Supertranslation-Invariant Formula for the Angular Momentum Flux in Gravitational Scattering

Reza Javadinezhad* and Massimo Porrati†
Center for Cosmology and Particle Physics, Department of Physics, New York University, 726 Broadway, New York, New York 10003, USA

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The angular momentum radiated in gravitational scattering can be changed by performing a supertranslation of the asymptotic metric, i.e., by adding radiation with infinite wavelength to the metric. This puzzling property can be avoided by adopting a supertranslation-invariant definition of the angular momentum flux in general relativity. Definitions currently available in the literature cannot reproduce the flux necessary to obtain the correct radiation reaction effects in gravitational scattering. They also disagree with computations of the flux performed using scattering amplitudes and soft-graviton theorems. In this Letter, we provide a new supertranslation-invariant definition of the angular momentum flux in gravitational scattering that uses only asymptotic metric data and reproduces the flux necessary to obtain the correct radiation reaction effects.

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Gravitational scattering produces radiation that carries away energy, momentum, and angular momentum. These quantities can be defined rigorously using the formalism developed in [1–4]. They are computed at future null infinity $\mathcal{I}^+$ in Bondi-Sachs coordinates

$$\begin{aligned} ds^2 = {} & -du^2 - 2du\,dr + r^2\left(h_{AB} + \frac{C_{AB}}{r}\right)d\Theta^A d\Theta^B \\ & + D^A C_{AB}\,du\,d\Theta^B + \frac{2m}{r}du^2 \\ & + \frac{1}{16r^2}C_{AB}C^{AB}\,dudr \\ & + \frac{1}{r}\left(\frac{4}{3}(N_A + u\partial_A m) - \frac{1}{8}\partial_A(C_{BD}C^{BD})\right)dud\Theta^A \\ & + \frac{1}{4}h_{AB}C_{CD}C^{CD}d\Theta^A d\Theta^B + \cdots, \end{aligned} \tag{1}$$

where the mass aspect $m(\Theta, u)$ is a scalar, the angular momentum aspect $N_A(\Theta, u)$ is a vector, and the shear $C_{AB}(\Theta, u)$ is a symmetric and traceless tensor. All these quantities are defined on the celestial sphere with coordinates Θ_A and round metric h_{AB} and also depend on the retarded time u. The dots in (1) denote subdominant terms in $1/r$. The coordinate system in Eq. (1) is left invariant by the asymptotic symmetries $u \to u + f(\Theta)$, called supertranslations [2]. Energy and angular momentum are defined in terms of m, N_A, and the three Killing vectors Y^A of the celestial sphere [2]

$$\begin{aligned} E(u) &= \frac{1}{4\pi G}\int d^2\Theta\sqrt{h}m(\Theta, u), \\ J_Y(u) &= \frac{1}{8\pi G}\int d^2\Theta\sqrt{h}Y^A N_A(\theta, u). \end{aligned} \tag{2}$$

By definition, the Killing vectors obey $D_A Y_B + D_B Y_A = 0$. When it is well defined, the total energy flux $\Delta E \equiv E(+\infty) - E(-\infty)$ is invariant under supertranslations, while the angular momentum flux $\Delta J_Y \equiv J_Y(+\infty) - J_Y(-\infty)$ is not, as first noticed in [5].

Supertranslation-invariant definitions of the angular momentum flux for finitely radiating systems were given in [6] and later, independently, in [7,8]. A definition of the angular momentum flux in terms of bulk integrals of soft-graviton dressed canonical degrees of freedom was given in [9]. All these definitions coincide and amount to a simple prescription: replace $N_A(u, \Theta)$ in Eq. (2) with $N_A(u, \theta) - 2m(u, \Theta)D_A C(\Theta)$,

$$J_Y^{\mathrm{BMS}}(u) = \frac{1}{8\pi G}\int d^2\Theta\sqrt{h}Y^A[N_A(\theta, u) - 2m(u, \Theta)D_A C(\Theta)]. \tag{3}$$

The superscript "BMS" denotes supertranslation-invariant quantities. The change in angular momentum flux implied by this prescription cannot be reabsorbed in a local redefinition of N_A because it uses the boundary graviton C, which is defined at $u = -\infty$ by

$$\lim_{u\to-\infty} C_{AB}(u, \Theta^A) = -2D_A D_B C(\Theta^A) + h_{AB}D^2 C(\Theta^A). \tag{4}$$

