices to show that log converges absolutely. Now, we consider the estimates (4.5.11.2) for

$$r = [Z_X(s)] = [q^{\dim X}] \pm [z_1] \pm \cdots \pm [z_N],$$

where $q^{\dim X - 1/2} \geqslant |z_i| \geqslant 1$. Using that p'_m acts additively and that

$$p'_m([z]) = \frac{1}{m} \sum_{d|m} \mu(m/d) p_d([z]) = \frac{1}{m} \sum_{d|m} \mu(m/d)[z^d],$$

(see Section 3.2) we obtain the estimate

$$||p'_m(r)||_H \leqslant q^{m \dim X}(1 + Nq^{-m/2}) \leqslant Cq^{m \dim X}. \tag{4.5.12.1}$$

Suppose $\Sigma := \sum_i |a_i| \leqslant q^{e \dim X}$. Then, we can bound (4.5.11.2) by

$$C \cdot (\Sigma^d q^{\dim X} + \sum_{m|d, m\neq 1} \Sigma^{d/m} q^{m \dim X}) \leqslant C \cdot (\Sigma^d q^{\dim X} + \sum_{m|d, m\neq 1} q^{(\frac{de}{m}+m)\dim X}).$$
$$\leqslant C \cdot \left(\Sigma^d q^{\dim X} + dq^{(\frac{de}{2}+2)\dim X}\right).$$

In particular, if $\Sigma < q^{e \dim X}$, we conclude the series converges absolutely for

$$||u_1||_H, \ldots, ||u_k||_H \leqslant q^{-e \dim X}.$$

For the point counting case, it suffices to show convergence for $\mathbb{F}_q$-points. Then, $p'_m([X])$ is just an integer, the number of closed points of degree m on $X/\mathbb{F}_q$. Thus for any M we can factor out the polynomial

$$\prod_{m=1}^{M} P(t_1^m, \ldots, t_k^m)^{p'_m([X])},$$

and then taking log of what remains gives a series with coefficients bounded by (4.5.11.2) but where the sums are over $m > M$. We then obtain absolute convergence by taking M large enough that $\Sigma^{1/M} \leqslant q^{e \dim X}$ and estimating as above. □

Remark 4.5.13. In fact, as can be seen from the proof of Lemma 4.5.12, if ϵ is such that $\Sigma < q^{e \dim X - \epsilon}$, then Hadamard convergence holds for $||u_i||_H \leqslant q^{-e \dim X + \eta}$ for $\eta < \epsilon$, and thus for $||t_i||_H \leqslant q^{-\dim X + \eta/e}$. In the same manner, we can get convergence of point counts for $|t_i| < q^{-\dim X + \delta}$ for some $\delta > 0$.

Proof of Theorem 4.5.8. The convergence in the weight topology follows from Corollary 4.3.8. For Hadamard and point counting convergence, we note that by the properties of the Möbius function, the power series

$$(1 - t_1) \cdots (1 - t_k) \left(1 + \sum_{a \in A} \mathbf{t}^{\vec{a}}\right) = \sum_{\vec{n}} \mu_V(\vec{n}) \mathbf{t}^{\vec{n}}$$

is a polynomial satisfying the assumptions of Lemma 4.5.12, and the latter combined with Theorem 4.3.5 allows us to conclude. □

4.6. Finite sets of allowable labels

In the previous section, we showed that in the case $A = A(V)$, we can prove Hadamard convergence of

$$Z_{X_A^{\vec{d}}}/_W Z_{\mathrm{Sym}^{\vec{d}}(X)}$$

for q sufficiently large, with the bound in q depending only on the sum of the absolute values of the values of the Möbius function of V. In some special cases, it is actually possible to improve this bound. The aim of this section is to give a sharp lower bound on q in the setting of Example 4.1.1-(2). Thus, for a fixed k we are looking at the behavior of $C^{(d_1,\ldots,d_k)}(X)$ as $d_1,\ldots,d_k \to \infty$, and the corresponding generating function is

$$\prod_{x\in X}(1+t_1+\ldots+t_k). \tag{4.6.0.1}$$

Remark 4.6.1. For any finite set of patterns A not necessarily of the form $A(V)$, the generating function for $X_A^{\vec{d}}$ is obtained by monomial substitutions from (4.6.0.1) for $k=|A|$. Similarly to the proof of Lemma 4.5.12, the bounds we obtain in the universal case considered here can also be used to study the case of arbitrary finite A.

Theorem 4.6.2. *Suppose $X/\mathbb{F}_q$ is geometrically irreducible. Then,*

$$Z_{C^{\vec{d}}(X)}/_W Z_{\mathrm{Sym}^{\vec{d}}(X)}$$

converges as $\vec{d} \in \mathbb{Z}^k_{\geqslant 0}$ goes to infinity in the weight and point-counting topologies on $\mathbb{Z}[\mathbb{C}^\times]$. If $k < q^{\dim X}$, it converges in the Hadamard topology.

Remark 4.6.3. This case is also covered by Theorem 4.5.8 by taking V to be the set of all vectors in $\mathbb{Z}^k_{\geqslant 0}$ with sum of coordinates equal to 2. However, the bound obtained is worse: it is of the form $f(k) < q^{2\dim X}$ for $f(k)$ exponential in k. For $k=2$, however, it gives the equivalent condition $\sqrt{5} < q^{\dim X}$.

Remark 4.6.4. The condition for Hadamard convergence is not just an artifact of the proof: When $X=\mathbb{A}^1/\mathbb{F}_q$ and $k=2$, we can use a computer to compute the limiting formal divisor in the weight topology to high precision by expanding the limiting value as described in Corollary 4.3.8 using the expansion of its logarithm given in Lemma 4.5.11 (note that we can identify powers and constant Euler products). The limit is of the form $\sum_{n\geqslant 0}(-1)^n a_n[q^{-n}]$, and we have verified that $a_n \geqslant 2^n$ for $n \leqslant 250$. Moreover, the computations strongly suggest that the ratios $|a_n|/|a_{n-1}|$ are a decreasing sequence for $n\geqslant 2$ with $\lim_{n\to\infty}|a_n/a_{n-1}|=2$. If this holds, then for $q=2$, any sequence of functions converging in the weight topology to this formal divisor has unbounded Hadamard norms. It is probably possible to prove the estimate $a_n \geqslant 2^n$ by expanding more carefully using the techniques below, but we leave this to the interested reader. In Appendix A, we give the first 250 terms of this formal divisor; for comparison, we also give the exact divisor of $Z_{C^{40,40}\mathbb{A}^1}(tq^{-80})$ along with its Hadamard and point-counting norm for $q=2$ (this can be computed in a similar way by expanding out the generating series $(1+t_1+t_2)^{[\mathbb{A}^1]}$ for $[C^{(d_1,d_2)}(\mathbb{A}^1)]$ via Lemma 4.5.11).

Note that this does not violate our meta-conjecture: limits exist for both the weight and point-counting topologies, but the limit in the weight topology is not a Hadamard function! In particular, the limit in the weight topology cannot even be compared to the limit in the point-counting topology, because a general formal divisor does not have a well-defined Taylor expansion.

To prove Theorem 4.6.2, we will use the same approach as for the proof of 4.5.8. We start by the following variant of Lemma 4.5.11.

Lemma 4.6.5. *If R is a pre-λ ring and $r \in R$, the coefficient of $\mathbf{t}^{\vec{d}}$ in*

$$\log\left(((1-t_1)(1-t_2)\cdots(1-t_k)(1+t_1+\cdots+t_k))^r\right) \in R\otimes_{\mathbb{Z}}\mathbb{Q}[[t_1,\ldots,t_k]]$$

is

$$\begin{cases} -\sum_{md'=d}\frac{1+(-1)^{d'}}{d'}p'_m(r) & \text{if } \mathbf{t}^{\vec{d}}=t_i^d \\ -\sum_{m\vec{d'}=\vec{d}}\binom{\sum\vec{d'}}{\vec{d'}}\frac{(-1)^{\sum\vec{d'}}}{\sum\vec{d'}}p'_m(r) & \text{otherwise.}\end{cases} \tag{4.6.5.1}$$

In particular, if $R\otimes\mathbb{Q}$ is normed, we find that the sum of the norms of the coefficients of $\mathbf{t}^{\vec{d}}$ for a fixed total degree $d=\sum\vec{d}$ is bounded by

$$\sum_{m|d,m\neq d}(k)^{d/m}||p'_m(r)||, \tag{4.6.5.2}$$

Proof. The proof is the same as for Lemma 4.5.11, except that in the final summation one needs to note that $m=d$ gives zero in the first case and cannot occur in the second case. □

Proof of Theorem 4.6.2. Convergence in the weight topology follows from Corollary 4.3.8, and convergence in the point counting topology from Lemma 4.5.12 and Theorem 4.3.5. For Hadamard convergence, we also apply Theorem 4.3.5, and proceed as in the proof of Lemma 4.5.12 to prove the required absolute convergence: it suffices to study convergence of the power series

$$((1-t_1)(1-t_2)\cdots(1-t_k)(1+t_1+\cdots+t_k))^{Z_X(s)}$$

using Lemma 4.6.5. The point of the latter is to exploit the factors $(1-t_1)\cdots(1-t_k)$ to cancel out the contribution from $p'_d(r)$ in (4.6.5.2), which otherwise would have given a term which would have obstructed convergence in the estimates below. Suppose $k\leqslant q^{\dim X}$. Then, using the estimate (4.5.12.1) on the Möbius-inverted power sums of $Z_X(s)$, and the fact that there is no $m=d$ term, we may bound (4.6.5.2) by

$$C\cdot(k^d q^{\dim X}+dq^{(2+d/2)\dim X}).$$

In particular, if $k < q^{\dim X}$, we conclude the series converges absolutely for

$$||t_1||_H, \ldots, ||t_k||_H \leqslant q^{-\dim X}.$$

This concludes the Hadamard case. $\square$

Remark 4.6.6. For $X/\mathbb{C}$ smooth, one can see that the Betti numbers

$$\dim H^i(C^{(d_1,d_2,\ldots,d_k)}X(\mathbb{C}),\mathbb{Q}) = \dim H^i(C^{\overbrace{(1,1,\ldots,1)}^{\sum d_i}}X(\mathbb{C}),\mathbb{Q})^{S_{d_1}\times S_{d_2}\times\cdots\times S_{d_k}}$$

stabilize as $(d_1, d_2, \ldots, d_k) \to \infty$ using representation stability for the cohomology of pure configuration spaces combined with the Pieri rule.

