

# The Quantum Union Bound made easy

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## Abstract

We give a short proof of Gao's Quantum Union Bound and Gentle Sequential Measurement theorems.

## 1 Introduction

Let  $\rho \in \mathbb{C}^{d \times d}$  be a quantum mixed state and let  $A_1, \dots, A_m \geq 0$  be (orthogonal) projectors on  $\mathbb{C}^d$ , which may be thought of as “quantum events”. We write  $\bar{A}_t = \mathbb{1} - A_t$ , where  $\mathbb{1}$  is the identity operator. For intuition, we think of the  $A_t$ 's as “good” events that happen with high probability: we write

$$\mathbf{E}_\rho[A_t] := \text{tr}(\rho A_t) = 1 - \epsilon_t,$$

and hence the “bad” event  $\bar{A}_t$  has  $\mathbf{E}_\rho[\bar{A}_t] = \epsilon_t$ . Suppose we now sequentially measure  $\rho$  with the two-outcome projective measurements  $(\bar{A}_1, A_1), (\bar{A}_2, A_2), \dots, (\bar{A}_m, A_m)$ . For  $0 \leq t \leq m$ , let  $\rho_t$  denote the state conditioned on outcomes  $A_1, \dots, A_t$  all occurring. The Quantum Union Bound question now asks, “What is the probability,  $\text{SUCC}$ , that all  $m$  “good” outcomes occur?” We may also ask the related question of Gentle Sequential Measurement: Conditioned on all good outcomes occurring, how far is the resulting state  $\rho_m$  from  $\rho$  (say, in trace distance)?

For full details of the history of these questions, see the discussion in [5]. An important milestone regarding the Quantum Union Bound came from Sen [7], who established  $\text{FAIL} \leq 2\sqrt{\text{LOSS}}$ , where we denote  $\text{FAIL} = 1 - \text{SUCC}$  and  $\text{LOSS} = \sum_t \epsilon_t$ . Subsequently, Gao [4] obtained the square of Sen's upper bound. His results were:

**THEOREM 1.1.** (Gao's Quantum Union Bound.)  $\text{FAIL} \leq 4\text{LOSS}$ . (♥)

**THEOREM 1.2.** (Gentle Sequential Measurement.)  $D_{\text{tr}}(\rho, \rho_m) \leq \sqrt{\text{LOSS}}$ . (◇)

Khabbazi Oskoui, Mancini, and Wilde [5] obtained a further improvement to (♥), discussed in Section 3.3. In this work we give a simple proof of a common generalization of Theorem 1.1 and Theorem 1.2. Denoting fidelity (Notation (1.1)) by  $F(\cdot, \cdot)$ , we show:

**THEOREM 1.3.**  $1 \leq \sqrt{\text{SUCC}}\sqrt{F(\rho, \rho_m)} + \sqrt{\text{FAIL}}\sqrt{\text{LOSS}}$ .

Then to deduce Theorem 1.1 from Theorem 1.3, we use  $F(\rho, \rho_m) \leq 1$  to get

$$(\heartsuit') \quad 1 \leq \sqrt{\text{SUCC}} + \sqrt{\text{FAIL}}\sqrt{\text{LOSS}} = \sqrt{1 - \text{FAIL}} + \sqrt{\text{FAIL}}\sqrt{\text{LOSS}} \implies \text{FAIL} \leq \frac{4\text{LOSS}}{(1 + \text{LOSS})^2} \leq 4\text{LOSS}.$$

Here  $\text{FAIL} \leq \frac{4\text{LOSS}}{(1 + \text{LOSS})^2}$  arises from solving the quadratic for  $\text{FAIL}$ ; it assumes  $\text{LOSS} \leq 1$ . One can also get (♥') via

$$\text{AM-GM: } \frac{1}{2}\text{FAIL} \leq 1 - \sqrt{1 - \text{FAIL}} \leq \sqrt{\text{FAIL}}\sqrt{\text{LOSS}} = \sqrt{\frac{1}{2}\text{FAIL}}\sqrt{2\text{LOSS}} \leq \frac{1}{2}(\frac{1}{2}\text{FAIL} + 2\text{LOSS}).$$

To deduce Theorem 1.2, we apply Cauchy–Schwarz to Theorem 1.3 obtaining

$$(\diamondsuit') \quad 1 \leq \sqrt{\text{SUCC} + \text{FAIL}} \sqrt{F(\rho, \rho_m) + \text{LOSS}} \implies \bar{F}(\rho, \rho_m) := 1 - F(\rho, \rho_m) \leq \text{LOSS},$$

stronger than Theorem 1.2 thanks to the Fuchs–van de Graaf [3] inequality  $D_{\text{tr}}(\rho, \sigma) \leq \bar{F}(\rho, \sigma)^{1/2}$ .