The angular momentum flux $\Delta J_Y^{\mathrm{BMS}}$ can be written also as a 3D integral over future null infinity [9,10]. Assuming that the only massless field relevant to the scattering is the graviton, we find

$$\Delta J_Y^{\rm BMS} = \frac{1}{8\pi G}\int_{-\infty}^{+\infty} du \int d^2\Theta \sqrt{h} Y^A(\Theta)\left[\frac{1}{2}\hat{C}_{AB} D_C N^{BC} - \frac{1}{4} D_B(\hat{C}^{BC} N_{CA})\right], \tag{5}$$

$$\hat{C}_{AB}(u,\Theta) \equiv C_{AB}(u,\Theta) - C_{AB}(-\infty,\Theta) = \int_{-\infty}^{+\infty} du N_{AB}, \quad N_{AB} = \partial_u C_{AB}. \tag{6}$$

The non-supertranslation-invariant definition of ΔJ_Y given in Eq. (2) is the same, except for the replacement $\hat{C}_{AB} \to C_{AB}$.

Equation (5) is a natural but not *unique* definition of the angular momentum flux. When the radiating system reverts back to empty space, it defines an angular momentum that, together with translations and with the definition of boosts given in [9], generates the Poincaré algebra by Poisson brackets and coincides with the usual definition of angular momentum for boosted Kerr black holes and Minkowski space [9]. Reference [11] proves that it is also cross section continuous. All would be well if it were not for a fly in the ointment, namely, the contribution of radiation reaction to gravitational scattering. It first appears at third post-Minkowskian, $\mathcal{O}(G^3)$ order and has been computed by Damour in [12] using linear response theory [13]. The computation has been confirmed by several independent computations used by various groups using different methods [14–18]. The linear response theory requires ΔJ_Y to be $\mathcal{O}(G^2)$, but the lowest order at which the Bondi news N_{AB} can be nonzero is $\mathcal{O}(G^2)$, so Eq. (5) implies $\Delta J_Y^{\rm BMS} = \mathcal{O}(G^3)$. Clearly, the angular momentum flux computed in [12] and derived by different methods in [19,20] is not the BMS-invariant one. It is instead the non-BMS-invariant expression computed using (2) in a particular BMS frame [21]. The special frame is defined perturbatively in G in terms of the initial scattering data, namely, the energies and momenta of the incoming particles [21]. In that frame, the boundary graviton $C(\Theta)$ is nonzero and is also defined in terms of particle energy and momenta. It was computed explicitly in [21] and is implicitly given in [19,20] (see also [22]). Because the BMS-invariant angular momentum in (5) precisely subtracts $C(\Theta)$, it differs from the one used in [12,19,20]. The explicit relation between the flux used by Damour, which we shall call ΔJ_Y^D, and the BMS-invariant one $\Delta J_Y^{\rm BMS}$ is

$$\Delta J_Y^D = \Delta J_Y^{\rm BMS} + \frac{1}{4\pi G}\int d^2\Theta \sqrt{h} Y^A \Delta m(\Theta) D_A \beta(\underline{p},\Theta). \tag{7}$$

Here $\Delta m(\Theta) = m(+\infty,\Theta) - m(-\infty,\Theta)$, while $\beta(\underline{p},\Theta)$ is the boundary graviton computed in [21]. These notations make clear that β is a function of the n initial four-momenta of the initial particles $\underline{p} = (p_1, \ldots, p_N)$. Explicitly [21], the metric is written as $g_{\mu\nu} = \eta_{\mu\nu} + \delta g_{\mu\nu}$, $\delta g_{\mu\nu} = \mathcal{O}(G)$, the signature is mostly plus, and

$$\beta(\underline{p},\Theta) = \sum_{I=1}^{N} 2G m_I (n\cdot v_I)\log(-n\cdot v_I). \tag{8}$$

Here $n\cdot v \equiv n^\mu v^\mu \eta_{\mu\nu}$, v_I^μ is the four-velocity of the Ith particle, m_I is its mass, and $n^\mu = (1, n^i)$, $x^i = r n^i$, and $\sum_{i=1}^3 n^i n^i = 1$. Notice that the sign of β in (8) is the opposite of Ref. [21] because for us β is the boundary graviton, while for [21] it is the supertranslation that sets the shear to zero.

The angular momentum flux ΔJ_Y^D is BMS invariant by construction since it is computed in a specific BMS frame, but it is not defined in terms of the true, BMS-independent asymptotic degrees of freedom $N_{AB}(u,\Theta)$ and $m(u,\Theta)$.

