

Exponential-Wrapped Distributions on Symmetric Spaces*

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Abstract. In many applications, the curvature of the space supporting the data makes the statistical modeling challenging. In this paper we discuss the construction and use of probability distributions wrapped around manifolds using exponential maps. These distributions have already been used on specific manifolds. We describe their construction in the unifying framework of affine locally symmetric spaces. Affine locally symmetric spaces are a broad class of manifolds containing many manifolds encountered in the data sciences. We show that on these spaces, exponential-wrapped distributions enjoy interesting properties for practical use. We provide the generic expression of the Jacobian appearing in these distributions and compute it on two particular examples: Grassmannians and pseudohyperboloids. We illustrate the interest of such distributions in a classification experiment on simulated data.

Key words. statistics on manifolds, exponential map, differential of the exponential map, affine locally symmetric spaces, Lie groups, wrapped distributions

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1. Introduction. Density estimation on manifolds has been the subject of theoretical studies for several decades (Hall, Watson, and Cabrera, 1987; Hendriks, 1990; Huckemann et al., 2010; Kim, 1998; Pelletier, 2005). More recently, probability densities on manifolds have also become tools of major interest in applied data science, from classification of video data on Grassmannian manifolds (Slama et al., 2015; Turaga et al., 2011), to modeling of hierarchical structures on hyperbolic spaces (Ding and Regev, 2020; Mathieu et al., 2019). An important difficulty is to define statistical models adapted to practical use for broad classes of manifolds; much of the focus has been on developing methods for specific manifolds, such as the sphere (Fisher, 1953; Hauberg, 2018; Kato and McCullagh, 2020). On a Riemannian

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manifold, a seemingly simple candidate family includes distributions whose densities with respect to the Riemannian measure are the normalized indicator functions

$$\varphi_{p,r} = \frac{1}{v(\mathcal{B}(p,r))} \mathbb{1}_{\mathcal{B}(p,r)},$$

where $\mathcal{B}(p,r)$ is the ball centered at p of radius r and v is the Riemannian volume. However, computing the normalization constant is typically nontrivial, as there are no closed form expressions for the volume of balls.

In this article, we focus on statistical models defined by pushing probability densities supported on tangent spaces to the manifold, using an exponential map. Since there exist various ways to push a density from a tangent space to the manifold, we refer to such densities as “exponential-wrapped densities.” Exponential-wrapped densities have been studied and used in many applications; see, for instance, Chevallier, Barbaresco, and Angulo (2015), Chevallier et al. (2016); Ding and Regev (2020); Falorsi et al. (2019); Jona-Lasino, Gelfand, and Jona-Lasino (2012); Kurtek et al. (2012); Mallasto and Feragen (2018); Mallasto, Hauberg, and Feragen (2019); Mathieu et al. (2019); Pelletier (2005); Slama, Wannous, and Daoudi (2014); Slama et al. (2015); Srivastava et al. (2005); Turaga et al. (2011). Most of these papers focus on individual manifolds, where the exponential-wrapped densities enjoy interesting properties. Our overarching contribution is to develop a unified framework, and corresponding theory and methodology, for exponential-wrapped modeling on affine locally symmetric spaces (ALSS). ALSS encompass most manifolds used in data science, including (pseudo-)Riemannian symmetric spaces and arbitrary Lie groups. For reasons mentioned later in the introduction, ALSS are likely to form the most general setting on which exponential-wrapped densities remain tractable.

Defining an exponential-wrapped density requires the existence of an exponential map. The exponential map is commonly defined for Riemannian manifolds using geodesics, or for Lie groups using one-parameter subgroups. However, both exponentials can be seen as exponential maps of an underlying affine connection. In this article, symmetric spaces refer to affine symmetric spaces in general, and not to Riemannian symmetric spaces.

Manifolds with affine connections are to Riemannian manifolds what affine spaces are to Euclidean vector spaces: they have a notion of straight lines but no distance. It is interesting to note that many statistical models on $\mathbb{R}^n$ do not depend on the Euclidean structure. For instance, defining a Gaussian distribution relies only on the affine structure and not on the distance. Similarly, exponential-wrapped models on Riemannian manifolds usually depend only on the affine connection associated with the metric. The main difference between the two settings is that the affine structure does not provide a notion of isotropic distributions.

In order to obtain tractable exponential-wrapped densities, it is important that the exponential map, its inverse, and its Jacobian determinant (see Figure 1.1), admit simple expressions. ALSS provide a broad class in which this is possible. First, exponential maps and their inverses on injectivity domains can be computed at a reasonable cost: they can be identified with a Lie group exponential. Second, we provide explicit expressions of the Jacobian determinants for arbitrary symmetric spaces. The differential of the exponential map is governed by a matrix second order differential equation: the equation of Jacobi fields. On locally symmetric spaces this equation has constant matrix coefficients, which enables the computation of the Jacobian determinant.

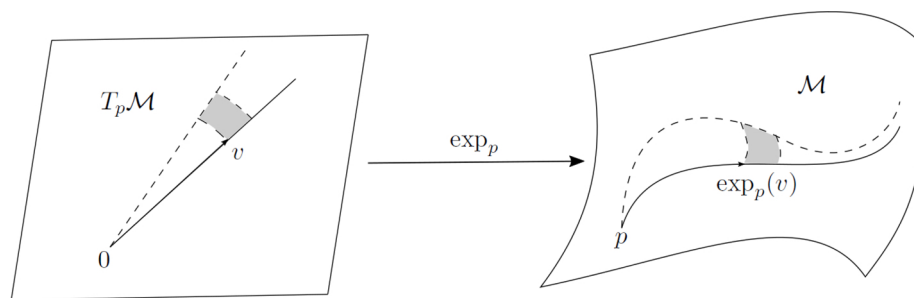


Figure 1.1. The infinitesimal volume change between the grey areas depends on how the neighboring geodesics are deviating or getting closer. It is given by the absolute value of the Jacobian determinant of the exponential map, and is determined by the curvature of the connection ∇ along the geodesic $\exp_p(tv)$.

Outside of ALSS, we expect this to happen on only a few specific manifolds. We are currently aware of only two examples of non-locally-symmetric manifolds appearing in data science where the exponential map, its inverse, and Jacobian determinant can also be computed easily: Gaussian distributions endowed with the Wasserstein Riemannian metric Chevallier, Kalunga, and Angulo (2017) and Kendall shape spaces Nava-Yazdani et al. (2020).

In section 2, we describe exponential-wrapped distributions on ALSS. This setting encompasses and generalizes most previously considered settings, while preserving all the advantages of wrapped distributions. In section 3 we give the formal definitions of ALSS and homogeneous symmetric spaces, and set some notations. In section 4 we provide the general expression of the Jacobian appearing in exponential-wrapped densities on ALSS, and compute it on two original examples: Grassmannian manifolds and pseudohyperboloids. In section 5, we present a classification experiment based on exponential-wrapped distributions. The experiment shows the interest of using multiple tangent spaces to model data. Section 6 concludes the paper.

2. Exponential-wrapped densities. Exponential-wrapped densities are traditionally used to define distribution on the circle S_1 ; see, for instance, Mardia (1972, p. 53). The density on the circle is obtained by taking a density on $\mathbb{R}$ and by wrapping it around a circle. Formally, if f is a density on $\mathbb{R}$, the wrapped density can be defined as

$$f_{2\pi}(\theta) = \sum_{k=-\infty}^{\infty} f(\theta + k2\pi).$$

Wrapped densities on a circle can sometimes be written in closed form; it is the case for instance when f is a Cauchy distribution. When the circle S_1 is viewed as a Riemannian manifold and $\mathbb{R}$ as a tangent space, the map $x \mapsto x \bmod 2\pi$ can be interpreted as a Riemannian exponential map. This point of view enables extension to more general manifolds endowed with an exponential map. In the vocabulary of measure theory, the exponential-wrapped probability is the pushforward of the probability in the tangent space by the exponential map. When the dimension of the space is greater than one, wrapping a density from a tangent space around the manifold usually requires taking into account a volume distortion. Indeed, the exponential map is generally not an area preserving map between the tangent

space with a Lebesgue measure and the reference measure on the manifold. In this paper, we focus on the cases where the probability distributions in the tangent spaces are contained in injectivity domains of the exponential maps. This is a restrictive assumption on manifolds such as spheres, where the injectivity domains are disks. However, as we will see in section 5.1, it holds, at least approximately, for most exponential-wrapped distributions used in practice. In this context, the difficulty does not lie in the computation of an infinite series as for most standard wrapped densities on circles, but in the computation of the volume distortion.