In more detail,[2] representation stability implies that for $d = \sum d_i \gg i$,

$$H^i(C^{\overbrace{(1,1,\ldots,1)}^{d}}X(\mathbb{C});\mathbb{Q}) \cong \bigoplus V(\lambda)^{\oplus c_\lambda},$$

where the direct sum is over partitions λ, the multiplicities c_λ are constants and $V(\lambda)$ is the representation of S_d whose corresponding Young diagram has shape $(d - \sum \lambda, \lambda)$. It suffices to check that $\dim V(\lambda)^{S_{d_1}\times S_{d_2}\times\cdots\times S_{d_k}}$ stabilizes as $(d_1, \ldots, d_k) \to \infty$. By the Pieri rule, this dimension is given by the number of ways of choosing subdiagrams with shapes

$$(d_1) = \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_k = (d - \sum \lambda, \lambda)$$

where each λ_i is obtained by adding d_i boxes to λ_{i-1}, no two in the same column. As long as each d_i is at least the length of the first row in λ, then this is equal to the number of ways to build λ in $k-1$ steps where at each step we add at most one box in each column (the remaining boxes that don't go towards building λ must go to the first row). This count only depends on λ and k. It would be interesting to explain the growth observed in Remark 4.6.4 from this perspective.

5. Rational curves on toric varieties

In this section we apply the results of Section 4 to generalize the main theorem of Bourqui's paper [4], which studies moduli spaces of rational curves on split toric varieties.

[2] We thank Nate Harman for explaining the following argument to us.

5.1. Geometric setting

We now introduce the necessary notation and give a brief overview of the geometric context of the theory of (split) toric varieties. We refer to the classical references on toric varieties (e.g. [14]) for details.

Let K be a field, $r \geqslant 1$ be an integer, and $U = \mathbb{G}_m^r$ a split torus of dimension r defined over K. We denote by $\mathcal{X}^*(U) = \mathrm{Hom}(U, \mathbb{G}_m)$ its group of characters, and $\mathcal{X}_*(U) = \mathrm{Hom}(\mathbb{G}_m, U)$ its group of cocharacters. Both are free $\mathbb{Z}$-modules of rank r, and there is a natural pairing

$$\langle \cdot, \cdot \rangle : \mathcal{X}^*(U) \times \mathcal{X}_*(U) \to \mathbb{Z}.$$

A projective and regular fan Σ of the $\mathbb{Z}$-module $\mathcal{X}_*(U)$ defines a smooth projective split toric variety X_Σ with open orbit U. We denote by $\Sigma(1)$ the set of the rays (that is, one-dimensional faces) of Σ. A generator ρ_α of such a ray $\alpha \in \Sigma(1)$ defines a U-invariant divisor D_α on X_Σ, and there is a short exact sequence

$$0 \to \mathcal{X}^*(U) \to \bigoplus_{\alpha \in \Sigma(1)} \mathbb{Z} D_\alpha \to \mathrm{Pic}(X_\Sigma) \to 0,$$

where the first map is given by sending $m \in \mathcal{X}^*(U)$ to

$$\sum_\alpha \langle m, \rho_\alpha \rangle D_\alpha.$$

From this, we in particular get the identity

$$\mathrm{rkPic}(X_\Sigma) = |\Sigma(1)| - r. \tag{5.1.0.1}$$

An anticanonical divisor is given by $\sum_{\alpha \in \Sigma(1)} D_\alpha$; we denote by $\mathcal{L}_0$ its class in the Picard group. The effective cone of X_Σ is the image in $\mathrm{Pic}(X_\Sigma) \otimes \mathbb{R}$ of the cone $\sum_\alpha \mathbb{R}_{\geqslant 0} D_\alpha$, so that in particular $\mathcal{L}_0$ lies in the interior of the effective cone of X_Σ.

Bourqui's proof introduces a regular fan Δ of the $\mathbb{Z}$-module $\mathrm{Pic}(X_\Sigma)^\vee$ whose support is the dual of the effective cone of X_Σ. The cones of maximal dimension of Δ have dimension $\rho = \mathrm{rkPic}(X_\Sigma)$. For every ray $i \in \Delta(1)$ we denote by m_i its generator. We write

$$a = \mathrm{lcm}\{\langle m_i, \mathcal{L}_0 \rangle,\ i \in \Delta(1)\}. \tag{5.1.0.2}$$

Since $\mathcal{L}_0$ is in the interior of the effective cone of X_Σ, this is a positive integer. In this setting, the invariant $\alpha^*(X_\Sigma)$ defined in Section 4.3 of Bourqui's paper may be expressed as:

$$\alpha^*(X_\Sigma) = \sum_{\substack{\delta \in \Delta \\ \dim(\delta) = \mathrm{rkPic}(X_\Sigma)}} \prod_{i \in \delta(1)} \frac{1}{\langle m_i, \mathcal{L}_0 \rangle}$$

(see [4, Remarque 5.23]). Note that $a^\rho\alpha^*(X_\Sigma)$ is a positive integer.

5.2. Möbius functions

By [4, Section 3.5], to every fan Σ there is a natural way to associate a subset B_Σ of $\{0,1\}^{\Sigma(1)}$ and a Möbius function $\mu^0_{B_\Sigma} : \{0,1\}^{\Sigma(1)} \to \mathbb{Z}$. Denoting by $B_\Sigma^{\min}$ the minimal elements of B_Σ (which by Bourqui's Lemme 3.8 contains only vectors of norm at least 2), it is straightforward from the definitions that our Möbius function $\mu_{B_\Sigma^{\min}} : \mathbb{Z}^{\Sigma(1)}_{\geqslant 0} \to \mathbb{Z}$ from Section 4.5.1 coincides with Bourqui's $\mu^0_{B_\Sigma}$ on $\{0,1\}^{\Sigma(1)}$, and is zero outside of $\{0,1\}^{\Sigma(1)}$.

We consider the elements $\mu_\Sigma(\vec{e}) \in \mathcal{M}_K$ such that

$$\prod_{x\in\mathbb{P}^1}\left(\sum_{\vec{n}}\mu^0_{B_\Sigma}(\vec{n})\mathbf{t}^{\vec{n}}\right) = \sum_{\vec{e}}\mu_\Sigma(\vec{e})\mathbf{t}^{\vec{e}}. \tag{5.2.0.1}$$

Since the answer to Bourqui's Question 3.5 is positive (see Remark 4.5.7), these are analogues in the Grothendieck ring of varieties of the elements $\mu^\chi_\Sigma(\vec{e})$ considered in Bourqui's proof.

Remark 5.2.1. Similarly to [3, Proposition 1-(3)] and [4, Proposition 5.18], we can show, using the universal torsor formalism, that

$$\sum_{\vec{n}}\mu^0_{B_\Sigma}(\vec{n})\mathbb{L}^{-|\vec{n}|} = (1-\mathbb{L}^{-1})^{\mathrm{rkPic}(X_\Sigma)}[X_\Sigma]\mathbb{L}^{-\dim(X_\Sigma)}.$$

Thus, analogously to the arithmetic case, the value at $\mathbb{L}^{-1}$ of our motivic Euler product (5.2.0.1) may be thought of as a product of local densities with convergence factors.

The main idea of the proof of Theorem C will be to reduce the convergence of the motivic height zeta function to the convergence of series of the form

$$\sum_{\vec{e}\in\mathbb{Z}^{\Sigma(1)}_{\geqslant 0}}\mu_\Sigma(\vec{e})W_{\vec{e}}T^{|\vec{e}|} \tag{5.2.1.1}$$

where the $W_{\vec{e}}$ are elements in the completed Grothendieck ring of varieties. We briefly discuss here how this convergence can be checked in the different topologies in play.

5.2.2. Dimensional topology

From the proof of Lemma 4.2.5, we see that

$$\dim\mu_\Sigma(\vec{e}) < \frac{|\vec{e}|}{2}.$$

Thus, convergence of the series (5.2.1.1) for $|T| < \mathbb{L}^{-1+\eta}$ follows as soon as one has estimates of the form

$$\dim W_{\vec{e}} \leqslant \epsilon|\vec{e}|$$

for ϵ such that $0 < \epsilon < \frac{1}{2}$. In particular, this gives us an analogue in the Grothendieck ring of varieties to Bourqui's Corollaire 3.4, which is sufficient to lift Bourqui's proof to the Grothendieck ring of varieties.

5.2.3. Hadamard convergence

Denote $M_\Sigma = \sum_{\vec{n}\neq 0} |\mu^0_{B_\Sigma}(\vec{n})|$ and let e_Σ be the minimal number of non-zero coordinates of a vector in B_Σ. According to Lemma 4.5.12 and Remark 4.5.13, if $q > M_\Sigma^{1/e_\Sigma}$, there exists $\delta > 0$ such that the series

$$\sum_{\vec{e}} \mu_\Sigma(\vec{e}) T^{|\vec{e}|}$$

converges absolutely for $||T||_H < q^{-1+\delta}$. We deduce that for ϵ such that $0 < \epsilon < \delta$, if we have bounds

$$||W_{\vec{e}}||_H < q^{\epsilon|\vec{e}|},$$

then the series (5.2.1.1) converges for $||T||_H < q^{-1+\delta-\epsilon}$.

5.2.4. Point counting convergence

Point counting convergence is handled similarly to Hadamard convergence: if $\delta > 0$ is such that for every prime power q the series of point counts

$$\sum_{\vec{e}} \#_{\mathbb{F}_q} \mu_\Sigma(\vec{e}) T^{|\vec{e}|}$$

converges for $|T| < q^{-1+\delta}$, then it suffices to have bounds

$$\#_{\mathbb{F}_q} W_{\vec{e}} < q^{\epsilon|\vec{e}|}$$

for some ϵ such that $0 < \epsilon < \delta$.

5.3. Statement

Now that we have introduced all of the data of the problem, we can state our result more precisely.