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## 1.1 Notation

NOTATION 1.1. For two states  $\rho, \sigma \in \mathbb{C}^{d \times d}$ , their fidelity is  $F(\rho, \sigma) = \|\sqrt{\rho}\sqrt{\sigma}\|_1^2 = (\text{tr} \sqrt{\sqrt{\rho}\sigma\sqrt{\rho}})^2$ , where we recall the Schatten 1-norm  $\|M\|_1 = \text{tr} \sqrt{MM^\dagger} = \text{tr} \sqrt{M^\dagger M}$ .

The fidelity between two states is at most 1; this is a consequence of the matrix Cauchy–Schwarz inequality  $\|M_1 M_2\|_1^2 \leq \text{tr}(M_1^\dagger M_1) \text{tr}(M_2^\dagger M_2)$ .

NOTATION 1.2. Let  $M$  be a  $d$ -column matrix with  $M^\dagger M \leq \mathbb{1}$ , thought of as a nondestructive measurement matrix (so that  $M^\dagger M$  is one element of a POVM). The probability of  $M$  occurring when  $\rho$  is measured is  $\mathbf{E}_\rho[M^\dagger M]$ , and we denote the resulting state conditioned on this outcome by  $\rho|M = (M\rho M^\dagger)/\mathbf{E}_\rho[M^\dagger M]$ . (We tacitly assume the denominator is nonzero.)

REMARK 1.1. We work over finite-dimensional Hilbert spaces for simplicity, but this is inessential; the proofs extend to any separable Hilbert space.

## 2 Proof

LEMMA 2.1. For quantum states  $\rho, \sigma \in \mathbb{C}^{d \times d}$  and  $A \in \mathbb{C}^{d \times d}$  with  $\bar{A} \geq 0$ ,

$$\sqrt{F(\rho, \sigma)} \leq \sqrt{\mathbf{E}_\sigma[A^\dagger A]} \sqrt{F(\rho, \sigma|A)} + \sqrt{\mathbf{E}_\sigma[\bar{A}]} \sqrt{\mathbf{E}_\rho[\bar{A}]}.$$

*Proof.* We have  $\sqrt{F(\rho, \sigma)} = \|\sqrt{\rho}\sqrt{\sigma}\|_1 = \|\sqrt{\rho}(A + \bar{A})\sqrt{\sigma}\|_1 \leq \|\sqrt{\rho}A\sqrt{\sigma}\|_1 + \|\sqrt{\rho}\bar{A}\sqrt{\sigma}\|_1$ . On one hand,

$$\|\sqrt{\rho}A\sqrt{\sigma}\|_1 = \text{tr} \sqrt{\sqrt{\rho}A\sigma A^\dagger \sqrt{\rho}} = \sqrt{\mathbf{E}_\sigma[A^\dagger A]} \sqrt{F(\rho, \sigma|A)}.$$

On the other hand, by matrix Cauchy–Schwarz we have

$$\|\sqrt{\rho}\bar{A}\sqrt{\sigma}\|_1 = \left\| \sqrt{\rho}\bar{A}^{1/2} \bar{A}^{1/2} \sqrt{\sigma} \right\|_1 \leq \sqrt{\mathbf{E}_\sigma[\bar{A}]} \sqrt{\mathbf{E}_\rho[\bar{A}]}.$$

(Remark: in Section 3.2 we note that in fact  $\|\sqrt{\rho}\bar{A}\sqrt{\sigma}\|_1^2 = F(\rho|\bar{A}^{1/2}, \sigma|\bar{A}^{1/2}) \cdot \mathbf{E}_\sigma[\bar{A}] \cdot \mathbf{E}_\rho[\bar{A}]$ .)  $\square$

For a geometric interpretation with pure states, see Section 3.5. We now prove Theorem 1.3.