Start by giving a precise definition of exponential-wrapped distributions. Let $\mathcal{M}$ be a manifold with a reference measure vol , and an exponential map $\exp_p : T_p\mathcal{M} \rightarrow \mathcal{M}$ at p , a point in $\mathcal{M}$. Given λ , a probability distribution on $T_p\mathcal{M}$, the corresponding exponential-wrapped distribution is defined as the pushforward of λ by the exponential

$$(2.1) \quad \Lambda = \exp_{p*} \lambda,$$

where the $*$ refers to the pushforward by $\exp_p$: $\Lambda(A) = \lambda(\exp_p^{-1}(A))$. In the rest of the paper, we assume that λ is supported on a domain $U \subset T_p\mathcal{M}$ on which $\exp_p$ is injective, and that it has a density h with respect to a Lebesgue measure ν_p of $T_p\mathcal{M}$. Under these assumptions, the density f of Λ can be expressed from h and a volume change term. When $q = \exp_u$, we have

$$(2.2) \quad f(q) = \frac{d\Lambda}{dvol}(q) = \frac{d\exp_{p*}(\nu_p)}{dvol} \frac{d\Lambda}{d\exp_{p*}(\nu_p)}(q) = \frac{d\exp_{p*}(\nu_p)}{dvol}(q) h(u),$$

and when $q \notin \exp_p(U)$, $f(q) = 0$. The volume change term is determined by the Jacobian determinant of the differential of the exponential map, expressed in a suitable basis. Its computation is addressed in section 4.

Note that the density with respect to vol given by

$$(2.3) \quad f(q) = h(u)$$

when $q = \exp_u$ and $f(q) = 0$ when $q \notin \exp_p(U)$, can also be turned into a probability density by adding a global normalization factor

$$(2.4) \quad f(q) = \frac{1}{\alpha} h(u) \quad \text{with } \alpha = \int_{q \in \exp_p(U)} h(\log_p(q)) dvol.$$

Equations (2.2), (2.3), and (2.4) have sometimes been confused in the literature; see, for instance, Slama, Wannous, and Daoudi (2014); Slama et al. (2015); Srivastava et al. (2005); Turaga et al. (2011). Before focusing on wrapped densities, it is interesting to note that after being normalized, the density of (2.4) enjoys interesting properties in specific contexts. For instance, when $\mathcal{M}$ is a noncompact Riemannian symmetric space of dimension d , the densities

$$(2.5) \quad f(q | p, \Sigma) \propto h(\log_p(q)) = e^{-\frac{1}{2} \log_p(q)^T \Sigma^{-1} \log_p(q)}, \quad p, q \in \mathcal{M}, \Sigma \in \text{SPD}(d),$$

where $\log_p(q)$ is a coordinate expression of the inverse of the exponential map, have two remarkable properties: (i) f is the maximum entropy distribution for fixed Frechet average and

covariance; see Pennec (2006), and (ii) when Σ is isotropic the maximum likelihood estimator of p is the empirical Frechet average; see Said et al. (2017a,b). However, probability densities obtained from (2.4) often suffer from several practical limitations. (1) The normalization constant can be computed explicitly only in exceptional cases. (2) Sampling from the distribution is not straightforward, and may require numerical approximations. (3) The link between the parameter Σ and the covariance of the distribution is not explicit.

As we will see, these practical limitations do not hold for exponential-wrapped densities on symmetric spaces, which makes them particularly adapted to many practical situations.

Densities are explicit. On an arbitrary Riemannian manifold, such densities are hard to compute, since the exponential map and its inverse have no explicit forms. However, as we will see in the next section, on ALSS exponential maps are locally identified with Lie groups exponentials and are hence efficiently computed. Furthermore, we show in section 4 that on these spaces, the volume distortion induced by the exponential map is always tractable. Hence, the density itself is tractable.

Sampling is straightforward. In order to sample from f , it suffices to sample from h : if $U_1, \dots, U_n$ are identical and independently distributed (i.i.d.) random variables on a tangent space $T_p\mathcal{M}$ following the density h , then $X_1 = \exp_p(U_1), \dots, X_n = \exp_p(U_n)$ are i.i.d. random variables on $\mathcal{M}$ following the density f . Since the exponential map can be computed in closed form, exponential-wrapped densities on ALSS are trivial to sample from as long as one can sample from the pullback density on the tangent plane. This is in sharp contrast to the very substantial problems that are often faced in sampling from distributions supported on manifolds.

Correspondence between moments of f and h . A mean, or exponential barycenter, of a probability density f on $\mathcal{M}$ can be defined as a point $\bar{p}$ satisfying

$$\mathbb{E}_f[\log_{\bar{p}}(q)] = \int_{q \in \mathcal{M}} \log_{\bar{p}}(q) f(q) d\text{vol} = 0 \in T_{\bar{p}}\mathcal{M};$$

see Pennec (2019). Hence, if the mean $\int uh(u)du$ of h is $0 \in T_p\mathcal{M}$, then it can be checked that p is a mean of f . Higher intrinsic moments of the density f at p are usually defined as

$$(2.6) \quad m_p^k = \mathbb{E}_f[\log_p(q)^{\otimes k}] = \int_{y \in \mathcal{M}} \log_p(q)^{\otimes k} f(q) d\text{vol} = \int_{u \in T_p\mathcal{M}} u^{\otimes k} h(u) du,$$

where the second equality is obtained by the change of variable $u = \log_p(q)$. Hence the higher moments of f at p are the same as those of h . An important consequence is that the moments of h can be estimated by the empirical moments of f . This property does not hold for densities defined from (2.4) due to the absence of the volume correction.

3. Symmetric spaces.

3.1. Affine connections and ALSS. Let $\mathcal{M}$ be a manifold endowed with an affine connection ∇ . Recall that the connection enables differentiation of vector fields: given two vector fields X and Y on $\mathcal{M}$, $(\nabla_Y X)(p) \in T_p\mathcal{M}$ defines the derivative of the field X in the direction of the field Y at p , a tangent vector at p . This connection enables transportation of a vector $u \in T_{c(0)}\mathcal{M}$ along a differentiable curve $c(t)$ by imposing $\nabla_{c'(t)} u(t) = 0$; this is parallel transport of u along the curve c . A path γ is called geodesic if $\gamma'(t)$ is the parallel transport of $\gamma'(0)$ along γ :

$$\nabla_{\gamma'(t)}\gamma'(t) = 0.$$

Assume that $\gamma(0) = p$. The geodesics define an exponential map from tangent spaces to the manifold: $\exp_p(\gamma'(0)) = \gamma(1)$.

Each affine connection has a torsion tensor T defined as

$$T(u, v) = \nabla_u v - \nabla_v u - [u, v] = 0,$$

where u, v are vector fields and $[\cdot, \cdot]$ the Lie bracket between vector fields. For every Riemannian manifold there is an affine connection which has the same geodesics and exponential maps. If the affine connection is chosen with null torsion, the connection is unique and called the Levi-Civita connection. In the rest of the paper, it is always assumed that the torsion T of ∇ is null:

$$T = 0.$$

Though the expression of the torsion tensor does not appear explicitly in the rest of the paper, this assumption plays an important role in our main result through the equation of Jacobi fields.

Affine connections also have a curvature tensor defined by

$$R(u, v)w = \nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[u, v]} w,$$

where u, v, w are three vector fields. ALSS are defined as manifolds with an affine connection such that the derivative of the curvature tensor with respect to any vector field is always null:

$$\nabla R = 0.$$

The assumptions $T = \nabla R = 0$ encompass a large variety of spaces. An important case that we will address in this paper is when the connection ∇ arise from a (pseudo-)Riemannian metric. The manifold is then called a (pseudo-)Riemannian ALSS. As is described in Pennec and Lorenzi (2020), ALSS also contain another important class of spaces: arbitrary Lie groups endowed with their 0-connection.

They are a particularly interesting class of spaces since exponentials and logarithms can be identified with matrix counterparts, and the Jacobian of the exponential can be computed explicitly.

3.2. Homogeneous symmetric spaces. Alternatively, a homogeneous symmetric space can be characterized algebraically. It is a homogeneous space G/K with an involution σ which has the following properties: G is a connected Lie group, σ is an involutive automorphism, and K is an open subgroup of the set of fixed points of σ . Such a homogeneous space has a unique canonical connection which verifies that ∇ is equivariant under the action of G , $T = 0$, and $\nabla R = 0$. Hence, homogeneous symmetric spaces are also ALSS. The Lie algebra of the Lie group G can be decomposed into a direct sum $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, where $\mathfrak{k}$ and $\mathfrak{m}$ are the $+1$ and -1 eigenspaces of $d\sigma$. Hence, $\mathfrak{k}$ is the Lie algebra of K , and $\mathfrak{m}$ can be identified with the tangent space at eK of the quotient manifold, $\mathfrak{m} \sim T_{eK}G/K$, where e is the identity of the group.

