An Eigen Decomposition Method for Finite Element Analysis of Electromagnetic Problems

Minyechil Mekonnen and Su Yan
Department of Electrical Engineering and Computer Science, Howard University, Washington, DC 20059, USA
minyechil.mekonnen@bison.howard.edu; su.yan@howard.edu

Abstract—**Finite element method (FEM) is flexible in modeling complex geometries and material compositions and accurate in numerical simulations of a wide range of electromagnetic problems. However, it suffers from the poor convergence when solved iteratively. In this paper, an eigen decomposition method is proposed by removing the null-space of the FEM matrix to speed up the iterative convergence of the FEM solution. The scattering analysis of a large resonant cavity is performed to demonstrate the efficiency of the proposed method.**

I. Introduction

Finite element method (FEM) is one of the most widely used numerical methods for various electromagnetic applications [1], such as the scattering analysis, antenna design, and microwave circuit evaluation. Based on an unstructured mesh of the geometry and a numerical discretization of the vector wave equation, FEM is able to model structures with arbitrary shapes and electromagnetic problems with complicated material compositions. In FEM, the vector wave equation is converted into a matrix equation, where the FEM matrix is sparse and symmetric. To solve such a matrix equation, both direct and iterative methods can be applied. When the matrix dimension is very large, iterative methods are usually preferred due to their lower computational complexities. However, the condition of the FEM matrix obtained by discretizing the wave equation is very poor, leading to an extremely slow iterative convergence. As a result, an efficient preconditioner is usually required to achieve a rapid convergence. Unfortunately, the application of preconditioners encounters two major issues. First, the preconditioners can be very expensive to construct and apply, and their performance varies from case to case. Second, there is no effective preconditioner for electromagnetic problems with physical resonance.

In this paper, an algorithm based on an eigen decomposition method is proposed to speed up the convergence of the iterative solution of the FEM matrix equation. Based on an efficient partial solution of an associated eigenvalue problem (EVP), the smallest eigenvalues and their eigenvectors are removed from the subspace in which the numerical solution is sought, leading to a better effective condition of the system and a faster iterative convergence. An example concerning the scattering from a large resonant cavity is presented to demonstrate the performance of the proposed method.

II. Eigen Decomposition Method

The standard finite element discretization of the vector wave equation results in the following matrix equation [1]

$$[K]\{E\}=\{b\} \tag{1}$$

where the FEM matrix $[K]$ is an N-by-N sparse symmetric matrix, $\{E\}$ and $\{b\}$ are N-by-1 vectors containing the unknown coefficients and right-hand side (RHS) excitations, respectively. When solved iteratively, (1) encounters a very slow convergence due to the bad conditioning of the FEM matrix. Mathematically speaking, the FEM matrix has zero or next-to-zero eigenvalues, especially for physically resonant problems. The corresponding eigenvectors span a null-space that cannot be efficiently described by an iterative solver. The presence of the null-space makes the FEM matrix either singular or near singular, which is extremely difficult to converge in an iterative solution. In this work, such a null-space is explicitly removed from the solution subspace of the FEM matrix, resulting in an eigen decomposition method that leads to a fast iterative convergence, even in problems with internal resonance.

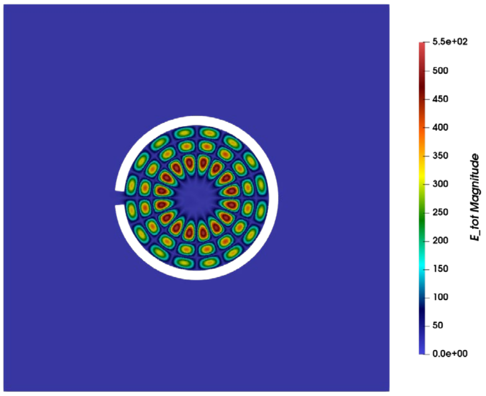
To this end, define the following EVP associated with (1)

$$[K][U]=[U][\Lambda] \tag{2}$$

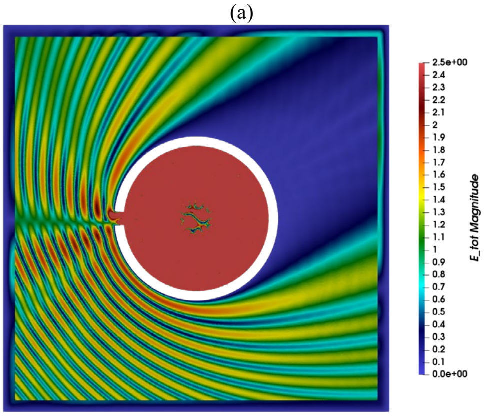
where matrix $[K]$ is obtained from the finite element discretization of the electromagnetic problem, which is the same as that in (1), $[U]$ stands for the matrix of eigenvectors, and $[\Lambda]$ stands for diagonal eigenvalue matrix. Since $[K]$ is symmetric, $[U]$ can be orthonormalized such that the solution can be expressed as