*Proof.* For  $0 \leq t \leq m$ , consider the event that the “good” outcomes  $A_1, \dots, A_t$  all occur. We write  $p_t$  for the probability of this event,  $\rho_t$  for the state  $\rho$  conditioned on this event, and  $r_t = \sqrt{p_t} \sqrt{F(\rho, \rho_t)}$ . For  $1 \leq t \leq m$  we write  $q_t$  for the probability that  $\bar{A}_t$  is the first “bad” outcome that occurs. Now

$$r_{t-1} - r_t = \sqrt{p_{t-1}} \left( \sqrt{F(\rho, \rho_{t-1})} - \sqrt{\mathbf{E}_{\rho_{t-1}}[A_t]} \sqrt{F(\rho, \rho_{t-1}|A_t)} \right) \leq \sqrt{p_{t-1}} \sqrt{\mathbf{E}_{\rho_{t-1}}[\bar{A}_t]} \sqrt{\mathbf{E}_\rho[\bar{A}_t]} = \sqrt{q_t} \sqrt{\epsilon_t},$$

where the inequality used Lemma 2.1 and  $A_t^\dagger A_t = A_t$ . Summing this for  $t = 1 \dots m$  yields

$$1 - \sqrt{\text{SUCC}} \sqrt{F(\rho, \rho_m)} = r_0 - r_m \leq \sum_{t=1}^m \sqrt{q_t} \sqrt{\epsilon_t} \leq \sqrt{\sum_{t=1}^m q_t} \sqrt{\sum_{t=1}^m \epsilon_t} = \sqrt{\text{FAIL}} \sqrt{\text{LOSS}},$$

where the last inequality is Cauchy–Schwarz.  $\square$

Anshu [1] has observed that if the above proof is written using subnormalized pure states, it becomes structurally very similar to Sen’s proof [7].

## 3 Additional commentary

**3.1 Simpler proof of Gentle Sequential Measurement** We remark that if one’s only goal is to prove Theorem 1.2, the proof is even simpler. Assuming  $A$  is a projector, applying Cauchy–Schwarz to Lemma 2.1 yields

$$\sqrt{F(\rho, \sigma)} \leq \sqrt{\mathbf{E}_\sigma[A]} \sqrt{F(\rho, \sigma|A)} + \sqrt{\mathbf{E}_\sigma[\bar{A}]} \sqrt{\mathbf{E}_\rho[\bar{A}]} \leq \sqrt{\mathbf{E}_\sigma[A] + \mathbf{E}_\sigma[\bar{A}]} \sqrt{F(\rho, \sigma|A) + \mathbf{E}_\rho[\bar{A}]}.$$

Squaring and rearranging yields:

PROPOSITION 3.1. If  $\rho, \sigma \in \mathbb{C}^{d \times d}$  are states and  $A \in \mathbb{C}^{d \times d}$  is a projector,  $\bar{F}(\rho, \sigma|A) \leq \bar{F}(\rho, \sigma) + \mathbf{E}_\rho[\bar{A}]$ .

Taking  $\sigma = \rho_{t-1}$  and  $A = A_t$  we get  $\bar{F}(\rho, \rho_t) \leq \bar{F}(\rho, \rho_{t-1}) + \epsilon_t$ , and hence  $\bar{F}(\rho, \rho_m) \leq \text{LOSS}$  by iterating.

**3.2 Fidelity and conditioning** We first recall some traditional matrix notation:

NOTATION 3.1. If  $M$  is any matrix, recall that  $|M|$  denotes  $\sqrt{M^\dagger M}$ , so  $\|M\|_1 = \text{tr } |M|$ .

FACT 3.1. For any  $M \in \mathbb{C}^{m \times \ell}$ ,  $N \in \mathbb{C}^{\ell \times n}$ , it is immediate that  $|M \cdot N| = | |M| \cdot N |$ . Taking trace on both sides and using  $\|X\|_1 = \|X^\dagger\|_1$ , we can infer  $\|M \cdot N\|_1 = \| |M| \cdot |N^\dagger| \|_1$ .

Now we introduce some additional notation:

NOTATION 3.2. For  $\rho \in \mathbb{C}^{d \times d}$  a quantum state and  $A \in \mathbb{C}^{d \times d}$ , we write  $\|A\|_\rho = \sqrt{\mathbf{E}_\rho[A^\dagger A]}$ .

FACT 3.2. If  $A$  is a projector then  $\|A\|_\rho^2 = \mathbf{E}_\rho[A] = F(\rho, \rho|A)$ . (The latter formula, basically the “Gentle Measurement Lemma” [8, 6], follows just by writing the definitions and using  $\sqrt{\rho}A\sqrt{\rho} \geq 0$ , since  $A \geq 0$ .)