A key feature for practical use of homogeneous symmetric spaces is that for $u \in \mathfrak{m}$, $\exp(u)K = \exp_{eK}(u)$, where the first exponential is the group exponential while the second is the exponential of the canonical affine connection; see Nomizu (1954, section 10). The exponential of the connection at an arbitrary point gK can be computed from the Lie group exponential by

$$\exp_{gK}(u) = g \exp(g^{-1}.u)K,$$

where $g^{-1}.u \in T_{eK}G/K \sim \mathfrak{m}$ is the action of g^{-1} on the tangent vector $u \in T_{gK}G/K$. Another important feature is that the action of $\exp(u \in \mathfrak{m})$ on a tangent vector $v \in T_{gK}G/K$ is the parallel transport of v from gK to $\exp(u)gK$ along $\exp(tu)gK$.

3.3. Identifications and notations. Nomizu showed in Nomizu (1954) showed that for an affine locally symmetric space $\mathcal{M}$, there is a neighborhood N_p around each $p \in \mathcal{M}$ such that N_p is isomorphic to a neighborhood of a homogeneous symmetric space. In the rest of the paper,

- $\mathcal{M}$ is a differentiable manifold with a connection ∇ such that $T=0$ and $\nabla R=0$, and p is an arbitrary reference point;
- N_p is a neighborhood of p identified with a neighborhood of a homogeneous symmetric space G/K with $p \sim eK$. The tangent space $T_p\mathcal{M}$ is identified with $\mathfrak{m}$, where $\mathfrak{m}$ is defined above in section 3.2.

4. The Jacobian of the exponential map.

4.1. Main ingredient. In this section we provide a general expression for the Jacobian determinant of the exponential map on ALSS. This expression is not entirely original, since it can be derived from Taniguchi (1984). However, it was never mentioned in the statistics and data science literature. Theorem 4.1 is expressed for an arbitrary point $q \in \mathcal{M}$. We have that $d\exp_q(u) : T_q\mathcal{M} \rightarrow T_{\exp_q(u)}\mathcal{M}$. On an arbitrary ALSS, there is no reference basis, scalar product, or volume measure in the tangent spaces. In order to define the Jacobian determinant

$$J_q(u) = \det(d\exp_q(u)),$$

we set an arbitrary basis $e_1, \dots, e_d$ of $T_q\mathcal{M}$ and parallel transport it to $T_{\exp_q(u)}\mathcal{M}$ along the geodesic $\exp_q(tu)$. Check that J_q is independent of the choice of basis $e_1, \dots, e_d$ of $T_q\mathcal{M}$. Note $\tau_t : T_q\mathcal{M} \rightarrow T_{\exp_q(tu)}\mathcal{M}$, the parallel transport between $T_q\mathcal{M}$ and $T_{\exp_q(tu)}\mathcal{M}$ along $\exp_q(tu)$. By definition the matrix of τ_1 in $e_1, \dots, e_d$ and $\tau_1(e_1), \dots, \tau_1(e_d)$ is the identity, hence

$$J_q(u) = \det(\tau_1 \circ \tau_1^{-1} \circ d\exp_q(u)) = \det(\tau_1^{-1} \circ d\exp_q(u)).$$

Since $\tau_1^{-1} \circ d\exp_q(u)$ is an endomorphism of $T_q\mathcal{M}$, its determinant is independent of a basis, hence $J_q(u)$ is independent of the basis of $T_q\mathcal{M}$.

Let $R_u : T_q\mathcal{M} \rightarrow T_q\mathcal{M}$ be the linear map given by $R_u(v) = R(v, u)u$, where R is the curvature tensor. Using the equation of Jacobi fields on ALSS the author of Taniguchi (1984) shows that the differential of the exponential is given by

$$d\exp_q(u) = \tau_1 \circ \sum_{n=0}^{\infty} \frac{(-R_u)^n}{(2n+1)!}.$$

Triangularizing the matrix of R_u over $\mathbb{C}$ leads to the following result.

Theorem 4.1. *Let R_u be the linear map defined above. Note its i th complex eigenvalue $\lambda_i(R_u)$ and its algebraic multiplicity n_i . The Jacobian determinant J_q of the exponential map at u in the basis $e_1, \dots, e_d$ and $\tau_1(e_1), \dots, \tau_1(e_d)$ is given by*

$$(4.1) \quad J_q(u) = \prod_i \left(\frac{\sinh\left(\sqrt{-\lambda_i(R_u)}\right)}{\sqrt{-\lambda_i(R_u)}} \right)^{n_i},$$

with $\frac{\sinh\left(\sqrt{-\lambda_i(R_u)}\right)}{\sqrt{-\lambda_i(R_u)}} = 1$ when $\lambda_i(R_u) = 0$.

The proof is provided in Appendix A.1. Recall that $\sinh(\mathbf{i}x) = \mathbf{i}\sin(x)$, hence the hyperbolic sine becomes a sine when the eigenvalue λ_i is real positive.

The formula for the case of Riemannian symmetric spaces, that can be found in Helgason (1979, p. 294), has a similar structure but the eigenvalues are roots of the complexified Lie algebra $\mathfrak{g}_{\mathbb{C}}$ of G . Since our formula derives directly from the equation of the Jacobi fields, it is naturally expressed using the curvature tensor. A benefit is that it can be used and understood without knowledge of roots systems of semisimple Lie algebras. Nonetheless, it is sometimes interesting to relate the λ_i to algebraic quantities. The curvature tensor at the point p relates to the Lie bracket of the Lie algebra of the group G in a simple way; see Nomizu (1954):

$$(4.2) \quad \forall u, v, w \in T_p \mathcal{M} \sim \mathfrak{m}, \quad R(u, v, w) = -[[u, v], w].$$

Recall also that $ad_u(v) = [u, v]$. Hence at the point $p \sim eK$, $R_u(v) = -[u, [u, v]] = -ad_u^2(v)$ and the eigenvalues of R_u are the eigenvalues of $-ad_u^2$ restricted to $\mathfrak{m}$. Due to the homogeneity of G/K , the Jacobian J_p determines the Jacobian of all other exponential maps $\exp_q$.

Corollary 4.2. *The Jacobian determinant of $\exp_q$ at v in a parallel transported basis is*

$$J_q(v) = J_p(kg^{-1}.v),$$

where k is arbitrary element of K . Here $kg^{-1}.v$ is understood as the differential of the action of kg^{-1} applied to v .

The proof is given in Appendix A.1. This formula enables one to always turn the computation of the Jacobian into a computation of eigenvalues of $-ad^2$. In the rest of the paper the Jacobian J_p is simply noted J .

The formula (4.1) is given in a parallel transported basis and does not rely on other properties of the connection ∇ other than $T = 0$ and $\nabla R = 0$. In sections 4.2 and 4.3 we give particular attention to two classes of symmetric spaces: (pseudo-)Riemannian locally symmetric spaces and Lie groups endowed with their Cartan–Schouten connection. In both contexts the additional structures enable one to state adapted results for the construction of exponential-wrapped probability densities. We address the use of the Jacobian for exponential-wrapped densities on arbitrary locally symmetric spaces in section 4.4.

4.2. Riemannian and pseudo-Riemannian symmetric spaces. Assume that the connection ∇ of the manifold $\mathcal{M}$ is the Levi-Civita connection of a Riemannian or pseudo-Riemannian

metric g . $\mathcal{M}$ has a natural volume measure vol induced by the metric. Let $e_1, \dots, e_d \in T_p \mathcal{M}$ be an orthonormal basis ($|g(e_i, e_j)| = \delta_{ij}$), and let ν_p denote the corresponding Lebesgue measure. Since parallel transport is an isometry, the Jacobian determinant J is related to the volume change of (2.2) in the following way,

$$\frac{d \exp_{p*}(\nu_p)}{d vol}(\exp(u)) = |J(u)|^{-1}.$$

We now provide the expression of the Jacobian on an example of a Riemannian symmetric space: real Grassmanian manifolds, and an example of a pseudo-Riemannian symmetric space, pseudohyperboloids. We are currently not aware of references containing these formulas. Moreover, the Jacobian on Grassmannians was omitted at several occasions in the densities of wrapped distributions; see, for instance, Slama, Wannous, and Daoudi (2014); Slama et al. (2015); Srivastava et al. (2005); Turaga et al. (2011).

4.2.1. Real Grassmanians.

The Grassmanian of vector subspaces. The Grassmanian $Gr_k(n)$ denotes the spaces of k dimensional vector subspaces of $\mathbb{R}^n$. We first describe the homogeneous symmetric structure of Grassmanians, as done in section 3.2 for the general case.

Let O denote the groups of orthogonal matrices and SO their subgroups of determinant 1. Clearly $O(n)$ acts transitively on subspaces of dimension k . Furthermore it is easy to see that block diagonal matrices with the first block in $O(k)$ and the second in $O(n-k)$ leave stable the vector spaces spanned by the k first basis vectors. Hence

$$Gr_k(n) \sim O(n)/(O(k) \times O(n-k)).$$

This quotient can be simplified to $Gr_k(n) \sim SO(n)/S(O(k) \times O(n-k))$, where $S(O(k) \times O(n-k))$ refers to the block diagonal matrices of determinant 1, with the first block in $O(k)$ and the second in $O(n-k)$.