Theorem 5.3.1. *Let K be a field and X_Σ a smooth and projective split toric variety over K with open orbit U. For every integer $d \geqslant 0$, we denote by $U_{0,d}$ the quasi-projective variety parameterizing K-morphisms $\mathbb{P}^1_K \to X_\Sigma$ with image intersecting U and of anticanonical degree d. Let ρ be the rank of the Picard group of X_Σ.*

(1) *There exists a real number $\eta > 0$ such that the series*

$$(1-(\mathbb{L}T)^a)^\rho \left(\sum_{d\geqslant 0}[U_{0,d}]T^d\right), \tag{5.3.1.1}$$

where a is the integer defined in (5.1.0.2), converges for $||T|| < ||\mathbb{L}||^{-1+\eta}$ in the dimensional topology. Its value at $\mathbb{L}^{-1}$ is non-zero and equal to

$$a^\rho \alpha^*(X_\Sigma)\mathbb{L}^r(1-\mathbb{L}^{-1})^{-\rho}\prod_{x\in\mathbb{P}^1}\left(1+\sum_{\vec{n}}\mu_{B^0_\Sigma}(\vec{n})T^{|\vec{n}|}\right)\Bigg|_{T=\mathbb{L}^{-1}}.$$

(2) *Assume now $K=\mathbb{F}_q$ finite. Then the convergence of (5.3.1.1) also holds in the point counting topology. If in the notation of 5.2.3 one has $q > M_\Sigma^{1/e_\Sigma}$, then it holds in the Hadamard topology.*

Remark 5.3.2. The statement in [4] is for the series

$$(1-\mathbb{L}T)^\rho\sum_{d\geqslant 0}[U_{0,d}]T^d.$$

Indeed, as we will see below, the proof consists in writing the series $Z(T)$ as a finite sum of terms of the form $C_i(T)R_i(T)$ where C_i is a rational function such that $(1-\mathbb{L}T)^\rho C_i(T)$ has no pole at $\mathbb{L}^{-1}$ and $R_i(T)$ is a series which converges for $||T|| < ||\mathbb{L}||^{-1+\eta}$. Thus, while multiplying by $(1-\mathbb{L}T)^\rho$ is enough to be able to evaluate at $\mathbb{L}^{-1}$ (and thus sufficient for Bourqui's purposes), to eliminate some potential other poles of the rational functions $C_i(t)$ and obtain convergence for all $||T|| < ||\mathbb{L}||^{-1+\eta}$ one needs to multiply by some additional factors.

5.4. Proof of the theorem

The rest of the section is devoted to a proof of Theorem 5.3.1. This requires a careful analysis of Bourqui's proof, checking that the convergence statements can be adapted to our more general setting. Additionally to the dimensional bounds from [4], we will also need some estimates from [3].

Following Bourqui, we denote

$$Z^{\mathrm{mot}}_{\mathbb{P}^1,U,h_0}(T)=\sum_{d\geqslant 0}[U_{0,d}]T^d.$$

Essentially, the proof consists in writing $Z^{\mathrm{mot}}_{\mathbb{P}^1,U,h_0}(T)$ as a finite sum of series of the form (5.2.1.1), and checking convergence for each of them following the discussion in Section 5.2.

The first important step in Bourqui's proof is a universal torsor argument, expressing each space $[U_{0,d}]$ in terms of certain spaces of zero-cycles on $\mathbb{P}^1$. This leads to identity (5.4) in [4], after which, in the beginning of section 5.4, motivic Möbius inversion is applied. The resulting series is decomposed into a finite number of series depending on different parameters, and the contributions of which are studied separately:

$$Z^{\mathrm{mot}}_{\mathbb{P}^1,U,h_0}(T) = (\mathbb{L}-1)^{-\rho} \sum_{A\subset\Sigma(1)} (-1)^{|A|} \sum_{\delta\in\Delta} Z_{A,\delta}(T) \tag{5.4.0.1}$$

where Δ is a fan with support the dual of the effective cone of X_Σ. We do not give more details here, because, as Bourqui remarks in the beginning of Section 5.4, all of his computations not involving convergence issues are valid in the Grothendieck ring of varieties $\mathcal{M}_K$.

To study convergence, one has to distinguish between different cases. We first study the case $A = \varnothing$.

For every δ, there is a further decomposition

$$Z_{\varnothing,\delta} = \sum_{J\subset\Sigma(1)} (-1)^{|J|} Z_{\varnothing,\delta,J}(T). \tag{5.4.0.2}$$

We are going to show, as in Bourqui's proof, that terms $Z_{\varnothing,\delta,J}(T)$ where $J = \varnothing$ and δ is of maximal dimension (that is, $\dim(\delta) = \rho$) give the main pole.

Proposition 5.4.1.

(1) *Let $\delta \in \Delta$. There exists a real number $\eta > 0$ such that the series*

$$(1-(\mathbb{L}T)^a)^{\dim(\delta)} Z_{\varnothing,\delta,J}(T)$$

converges for $||T|| < ||\mathbb{L}||^{-1+\eta}$ in the dimensional topology. If K is finite, its specialization via the zeta measure converges in the point counting topology. When $q > M_\Sigma^{1/e_\Sigma}$, it converges in the Hadamard topology.

(2) *If J is nonempty and δ is of maximal dimension, the value of the series from (1) at $\mathbb{L}^{-1}$ is zero.*

(3) *The value of the series*

$$(1-(\mathbb{L}T)^a)^{\rho} \sum_{\substack{\delta\in\Delta\\ \dim(\delta)=r}} Z_{\varnothing,\delta,\varnothing}(T)$$

at $T = \mathbb{L}^{-1}$ is

$$a^\rho \alpha^*(X_\Sigma) \mathbb{L}^{|\Sigma(1)|} \sum_{\vec{e}\in\mathbb{Z}^{\Sigma(1)}_{\geqslant 0}} \mu_\Sigma(\vec{e})\mathbb{L}^{-|\vec{e}|},$$

which is a non-zero element of $\widehat{\mathcal{M}}_K$.

Proof. The key step in Bourqui's argument is to decompose

$$Z_{\varnothing,\delta,J}(T) = \sum_{\vec{e}} \mu_\Sigma(\vec{e}) Z_{\varnothing,\delta,J,\vec{e}}(T)$$

and rewrite $Z_{\varnothing,\delta,J,\vec{e}}(T)$, as in the statement of [4, Lemme 5.26, (ii)]; it is a geometric series over a truncated cone denoted by $\mathcal{C}(\delta(1))_{J,\vec{e}}$, and one separates it into a product of infinite and finite geometric series. What is important for us here is that through this procedure, we can write

$$Z_{\varnothing,\delta,J,\vec{e}}(T) = \mathbb{L}^{|\Sigma(1)|-|\vec{e}|} C_\delta(T) Q_{\delta,\vec{e}}(T)$$

where $C_\delta(T)$ is a rational function (coming from the factors which are infinite geometric series) not depending on $\vec{e}$, but only on J and on δ, and $Q_{\delta,\vec{e}}(T)$ is a polynomial (coming from the finite geometric sums), of the form

$$Q_{\delta,\vec{e}}(T) = \sum_{y} (\mathbb{L}T)^{\langle y, \mathcal{L}_0\rangle}.$$

Here the summation is over elements y of the set denoted by $\mathcal{C}(I_{J,2})_{J,\vec{e}}$ by Bourqui, the size of which is bounded by $|\vec{e}|^{|\Sigma(1)|}$ according to the proof of Lemme 3 in [3] (see bottom of page 192). Note that Bourqui's polynomial $P_{I'}(T)$ is what we denote $\mathbb{L}^{|\Sigma(1)|}\mathbb{L}^{-|\vec{e}|}Q_{\delta,\vec{e}}(T)$.

We have

$$(1-(\mathbb{L}T)^a)^{\dim(\delta)} Z_{\varnothing,\delta,J}(T) = (1-(\mathbb{L}T)^a)^{\dim(\delta)} C_\delta(T) \mathbb{L}^{|\Sigma(1)|} \sum_{\vec{e}} \mu_\Sigma(\vec{e}) \mathbb{L}^{-|\vec{e}|} Q_{\delta,\vec{e}}(T).$$

We see from the expression

$$C_\delta(T) = \prod_{i\in I(\delta)} \left(\frac{1}{1-(\mathbb{L}T)^{\langle m_i, \mathcal{L}_0\rangle}} - 1 \right)$$

where $I(\delta)$ is a subset of $\delta(1)$ (the set of rays of the cone δ), and from the definition of a, that the rational function

$$(1-(\mathbb{L}T)^a)^{\dim(\delta)} C_\delta(T)$$

has no poles, and we therefore may turn to the analysis of the series

$$\sum_{\vec{e}} \mu_\Sigma(\vec{e}) \mathbb{L}^{-|\vec{e}|} Q_{\delta,\vec{e}}(T).$$

Let us first consider the dimensional topology. By the top of page 193 in the proof of Lemme 3 in [3], for $y \in \mathcal{C}(I_{J,2})_{J,\vec{e}}$, we have bounds

$$0 \leqslant \langle y, \mathcal{L}_0 \rangle \leqslant C|\vec{e}| \tag{5.4.1.1}$$

for an explicit positive constant C, so that for $\eta > 0$ sufficiently small and $||T|| < ||\mathbb{L}||^{-1+\eta}$, the polynomial $Q_{\delta,\vec{e}}(T)$ takes values with dimension bounded by $\epsilon|e|$ for some small $\epsilon > 0$, and using the discussion in 5.2.2 we may conclude.

We now turn to the Hadamard and point counting topologies. Using (5.4.1.1) together with the fact that the polynomial $Q_{\delta,\vec{e}}(T)$ has polynomially many terms, for η sufficiently small and $||T||_H < ||\mathbb{L}||_H^{-1+\eta}$, the values of $Q_{\delta,\vec{e}}(T)$ are bounded by $q^{\epsilon|e|}$ for some small $\epsilon > 0$. By the discussion in 5.2.3, we see that our series converges for $||T||_H < ||\mathbb{L}||_H^{-1+\eta}$ for some $\eta > 0$. We proceed similarly in the point counting case. This proves the first statement.

We now come to the second statement. From [4, Lemme 5.26,(ii)], we see that if J is nonempty and δ of maximal dimension, then $I(\delta)$ is a strict subset of $\delta(1)$, so that $C_\delta(T)$ comprises strictly less than $\dim(\delta)$ factors, and so

$$(1 - (\mathbb{L}T)^a)^{\dim(\delta)} C_\delta(T)$$

has a zero at $\mathbb{L}^{-1}$. This together with the convergence proved above yields the result.