REMARK 3.1. Notation (1.2) may alternately be written as  $\sqrt{\rho|M|} = \frac{|\sqrt{\rho}M^\dagger|}{\|M\|_\rho}$ .

Although Theorem 1.3 looks neat as stated, we actually prefer the definition of fidelity that doesn’t have the square built in (as in, e.g., the Nielsen–Chuang text). For lack of better symbols, we introduce the following notation for it:

NOTATION 3.3. We write  $\mathfrak{f}(\rho, \sigma) = \sqrt{F(\rho, \sigma)} = \|\sqrt{\rho}\sqrt{\sigma}\|_1$ , and  $\bar{\mathfrak{f}}(\rho, \sigma) = \sqrt{\bar{F}(\rho, \sigma)} = \sqrt{1 - \mathfrak{f}(\rho, \sigma)^2}$ .

Now the following is an immediate consequence of Fact (3.1) and Remark (3.1):

PROPOSITION 3.2. If  $\rho, \sigma \in \mathbb{C}^{d \times d}$  are states and  $M, N \in \mathbb{C}^{d \times d}$ , then  $\mathfrak{f}(\rho|M, \sigma|N) = \frac{\|\sqrt{\rho}M^\dagger N\sqrt{\sigma}\|_1}{\|M\|_\rho \|N\|_\sigma}$ .

This formula is quite useful. In particular ( $M = \mathbb{1}$ ,  $N = A$ ) it implies  $\mathfrak{f}(\rho, \sigma|A) = \|\sqrt{\rho}A\sqrt{\sigma}\|_1 / \|A\|_\sigma$ , which is identical to the first fact derived in our main Lemma 2.1. Note furthermore that if  $A \geq 0$ ,

$$\|\sqrt{\rho}A\sqrt{\sigma}\|_1 = \|\sqrt{\rho}\sqrt{A}\sqrt{A}\sqrt{\sigma}\|_1 = \mathfrak{f}(\rho|\sqrt{A}, \sigma|\sqrt{A}) \cdot \sqrt{\mathbf{E}_\rho[A]} \sqrt{\mathbf{E}_\sigma[A]}$$

where we used Proposition 3.2 again ( $M = N = \sqrt{A}$ ). This shows the *second* fact derived in our main Lemma 2.1 (more precisely, it shows the “Remark” at the end, after replacing  $A$  with  $\bar{A}$ ). Finally, putting these two implications together yields:

COROLLARY 3.1. If  $A \geq 0$ , then  $\mathfrak{f}(\rho, \sigma|A) \leq \frac{\sqrt{\mathbf{E}_\rho[A]} \sqrt{\mathbf{E}_\sigma[A]}}{\|A\|_\sigma}$ . If  $A$  is furthermore a projector, the right-hand side simplifies to  $\sqrt{\mathbf{E}_\rho[A]}$ ; i.e.,  $\mathfrak{f}(\rho, \sigma|A) \leq \mathfrak{f}(\rho, \rho|A)$ .

**3.3 Obtaining the bound from [5]** The proof given by Khabbazi Oskouei, Mancini, and Wilde [5] included an improvement to Gao’s Quantum Union Bound: they showed that

$$(\clubsuit) \quad \text{FAIL}^* := \text{FAIL} - \epsilon_1 \leq p'\epsilon_1 + (p + p') \sum_{1 < t < m} \epsilon_t + p\epsilon_m$$

for any (positive)  $p, p'$  with  $1/p + 1/p' = 1$ . (Gao’s bound is implied by the  $p = p' = 2$  case.) They also gave an application where it is essential that  $p$  may be made arbitrarily close to 1. We can obtain the same bound by slightly modifying our proof of Theorem 1.3.