The involutive automorphism of the symmetric structure is given by

$$(4.3) \quad \sigma(X) = \begin{pmatrix} I_k & 0 \\ 0^T & -I_{n-k} \end{pmatrix} X \begin{pmatrix} I_k & 0 \\ 0^T & -I_{n-k} \end{pmatrix}.$$

It can be checked that $S(O(k) \times O(n-k))$ is an open subgroup of the set of fixed points of σ , hence the involution makes $SO(n)/S(O(k) \times O(n-k))$ a homogeneous symmetric space. Since $S(O(k) \times O(n-k))$ is compact, the quotient admits an invariant Riemannian metric and is a homogeneous Riemannian symmetric space, see Helgason (1979).

The Lie algebra of $SO(n)$ is decomposed on eigenspaces of $d\sigma$ at identity,

$$\mathfrak{so}(n) = \mathfrak{k} \oplus \mathfrak{m}.$$

It can be checked that the -1 eigenspace $\mathfrak{m}$ is given by,

$$\mathfrak{m} = \left\{ X_B = \begin{pmatrix} 0 & B \\ -B^T & 0 \end{pmatrix}, \quad B \in Mat_{k,n-k}(\mathbb{R}) \right\},$$

where $Mat_{k,n-k}(\mathbb{R})$ are real k by $n-k$ matrices. Recall that on matrix groups $ad_X(Y) = XY - YX$. The computations shown in Appendix B.1 of the eigenvalues of the adjoints ad^2 restricted to $\mathfrak{m}$ lead to the following Jacobian at $p \sim I_n \cdot S(O(k) \times O(n-k))$,

$$(4.4) \quad J(X_B) = \prod_{i < j} \frac{\sin(\sigma_i - \sigma_j)}{\sigma_i - \sigma_j} \frac{\sin(\sigma_i + \sigma_j)}{\sigma_i + \sigma_j} \prod_i \left(\frac{\sin(\sigma_i)}{\sigma_i} \right)^{|n-2k|},$$

where σ_i are the singular values of B counted with multiplicity one, and where each fraction is replaced by 1 when the denominator is 0.

Note that $S(O(k) \times O(n-k))$ has two components and that the identity component $SO(k) \times SO(n-k)$ is also an open subgroup of the set of fixed points of σ . Hence $SO(n)/(SO(k) \times SO(n-k))$ is another homogeneous symmetric space: the oriented real Grassmanian. Since $\mathfrak{m}$ and ad_{X_B} remain the same, the Jacobian also has the same expression.

The Grassmanian of affine subspaces. Let $\text{Graff}_k(n)$ be the set of affine subspaces of dimension k of $\mathbb{R}^n$. It is clear that the set of isometries of $\mathbb{R}^n$, noted $E(n)$, acts transitively on $\text{Graff}_k(n)$. Furthermore, the stabilizer of the subspace generated by the first k vectors is given by $E(k) \times O(n-k)$: a rigid motion of the subspace and a rotation of the complement. Hence, $\text{Graff}_k(n)$ is a homogeneous space,

$$\text{Graff}_k(n) \sim E(n)/(E(k) \times O(n-k)).$$

Authors of Lim, Wong, and Ye (2021) show that the geometry of this quotient is nicely described by an embedding in the Grassmanian of vector subspaces $\text{Gr}_{k+1}(n+1)$. Let V be a k dimensional vector subspace and b be a vector of $\mathbb{R}^n$. The following map j ,

$$j((V, b)) = \text{span}(V \cup \{b, e_{n+1}\}),$$

where e_{n+1} is the last basis vector of $\mathbb{R}^{n+1}$, embeds $\text{Graff}_k(n)$ in $\text{Gr}_{k+1}(n+1)$. The canonical Riemannian metric on $\text{Graff}_k(n)$ is then the metric induced by $\text{Gr}_{k+1}(n+1)$. Furthermore, $j(\text{Graff}_k(n))$ is an open subset of $\text{Gr}_{k+1}(n+1)$. Since $\text{Gr}_{k+1}(n+1)$ is homogeneous Riemannian symmetric, this embedding makes $\text{Graff}_k(n)$ a Riemannian locally symmetric space. Hence we have locally $\text{Graff}_k(n) \sim \text{Gr}_{k+1}(n+1) \sim O(n+1)/(O(k+1) \times O(n-k))$ and the Jacobian can be computed with (4.4).

4.2.2. Pseudohyperboloids. We now provide the Jacobian on pseudohyperboloids. They are pseudo-Riemannian manifolds recently used in Law and Stam (2020), where the authors show their relevance for graph embedding problems. Let us start by describing pseudohyperboloids, following the approach of Law and Stam (2020).

For $p, q \geq 0$, let $\mathbb{R}^{p,q+1}$ be the space $\mathbb{R}^p \times \mathbb{R}^{q+1}$ endowed with the pseudo-Euclidean scalar product

$$\langle x, y \rangle = \sum_{i=1}^p x_i y_i - \sum_{j=p+1}^{p+q+1} x_j y_j.$$

Define the pseudohyperboloid $\mathcal{Q}_\beta^{p,q}$ as

$$\mathcal{Q}_\beta^{p,q} = \{x \in \mathbb{R}^{p,q+1}, \langle x, x \rangle = \beta\},$$

where $\beta < 0$. Pseudospheres are defined with $\beta > 0$, but note that $\mathcal{Q}_\beta^{p,q}$ and $\mathcal{Q}_{-\beta}^{q+1,p-1}$ are anti-isometric. Furthermore, since all $\beta < 0$ lead to homotetic pseudohyperboloids, we set $\beta = -1$.

As described in section 3.2, we can now exhibit the symmetric space structure of $\mathcal{Q}_\beta^{p,q}$ and compute the Jacobian determinant of the exponential map. Let $O(p, q+1)$ be the indefinite orthogonal group which preserves the pseudoscalar product of $\mathbb{R}^{p,q+1}$. The group $O(p, q+1)$ acts transitively by isometries on $\mathcal{Q}_\beta^{p,q}$. Since the stabilizer of the last basis vector $e_{p+q+1} \in \mathcal{Q}_\beta^{p,q}$ is the subgroup $O(p, q)$, we have that $\mathcal{Q}_\beta^{p,q} \sim O(p, q+1)/O(p, q)$. Consider an involution similar to the one defined in (4.3):

$$(4.5) \quad \sigma(X) = \begin{pmatrix} -I_{p+q} & 0 \\ 0 & 1 \end{pmatrix} X \begin{pmatrix} -I_{p+q} & 0 \\ 0 & 1 \end{pmatrix}.$$

σ is an involution of $O(p, q+1)$ and it can be checked that $O(p, q)$ is an open subgroup of the set of fixed points of σ . Hence it gives $O(p, q+1)/O(p, q)$, a homogeneous symmetric structure. The Lie algebra can be decomposed on the eigenspaces of $d\sigma$ at identity,

$$\mathfrak{o}(p, q+1) = \mathfrak{k} \oplus \mathfrak{m},$$

and it can be checked that the -1 eigenspace $\mathfrak{m}$ is given by,

$$\mathfrak{m} = \left\{ X_{v,w} = \begin{pmatrix} 0 & 0 & v \\ 0 & 0 & w \\ v^T & -w^T & 0 \end{pmatrix}, \quad v \in \mathbb{R}^p, w \in \mathbb{R}^q \right\}.$$

Again, on a matrix group $ad_X(Y) = XY - YX$. The computations of the eigenvalues of the ad^2 restricted to $\mathfrak{m}$ given in Appendix B.2 lead to the following Jacobian at $p \sim I_{p+q+1}.O(p, q)$,

$$(4.6) \quad J(X_{v,w}) = \begin{cases} \left(\frac{\sinh(\|v\|^2 - \|w\|^2)}{\|v\|^2 - \|w\|^2} \right)^{p+q-1} & \text{if } \|v\| \neq \|w\|, \\ 1 & \text{if } \|v\| = \|w\|, \end{cases}$$

where $\|v\|$ and $\|w\|$ are the Euclidean norms of v and w .

4.3. Lie groups. As pointed out by Taniguchi (1984), the differential of the exponential map on symmetric spaces can be used to derive the differential of the exponential map on Lie groups. We describe here how the Jacobian determinants relate to each other.

Remarkably, every Lie group has an affine connection ∇ compatible with the group structures, called the 0-connection, which makes it an ALSS. Since the symmetric structure of the 0-connection was only described very recently in the data science literature (see Pennec and Lorenzi, 2020), we recall the most important facts. Let $\mathcal{X}$ be a manifold equipped with a Lie group structure with identity $o \in \mathcal{X}$.