It remains to prove the last statement. Assume J is empty, and let δ be of maximal dimension. In this case, from [4, Lemme 5.26,(ii)], we see that $I(\delta) = \delta(1)$ and that our polynomial $Q_{\delta,\vec{e}}$ is in fact constant equal to 1. A quick computation then shows that the value of

$$(1 - (\mathbb{L}T)^a)^{\mathrm{rkPic}(X_\Sigma)} \sum_{\substack{\delta \in \Delta \\ \dim(\delta) = \mathrm{rkPic}(X_\Sigma)}} C_\delta(T)$$

at $T = \mathbb{L}^{-1}$ equals $a^\rho \alpha^*(X_\Sigma)$. From this we deduce the value of the limit. It follows from Lemma 4.2.5 that the limit is non-zero. $\square$

We now come to the case $A \neq \varnothing$. The argument is similar: there is a decomposition

$$Z_{A,\delta}(T) = \sum_{J \subset \Sigma(1) \setminus A} (-1)^{|J|} Z_{A,\delta,J}(T), \tag{5.4.1.2}$$

where

$$Z_{A,\delta,J}(T) = \sum_{\vec{e}} \mu_\Sigma(\vec{e}) Z_{A,\delta,J,\vec{e}}(T).$$

Proposition 5.4.2. *Let δ be a cone of Δ, A a non-empty subset of $\Sigma(1)$ and J a subset of $\Sigma(1)\setminus A$. There exists $\eta > 0$ such that the series*

$$(1 - (\mathbb{L}T)^a)^{\dim(\delta)} Z_{A,\delta,J}(T)$$

converges for $||T|| < ||\mathbb{L}||^{-1+\eta}$ in the dimensional topology. If $k = \mathbb{F}_q$ is finite, it converges in the point counting topology. If moreover $q > M_\Sigma^{1/e_\Sigma}$, then it also converges in the Hadamard topology. If $\dim(\delta) = \rho$, then the value of this series at $\mathbb{L}^{-1}$ is zero.

Proof. The proof proceeds similarly to the one of Proposition 5.4.1. According to [4, Lemme 5.28 (ii)], we may write

$$Z_{A,\delta,T,\vec{e}}(T) = C_\delta(T)R_{\delta,\vec{e}}(T),$$

where $C_\delta(T)$ is a rational function which does not depend on $\vec{e}$, and $R_{\delta,\vec{e}}(T)$ (denoted by $R_{I'}(T)$ in Bourqui's paper) is a power series with coefficients in $Z[\mathbb{L}]$. As in the previous proposition, we again have that the rational function

$$(1 - (\mathbb{L}T)^a)^{\dim(\delta)} C_\delta(T)$$

has no poles, and has a zero at $\mathbb{L}^{-1}$ if δ is of maximal dimension. Thus, essentially the only difference with the previous case is that the polynomial factor $Q_{\delta,\vec{e}}$ has been replaced with a non-polynomial one, and we need to work a little bit more to get sufficient bounds.

Writing $R_{\delta,\vec{e}}(T)$ out explicitly, we see that we are interested in the convergence properties of the series

$$\sum_{\vec{e}} \mu_\Sigma(\vec{e})\mathbb{L}^{-|\vec{e}|} \sum_{\substack{(h_\alpha)_{\alpha\in A}\\ h_\alpha \geqslant e_\alpha}} \mathbb{L}^{-\sum_{\alpha\in A}(h_\alpha - e_\alpha)} \sum_y (\mathbb{L}T)^{\langle y, \mathcal{L}_0\rangle} \tag{5.4.2.1}$$

where the sum over y is taken over a finite subset of the dual of the effective cone, the size of which is bounded polynomially in the h_α and e_α, according to the middle of page 197 in the proof of Lemme 4 in [3]. In the same reference, we also see that for y in this set, there is a positive constant C such that

$$0 \leqslant \langle y, \mathcal{L}_0\rangle \leqslant C\left(\sum_{\alpha\in A}(h_\alpha - e_\alpha) + |\vec{e}|\right).$$

Using the latter, we see that for $||T|| < ||\mathbb{L}||^{-1+\eta}$ the dimension of the term corresponding to (h_α) is bounded by

$$-(1-\epsilon)\left(\sum_{\alpha\in A}(h_\alpha - e_\alpha)\right) + \epsilon|\vec{e}|$$

for some small ϵ. From this we see that the $\vec{e}$-term of (5.4.2.1) converges in the dimensional topology for $||T|| < ||\mathbb{L}||^{-1+\eta}$ and takes values with dimension bounded by $\epsilon|\vec{e}|$. By 5.2.2, we have the desired convergence of the series (5.4.2.1) in the dimensional topology.

In the same way, the Hadamard norm of the $\vec{e}$-term of (5.4.2.1) is bounded by

$$\sum_{\substack{(h_\alpha)_{\alpha\in A}\\ h_\alpha\geqslant e_\alpha}} q^{-(1-\epsilon)\sum_{\alpha\in A}(h_\alpha-e_\alpha)}q^{\epsilon|\vec{e}|}=\left(\frac{1}{1-q^{-1+\epsilon}}\right)^{|A|}q^{\epsilon|\vec{e}|}$$

which by 5.2.3 is enough to deduce Hadamard convergence. Point counting convergence is handled in the same way. □

To conclude the proof of Theorem 5.3.1, we combine the decompositions in (5.4.0.1), (5.4.0.2) and (5.4.1.2) with Propositions 5.4.1 and 5.4.2, to show that

$$(1-(\mathbb{L}T)^a)^{\mathrm{rkPic}(X_\Sigma)}Z^{\mathrm{mot}}_{\mathbb{P}^1,U,h_0}(T)$$

is a finite sum of series that converge for $||T||<||\mathbb{L}||^{-1+\eta}$. Moreover, the only ones that give a non-zero contribution to the value at $\mathbb{L}^{-1}$ are those corresponding to $A=\varnothing$, $J=\varnothing$ and δ of maximal dimension, and their contribution is given by Proposition 5.4.1-(3). Using relation (5.1.0.1), we conclude that the value of our series at $T=\mathbb{L}^{-1}$ is

$$a^\rho\alpha^*(X_\Sigma)\mathbb{L}^r(1-\mathbb{L}^{-1})^{-\rho}\sum_{\vec{e}\in\mathbb{Z}^{\Sigma(1)}_{\geqslant 0}}\mu_\Sigma(\vec{e})\mathbb{L}^{-|\vec{e}|}.$$

Remark 5.4.3. In fact, the proof of Lemma 4.2.5 allows us to deduce that this value is of the form

$$a^\rho\alpha^*(X_\Sigma)\mathbb{L}^\rho+\text{terms of lower dimension},$$

where we recall that $a^\rho\alpha^*(X_\Sigma)$ is a positive integer.

6. The configuration random variable

In this section we prove the following generalization of Theorem B, which also strengthens and provides a more natural formulation of [19, Corollary B].

Theorem 6.0.1. *Let X be a geometrically irreducible variety over a field K. If*

(1) *K is arbitrary, X is stably rational, and ϕ is the measure to $\widehat{\mathcal{M}_K}$ or*
(2) *$K=\mathbb{C}$ and ϕ is the Hodge measure to $\widehat{K_0(\mathrm{HS})}$, or*
(3) *$K=\mathbb{F}_q$ and ϕ the zeta measure to $\mathcal{H}_1$,*

then

$$\lim_{d\to\infty}\frac{[C^{\lambda\cdot *^d}(X)]_\phi}{[C^{|\lambda|+d}(X)]_\phi}=C^\lambda_X\left(\frac{1}{1+\mathbb{L}_\phi^{\dim X}}\right).$$

Here the element $C_X^\lambda\left(\frac{1}{1+\mathbb{L}_\phi^{\dim X}}\right)$ will be made sense of by showing that the map

$$a \mapsto C_X^\lambda(a)_\phi$$

extends by continuity to the closure of $\mathbb{Z}[\mathbb{L}^{\pm 1}] \cong \phi(\mathbb{Z}[\mathbb{L}^{\pm 1}])$, which contains

$$\frac{1}{1+\mathbb{L}_\phi^{\dim X}} = \mathbb{L}_\phi^{-\dim X}\frac{1}{1+\mathbb{L}_\phi^{-\dim X}} = \mathbb{L}_\phi^{-\dim X} - \mathbb{L}_\phi^{-2\dim X} + \dots.$$

We also note that in case (1) of the theorem, the assumption that X is stably rational is only there to ensure $Z_X^{\mathrm{Kap}}(t)$ is weakly rational, and could be replaced with that condition.

6.1. Continuity of labeled configuration spaces

We now prove a lemma giving the continuity properties required to make reasonable sense of $C^\lambda\left(\frac{1}{1+\mathbb{L}_\phi^{\dim X}}\right)$ and similar quantities. In the dimension topology (and thus also for the Hodge measure) a very strong continuity on the entire Grothendieck ring follows immediately from the definitions. The case of the Hadamard topology is more subtle because we do not know any suitable general bounds on the Hadamard norm of a labeled configuration space. However, if we restrict to $\mathbb{Z}[\mathbb{L}^{\pm 1}]$, which is enough for our purposes here, a simple estimate will suffice.

Lemma 6.1.1. *Suppose X is a variety over a field K.*

(1) *The map*

$$\mathcal{M}_X \to \mathcal{M}_{C^\lambda(X)},\ a \mapsto C_X^\lambda(a)$$

is continuous for the dimension topologies on both sides and thus induces a continuous map

$$\widehat{\mathcal{M}_X} \to \widehat{\mathcal{M}_{C^\lambda(X)}},\ a \mapsto C_X^\lambda(a).$$

(2) *If $K = \mathbb{C}$ and ϕ is the Hodge measure to $\widehat{K_0(\mathrm{HS})}$, then composition of the arrow from (1) with the forgetful map $\widehat{\mathcal{M}_{C^\lambda(X)}} \to \widehat{\mathcal{M}_{\mathbb{C}}}$ and ϕ induces a continuous map*

$$\widehat{\mathcal{M}_X} \to \widehat{K_0(\mathrm{HS})},\ a \mapsto C_X^\lambda(a)_\phi.$$

(3) *If $K = \mathbb{F}_q$ and ϕ is the zeta measure, the map*

$$\mathbb{Z}[\mathbb{L}^{\pm}] = \phi(\mathbb{Z}[\mathbb{L}^{\pm}]) \to \mathcal{H}_1,\ a \mapsto C_X^\lambda(a)_\phi = Z_{C_X^\lambda(a)}(t)$$

extends to a continuous map

$$\widehat{\mathbb{Z}[\mathbb{L}^{\pm 1}]} = \overline{\phi(\mathbb{Z}[\mathbb{L}^{\pm 1}])} \to \mathcal{H}_1,\ a \mapsto C_X^{\lambda}(a)_{\phi}$$

where the domain is the completion of $\mathbb{Z}[\mathbb{L}^{\pm 1}]$ *for the induced norm, or equivalently the closure of* $\phi(\mathbb{Z}[\mathbb{L}^{\pm 1}])$ *in* $\mathcal{H}_1$*.*

Proof. The first two statements are immediate from the definitions via simple dimension estimates. We now treat the third statement; in the proof, we denote the Hadamard norm by $||\cdot||$.