The purpose of this Letter is to present a new formula for the angular momentum, which we shall call $J_Y^{\rm new}(u)$, possessing the following properties: (a) It is written purely in terms of asymptotic data. (b) It gives a supertranslation-invariant total flux $\Delta J_Y^{\rm new} = J_Y^{\rm new}(+\infty) - J_Y^{\rm new}(-\infty)$ such that $\Delta J_Y^{\rm new} = \Delta J_Y^D$ for a system of massive point particles. (c) The $J_Y^{\rm new}(-\infty)$ generate the rotation algebra. (d) When the limit $u \to -\infty$ is well defined, in particular for stationary spacetimes, $J_Y^{\rm new}(-\infty) = J_Y^{\rm BMS}(-\infty) = J_Y(-\infty)$.

We shall make no attempt to justify our formula from any principle. In particular, we shall not try to understand why the Damour formula for angular momentum flux is the correct one for radiation reaction computations. We refer to [21] for a discussion on this point.

To begin with, we need some identities valid for any velocity vector v and $\gamma = 1/\sqrt{1-v^2}$,

$$D_A D_B(n\cdot v) = -h_{AB}(\gamma + n\cdot v), \tag{9}$$

$$D_A(n\cdot v) D^A(n\cdot v) = -[1 + (n\cdot v)^2 + 2\gamma n\cdot v], \tag{10}$$

$$D_B D_A\left(\frac{1}{n\cdot v}\right) = \frac{h_{AB}(\gamma + n\cdot v)}{(n\cdot v)^2} + 2\frac{D_A(n\cdot v) D_B(n\cdot v)}{(n\cdot v)^3}, \tag{11}$$

$$D_A D^A\left(\frac{1}{n\cdot v}\right) = \frac{-2}{(n\cdot v)^3}(1 + \gamma n\cdot v), \tag{12}$$

$$D^2 \log(-n\cdot v) = -1 + \frac{1}{(n\cdot v)^2}. \tag{13}$$

The shear tensor obtained from Eqs. (4) and (8) is [12,21]

$$C_{AB} = (-2D_A D_B + h_{AB} D^2) 2G \sum_{I=1}^{N} m_I n \cdot v_I \log(-n \cdot v_I)$$
$$= -G \sum_{I=1}^{N} m_I \left[\left(\gamma_I + \frac{1}{2} n \cdot v_I + \frac{1}{2n \cdot v_I} \right) h_{AB} + \frac{D_A(n \cdot v_I) D_B(n \cdot v_I)}{n \cdot v_I} \right], \tag{14}$$

therefore,

$$D^A C_{AB} = -2G \sum_{I=1}^{N} m_I \left(-3 + \frac{1}{(n \cdot v_I)^2} - \frac{\gamma_I}{n \cdot v_I} \right) D_B(n \cdot v_I), \tag{15}$$

and from this we can easily calculate $D^A D^B C_{AB}$,

$$D^B D^A C_{AB} = -4G \sum_{I=1}^{N} m_I \left[4\gamma_I + 3n \cdot v_I + \frac{1}{(n \cdot v_I)^3} \right]. \tag{16}$$

The first two terms are essential so that the sum of all three terms in the brackets has zero monopole and dipole moments, as it can be checked explicitly.

We calculate next the Bondi mass aspect. For this we need to compute g_{uu} in the Bondi gauge. In Minkowski space and in the notations of Ref. [21] the metric is given by

$$g^{\mu\nu} = \eta^{\mu\nu} - 4G \sum_{I=1}^{N} \frac{m_I (v_I^\mu v_I^\nu + \frac{1}{2}\eta^{\mu\nu})}{\Gamma_I(x)}. \tag{17}$$

We need to make a coordinate transformation to write the metric in the Bondi gauge. Metric (17) can be written in terms of retarded time $U = t - \rho$, radius ρ, and angular coordinates θ^A, in which a useful expansion for the functions Γ is

$$\frac{1}{\Gamma} = \frac{H(\theta^A)}{\rho} + \frac{K(U, \theta^A)}{\rho^2} + \mathcal{O}\left(\frac{1}{\rho^3}\right), \tag{18}$$

where $H(\theta^A) = -1/(n \cdot v)$ and

$$K(U, \theta^A) = -\frac{u(1 + \gamma n \cdot v)}{(n \cdot v)^3} + h(\theta^A). \tag{19}$$

We consider a coordinate transformation from the original coordinates U, ρ, θ^A to Bondi gauge coordinates u, r, Θ^A of the form

$$u = U + \delta U, \quad r = \rho + \delta\rho, \quad \Theta^A = \theta^A + \delta\theta^A. \tag{20}$$