Proposition 4.3. *Let ∇ be the bi-invariant connection defined by*

$$\nabla_{\tilde{u}} \tilde{v} = \frac{1}{2} [u, v], \quad u, v \in T_o \mathcal{X},$$

where $\tilde{u}$ and $\tilde{v}$ are the left invariant vector fields generated by u and v , and $[\cdot, \cdot]$ is the Lie bracket associated with the Lie group structure on $\mathcal{X}$. ∇ is called the 0-connection, or 0-Cartan-Schouten connection. We have,

- (i) $(\mathcal{X}, \nabla)$ is an affine locally symmetric space;
- (ii) one parameter subgroups are geodesics: at o , the group exponential and the exponential of the connection coincide;
- (iii) the curvature and the Lie bracket are related by

$$\forall u, v, w \in T_o \mathcal{X} \quad R(u, v, w) = -\frac{1}{4}[[u, v], w],$$

- (iv) the parallel transport from o to $\exp(u)$ of the vector v is given by

$$\Pi_o^{\exp(u)} v = dL_{\exp(\frac{u}{2})} \circ dR_{\exp(\frac{u}{2})}(v),$$

where dL and dR are the differential of the left and right multiplications.

Proofs can be found in Pennec and Lorenzi (2020). Assume now that the manifold $\mathcal{M}$ has a Lie group structure of identity p , and that ∇ is the 0-Cartan-Shouten connection. We show how Theorem 4.1 leads to the formula of the Jacobian on Lie groups, given for instance in Falorsi et al. (2019). Note that the Lie bracket $[\cdot, \cdot]$ in Proposition 4.3 is not the same as the one coming from the identification $T_p \mathcal{M} \sim \mathfrak{m} \subset T_e G$, where G is the Lie group involved in the local identification $\mathcal{M} \sim G/H$. Similarly to (4.2), using (iii) we can write $R_u(v) = -\frac{1}{4}[u, [u, v]] = -\frac{1}{4}ad_u^2(v)$, but where ad_u is now the adjoint map of the Lie algebra of $\mathcal{M}$. This relation enables us to obtain an algebraic expression of the Jacobian which involves only the structure of $\mathcal{M}$ and not of the Lie group G .

On a Lie group the differential of the group exponential is usually computed in a basis transported by left (or right) multiplication. As a result, the Jacobian determinant is a volume change between a Lebesgue measure on the Lie algebra and a Haar measure. As (iii) shows, parallel transported bases are not simply obtained by left or right multiplication, hence the Jacobian determinant of Theorem 4.1 is not a volume change with respect to a Haar measure. However, a simple calculation shown in Appendix A.3 enables us to relate the two Jacobians and to obtain the following corollary of Theorem 4.1.

Corollary 4.4. *Set a basis $B = (e_1, \dots, e_d = u)$ of $T_p \mathcal{M}$ and let $u \in T_p \mathcal{M}$. Let B_L be the basis of $T_{\exp(u)} \mathcal{M}$ obtained by left multiplication of B . The Jacobian determinant of the exponential map expressed in B and B_L is given by*

$$\tilde{J}_p(u) = \prod_i \left(\frac{1 - e^{\alpha_i(u)}}{\alpha_i(u)} \right)^{m_i},$$

where $\alpha_i(u)$ are the eigenvalues of ad_u and m_i their multiplicities.

The proof is given in Appendix A.3. Let ν_p be the Lebesgue measure on $T_p \mathcal{M}$ and vol be the left Haar measure on $\mathcal{M}$ generated by the basis B . We have

$$\frac{d \exp_*(\nu_p)}{d vol}(\exp(u)) = |\tilde{J}_p(u)|^{-1}.$$

Similarly to Corollary 4.2 expressing the Jacobian at arbitrary points, it can be checked that on Lie groups, the Jacobian in a left-transported basis computed at an arbitrary q is given by

$$\tilde{J}_q(u) = \tilde{J}_p(dL_{q^{-1}}.u).$$

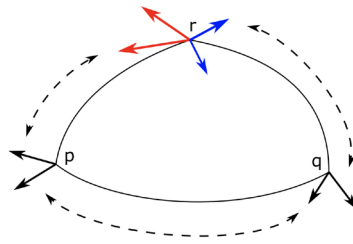


Figure 4.1. The dotted lines represent the parallel transport between p , q , and r . The term H in (4.7) is the determinant of the red basis in the blue basis.

4.4. The general case. We are currently not aware of practical problems in data science or physics involving a random phenomenon on a symmetric space which is not Riemannian, pseudo-Riemannian, or a Lie group. However, such spaces remain an interesting class, with some interesting special cases. For example, the connection ∇ on $\mathbb{R}^2$ whose Christoffel's coefficients are all zeros except $\Gamma_{11}^2(x, y) = y$ is symmetric but does not correspond to a Riemannian or Lie structure. We outline the use of the Jacobian on a general affine locally symmetric space but we do not provide proofs of the results in this paper.

In both the Riemannian and Lie group settings we interpreted the Jacobian as a volume change between a Lebesgue measure of the tangent space and a reference measure on the manifold. On general symmetric spaces there might not be such a reference measure. In that case, exponential-wrapped probability distributions do not have a natural notion of density, even when they are absolutely continuous with respect to the Lebesgue measures of the charts of $\mathcal{M}$. However, relative densities between exponential-wrapped probability distributions can still be computed. Let $p, q \in \mathcal{M}$ and $\mathcal{U} \subset \mathcal{M}$ be such that $\log_p$ and $\log_q$ are well defined on $\mathcal{U}$. Let λ_p and λ_q be two probability distributions supported on $\log_p(\mathcal{U})$ and $\log_q(\mathcal{U})$, respectively. Set an arbitrary reference basis on $T_p\mathcal{M}$ and parallel transport it to $T_q\mathcal{M}$. If λ_p and λ_q have densities h_p and h_q with respect to the corresponding Lebesgue measures, then for any $r \in \mathcal{U}$ it can be shown that

$$(4.7) \quad \frac{d \exp_{q*} \lambda_q}{d \exp_{p*} \lambda_p}(r) = H \cdot \frac{J_p(\log(r))}{J_q(\log_p(r))} \frac{h_q(\log_q(r))}{h_p(\log_p(r))},$$

where H is the determinant of the holonomy map along the geodesic triangle $r \rightarrow q \rightarrow p \rightarrow r$; see Figure 4.1. Furthermore, as mentioned in section 3.2, the parallel transport on a symmetric space is obtained by the action of elements of $\exp(\mathfrak{m})$. This enables us to compute H explicitly.

5. A classification experiment using exponential-wrapped distributions. Outside cases where laws are modeled using a fixed tangent space, the analysis of the convergence of density estimators based on exponential-wrapped distributions is still in early development.

As suggested in section 2, exponential-wrapped distributions are sometimes conveniently estimated with moment matching estimators. The study of the theoretical properties of moment matching estimators is out of the scope of this paper. However, the module `frechet_mean` of the python package **Geomstats** (see Miolane et al., 2020), now enables the computation of the empirical moments on several symmetric spaces in a simple way.

Taking advantage of this python package, we present a classification experiment on simulated data drawn in two Riemannian symmetric spaces: the real Grassmannian of two-dimensional subspaces of $\mathbb{R}^4$ and the space of 2×2 symmetric positive definite matrices.

On both spaces, we consider four equiprobable classes. For a class C_i , a training set and a test set are drawn from an exponential-wrapped density f_i . Each training set is then modeled by an estimated exponential-wrapped distribution $\hat{f}_i$, and samples from test sets are classified according to the maximum a posteriori probability. Several approaches are compared, depending on the number, and location, of tangent spaces used to model the data. In *model 1*, the training sets are modeled with exponential wrapped distributions originating from different tangent spaces, while in *models 2* and *3*, the training sets are modeled with exponential-wrapped distributions originating from the same tangent space. In models 2 and 3, the Jacobians between the tangent spaces and the manifold are not involved in the classification, since all the data are classified in the same tangent space. The classification results show the interest of model 1 over models 2 and 3. All the computations necessary to the classification are achieved with the package **Geomstats**.

The training set and test set of the class C_i are obtained by sampling from an isotropic exponential-wrapped normal density $f_i = f(\cdot; p_i, v_i)$, which we describe in the next paragraph.