In the modified proof, we simply save on the first term since we know that  $q_1 = \epsilon_1$ . This gives

$$1 - \sqrt{\text{SUCC}} \sqrt{F(\rho, \rho_m)} \leq \epsilon_1 + \sqrt{\text{FAIL}^*} \sqrt{\text{LOSS} - \epsilon_1}.$$

But from Corollary 3.1 we obtain  $F(\rho, \rho_m) = F(\rho, \rho_{m-1}|A_m) \leq F(\rho, \rho|A_m) = 1 - \epsilon_m$ . Thus

$$1 - \sqrt{1 - \text{FAIL}^* - \epsilon_1} \sqrt{1 - \epsilon_m} \leq \epsilon_1 + \sqrt{\text{FAIL}^*} \sqrt{\sum_{t>1} \epsilon_t}.$$

One can solve the associated quadratic equation for  $\text{FAIL}^*$  to get a sharp, but messy, bound. More simply, we can use AM-GM twice to get  $1 - \sqrt{1 - \text{FAIL}^*} - \epsilon_1 \sqrt{1 - \epsilon_m} \geq \frac{1}{2}(\text{FAIL}^* + \epsilon_1 + \epsilon_m)$ , and

$$\epsilon_1 + \sqrt{\text{FAIL}^*} \sqrt{\sum_{t>1} \epsilon_t} = \epsilon_1 + \sqrt{p^{-1} \text{FAIL}^*} \sqrt{p \sum_{t>1} \epsilon_t} \leq \epsilon_1 + \frac{1}{2} p^{-1} \text{FAIL}^* + \frac{1}{2} p \sum_{t>1} \epsilon_t.$$

Putting these together yields  $\text{FAIL}^* + \epsilon_1 + \epsilon_m \leq 2\epsilon_1 + p^{-1} \text{FAIL}^* + p \sum_{t>1} \epsilon_t$ , which yields  $(\clubsuit)$  after multiplication by  $p'$  and rearrangement (note that  $p + p' = pp'$ ).

**3.4 Intuition I: Bhattacharyya coefficient** A useful way of discovering results concerning quantum fidelity is via analogy with its easier-to-understand classical counterpart:

**NOTATION 3.4.** Recall that for two probability distributions  $p, q$  on  $[d]$ , their Bhattacharyya coefficient is  $\text{BC}(p, q) = \sum_{i=1}^d \sqrt{p_i} \sqrt{q_i} \in [0, 1]$ . (This equals  $\mathfrak{f}(\text{diag}(p), \text{diag}(q))$ .)

The well-known classical analogue (indeed, consequence) of the Fuchs–van de Graaf inequality is:

**FACT 3.3.** The total variation distance  $d_{\text{TV}}(p, q) = \frac{1}{2} \|p - q\|_1$  satisfies  $d_{\text{TV}}(p, q) \leq \sqrt{1 - \text{BC}(p, q)^2}$ . (This is slightly sharper than bounding total variation distance by Hellinger distance.)

An event  $A \subseteq [d]$  is the analogue of a projector, so the following can be compared to Fact (3.2):

**FACT 3.4.** If  $A \subseteq [d]$  is an event, then  $p(A) = \mathbf{Pr}_p[A] = \mathbf{E}_p[1_A] = \text{BC}(p, p|A)^2$ .

The analogue of our main Lemma 2.1 is also natural in the classical case:

**LEMMA 3.1.** If  $A \subseteq [d]$ , then  $\text{BC}(p, q) = \sqrt{q(A)} \text{BC}(p, q|A) + \sqrt{q(\bar{A})} \sqrt{p(\bar{A})}$ .

*Proof.* Since  $(q|A)_i$  is  $q_i/q(A)$  if  $i \in A$ , and is 0 if  $i \in \bar{A}$ , we get

$$\text{BC}(p, q) = \sum_{i \in A} \sqrt{p_i} \sqrt{q_i} + \sum_{i \in \bar{A}} \sqrt{p_i} \sqrt{q_i} = \sqrt{q(A)} \text{BC}(p, q|A) + \sum_{i \in \bar{A}} \sqrt{p_i} \sqrt{q_i},$$

and the result follows by applying Cauchy–Schwarz to the second term.  $\square$

**3.5 Intuition II: Pure states and geometry** As observed by Gao [4], a purification argument immediately shows that to prove quantum union bounds, it suffices to consider pure states. This can assist with geometric intuition, particularly if one imagines — with only mild loss of generality — that all states and projectors are real.