$$\{E\}=[K]^{-1}\{b\}=[U][\Lambda]^{-1}[U]^{\mathrm{T}}\{b\}. \tag{3}$$

However, the direct solution of (2) for all the eigenvalues and eigenvectors is very expensive and thus, impractical. Fortunately, an efficient partial solution of (2) based on an oblique projection method [2] is possible. This permits a decomposition of (1) into two problems based on the moduli of the eigenvalues. First, the first M $(M \ll N)$ smallest eigenvalues $[\Lambda_M]$ and their associated eigenvectors $[U_M]$ are



(a)



(b)

Figure 1. Total electric field distribution with the range of the color bar set as (a) $E \in [0, 550]$ V/m and (b) $E \in [0, 2.5]$ V/m.

calculated by partially solving (2). These eigenvectors span the null-space of the FEM matrix. The RHS vector $\{b\}$ can then be expressed as the summation of $\{b_M\}$ spanned by the null-space and the remaining part $\{b_R\}$

$$\{b\} = \{b_M\} + \{b_R\} = [U_M][U_M]^{\mathrm{T}}\{b\} + \{b_R\}. \tag{4}$$

With (4), the matrix solution can be reformulated as

$$\begin{aligned} \{E\} &= [K]^{-1}(\{b_M\} + \{b_R\}) \\ &= [U_M][\Lambda_M]^{-1}[U_M]^{\mathrm{T}}\{b\} + [K]^{-1}\{b_R\} \\ &= \sum_{i=1}^{M} \lambda_i^{-1}\{u_i\}\{u_i\}^{\mathrm{T}}\{b\} + \{E_R\} \end{aligned} \tag{5}$$

where the first term can be constructed directly from the partial solution of the EVP (2) and the second term $\{E_R\}$ can be solved iteratively with a much faster convergence since it contains no components in the null-space anymore.

III. Numerical Example

In this section, electromagnetic scattering from a large perfectly electric conducting (PEC) cavity is simulated to demonstrate the computational performance of the proposed method. Considered here is a two-dimensional PEC cylindrical cavity with an inner radius of 3.1 m, a thickness of 0.4 m, and a 10-degree aperture. This structure is so chosen that once the incident plane wave enters the cavity, the

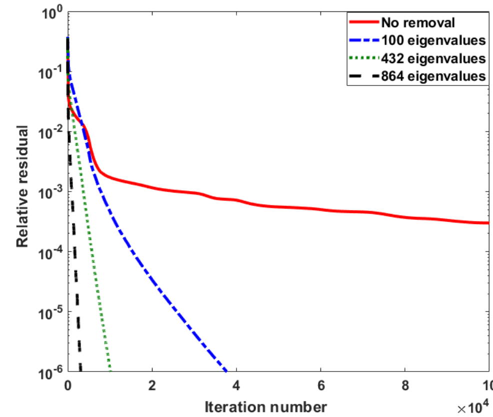


Figure 2. Iterative convergence history of the FEM solution.

electromagnetic energy will be trapped inside, and as a result, a strong physical internal resonance is induced. The simulation domain is chosen to be a 15.5-m-by-15.5-m square region surrounded by a perfectly matched layer (PML) with a thickness of 0.5 m in each direction. Upon discretization with 0.05 m equilateral triangular elements, the PML is meshed into 12 layers in each direction and the total number of unknowns in the entire simulation domain is 113,115. A 302,010,910-Hz transverse magnetically (TM) polarized wave illuminates the cavity from the lower left direction at a 30-degree angle. The frequency is carefully chosen to excite the physical internal resonance inside the cavity.

This problem is solved iteratively using the generalized minimal residual (GMRES) solver [3]. The distribution of the total electric field is shown in Fig. 1. It can be seen from Fig. 1(a) that the field inside the cavity follows very nicely the $\mathrm{TM}_{3,8}$ mode distribution with a maximum intensity of 550 V/m, which is much stronger than the maximum intensity of the field outside the cavity that is about 2.5 V/m [Fig. 1(b)]. This clearly demonstrates the internal resonance where most of the energy gets trapped inside the cavity. Figure 2 presents the convergence histories of the GMRES iterative solutions with and without the eigen decomposition method. Apparently, without the removal of the null-space, the solver cannot converge after 100,000 iterations that takes more than 1,400 minutes. By removing 864 smallest eigenvalues out of the total of 113,115 eigenvalues, the iterative solver converges in 3,100 iterations, with a total solution time (including the partial solution of the EVP) of 33 minutes.

References

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