5.1. Isotropic exponential-wrapped normal distributions. Define the distribution $\mathcal{N}_{\mathcal{M}}(p, v)$ as

$$\mathcal{N}_{\mathcal{M}}(p, v) = \exp_{p*} \left(\mathcal{N} \left(0, \frac{v}{d} \langle \cdot, \cdot \rangle_p \right) \right),$$

where $\mathcal{N}$ is a multivariate normal distribution, $\langle \cdot, \cdot \rangle_p$ is the inner product of $T_p \mathcal{M}$, and d the dimension of $\mathcal{M}$. Note $f(\cdot; p, v)$, the density of $\mathcal{N}_{\mathcal{M}}(p, v)$. When the manifold $\mathcal{M}$ is a space of symmetric positive definite matrices, the exponential map is a bijection between each tangent space and $\mathcal{M}$. After particularizing (2.2), we obtain that the density $f_i(\cdot) = f(\cdot; p_i, v_i)$ is given by

$$(5.1) \quad f_i(q) = \frac{1}{J_{p_i}(\log_{p_i}(q))} \frac{1}{\sqrt{(2\pi w_i)^d}} e^{-\frac{d(q, p_i)^2}{2w_i}},$$

where $w_i = \frac{v_i}{d}$. When $\mathcal{M}$ is a real Grassmannian manifold, the exponential maps are surjective but not injective. In the current experiment, the normal distributions on the Grassmannian are taken with small variances, which enables us to neglect the mass outside the injectivity radius. This hypothesis is often made in practice; see Falorsi et al. (2019); Fletcher et al. (2003); Mallasto and Feragen (2018), and avoid the technicalities of truncated normal distributions used in Slama et al. (2015); Turaga et al. (2011). This assumption enables us to approximate the density by using (5.1).

As pointed out in the end of section 2, an important aspect of such an exponential-wrapped normal density, with respect to other types of normal densities on manifolds, is that the parameters p and v correspond to empirical moments of $\mathcal{N}_{\mathcal{M}}(p, v)$. Indeed, the change of variable $u = \log_p(q)$ leads to

$$\int_{\mathcal{M}} \log_p(q) f(q; p, v) d\text{vol} = \frac{1}{\sqrt{(2\pi w)^d}} \int_{T_p \mathcal{M}} u e^{-\frac{\|u\|^2}{2w}} du = 0,$$

where vol is the Riemannian volume and $w = \frac{v}{d}$. Hence p is a mean of $\mathcal{N}_{\mathcal{M}}(p, v)$.

The same change of variable also gives

$$\int_{\mathcal{M}} d(q, p)^2 f(q; p, v) d\text{vol} = \frac{1}{\sqrt{(2\pi w)^d}} \int_{T_p \mathcal{M}} \|u\|^2 e^{-\frac{\|u\|^2}{2w^2}} du = v.$$

Hence v is the variance of $\mathcal{N}_{\mathcal{M}}(p, v)$. This allows us to estimate the parameters p and v by empirical moments. Note that on symmetric spaces with positive curvature, such as Grassmannian manifolds, the uniqueness of the mean is not guaranteed when the distribution is not sufficiently concentrated. Hence the convergence of the estimation of p by an empirical mean is also not guaranteed. The small variance hypothesis enables us to neglect this phenomenon.

5.2. The Grassmannian $Gr_2(4)$. We now give the expression of the Jacobian on the Grassmannian of two-dimensional vector subspaces of $\mathbb{R}^4$, noted as $Gr_2(4)$, as well as the parameters of the four classes C_i . $Gr_2(4)$ is a four-dimensional manifold described in section 4.2.1. It is identified with the quotient

$$O(4)/(O(2) \times O(2)),$$

and its tangent space at $I.O(2) \times O(2)$ is identified with

$$\mathfrak{m} = \left\{ X_B = \begin{pmatrix} 0 & B \\ -B^T & 0 \end{pmatrix}, \quad B \in \text{Mat}_{2,2}(\mathbb{R}) \right\}.$$

The Jacobian becomes

$$J(X_B) = \frac{\sin(\sigma_1 - \sigma_2)}{\sigma_1 - \sigma_2} \frac{\sin(\sigma_1 + \sigma_2)}{\sigma_1 + \sigma_2},$$

where σ_1 and σ_2 are the singular values of B . On $Gr_2(4)$ the parameters of the distributions of the four classes are chosen as $p_i = \exp_I(X_{B_i})$ with

$$B_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, B_2 = \begin{pmatrix} 0 & 0 \\ -\frac{\pi}{2} & 0 \end{pmatrix}, B_3 = \begin{pmatrix} 0 & -\frac{\pi}{2} \\ 0 & 0 \end{pmatrix}, B_4 = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{\pi}{2} \end{pmatrix},$$

and

$$v_1 = v_2 = v_3 = v_4 = 0.6.$$

For this choice of variance, a Monte Carlo sampling shows that in the tangent spaces, 99.8% of the mass lies in the injectivity ball $B(\frac{\pi}{2})$ and 60% lie in the ball $B(\frac{\pi}{4})$, $\frac{\pi}{2}$ being the injectivity radius of $Gr_2(4)$. This distribution of mass is consistent with the approximation made in (5.1), and ensures in practice the uniqueness of the mean.

5.3. The space of 2×2 symmetric positive definite matrices. Before providing the expression of the Jacobian and the parameters of the classes C_i , we start with a brief description of the structure of symmetric space. Note $\text{Sym}(2)$ and $\text{SPD}(2)$ are the spaces of 2×2 symmetric and symmetric positive definite matrices. Since $\text{SPD}(2)$ is an open subset of the vector space $\text{Sym}(2)$, all the tangent spaces of $\text{SPD}(2)$ are identified with $\text{Sym}(2)$. Endow $\text{SPD}(2)$ with the following Riemannian metric

$$g_{\Sigma}(X, Y) = \text{trace}(\Sigma^{-1} X \Sigma^{-1} Y),$$

where $\Sigma \in \text{SPD}(2)$ and $X, Y \in \text{Sym}(2)$. The metric g makes $\text{SPD}(2)$ a Riemannian symmetric space, whose detailed presentation can be found in Terras (1984). Let us simply give the identifications introduced in section 3.3. $\text{SPD}(2)$ is identified with $GL(2)/O(2)$ by the map $\Sigma \mapsto \Sigma^{1/2}.O(2)$, where $\Sigma^{1/2}$ is the symmetric square root of Σ , and the tangent space of $GL(2)/O(2)$ at $I.O(2)$ is itself identified with

$$\mathfrak{m} = \text{Sym}(2).$$

This leads to an identification of $T_I \text{SPD}(2)$ and $\mathfrak{m}$ given by $X \mapsto \frac{1}{2}X$. For X in $T_I \text{SPD}(2)$, the computation of the eigenvalues of $ad_{\frac{1}{2}X}^2 : \mathfrak{m} \rightarrow \mathfrak{m}$ gives the following Jacobian,

$$J(X) = 2 \frac{\sinh(\frac{\sigma_1 - \sigma_2}{2})}{\sigma_1 - \sigma_2},$$

where σ_1 and σ_2 are the eigenvalues of X : see also Chevallier, Kalunga, and Angulo (2017). On $\text{SPD}(2)$, the parameters of the distributions of the four classes are chosen as

$$p_1 = \exp_I \left(\begin{pmatrix} 3 & 0 \\ 0 & -3 \end{pmatrix} \right), p_2 = \exp_I \left(\begin{pmatrix} 3 & 0.3 \\ 0.3 & -3 \end{pmatrix} \right),$$

$$p_3 = \exp_I \left(\begin{pmatrix} -3 & 0 \\ 0 & 3 \end{pmatrix} \right), p_4 = \exp_I \left(\begin{pmatrix} -3 & -0.3 \\ -0.3 & 3 \end{pmatrix} \right),$$

and

$$v_1 = v_2 = v_3 = v_4 = 2.$$

5.4. Estimation of exponential-wrapped normal distributions. The test sets are modeled according to three procedures.

- In model 1, the parameters of the density $f_i = f(\cdot; p_i, v_i)$ of the class C_i are estimated by the empirical mean and variance $\hat{p}_i, \hat{v}_i$. The test set of the class C_i is then modeled by the density $\hat{f}_i = f(\cdot; \hat{p}_i, \hat{v}_i)$.
- In model 2, all the data points are first lifted into a single tangent space $T_{p_0}\mathcal{M}$ by the logarithm $\log_{p_0}$. The point p_0 is chosen to be a mean of all the training sets. Each lifted training set is then modeled by an isotropic normal density $h(\cdot; \hat{\mu}_i, \hat{v}_i)$ on $T_{p_0}\mathcal{M}$ of parameters

$$\hat{\mu}_i = \frac{1}{N} \sum_i \log_{p_0}(q_i) \text{ and } \hat{v}_i = \frac{1}{N} \sum_i \|\log_{p_0}(q_i) - \mu_i\|_{p_0}^2,$$

where N is the size of the training sets.

- Model 3 differs from model 2 in the choice of the lifting point p_0 , which is now set as the mean of the training set of the first class.