In this case, let  $|\psi_t\rangle$  denote the unit vector in  $\mathbb{R}^d$  obtained by conditioning on the first  $t$  projective measurements succeeding. Then if  $H = H_{t+1}$  denotes the subspace onto which  $A_{t+1}$  projects, the analysis of the  $(t+1)$ th measurement really only depends on four vectors, namely  $\text{Proj}_H |\psi_0\rangle$ ,  $\text{Proj}_H |\psi_t\rangle$ ,  $\text{Proj}_{H^\perp} |\psi_0\rangle$ , and  $\text{Proj}_{H^\perp} |\psi_t\rangle$ . So without loss of generality we may project everything into  $\mathbb{R}^4$ , with the first three vectors spanning  $\mathbb{R}^3$ . We can then picture a globe in  $\mathbb{R}^3$  of unit radius, with  $H_{t+1}$  being the plane of the equator,  $|\psi_0\rangle$  and  $|\psi_{t+1}\rangle$  lying on the globe’s surface, and  $|\psi_t\rangle = r|\tilde{\psi}_t\rangle + |\tilde{\psi}_t^\perp\rangle$  for some  $|\tilde{\psi}_t\rangle$  on the globe’s surface, with  $0 \leq r \leq 1$  and  $|\tilde{\psi}_t^\perp\rangle$  pointing into the fourth dimension. For  $j \in \{0, t, t+1\}$ , we’ll write  $(\lambda_j, \phi_j)$  for the longitude/latitude of  $|\psi_j\rangle$  (or  $|\tilde{\psi}_j\rangle$  when  $j = t$ ). We may assume that  $\lambda_t = \lambda_{t+1} = 0$ , and hence  $|\psi_{t+1}\rangle = (0, 0)$ . (See the left image in Figure 1.)

For  $j \in \{t, t+1\}$ , let us write  $\Delta_j$  for the angle between  $|\psi_0\rangle$  and  $|\psi_j\rangle$ , and also write  $\tilde{\Delta}_t$  for the angle between  $|\psi_0\rangle$  and  $|\tilde{\psi}_t\rangle$  (equivalently,  $r|\tilde{\psi}_t\rangle$ ). We claim that

$$\cos \Delta_{t+1} = \cos \phi_0 \cos \lambda_0, \quad \cos \tilde{\Delta}_t = \cos \phi_t \cos \phi_0 \cos \lambda_0 + \sin \phi_t \sin \phi_0, \quad \cos \Delta_t \leq \cos \tilde{\Delta}_t.$$

The first formula is the spherical Pythagorean Theorem applied to the triangle with vertices  $|\psi_0\rangle$ ,  $(\lambda_0, 0)$ , and  $|\psi_{t+1}\rangle$ . The second is the great-circle distance formula; equivalently, the spherical Cosine Law applied to the triangle formed by  $|\psi_0\rangle$ , the north pole (blue dot), and  $|\tilde{\psi}_t\rangle$ . Finally, the inequality holds because the angle,  $\Delta_t$ ,

that  $|\psi_0\rangle$  makes with  $|\psi_t\rangle$  is at least the angle,  $\tilde{\Delta}_t$ , it makes with  $r|\tilde{\psi}_t\rangle$ , since the former is equal to the latter plus a vector  $|\tilde{\psi}_t^\perp\rangle$  that is orthogonal to both  $|\psi_0\rangle$  and  $r|\tilde{\psi}_t\rangle$ . Combining the above three results now yields

$$(3.1) \quad \cos \Delta_t \leq \cos \phi_t \cos \Delta_{t+1} + \sin \phi_t \sin \phi_0,$$

which is exactly the relationship derived in our main Lemma 2.1 (with  $\rho$  being  $|\psi_0\rangle$  and  $\sigma$  being  $|\psi_t\rangle$  and  $A$  being projection onto  $H_{t+1}$ ).