Now the mass aspect can be read from the following equation:

$$g_{uu} = g_{UU} \left(1 - \frac{\partial \delta U}{\partial u} \right)^2 + 2 g_{U\rho} \left(1 - \frac{\partial \delta U}{\partial u} \right) \left(-\frac{\partial \delta\rho}{\partial u} \right). \tag{21}$$

We have omitted subleading terms and other terms that do not contribute to g_{uu} at order $\mathcal{O}(1/r)$ and hence do not change the mass aspect. Equation (21) can be simplified to

$$\delta m = r \partial_u (\delta U + \delta\rho). \tag{22}$$

To find $\delta\rho$ and δU we use the Bondi gauge conditions and the formulas for δU, $\delta\rho$, and $\delta\theta^A$ given in [21]. The result is

$$\delta U = 2G \sum_{I=1}^{N} m_I \left[-n \cdot v_I \log(r) - \frac{u}{r} \frac{1 + \gamma_I n \cdot v_I}{n \cdot v_I} + \mathcal{O}\left(\frac{1}{r^2}\right) \right],$$
$$\delta\rho = \frac{Gu}{r} \sum_{I=1}^{N} \frac{m_I}{(n \cdot v_I)^2} \left((1 + \gamma_I n \cdot v_I)(3n \cdot v_I + 4\gamma_I) + \frac{1 + \gamma_I n \cdot v_I}{n \cdot v_I} - \frac{(2 + \gamma_I n \cdot v_I)[1 + 2\gamma_I n \cdot v_I + (n \cdot v_I)^2]}{n \cdot v_I} \right). \tag{23}$$

From these equations and after some simplifications, we find that δm is

$$\delta m = G \sum_{I=1}^{N} m_I \left(-\frac{1}{(n \cdot v_I)^3} + \frac{2\gamma_I^2 - 1}{n \cdot v_I} \right). \tag{24}$$

The Bondi mass aspect is then

$$m(-\infty, \Theta) = -G \sum_{I=1}^{N} \frac{m_I}{(n \cdot v_I)^3}. \tag{25}$$

This equation coincides as it should with Eq. (5.16) of [6]. The relation between mass aspect and the double divergence of the shear tensor is thus

$$m(-\infty, \Theta) = \frac{1}{4} D^A D^B C_{AB} + \text{monopole and dipole terms}. \tag{26}$$

This equation motivates our proposal for a supertranslation-invariant angular momentum flux. The key observation is that all the coefficients with $l > 1$ in the expansion of $m(u, \Theta)$ in spherical harmonics of the celestial sphere can be changed by soft radiation without any radiative loss of energy. This is because the mass aspect obeys the equation

$$\partial_u m = \frac{1}{4} D^A D^B N_{AB} - T_{uu}. \tag{27}$$

When no hard radiation is present, $T_{uu} = 0$ so Eq. (27) becomes

$$m(+\infty,\Theta) - m(-\infty,\Theta) = \frac{1}{4} D^A D^B C_{AB}(+\infty,\Theta) - \frac{1}{4} D^A D^B C_{AB}(-\infty,\Theta), \tag{28}$$

whose right-hand side has no $l = 0$, 1 components by construction. Equation (26) shows that soft gravitational reaction can make both the shear and the $l > 1$ harmonics of the mass aspect used in the computation of ΔJ_Y^D vanish simultaneously without producing any hard radiation ($T_{uu} = 0$), in particular, without radiating out any energy. We will take this as the key hint to define the specific boundary graviton to use in the definition of an angular momentum obeying the properties (a)–(d) listed above.

The most direct and physically motivated procedure to define a preferred boundary graviton would have been to characterize it in terms of initial data at past null infinity $\mathcal{I}^-$ by requiring that no incoming radiation crosses $\mathcal{I}^-$. To extract data at future null infinity $\mathcal{I}^+$ one would need to solve the scattering problem of a matter-gravity coupled system. This is too tall an order, so we choose instead to select the preferred boundary graviton by *demanding* that it satisfies Eq. (26). This definition guarantees that Eq. (28) can be satisfied with $C_{AB}(+\infty,\Theta) = 0$ and $m(+\infty,\Theta)$ equal to only a monopole plus dipole term without any $l > 1$ harmonic. Instead of adding a term to the BMS-invariant angular momentum flux, we could have defined the flux ΔJ_Y in a specific BMS frame. Our construction is equivalent to selecting such BMS frame by requiring that the shear of a system whose mass aspect at $u = -\infty$ contains no harmonics higher than $l = 1$ vanishes,

$$m(-\infty,\Theta) = c + \sum_{m=-1}^{1} c_m Y_{1m}(\Theta) \Rightarrow C_{AB}(-\infty,\Theta) = 0. \tag{29}$$