5.5. Classification results. Data are classified according to the maximum a posteriori probability. Since we consider equiprobable classes, maximizing the posterior probability is equivalent to maximizing the likelihood of the observation. Hence, a data point at $q \in \mathcal{M}$ is classified as

$$C(q) = \begin{cases} \operatorname{argmax}_i f(q; \hat{p}_i, v_i) & \text{(model 1),} \\ \operatorname{argmax}_i h(q; \hat{\mu}_i, \hat{v}_i) & \text{(models 2 and 3).} \end{cases}$$

On both spaces, we consider four equiprobable classes with a training set of size 4×50 and a test set of size 4×50 . The classification is repeated 5000 times. The following table shows the average rate of good classifications, plus or minus a standard deviation.

	model 1	model 2	model 3
$Gr_2(4)$	0.838 ± 0.026	0.777 ± 0.031	0.501 ± 0.052
$SPD(2)$	0.816 ± 0.028	0.682 ± 0.050	0.618 ± 0.035

On both spaces, the results illustrate the advantage of working with multiple tangent spaces over a global linearization of the space. In the case of a global linearization, choosing an off-centered tangent space (model 3) led to lower classification results than those obtained with a centered tangent space (model 2).

6. Discussion. Exponential-wrapped distributions had previously been defined and used on specific manifolds. In this paper we showed that ALSS are a broad class of manifolds where exponential-wrapped densities can be computed in closed form, under an injectivity condition. These distributions have then been used in a classification experiment on simulated data. Further studies should more deeply investigate the impact of the various factors affecting the classification results, such as the curvature tensor of the manifold or the number and locations of classes. In order to provide a theoretical background to these results, future work will also focus on the study of the convergence of estimators based on exponential-wrapped distributions. An important problem remains open in the case where the tangent space used to model data is not fixed in advance: differentiating the likelihood of densities with respect to the base point of the tangent space. The differentiation involves the double exponential expansion, whose expression on arbitrary affine manifolds can be found in Pennec (2019, section 3.2) and Gavrilov (2007). Our future efforts will focus on understanding the implications of this formula for density estimation with exponential wrapped densities.

Appendix A. Proofs of the general forms of Jacobians.

A.1. Proof of Theorem 4.1. The main part of the proof is similar to Taniguchi (1984). Chose a basis $e_1, \dots, e_d$ of $T_q \mathcal{M}$ with $e_d = u$. Expressed in $e_1(0), \dots, e_d(0)$ and $e_1(t), \dots, e_d(t)$, we have

$$J(tu) = \det(d \exp_p(tu)) = \det \left(\frac{\partial \exp}{\partial e_1}(te_d), \dots, \frac{\partial \exp}{\partial e_d}(te_d) \right).$$

On manifolds with null torsion, the

$$Y_i(t) = t \frac{\partial \exp}{\partial e_i}(te_d), \quad i = 1, \dots, d,$$

are solutions of the Jacobi equations $Y_i''(t) + R_{e_d(t)}(Y_i(t)) = 0$ with initial conditions $Y_i(0) = 0$ and $Y_i'(0) = e_i$. $Y''(t)$ refers here to the second covariant derivative along the geodesics $\exp(tu)$ and $R_{e_d(t)}$ to the map $R(e_d(t), \cdot)e_d(t)$. Given a tensor field T along $\exp(tu)$, note that $[T]$ is its coordinates in the basis $e_1(t), \dots, e_d(t)$. Since the manifold is locally symmetric, $[R_{e_d(t)}] = [R_{e_d(0)}] = [R_u]$. The Jacobi equation becomes a second order differential equation in $\mathbb{R}^d$ with constant coefficients:

$$[Y_i]'' + [R_u][Y_i] = 0.$$

In the rest of the proof, the matrix $[R_u]$ is simply noted as R . Let $F_t: \mathbb{R}^d \rightarrow \mathbb{R}^d$ be the linear map defined by $F_t(X) = v(t)$ with v the unique solution of the Cauchy problem

$$\begin{cases} v'' + Rv = 0, \\ v(0) = 0, \\ v'(0) = X. \end{cases}$$

It can be checked that $\frac{1}{t^d}F_t$ is the matrix expression of linear map $d\exp_p(tu)$, hence $J(u) = \det(F_1)$. First turn the differential equation into a first order differential equation. We get

$$\begin{pmatrix} v' \\ v'' \end{pmatrix} = \begin{pmatrix} 0 & I \\ -R & 0 \end{pmatrix} \begin{pmatrix} v \\ v' \end{pmatrix}.$$

Letting $A = \begin{pmatrix} 0 & I \\ R & 0 \end{pmatrix}$, the solution is given by $\begin{pmatrix} v(t) \\ v'(t) \end{pmatrix} = e^{tA} \begin{pmatrix} v(0) \\ v'(0) \end{pmatrix}$. It is easy to check that $e^{tA} = \begin{pmatrix} E_t & F_t \\ G_t & H_t \end{pmatrix}$, where F_t is the linear map defined previously. Hence, we want to compute the determinant of the upper right block of e^A . Compute first the powers of A . It can be checked by induction that for $k \in \mathbb{N}$,

$$A^{2k} = t^{2k} \begin{pmatrix} (-R)^k & 0 \\ 0 & (-R)^k \end{pmatrix} \text{ and } A^{2k+1} = A^{2k+1} \begin{pmatrix} 0 & (-R)^k \\ (-R)^{k+1} & 0 \end{pmatrix}.$$

We can deduce that $F_1 = 0 + I + 0 + \frac{-R}{3!} + 0 + \frac{(-R)^2}{5!} + \dots + 0 + \frac{(-R)^{2k}}{(2k+1)!} + \dots$, which is analogous to the formula provided in Taniguchi (1984). Hence we have that the matrix of the differential of the exponential map at the tangent vector u in a parallel transported basis is given by

$$d\exp_p(u) = \sum_0^\infty \frac{(-R)^n}{(2n+1)!}.$$

Recall that a matrix can always be triangularized over $\mathbb{C}$. Let $R = PTP^{-1}$ with T an upper triangular matrix. Recall also that the diagonal elements of T are the complex eigenvalues λ_i of R . We have $d\exp_p(u) = P \left(\sum_0^\infty \frac{(-T)^n}{(2n+1)!} \right) P^{-1}$ and $\det(d\exp_p(u)) = \det \left(\sum_0^\infty \frac{(-T)^n}{(2n+1)!} \right)$. Since $\sum_0^\infty \frac{(-T)^n}{(2n+1)!}$ is upper triangular we have that

$$(A.1) \quad J(u) = \det(d\exp_p(u)) = \prod_i \left(\sum_0^\infty \frac{(-\lambda_i)^n}{(2n+1)!} \right)^{n_i} = \prod_i \left(\frac{\sinh(\sqrt{-\lambda_i})}{\sqrt{-\lambda_i}} \right)^{n_i},$$

where n_i is the multiplicity of λ_i , and where $\frac{\sinh(0)}{0} = 1$. Note that since $\sinh$ is odd, it is clear that changing the choice of square root $\sqrt{-\lambda_i}$ to $-\sqrt{-\lambda_i}$ does not affect the determinant.

A.2. The Jacobian at an arbitrary base-point : Proof of Corollary 4.2. Recall that the connection ∇ is equivariant under the action of G . Hence $g.\exp_q(u) = \exp_{g.q}(g.u)$ and $g.\mathrm{d}\exp_q(u) = \mathrm{d}\exp_{g.q}(g.u)$. Let B be a basis of $T_p\mathcal{M}$ and B' its parallel translation to $T_{\exp_p(u)}\mathcal{M}$ along $\exp_p(u)$. We have that $g.B$ and $g.B'$ are bases of $T_{g.q}\mathcal{M}$ and $T_{g.\exp_p(g.u)}\mathcal{M}$ and that the determinant of $\mathrm{d}\exp_q(u)$ in B and B' is the same as the determinant of $g.\mathrm{d}\exp_q(u)$ in $g.B$ and $g.B'$. Moreover, the equivariance of the connection gives that $g.B$ and $g.B'$ are also related by parallel transport. In other words, $J_q(u) = J_{g.q}(g.u)$. In section 4.1, the function J is defined at $p \sim eK$. Hence if $q \sim gK$, we have $kg^{-1}.gK = eK$ for all $k \in K$ and

$$J_q(u) = J(kg^{-1}.u).$$

A.3. The Lie group formula : Proof of Corollary 4.4. At $\exp(u)$, the two bases B_L and B_T obtained, respectively, by left invariance and parallel transport are given by: $B_L = \mathrm{d}L_{\exp(u)}.B$ and $B_T = \mathrm{d}L_{\exp(\frac{u}{2})}\mathrm{d}R_{\exp(\frac{u}{2})}.B$. We have $\tilde{J}(u) = J(u).\det_{B_L}(B_T)$, hence we need to compute $\det_{B_L}(B_T)$:

$$\det_{B_L}(B_T) = \frac{\det}{\mathrm{d}L_{\exp(-u)}B_L}(\mathrm{d}L_{\exp(-u)}B_T) = \frac{\det}{e_1, \dots, e_d}(\mathrm{d}L_{\exp(-\frac{u}{2})}\mathrm{d}R_{\exp(\frac{u}{2})}.B) = \det(\mathrm{Ad}_{\exp(-\frac{u}{2})}).$$

Since $\mathrm{Ad}_{\exp(-\frac{u}{2})} = e^{-\frac{1}{2}\mathrm{ad}_u}$, $\det_{B_L}(B_T) = e^{-\frac{1}{2}\sum_i \alpha_i(u)}$. On the other hand,

$$e^{-\frac{1}{2}\alpha_i(u)} \frac{\sinh(\frac{1}{2}\alpha_i(u))}{\frac{1}{2}\alpha_i(u)} = e^{-\frac{1}{2}\alpha_i(u)} 2 \frac{e^{\frac{1}{2}\alpha_i(u)} - e^{-\frac{1}{2}\alpha_i(u)}}{2\alpha_i(u)} = \frac{1 - e^{-\alpha_i(u)}}{\alpha_i(u)},$$

which lead to the desired formula.