Fix $a \in \mathbb{Z}[\mathbb{L}^{\pm 1}]$, and consider a perturbation $a + h$. We write h in the slightly unusual expansion $h = \sum_{i=1}^{N} \epsilon_i \mathbb{L}^{k_i}$ for $\epsilon_i \in \{\pm 1\}$ and $\epsilon_i = \epsilon_j$ if $k_i = k_j$; for example, we expand

$$2\mathbb{L}^2 - 3\mathbb{L}^{-1} = \mathbb{L}^2 + \mathbb{L}^2 - \mathbb{L}^{-1} - \mathbb{L}^{-1} - \mathbb{L}^{-1}.$$

Note that as a consequence, we may express the Hadamard norm of h in the following way:

$$||h|| = \sum_{i=1}^{N} ||\mathbb{L}||^{k_i}.$$

Using the definition (2.2.4.2) of C_X^{λ}, the symmetric power addition formula (2.2.4.1), and the identity $\mathrm{Sym}_X^k(\mathbb{L}^{\ell}) = \mathbb{L}^{k\ell}$ valid in any relative Grothendieck ring, we find

$$C_X^{\lambda}(a+h) - C_X^{\lambda}(a) = \sum_{\substack{\sum_{j=0}^{N} \lambda_j = \lambda \\ \lambda_0 \neq \lambda}} \left(\mathrm{Sym}_X^{\lambda_0}(a) \prod_{j=1}^{N} \mathrm{Sym}_X^{\lambda_j}(\epsilon_i) \right)_{*,X} \mathbb{L}^{\sum_{j=1}^{N} |\lambda_j| k_j}$$

Here the sum is over tuples of partitions $(\lambda_0, \ldots, \lambda_N)$ such that their multiplicity vectors sum up to the multiplicity vector of λ, and such that the partition λ_0 is not equal to λ.

The key point is that, since λ and a are fixed, the terms $(\ldots)_{*,X}$ appearing vary over a finite set of classes (for all h). We can thus bound their Hadamard norms above by a real number M, so that we obtain

$$\begin{aligned} ||C_X^{\lambda}(a+h) - C^{\lambda}(a)|| &\leqslant M \sum_{\substack{\sum_{j=0}^{N} \lambda_j = \lambda \\ \lambda_0 \neq \lambda}} ||\mathbb{L}||^{\sum_{j=1}^{N} |\lambda_j| k_j} \\ &\leqslant M \left(\left(1 + \sum_{j=1}^{N} ||\mathbb{L}||^{k_j} \right)^{|\lambda|} - 1 \right) \\ &= M \left((1 + ||h||)^{|\lambda|} - 1 \right) \end{aligned}$$

To go from the first to the second line, observe that if the multiplicities of λ are $m_1, \ldots,$ m_n, then

$$\sum_{\substack{\sum \lambda_j = \lambda \\ \lambda_0 \neq \lambda}} ||\mathbb{L}||^{\sum_{j=1}^N |\lambda_j| k_j} = \left(\prod_{l=1}^{n} \left(\sum_{\sum_{j=1}^N a_j \leqslant m_l} ||\mathbb{L}||^{\sum a_j k_j} \right) \right) - 1$$

and the l-th term inside the product is bounded above by

$$\left(1 + \sum_{j=1}^{N} ||\mathbb{L}||^{k_j}\right)^{m_l} = \sum_{\sum_{j=1}^N a_j \leqslant m_l} \binom{m_l}{a_1,\, a_2, \ldots, a_N,\, m_l - \sum_{j=1}^N a_j} ||\mathbb{L}||^{\sum_{j=1}^N a_j k_j}.$$

This verifies continuity at a, and we conclude since a was arbitrary. $\square$

6.2. Configuration spaces with power series labels

Let now

$$f(s) = a_0 + a_1 s + \ldots \in \mathcal{M}_X[[s]]$$

Using property (2.2.4.1), for any generalized partition $\lambda = (n_i)_i$, there is a natural way of defining a power series

$$C_X^\lambda(f(s)) \in \mathcal{M}_{C^\lambda(X)}[[s]].$$

Explicitly, we have, denoting by $()_{*,X}$ the pullback to $C^\lambda(X)$,

$$\begin{aligned} C_X^\lambda(f(s)) &= \left(\prod_{j \geqslant 1} \mathrm{Sym}_X^{n_j} \left(\sum_{i \geqslant 0} a_i s^i \right) \right)_{*,X} \\ &= \left(\prod_{j \geqslant 1} \left(\sum_{\substack{(n_{i,j})_i \\ \sum_i n_{i,j} = n_j}} \left(\prod_i \mathrm{Sym}_X^{n_{i,j}}(a_i) \right) s^{\sum_i i n_{i,j}} \right) \right)_{*,X} \\ &= \sum_{\substack{(n_{i,j})_{i,j} \\ \sum_i n_{i,j} = n_j}} \left(\prod_{i,j} \mathrm{Sym}_X^{n_{i,j}}(a_i) \right)_{*,X} s^{\sum_{i,j} i n_{i,j}}. \end{aligned} \tag{6.2.0.1}$$

Arguing similarly to our proof of Lemma 6.1.1, we find

Lemma 6.2.1. *In the settings of Lemma 6.1.1, if $f(s) \in \mathbb{Z}[[s]]$ and if f converges absolutely at $\mathbb{L}_\phi^r$, then so does $C_X^\lambda(f(s))$ and*

$$C_X^\lambda(f(\mathbb{L}_\phi^r)) = C_X^\lambda(f(s))\big|_{s = \mathbb{L}_\phi^r}.$$

In the dimension topology, the statement holds for $f(s) \in \mathcal{M}_X[[s]]$, but we will not use this added generality here.

Remark 6.2.2. By expanding, one can see that the coefficient of t^λ in the motivic Euler product

$$\prod_{x\in X}(1+f_x(s)(t_1+t_2+\cdots)) = \prod_{x\in X}\left(1+\sum_{i,j}a_{i,x}s^it_j\right)$$

is exactly the image in $\mathcal{M}_K[[s]]$ of $C_X^\lambda(f(s))$.

6.2.3. An alternative expression for configuration spaces with power series labels

To motivate what we want to establish in this section, let us discuss quickly the classical set-up that we are trying to imitate. When X is a finite set and $(f_x(s))_{x\in X}$ is a family of formal power series indexed by X, the expansion of the finite product

$$\prod_{x\in X}(1+f_x(s)t) \tag{6.2.3.1}$$

can be written as

$$\sum_{n\geqslant 0}\left(\sum_{c\in C^n(X)}\prod_{x\in c}f_x(s)\right)t^n,$$

where $C^n(X)$ is the set of configurations of n distinct points of X. In other words, the family $(f_x(s))_{x\in X}$ defines a function on $C^n(X)$ given by $c \mapsto \prod_{x\in c}f_x(s)$, and the coefficient of t^n in the expansion of (6.2.3.1) is the summation of this function over $C^n(X)$.

In the usual Grothendieck ring dictionary, elements of $\mathcal{M}_{C^\lambda(X)}$ can be thought of as motivic functions defined on $C^\lambda(X)$, and taking the class of such an element in $\mathcal{M}_K$ may be thought of as summation over $C^\lambda(X)$. In view of Remark 6.2.2, if one replaces finite products with motivic Euler products, one should expect $C_X^\lambda(f(s))$ to be equal in $\mathcal{M}_{C^\lambda(X)}$ to a motivic Euler product relatively to $C^\lambda(X)$: to reproduce the fact that above every configuration we take the product over points of that configuration, the product will be over the universal configuration.

For X a variety over K and λ a partition, let $\mathbf{c}_\lambda/C^\lambda X$ denote the universal configuration,

$$\mathbf{c}_\lambda = \{(c,x)|x\in c\} \subset C^\lambda X\times X$$

Denote by $j_\lambda : \mathbf{c}_\lambda \to X$ the projection. Given $f(s)\in\mathcal{M}_X[[s]]$, let $j_\lambda^* f$ be the corresponding series in $\mathcal{M}_{\mathbf{c}_\lambda}[[s]]$ given by pullback of coefficients along j_λ.

Proposition 6.2.4. *We have the equality*

$$C_X^\lambda(f(s)) = \prod_{y \in \mathbf{c}_\lambda / C^\lambda X} (j_\lambda^* f)_y(s)$$

in $\mathcal{M}_{C^\lambda(X)}[[s]]$.