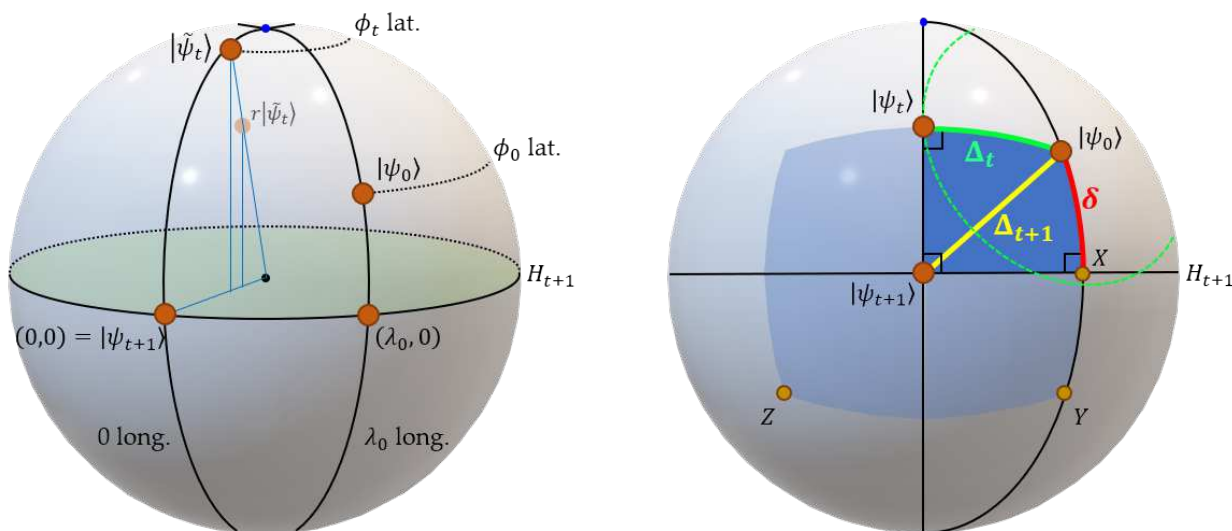


Figure 1: On the left, justifying Lemma 2.1 for pure states. On the right, tightness for Theorem 1.2.

**3.6 Tightness** The factor of 4 appearing in the Quantum Union Bound is tight, even in the case of one pure qubit with real amplitudes. To see this, fix a large  $m$  and then consider  $\delta \rightarrow 0^+$ . Now suppose the initial state of the qubit is  $|0\rangle$ , and  $A_t$  projects onto the line in  $\mathbb{R}^2$  making an angle of  $(-1)^t \cdot \delta$  with  $|0\rangle$ . Then one hand,  $\epsilon_t = \sin^2(\pm\delta) \sim \delta^2$  for each  $t$ , so  $\text{LOSS} \sim m\delta^2$ . On the other hand,

$$\text{FAIL} = 1 - (1 - \sin^2 \delta)(1 - \sin^2 2\delta)(1 - \sin^2 2\delta) \cdots (1 - \sin^2 2\delta) \sim (4m - 3)\delta^2.$$

From this we see that the constant “4” in Theorem 1.1’s  $\text{FAIL} \leq 4\text{LOSS}$  cannot be replaced by any smaller constant.

In fact, the same idea can be used to show that the refined bound denoted  $(\clubsuit)$  in Section 3.3 is asymptotically tight for all fixed  $m \geq 2$  and  $p, p'$ . To see this, let  $\delta_t = a_t \delta$  for constants  $a_1, \dots, a_m$ , and let  $A_t$  project onto the line in  $\mathbb{R}^2$  making an angle of  $(-1)^t \cdot \delta_t$  with  $|0\rangle$ . Then on one hand,  $\epsilon_t = \sin^2(\pm\delta_t) \sim a_t^2 \delta^2$ , and hence the bound from  $(\clubsuit)$  is

$$(\clubsuit') \quad \text{FAIL} \lesssim \left( a_1^2 + p' a_1^2 + (p + p') \sum_{1 < t < m} a_t^2 + p a_m^2 \right) \cdot \delta^2$$

On the other hand,

$$\begin{aligned} \text{FAIL} &= 1 - (1 - \sin^2 \delta_1)(1 - \sin^2(\delta_1 + \delta_2))(1 - \sin^2(\delta_2 + \delta_3)) \cdots (1 - \sin^2(\delta_{m-1} + \delta_m)) \\ &\sim \left( a_1^2 + (a_1 + a_2)^2 + (a_2 + a_3)^2 + \cdots + (a_{m-1} + a_m)^2 \right) \cdot \delta^2. \end{aligned}$$

But note that whenever  $1/p + 1/p' = 1$ , it is possible for  $a_t, a_{t+1}$  to satisfy  $(a_t + a_{t+1})^2 = p' a_t^2 + p a_{t+1}^2$ . (Specifically, this happens if  $a_{t+1}/a_t = p'/p$ .) So if this identity is always satisfied, then  $(\clubsuit')$  is indeed tight up to lower-order  $O(\delta^4)$  terms.