Appendix B. Eigenvalues of ad_X^2 on specific examples.

B.1. The real Grassmannian. Let $X_{B_1} = \begin{pmatrix} 0 & B_1 \\ -B_1^T & 0 \end{pmatrix}$ and $X_{B_2} = \begin{pmatrix} 0 & B_2 \\ -B_2^T & 0 \end{pmatrix}$. Let us first compute $\mathrm{ad}_{X_{B_1}}(X_{B_2})$. We obtain

$$\begin{aligned} \mathrm{ad}_{X_{B_1}}(X_{B_2}) &= X_{B_1}X_{B_2} - X_{B_2}X_{B_1} \\ &= \begin{pmatrix} B_2B_1^T - B_1B_2^T & 0 \\ 0 & B_2^TB_1 - B_1^TB_2 \end{pmatrix}. \end{aligned}$$

Hence,

$$\begin{aligned} \mathrm{ad}_{X_{B_1}}(\mathrm{ad}_{X_{B_1}}(X_{B_2})) &= \begin{pmatrix} 0 & 2B_1B_2^TB_1 - B_1B_1^TB_2 - B_2B_1^TB_1 \\ B_1^TB_1B_2^T + B_2^TB_1B_1^T - 2B_1^TB_2B_1^T & 0 \end{pmatrix}. \end{aligned}$$

Let $\varphi_{B_1}(B_2) = 2B_1B_2^TB_1 - B_1B_1^TB_2 - B_2B_1^TB_1$. Let $B_1 = UDV$ be the singular value decomposition of B_1 . We have

$$\begin{aligned} \varphi_{B_1}(B_2) &= 2B_1B_2^TB_1 - B_1B_1^TB_2 - B_2B_1^TB_1 \\ &= 2UDVB_2^TUDV - UDVV^{-1}D^TU^{-1}B_2 - B_2V^{-1}D^TU^{-1}UDV, \end{aligned}$$

$$\begin{aligned}
\varphi_{B_1}(UB_2V) &= 2UDVV^{-1}B_2^T U^{-1}UDV - UDD^T U^{-1}UB_2V - UB_2VV^{-1}D^T DV \\
&= 2UDB_2^T DV - UDD^T B_2V - UB_2D^T DV \\
&= U(2DB_2^T D - DD^T B_2 - B_2D^T D)V = U\varphi_D(B_2)V,
\end{aligned}$$

which shows that the eigenvalues of φ_{B_1} and φ_D are the same. We assume now that $B_1 = D$ with diagonal $\sigma_1, \dots, \sigma_q$, where $q = \min(k, n-k)$. Let $E_{i,j}$ be the canonical basis of k by $n-k$ matrices. Assuming $i, j \leq q$, a short calculation shows that

$$\begin{aligned}
\varphi_D(E_{ii}) &= 0, \\
\varphi_D(E_{ij} + E_{ji}) &= -(\sigma_i - \sigma_j)^2(E_{ij} + E_{ji}), \\
\varphi_D(E_{ij} - E_{ji}) &= -(\sigma_i + \sigma_j)^2(E_{ij} - E_{ji}).
\end{aligned}$$

When $k > n-k$, we can have $i > n-k$ and $\varphi_D(E_{ij}) = -\sigma_j^2 E_{ij}$, while when $k < n-k$ and $j > k$, $\varphi_D(E_{ij}) = -\sigma_i^2 E_{ij}$. Hence the singular value σ_i appears $k - (n-k)$ times or $(n-k) - k$ times. Since $R_u(v) = -\text{ad}_u^2(v)$, (4.1) can be rewritten with $\sqrt{-\lambda_i(R_u)} = \sqrt{\lambda_i(\text{ad}_u^2)}$ and the Jacobian becomes

$$(B.1) \quad J(X_B) = \prod_{i < j} \frac{\sin(\sigma_i - \sigma_j)}{\sigma_i - \sigma_j} \frac{\sin(\sigma_i + \sigma_j)}{\sigma_i + \sigma_j} \prod_i \left(\frac{\sin(\sigma_i)}{\sigma_i} \right)^{|n-2k|}.$$

B.2. Pseudohyperboloids. Let $X_{v_1, w_1} = \begin{pmatrix} 0 & 0 & v_1 \\ 0 & 0 & w_1 \\ v_1^T & -w_1^T & 0 \end{pmatrix}$ and $X_{v_2, w_2} = \begin{pmatrix} 0 & 0 & v_2 \\ 0 & 0 & w_2 \\ v_2^T & -w_2^T & 0 \end{pmatrix}$. Let us first compute $\text{ad}_{X_{v_1, w_1}}(X_{v_2, w_2})$. We obtain

$$\begin{aligned}
\text{ad}_{X_{v_1, w_1}}(X_{v_2, w_2}) &= X_{v_1, w_1} X_{v_2, w_2} - X_{v_2, w_2} X_{v_1, w_1} \\
&= \begin{pmatrix} v_1 v_2^T - v_2 v_1^T & v_2 w_1^T - v_1 w_2^T & 0 \\ w_1 v_2^T - w_2 v_1^T & w_2 w_1^T - w_1 w_2^T & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\end{aligned}$$

Hence we have $\text{ad}_{X_{v_1, w_1}}(\text{ad}_{X_{v_1, w_1}}(X_{v_2, w_2})) = \dots$

$$\begin{aligned}
&\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ v_1^T v_1 v_2^T - v_1^T v_2 v_1^T - w_1^T w_1 v_2^T + w_1^T w_2 v_1^T & v_1^T v_2 w_1^T - v_1^T v_1 w_2^T - w_1^T w_2 w_1^T + w_1^T w_1 w_2^T & 0 \end{pmatrix} + \\
&\begin{pmatrix} 0 & 0 & -v_1 v_2^T v_1 + v_2 v_1^T v_1 - v_2 w_1^T w_1 + v_1 w_2^T w_1 \\ 0 & 0 & -w_1 v_2^T v_1 + w_2 v_1^T v_1 - w_2 w_1^T w_1 + w_1 w_2^T w_1 \\ 0 & 0 & 0 \end{pmatrix}.
\end{aligned}$$

Note φ is the map on vectors $\begin{pmatrix} v_2 \\ w_2 \end{pmatrix} \in \mathbb{R}^{p+q}$ induced by $\text{ad}_{X_{v_1, w_1}}^2$. The matrix of φ is

$$M_\varphi = \begin{pmatrix} -v_1 v_1^T + (\|v_1\|^2 - \|w_1\|^2)I_p & v_1 w_1^T \\ -w_1 v_1^T & w_1 w_1^T + (\|v_1\|^2 - \|w_1\|^2)I_q \end{pmatrix}.$$

Hence, $M_\varphi = \begin{pmatrix} v_1 \\ w_1 \end{pmatrix} \begin{pmatrix} -v_1 \\ w_1 \end{pmatrix}^T + (\|v_1\|^2 - \|w_1\|^2)I$. The matrix $A = \begin{pmatrix} v_1 \\ w_1 \end{pmatrix} \begin{pmatrix} -v_1 \\ w_1 \end{pmatrix}^T$ is rank one and has 0 as an eigenvalue with multiplicity at least $p+q-1$. When $(\|w_1\|^2 - \|v_1\|^2) \neq 0$, the matrix A can be diagonalized with $(\|w_1\|^2 - \|v_1\|^2)$ in the first index and 0 on the rest of the diagonal. M_φ can then be diagonalized with 0 in the first index and $(\|v_1\|^2 - \|w_1\|^2)$ on the $p+q-1$ remaining indices. When $(\|w_1\|^2 - \|v_1\|^2) = 0$, the matrix A is not diagonalizable and 0 is its only eigenvalue. Hence M_φ only has eigenvalue 0. Since the 0 eigenvalues do not affect the Jacobian, it can always be written as

$$J(X_{v,w}) = \left(\frac{\sinh(\|v\|^2 - \|w\|^2)}{\|v\|^2 - \|w\|^2} \right)^{p+q-1}.$$

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