Proof. We start by expanding the right-hand side. For every i, denote $b_i = j_\lambda^* a_i$. For every $j \geqslant 1$, there is a projection map

$$\pi_j : C^\lambda X = \left(\prod_{i \geqslant 1} \mathrm{Sym}^{n_i} X \right)_{*,X} \to (\mathrm{Sym}^{n_j} X)_{*,X},$$

using which we introduce

$$\mathbf{c}_\lambda^{(j)} = \{(x, c) \in \mathbf{c}_\lambda,\ x \in \pi_j(c)\}.$$

By definition, $\mathbf{c}_\lambda$ is the disjoint union of the $\mathbf{c}_\lambda^{(j)}$, $j \geqslant 1$. We also define $b_i^{(j)}$ to be the restriction of b_i to $c_i^{(j)}$, so that in $\mathcal{M}_{\mathbf{c}_\lambda}$, we have

$$b_i = \sum_j b_i^{(j)}.$$

In other words, $b_i^{(j)}$ is the pullback of a_i to $\mathbf{c}_\lambda^{(j)}$. We now expand

$$\begin{aligned} \prod_{y \in \mathbf{c}_\lambda / C^\lambda X} (j_\lambda^* f)_x(s) &= \prod_{j \geqslant 1} \prod_{y \in \mathbf{c}_\lambda^{(j)} / C^\lambda X} \left(b_{0,y}^{(j)} + b_{1,y}^{(j)} s + b_{2,y}^{(j)} s^2 + \cdots \right) \\ &= \sum_{\substack{(n_{i,j})_{i,j} \\ \sum_i n_{i,j} = n_j}} \left(\prod_{i,j} \mathrm{Sym}^{n_{i,j}} (b_i^{(j)} / C^\lambda X) \right)_* s^{\sum_{i,j} n_{i,j} i} \end{aligned}$$

in $\mathcal{M}_{C^\lambda(X)}$. Note that only terms satisfying $\sum_i n_{i,j} = n_j$ for every j will contribute since for each j, the above product over $\mathbf{c}_\lambda^{(j)}$ relatively to $C^\lambda X$ is finite, with n_j factors. Using the expansion (6.2.0.1) of $C_X^\lambda(f(s))$, it remains to compare, for every collection of integers $(n_{i,j})_{i,j}$ such that $n_j = \sum_{i \geqslant 0} n_{i,j}$ for every $j \geqslant 1$, the classes of

$$\left(\prod_{i,j} \mathrm{Sym}^{n_{i,j}} (a_i) \right)_* \quad \text{and} \quad \left(\prod_{i,j} \mathrm{Sym}^{n_{i,j}} (b_i^{(j)} / C^\lambda X) \right)_*$$

in $\mathcal{M}_{C^\lambda(X)}$. For this, observe that the projections $\mathbf{c}_\lambda^{(j)} \to X$ induce the projection

$$\left(\prod_{i,j} {}_{/C^\lambda X} \mathrm{Sym}^{n_{i,j}}(\mathbf{c}_\lambda^{(j)}/C^\lambda X)\right)_* \to \left(\prod_{i,j} \mathrm{Sym}^{n_{i,j}}(X)\right)_*,$$

(where the product on the left is taken relatively to $C^\lambda(X)$), so that

$$\left(\prod_{i,j} \mathrm{Sym}^{n_{i,j}}(b_i^{(j)}/C^\lambda X)\right)_* \in \mathcal{M}_{C^\lambda X}$$

will be the pullback of $\left(\prod_{i,j} \mathrm{Sym}^{n_{i,j}}(a_i)\right)_*$ via this map. On the other hand, this map is actually the identity: since for every j, we have $\sum_{i,j} n_{i,j} = j$, a point $(c_{i,j})_{i,j} \in \left(\prod_{i,j} \mathrm{Sym}^{n_{i,j}}(X)\right)_*$ completely determines the configuration above it. $\square$

6.3. Proof of Theorem 6.0.1

We proceed as in the proof of Theorem 4.3.5 to show that the limit can be expressed as the value of a certain series; the results of 6.2 will then allow us to conclude. We write $n = \dim X$. By weak rationality,

$$(1-\mathbb{L}^n t)\sum_{d\geqslant 0}[C^d(X)]t^d = (1-\mathbb{L}^n t)\prod_{x\in X}(1+t) = (1-\mathbb{L}^n t)\frac{Z_X^{\mathrm{Kap}}(t)}{Z_X^{\mathrm{Kap}}(t^2)} \tag{6.3.0.1}$$

converges absolutely at $t = \mathbb{L}_\phi^{-n}$ to an invertible element. In particular, the sequence of partial sums $[C^d(X)]_\phi \mathbb{L}_\phi^{-nd}$ converges to an invertible element as $d \to \infty$.

On the other hand, denoting by $\mathbf{c}_\lambda \to C^\lambda(X)$ the universal configuration, note that the generating series of $C^{\lambda\cdot *^d}(X)$ in $\mathcal{M}_{C^\lambda(X)}$ has the following motivic Euler product decomposition

$$\sum_{d\geqslant 0}[C^{\lambda\cdot *^d}(X)]t^d = \prod_{x\in(X\times C^\lambda X - \mathbf{c}_\lambda)/C^\lambda X}(1+t).$$

Consider the quotient of power series with coefficients in $\mathcal{M}_{C^\lambda(X)}$

$$\begin{aligned}\frac{\sum_{d\geqslant 0}[C^{\lambda\cdot *^d}(X)]t^d}{\sum_{d\geqslant 0}[C^d(X)\times C^\lambda(X)]t^d} &= \frac{\prod_{x\in(X\times C^\lambda X-\mathbf{c}_\lambda)/C^\lambda X}(1+t)}{\prod_{x\in(X\times C^\lambda X)/C^\lambda X}(1+t)} \\ &= \prod_{x\in \mathbf{c}_\lambda/C^\lambda X}\frac{1}{1+t}.\end{aligned} \tag{6.3.0.2}$$

Applying Proposition 6.2.4 and integrating over $C^\lambda(X)$, we obtain an identity of power series with coefficients in $\mathcal{M}_K$

$$\frac{\sum_{d\geqslant 0}[C^{\lambda\cdot *^d}(X)]t^d}{\sum_{d\geqslant 0}[C^d(X)]t^d} = C^\lambda_X\left(\frac{1}{1+t}\right). \tag{6.3.0.3}$$

By Lemma 6.2.1, this power series converges absolutely at $t=\mathbb{L}_\phi^{-n}$ to $C^\lambda_X\left(\frac{1}{1+\mathbb{L}_\phi^{-n}}\right)$.

Multiplying the left-hand side of (6.3.0.3) by $(1-\mathbb{L}^n t)\sum_{d\geqslant 0}[C^d(X)]t^d$, which as observed above also converges absolutely at $t=\mathbb{L}_\phi^{-n}$, we conclude that the series

$$(1-\mathbb{L}^n t)\left(\sum_{d\geqslant 0}[C^{\lambda\cdot *^d}(X)]t^d\right)$$

also converges absolutely at $t=\mathbb{L}_\phi^{-n}$. In particular, the sequence of partial sums $[C^{\lambda\cdot *^d}(X)]\mathbb{L}^{-nd}$ converges, and we apply our usual trick to compute

$$\begin{aligned}
\lim_{d\to\infty}\frac{[C^{\lambda\cdot *^d}(X)]_\phi}{[C^{|\lambda|+d}(X)]_\phi} &= \mathbb{L}_\phi^{-n|\lambda|}\frac{\lim_{d\to\infty}[C^{\lambda\cdot *^d}(X)]_\phi\mathbb{L}_\phi^{-nd}}{\lim_{d\to\infty}[C^d(X)]_\phi\mathbb{L}_\phi^{-nd}}\\
&= \mathbb{L}_\phi^{-n|\lambda|}\frac{\left((1-\mathbb{L}^n t)\sum_{d\geqslant 0}[C^{\lambda\cdot *^d}(X)]t^d\right)|_{t=\mathbb{L}_\phi^{-n}}}{\left((1-\mathbb{L}^n t)\sum_{d\geqslant 0}[C^d(X)]t^d\right)|_{t=\mathbb{L}_\phi^{-n}}}\\
&= \mathbb{L}_\phi^{-n|\lambda|}\left(\frac{\sum_{d\geqslant 0}[C^{\lambda\cdot *^d}(X)]t^d}{\sum_{d\geqslant 0}[C^d(X)]t^d}\right)\Bigg|_{t=\mathbb{L}_\phi^{-n}}\\
&= \mathbb{L}_\phi^{-n|\lambda|}\left(C^\lambda_X\left(\frac{1}{1+t}\right)\right)\Bigg|_{t=\mathbb{L}_\phi^{-n}}\\
&= \mathbb{L}_\phi^{-n|\lambda|}C^\lambda_X\left(\frac{1}{1+\mathbb{L}_\phi^{-n}}\right)\\
&= C^\lambda_X\left(\frac{1}{1+\mathbb{L}_\phi^{n}}\right).
\end{aligned}$$

The last equality follows from an application of eq. (2.2.4.3) (which extends by continuity to the present setting) to move the coefficient $\mathbb{L}_\phi^{-n|\lambda|}$ into the labels.

7. Hadamard convergence and cohomological stability

It is by now well-known (cf., e.g., [11,7,12]) that for a sequence of smooth varieties over $\mathbb{F}_q$, cohomological stability combined with suitable bounds on Betti numbers implies stabilization of point-counts through the Grothendieck-Lefschetz trace formula. By essentially the same computation, we show in Theorem 7.0.1 below that weight stabilization combined with suitable dimension bounds on the cohomology implies Hadamard stabilization.

Note that cohomological stabilization implies weight stabilization (as long as the stabilization is as Galois representations; e.g., if the stabilization is realized by maps of algebraic varieties). In particular, point-counting results previously established via stable cohomology can be upgraded automatically to Hadamard stabilization: see Corollary 7.0.3 for a precise statement. For example, this gives an alternate proof of Theorem B for varieties admitting compactifiable lifts[3] to characteristic zero by applying the étale representation stability and dimension bounds of Farb-Wolfson [12]. Moreover, this also furnishes a natural strategy that may be useful in proving further cases of our meta-conjecture — when weight convergence to a Hadamard function is known, to establish Hadamard convergence it will suffice to establish bounds on the Betti numbers.

Theorem 7.0.1. *Suppose $X_n/\mathbb{F}_q$ is a sequence of smooth varieties such that*

(1) *There is a Hadamard function $Z_\infty(t)$ such that, in the weight topology,*

$$\lim_{n\to\infty} Z_{X_n}(tq^{-\dim X_n}) = Z_\infty(t). \tag{7.0.1.1}$$

(2) *There exist real numbers $C > 0$ and $1 \leqslant \lambda < \sqrt{q}$ such that, for any n, there exists a prime ℓ coprime to q such that*

$$\dim_{\mathbb{Q}_\ell} H^i(X_{n,\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell) \leqslant C\lambda^i$$

Then (7.0.1.1) holds also in the Hadamard topology.