Equation (4) plus the requirement that the initial shear vanishes when the mass aspect has no dipoles higher than $l = 1$ (i.e., that all higher moments in the mass aspect and in the shear can be radiated away without radiating energy) implies $\mathcal{C}_{AB}(-\infty) = (h_{AB} D^2 - 2 D_A D_B)\mathcal{C}$, with $\mathcal{C}$ given by

$$-(2 + D^2) D^2 \mathcal{C} = D^A D^B \mathcal{C}_{AB} = 4m(-\infty,\Theta) + \text{monopole and dipole terms}. \tag{30}$$

Expanding both $\mathcal{C}$ and m in spherical harmonics, we find an explicit formula relating their angular coefficients,

$$l(1 - l^2)(l + 2)\mathcal{C}_{lm} = 4m_{lm}, \quad \forall\ l > 1. \tag{31}$$

We set the undefined coefficients $\mathcal{C}_0 = \mathcal{C}_{1\pm1} = \mathcal{C}_{10} = 0$. Finally, our definition of a new angular momentum is

$$J_Y^{\text{new}}(u) = J_Y^{\text{BMS}}(u) + V_Y(u),$$
$$V_Y(u) = \frac{1}{4\pi G} \int d^2\Theta \sqrt{h} Y^A m(u,\Theta) D_A \mathcal{C}(\Theta), \tag{32}$$

with $\mathcal{C}$ defined in Eq. (31).

This definition obeys property (a) by construction. To check (b) we compute

$$\Delta J_Y^{\text{new}} = \Delta J_Y^{\text{BMS}} + \frac{1}{4\pi G} \int d^2\Theta \sqrt{h} Y^A \Delta m(\Theta) D_A \mathcal{C}(\Theta), \tag{33}$$

which obviously coincides with ΔJ_Y^D in Eq. (7) when $\mathcal{C} = \beta$. Since the Bondi mass aspect is a scalar under rotation of the celestial sphere, the new term $V_Y(u)$ in (32) transforms as a vector. Moreover, at $u = -\infty$, $V_Y(-\infty)$ and $V_X(-\infty)$ commute for arbitrary rotations X, Y; hence, $[J_Y^{\text{BMS}}(-\infty) + V_Y(-\infty), J_X^{\text{BMS}}(-\infty) + V_X(-\infty)] = J_{[Y,X]}^{\text{BMS}}(-\infty) + 2V_{[Y,X]}(-\infty)$. So property (c) is true if $V_Y(-\infty) = 0$, that is, if $V_Y(-\infty)$ satisfies property (d). Finally, to check property (d), we notice first that the operator $Y^A D_A$ acts as an infinitesimal rotation on any scalar function, so it maps an lth spherical harmonic into a linear combination of harmonics with the same l. Since $\mathcal{C}_l = 0$ for $l = 0$, 1, the $l = 0$, 1 harmonics of $m(u,\Theta)$ do not contribute to Eq. (32). This allows us to substitute $m(-\infty) \to -(2 + D^2) D^2 \mathcal{C}/4$ in (32), evaluate it at $u = -\infty$, and find

$$J_Y^{\text{new}}(-\infty) = J_Y^{\text{BMS}}(-\infty) - \frac{1}{16\pi G} \int d^2\Theta \sqrt{h} Y^A [(2 + D^2) D^2 \mathcal{C}] D_A \mathcal{C}. \tag{34}$$

The last term vanishes, as it can be proven by expanding $\mathcal{C}$ in spherical harmonics $\mathcal{C} = \sum_l C^l$ and using again that $Y^A D_A$ is a rotation, so

$$\int d^2\Theta \sqrt{h} Y^A [(2 + D^2) D^2 \mathcal{C}] D_A \mathcal{C} = \sum_l l(l^2 - 1)(l + 2) \int d^2\Theta \sqrt{h} Y^A \mathcal{C}^l D_A \mathcal{C}^l = \frac{1}{2} l(l^2 - 1)(l + 2) \int d^2\Theta \sqrt{h} Y^A D_A (\mathcal{C}^l)^2 = 0. \tag{35}$$

In the last step, we integrated by parts and used $D_A Y^A = 0$.

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*rj1154@nyu.edu
†mp9@nyu.edu

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