Next we show that Gentle Sequential Measurement bounds ( $\diamond'$ ) are exactly tight (assuming  $\sum_t \epsilon_t \leq 1$ ), even in the case of pure state qutrits with real amplitudes. This also implies exact tightness of ( $\diamond$ ), since  $D_{\text{tr}}(\rho, \sigma) = \bar{F}(\rho, \sigma)^{1/2}$  for pure states  $\rho, \sigma$ . To show this, suppose  $|\psi_0\rangle$  and  $|\psi_t\rangle$  are states in  $\mathbb{R}^3$  at angle  $\Delta_t$ , and let angle  $\delta = \delta_{t+1}$  be given. (One may imagine that  $\sin^2 \Delta_t = \sum_{1 \leq i \leq t} \epsilon_i$  already, and  $\sin^2 \delta = \epsilon_{t+1}$ .) We will show that there is a two-dimensional subspace  $H_{t+1}$  (the image of  $A_{t+1}$ ) such that: (i)  $H_{t+1}$  makes an angle of  $\delta$  with  $|\psi_0\rangle$ ; (ii) the state  $|\psi_{t+1}\rangle$  resulting from a successful measurement of  $|\psi_t\rangle$  by  $A_{t+1}$  has an angle  $\Delta_{t+1}$  from  $|\psi_0\rangle$  satisfying

$$(3.2) \quad \sin^2 \Delta_{t+1} = \sin^2 \Delta_t + \sin^2 \delta.$$

As we can arrange this for every  $t$ , we conclude that ( $\diamond'$ ) can be exactly tight.

It is not hard to see that to maximize  $\Delta_{t+1}$ , we should choose  $H_{t+1}$  to ensure that the (great-circle) arc connecting  $|\psi_t\rangle$  to  $|\psi_{t+1}\rangle$  is orthogonal to the arc connecting  $|\psi_0\rangle$  and  $|\psi_t\rangle$ , as in the image on the right of Figure 1. (In that image, one might imagine that  $|\psi_t\rangle$  could have been any point on the green dashed small circle of radius  $\Delta_t$  around  $|\psi_0\rangle$ ; to maximize  $\Delta_{t+1}$  we want the arc connecting  $|\psi_t\rangle$  to  $H_{t+1}$  to be tangent to this green circle.)

Thus it remains to verify that Equation (3.2) holds for the dark blue “Lambert (three-right-angle) quadrilateral” with corners  $|\psi_0\rangle$ ,  $|\psi_t\rangle$ ,  $|\psi_{t+1}\rangle$ , and  $X$  (the state if  $|\psi_0\rangle$  were successfully measured by  $A_{t+1}$ ). This is an elementary (though perhaps lesser-known) fact of spherical geometry. To verify it, one may form the three pale blue reflections of the Lambert quadrilateral, giving a centrally symmetric spherical quadrilateral. Then it is easy to verify that the triangle formed by  $|\psi_0\rangle$  and the points depicted as  $Y$  and  $Z$  form a so-called half-sum triangle (a *right* right triangle, in the terminology of [2]), with the triangular angles at  $|\psi_0\rangle$  and  $Z$  summing to the angle at  $Y$ . But then Equation (3.2) is immediate from Dickinson and Salmassi’s “Preferred Spherical Pythagorean Theorem” [2].

**3.7 How we discovered our proof** The proof we gave is short enough that one might imagine just discovering it from scratch. Alternatively, one might imagine discovering it by trying to prove the classical Union Bound while working exclusively with Bhattacharyya coefficient. But in fact, we essentially came up with our proof by iteratively refining and unifying the original proofs of Gao and of Khabbazi Oskouei–Mancini–Wilde. (Indeed, along the way we had a version of our proof that was roughly  $\alpha$  pages long, for each real number  $0.5 \leq \alpha \leq 4.0$ .)

The parallels are as follows: As noted, our main Lemma 2.1 essentially becomes the geometric equality Inequality (3.1) when reduced to the pure state case. In turn, this is equivalent to “inequality (10)” in [4]. Gao proves this inequality in a different but straightforward fashion, and his deduction of Theorem 1.2 from it is also relatively straightforward. (His “Lemma 1” parallels our Proposition 3.1.) Then like our proof, Gao’s proof of Theorem 1.1 is inductive and uses Inequality (3.1) (his “(10)”), but the inequalities he invokes are significantly more complicated. It seems that introducing our quantity “ $r_t$ ” is important for getting a slick proof. As for the Khabbazi Oskouei–Mancini–Wilde proof, the steps in it are all individually straightforward; however, it seems that working explicitly with fidelity, as we do, helps to get a clean proof. Our key Lemma 2.1 may be viewed as hidden in the proof of [5, “Lemma 3.3”]; one can extract it upon converting their calculational/iterative proof into an induction.

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