Remark 7.0.2. We give the proof below, but first, some comments:

(1) The flexibility of allowing ℓ to vary with n in (2) can be useful — for example, this variation appears in the bounds for the Betti numbers of Hurwitz spaces established in [11] (there the restriction $\ell > n$ arises because at a point in the argument one needs the derived S_n-invariants in $\mathbb{Z}/\ell$ cohomology to be equal to the S_n-invariants).
(2) The statement strikes a balance between brevity and utility, but the same method applies more generally: for example, for varieties that are not smooth with suitable bounds on compactly supported cohomology instead of cohomology, or to directly deduce the stabilization of L-functions in Remark 4.6.6 from cohomological stability and Betti bounds for the corresponding local systems as established (under lifting hypotheses) in [12].
(3) Ekedahl [10] has defined a topology of polynomial growth refining the dimensional topology on the Grothendieck ring of varieties, and a slight modification of the proof of Theorem 7.0.1 shows that the zeta measure to the ring of Hadamard functions is

[3] Ho [17] has established étale homological stability for configuration spaces in positive characteristic without a lifting hypothesis, but for our result one would need the same for all colored configuration spaces as well.

continuous in this topology. The definition of Ekedahl's topology is a bit ad hoc, and one of our motivations for introducing the Hadamard topology was to find a more natural way to express a similar constraint.

(4) It is not possible in general to obtain bounds on the Betti numbers from Hadamard convergence, and in fact it is easy to construct sequences of varieties that are equal in $K_0(\mathrm{Var}/K)$ but have unbounded Betti numbers: a very simple example is $X_n = (\mathbb{A}^1 - \{n \text{ points}\}) \sqcup \{n \text{ points}\}$. For a connected example, one can take X_n to be $\mathbb{P}^2$ with n lines intersecting at a single point removed and then blow up at n points (if working over a non-algebraically closed field like $\mathbb{F}_q$, one should take care to match up lines and points with the same fields of definition); for any $n \geqslant 1$, $X_n = [\mathbb{P}^2] - 1$. The issue is that cohomology classes of the same weight can cancel if they appear in odd and even degrees. For a sequence of smooth projective varieties, where the cohomological degree determines the weight, stabilization in the weight topology already implies stabilization of Betti numbers.

Proof. We argue with divisors, i.e. elements of the completion of $\mathbb{Z}[\mathbb{C}^\times]$. So, write D_n for the divisor attached[4] to $Z_{X_n}(tq^{-\dim X_n})$ and D_∞ for the divisor attached to $Z_\infty(t)$. For any divisor D, we write $\tau_m(D)$ for the part supported in the region $|z| \geqslant q^{-m/2}$. In particular, it is easy to see that

$$\lim_{m\to\infty} \tau_m(D_\infty) = D_\infty$$

in the Hadamard topology.

On the other hand, by the definition of convergence in the weight topology, for any $m > 0$, there exists $N > 0$ such that for all $n \geqslant N$,

$$\tau_m(D_n) = \tau_m(D_\infty). \tag{7.0.2.1}$$

Now, for any such n, taking ℓ as in (2) and fixing an embedding $\mathbb{Q}_\ell \to \mathbb{C}$, the Grothendieck-Lefschetz fixed point formula combined with Poincaré duality gives

$$D_n = \sum_{i=0}^{2\dim X_n} (-1)^i [H^i(X_{n,\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)^* \otimes_{\mathbb{Q}_\ell} \mathbb{C}],$$

where the brackets denote taking the class in $K_0(\mathrm{Rep}_\mathbb{Z})$ (using the identification with $\mathbb{Z}[\mathbb{C}^\times]$ explained in Section 3.1). Then, combining (7.0.2.1) with Deligne's [8, Théorème I] eigenvalue bounds which give that any eigenvalue α of Frobenius on $H^i(X_{n,\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)^*$ satisfies $|\alpha| \leqslant q^{-i/2}$, we obtain

[4] Recall from Section 3.1 that to a meromorphic function f we assign the divisor of $\frac{1}{f(1/t)}$, a normalization chosen so that the zeta function $\frac{1}{1-qt}$ of $\mathbb{A}^1_{\mathbb{F}_q}$ corresponds to $[q]$.

$$||D_n - \tau_m(D_\infty)||_H \leqslant q^{-m/2} \sum_{i=0}^{m} \dim_{\mathbb{Q}_\ell} H^i(X_{n,\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell) + \sum_{i=m+1}^{2\dim X_n} q^{-i/2} \dim_{\mathbb{Q}_\ell} H^i(X_{n,\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell).$$

Invoking the bounds in hypothesis (2), we find this sum is bounded above by

$$C\left(\frac{\lambda}{\sqrt{q}}\right)^m \left(m+1+\frac{1}{1-\frac{\lambda}{\sqrt{q}}}\right),$$

and this bound goes to zero as $m \to \infty$ because $\lambda < \sqrt{q}$. Thus, we conclude that also $D_n \to D_\infty$ in the Hadamard topology, as desired. $\square$

In particular, we obtain the following corollary, making precise the statement that cohomological stabilization plus Betti bounds gives Hadamard convergence:

Corollary 7.0.3. *Suppose $X_n/\mathbb{F}_q$ is a sequence of smooth varieties and $\ell \neq \mathrm{char}(\mathbb{F}_q)$ is a prime such that*

(1) *For each $i \geqslant 0$, there is an $N \geqslant 0$ such that for all $n \geqslant N$ the $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$-representations $H^i(X_{n,\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)^{\mathrm{ss}}$ are isomorphic to a fixed representation H^i_∞ (here the superscript denotes semisimplification).*
(2) *There exists $C > 0$ and $\lambda < \sqrt{q}$ such that, for all n and i*

$$\dim_{\mathbb{Q}_\ell} H^i(X_n, \mathbb{Q}_\ell) \leqslant C\lambda^i.$$

Then, $Z_{X_n}(q^{-\dim X_n} t)$ converges in the Hadamard topology to

$$\sum_{i=0}^{\infty} (-1)^i [H_{i,\infty}],$$

where here $H_{i,\infty}$ denotes the dual of H^i_∞, with scalars extended to give a complex vector space by the choice of any embedding $\mathbb{Q}_\ell \to \mathbb{C}$.

Appendix A. Computations

In Table 1 we give the first 250 terms for the divisor

$$\lim_{(d_1,d_2)\to\infty} Z_{C^{(d_1,d_2)}(\mathbb{A}^1_{\mathbb{F}_q})} / W Z_{\mathrm{Sym}^{(d_1,d_2)}(\mathbb{A}^1_{\mathbb{F}_q})} = \lim_{(d_1,d_2)\to\infty} Z_{C^{(d_1,d_2)}(\mathbb{A}^1_{\mathbb{F}_q})} \left(tq^{-(d_1+d_2)}\right),$$

where here the limits are in the weight topology and we have used the identity

Table 1
First 250 terms of $\lim_{(d_1,d_2)\to\infty} Z_{C^{(d_1,d_2)}(\mathbb{A}^1_{\mathbb{F}_q})}\left(tq^{-(d_1+d_2)}\right)$.

i	Coefficient of $[q^{-i}]$
0	1
1	-3
2	5
3	-10
4	24
5	-55
6	118
7	-250
8	540
9	-1166
10	2475
11	-5218
12	11028
13	-23267
14	48830
15	-102167
16	213525
17	-445513
18	927444
19	-1927166
20	3999248
21	-8288404
22	17153790
23	-35457313
24	73212391
25	-151015163
26	311189028
27	-640657585
28	1317827566
29	-2708586539
30	5562810556
31	-11416477207
32	23413972647
33	-47988657094
34	98296020099
35	-201224291653
36	411703666030
37	-841899534112
38	1720748369045
39	-3515328234048
40	7178192714838
41	-14651215348621
42	29891622362909
43	-60960729520648
44	124274709833930
45	-253252619275830
46	515905274269151
47	-1050598369362088
48	2138748809597243
49	-4352556333294442
50	8855142419299783
51	-18010175104285365
52	36619803977908694
53	-74437884037740152
54	151271098981190102
55	-307330496794545563
56	624233017196670858
57	-1267601222149736713
58	2573455649992469320
59	-5223384420459129280

(*continued on next page*)

Table 1 (*continued*)

i	Coefficient of $[q^{-i}]$
60	10599650504339968588
61	-21504939006993240476
62	43620910664324846165
63	-88463413927558983487
64	179369094441716234105
65	-363620936146947211139
66	737003893336634408989
67	-1493525070407312843889
68	3026071553583774207695
69	-6130160318373268357117
70	12416305165787441941888
71	-25144482169769700475430
72	50912518513077694908444
73	-103071775867327392158564
74	208636308694120684565878
75	-422256725089154019803026
76	854478921031350689093305
77	-1728883425555349537772147
78	3497607554122553346247355
79	-7074876073480110138967125
80	14309034136324437898603161
81	-28936554185525659439497151
82	58509927574525580489119380
83	-118293195094794161305291884
84	239132486275311262352998014
85	-483356022483243207472458684
86	976891909453073575341693979
87	-1974139172168771694130837742
88	3988980483500702718090433305
89	-8059348467160056857850301254
90	16281439069254327401112771720
91	-32888298017019215826487386452
92	66427309605631891852427285538
93	-134155799021691025521771284468
94	270913472774442897719424024492
95	-547029744205693924330614993253
96	1104463220953743112679968006155
97	-2229730248996000703271900417220
98	4501060935168121797776308252779
99	-9085308660778996913683241591916
100	18336963259621884186312937330536
101	-37006564130480842135420881234823
102	74678296938835045366712451653048
103	-150686722006049417188882164883466
104	304033289328605736801837757889052
105	-613385462035346120893462386389806
106	1237407104790484830103451785458541
107	-2496083454393123235346360455263145
108	5034699273149624209432668615272245
109	-10154451234898616906382452153763994
110	20478984176727698229478103214975622
111	-41298085225883776492838324830483147
112	83276312671638650634105812715706158
113	-167912729770850527045567532995866050
114	338545314297377669135739970834099721
115	-682529641465357484326226939785869136
116	1375935937942737803716090764040548543
117	-2773622488464218826498184276726243059
118	5590740893108400697907757830485514668
119	-11268463168356409496415532461615555613
120	22710868528096610367194049445753934421

Table 1 (*continued*)

i	Coefficient of $[q^{-i}]$
121	-45769580932908800976831871086499330994
122	92234769856874390273795578514079010526
123	-185860598612913044801841864730269995334
124	374503005630066811126878392442709025314
125	-754569018607279234423667610817191802785
126	1520262851219368816074332126572089460056
127	-3062772964719969636087115981981018953541
128	6170035792263619351050455292566978783598
129	-12429042123060060151003262179674049196038
130	25036007215716677420539355986395133276393
131	-50427824677622751805359126326377802814807
132	101567199482982873264392954021879354129190
133	-204557360491172243478624563111868531696823
134	411960370485300375547687520519502226821470
135	-829611500542651667331806049712787706316782
136	1670603456362831206847362302632157611640072
137	-3363965724925806599011266967125981954243654
138	6773444935084913876696775096583596298183108
139	-13637908675027441436057116768747781739434355
140	27457838849409380315413796530515268407489675
141	-55279688909583876580585833350065956851424889
142	111287337512971930313744554502136783509139787
143	-224030470620565185138114187204885173045566046
144	450972288103407951076429498671472480900894500
145	-907766787667089046300297874897356031995072893
146	1827177046255434145173073629090501998350189480
147	-3677639154926299735399831830273114930706933417
148	7401844724023358326931783359492391415803949804
149	-14896814949606945772712282120059272199690249898
150	29979866434245490844583144348449344360693631079
151	-60332177366298214408691600910066186887843803897
152	121409187297844700570116626172426598599222626198
153	-244307928267034553210257870812929829060224442649
154	491594743709525111581865915372346686734405187936
155	-989146826836016362658122074912175022534944715678
156	1990207473240877597531413403273254338876802547040
157	-4004240887908066523314775718374722296383353278432
158	8056130304128719786977903098713542750921357140249
159	-16207551036094612708384128936868306743927319581255
160	32605669827995537882141376003470500998619660883838
161	-65592449277668135923145148203994025195285602114161
162	131947066627455132690094280750865960630570254544336
163	-265418368252534732163388638198968661024046332532369
164	533885006556978841673859509747827333034790892137060
165	-1073866157949160528842644873292715224609980727316442
166	2159923824107976255787679933553822112189909230569860
167	-4344229119554769725531975946302444010351029324981531
168	8737218126698544324983354343181783399234144113040996
169	-17571949987350926356148244810740733916329389383280610
170	35338915495737920749569941345730118033124022425352917
171	-71067838406403965873181708846730412674466081241594550
172	142915644912669381123934767287154566227048796839910300
173	-287391133596243553978025526428865096061369492892216231
174	577901771904942496377897298680702776641561566472346509
175	-1162042191566025096073333560418006767663163414264761452
176	2336560963156321258206594903332568731935777435716478404
177	-4698073670829204910429951149497168205623270962654074924
178	9446048030016615435964025795014143220692909203774526167
179	-18991891895201248092411829357483627878037061328499916676
180	38183364495333086204204682114930004931552019076194474230
181	-76765868701357294714795824370173618727343318391694633802

(*continued on next page*)

Table 1 (*continued*)

i	Coefficient of $[q^{-i}]$
182	1543299738817128363232226800641053107211397414498887 09502
183	-3102563357342233242160766656419694597055886654971738838 04
184	6237052792026962892950665213173417178858869618759065831 59
185	-125379544093830051608914230066957982626342558882790082 3773
186	252035998792170249565975465975841655638934630915693558 8716
187	-506625688338208779356496499361861513725197897190239754 7338
188	101835849993049061340084664643449620265979769939248557 70002
189	-204693073509408351517470491993200449983193340899607692 99637
190	411428792841013831161845940189413788890333304769186559 28999
191	-826942676907643868273639626249209699681620356538276511 21126
192	166205513942814486717606830876444989743453016535327886 029097
193	-334044906428413611391355165977888136830209140798816585 873928
194	671357414335178326430744817824055982152630817133280117 748150
195	-134924972347141688231685558402681837162012743706800302 0111147
196	271156862626123216214567930855088587880462816786768965 3316398
197	-544927440247041067725148243887892350519500144175214526 4632550
198	109508207108700156161768404667485514691417066507502915 15619432
199	-220061804918080748212266332716487803993073607173523676 04021410
200	442214297311259308668187004241179066951455981037804118 15103936
201	-888609783513152533129577852213146085505087877882962965 18972158
202	178558157622312958198293390705731236882640973045841129 901489278
203	-358788644063983246567535867359362571245002544503178214 240054286
204	720921801100698726386694533287526587431214885606660302 982340624
205	-144853234152424644234224898697043919066460786588674806 3901262327
206	291044159379167741057142369550941690285688624517984108 3766133211
207	-584763588380100343402564810716816505920070068556811413 8941772899
208	117487749302797552972465375653151746300661977777344306 57895529207
209	-236045517673850685948214637481086053143196477588718796 36797902301
210	474230985922035765343554586969519838780510454330430639 55216443887
211	-952741699463081478407804033606821570594410795658690265 59277533215
212	191404251025076306444836648744474710925748794337580471 279140380135
213	-384520256153027720832343588144677015994231262916517617 702578354310
214	772463885412221247750959518172309275593339223655535204 987540150838
215	-155177424925367369673889523955154417361393302663944203 7615040708270
216	311724092456703975624525606873245316864062786544362886 4456247013851
217	-626186564906367870796839245509111249743029674009747256 8689378905507
218	125784968121727875743294590261046075210949088510616880 80277086644833
219	-252665205174499083635130587691915192387096044873731718 85812655365167
220	507520866234534995994958145417136583757638037733803554 10356385019761
221	-101942248171446376404811306396751761494336671084842843 72353399446503 4
222	204760613968597157212010029876713486804465127371302600 56317112402258 6
223	-411273399298062141627923580121235278530544185511903002 50425904006483 2
224	826051014634815005139464364267018370249005956581169493 76077095263326 1
225	-165911021014873329025207221062343414665201492278756420 18137141750780 37
226	333223628393918320322870714982695097748172052775405547 84488838979156 65
227	-669250330737973264439332559039079216575828735558696285 52093398730651 99
228	134410661348851573749411424651509961708740016296651696 04682521868959 572
229	-269942474252195997040220486592606100651380466102466648 66230534368138 664
230	542127174568073338034464682264854665635652903560897928 03625447444191 968
231	-108873864845805016850142237474283157059409319666122162 16579756522447 5771
232	218644559248718136649200739632525468463253521897267370 72529735683249 2893
233	-439082717498431167805788938314933663970201549385476217 02292994464089 0819
234	881752513762637047350955880772283529471535676582889977 85583010406490 4050
235	-177067894854041740465691170495244635782212584789776473 57769250451603 37542
236	355570526994229711706325215857602348133086146717430231 19257643237536 47955
237	-714010435490329676814259210767311268203655168331979375 58805282086576 22002
238	143375944935163870185874426049819801522556072643275042 28993603568963 746649
239	-287899568915927931625807019732061921934518873856131536 85521881169643 775108
240	578094423254575985026240054648304032807550018804079912 96344942812757 465951
241	-116077927318088147658885109110973474677707297047338797 61732201411567 0291868
242	233073911365640918583982238401098113091737014051890351 05144804752511 0442208

Table 1 (*continued*)

i	Coefficient of $[q^{-i}]$
243	-467983879434681066427015485809761438819926018767785270070916286071708841159
244	939639759662761645826607434203471991500948183010290880113340609940625137998
245	-1886623297459580540003449408545708292094079321179393559084522176323337329589
246	3787933707788042453135351202393917950224275817218928913807658304446716058235
247	-7605240761445472998617207473729266666313927167438658855852176879694593013499
248	15269226468590914708207748063222774107557618777595720957055954826812698597855
249	-30655939363580606732561996894410171491023294598074730332881732843263810990419
250	61546844703480411367930079662454293248852333175511986470297385611751909013867

Table 2
The divisor $Z_{C^{\bullet d_1 \star d_2}(\mathbb{A}^1_{\mathbb{F}_q})}\left(tq^{-(d_1+d_2)}\right)$ for $d_1 = d_2 = 40$.

i	Coefficient of $[q^{-i}]$	i	Coefficient of $[q^{-i}]$
0	1	40	7178192706102
1	-3	41	-14651215147355
2	5	42	29891619749371
3	-10	43	-60960704596332
4	24	44	124274516700328
5	-55	45	-253251337471208
6	118	46	515897749760655
7	-250	47	-1050558448626228
8	540	48	2138554410364751
9	-1166	49	-4351677301167434
10	2475	50	8851418068846937
11	-5218	51	-17995282472068951
12	11028	52	36563266171699586
13	-23267	53	-74233098656280122
14	48830	54	150560424516434836
15	-102167	55	-304959028474877462
16	213525	56	616599979165804930
17	-445513	57	-1243838077427284749
18	927444	58	2501726843328379367
19	-1927166	59	-5013001590559463419
20	3999248	60	9998887821446433255
21	-8288404	61	-19831768033161995124
22	17153790	62	39068695645092664153
23	-35457313	63	-76346502518625937950
24	73212391	64	147772610374307278073
25	-151015163	65	-282802569405937507518
26	311189028	66	533995597251319358859
27	-640657585	67	-992196836341429077092
28	1317827566	68	1807725664871875879551
29	-2708586539	69	-3213777325101403051314
30	5562810556	70	5535545342239809752109
31	-11416477207	71	-9140023894242417884359
32	23413972647	72	14237505104399687238881
33	-47988657094	73	-20437644238323518538138
34	98296020099	74	26158394219496320597829
35	-201224291653	75	-28551600325000321382432
36	411703666030	76	25039544129795914295127
37	-841899534112	77	-16225470533207349260900
38	1720748369045	78	6777326492252181076930
39	-3515328234048	79	-1343840109164979124000

$$[\mathrm{Sym}^{(d_1,d_2)}(\mathbb{A}^1_{\mathbb{F}_q})] = [\mathrm{Sym}^{d_1}(\mathbb{A}^1_{\mathbb{F}_q})][\mathrm{Sym}^{d_2}(\mathbb{A}^1_{\mathbb{F}_q})] = \mathbb{L}^{d_1}\mathbb{L}^{d_2} = \mathbb{L}^{d_1+d_2}$$

to identify the division with renormalization of the variable t by $q^{-(d_1+d_2)}$.

When $q = 2$, this limiting divisor does not appear to correspond to a Hadamard function (cf. Remark 4.6.4), but nonetheless our results show the sequence also converges in the point-counting topology. To further illustrate how this can occur, in Table 2 we give the exact formula of the divisor when $d_1 = d_2 = 40$. One can then compute to see the cancellation for point-counting: the $q = 2$ Hadamard norm is 395.538829916911 but the $q = 2$ point-counting semi-norm is 0.181319714263592.

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