Stability of Dining Clubs in the Kolkata Paise Restaurant Problem with and without Cheating

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We introduce the idea of a dining club to the Kolkata Paise Restaurant Problem. In this problem, N agents choose (randomly) among N restaurants, but if multiple agents choose the same restaurant, only one will eat. Agents in the dining club will coordinate their restaurant choice to avoid choice collision and increase their probability of eating. We model the problem of deciding whether to join the dining club as an evolutionary game and show that the strategy of joining the dining club is evolutionarily stable. We then introduce an optimized member tax to those individuals in the dining club, which is used to provide a safety net for those group members who don't eat because of collision with a non-dining club member. When non-dining club members are allowed to cheat and share communal food within the dining club, we show that a new unstable fixed point emerges in the dynamics. A bifurcation analysis is performed in this case. To conclude our theoretical study, we then introduce evolutionary dynamics for the cheater population and study these dynamics. Numerical experiments illustrate the behaviour of the system with more than one dining club and show several potential areas for future research.

I. INTRODUCTION

The Kolkata Paise Restaurant Problem (KPRP) was first introduced in 2007 [1] during work on the Kolkata Paise Hotel Problem. Since then, it has been studied extensively [1–17] in the econophysics literature. In its simplest form, we assume $N \gg 1$ agents will choose among N restaurants. Choice is governed by a distribution determined by an implicit ranking of the restaurants. The ranking represents the payoff of eating at a given restaurant. If two or more agents select the same restaurant, then the restaurant randomly chooses which agent to serve.

A broad overview of KPRP can be found in [3, 7, 11]. When all restaurants are ranked equally (i.e., have payoff 1) and agents choose a restaurant at random, the expected payoff to each agent is easily seen to be approach $1 - 1/e$ as $N \to \infty$. Using stochastic strategies and resource utilization models, the mean payoff can be increased to ~ 0.8 [18]. Identifying strategies to improve on the uncoordinated outcome is a central problem in KPRP.

KPRP is an example of an anti-coordination game (such as Hawk-Dove) [19]. Other examples of this class of game are minority games [20, 21] and the El Farol bar problem [22–25]. These types of games also emerge in models of channel sharing in communications systems [26–28].

Learning in KPRP is considered in [12, 18, 29] with both classical and quantum learning considered in [12]. Quantum versions of the problem are considered in [12, 15, 16] and its relevance to other areas of physical modelling are considered in [8, 10, 14, 17] with phase transitions considered recently in [2, 9]. Distributed and coordinated solutions to optimizing agent payoff are discussed in [4–6, 13].

In this paper, we use evolutionary game theory to study a group formation problem within the context of KPRP. We assume that some subset of the population of N individuals forms a dining club. Individuals in the dining club coordinate their actions and will choose distinct restaurants from each other, thus increasing the odds that any individual within the dining club will eat. In this context, we show the following results:

1. When all restaurants are ranked equally, membership in the dining club is globally stable. That is, asymptotically all players join the dining club (in the limit as $N \to \infty$).
2. When the dining club taxes its members by collecting food for redistribution to those members who did not eat, there is an optimal tax rate that ensures all members are equally well-fed.
3. When non-club members can choose to deceptively share in the communal food (freeload) of the dining club, a new unstable fixed point emerges. The fixed point corresponding to a population where all members join the

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dining club remains stable, but is no longer globally stable. We characterize the basin of attraction in this case. This effectively introduces a public goods game into the KPRP.

4. We then use numerical analysis to study the case where two dining clubs are active. We numerically illustrate the existence of equilibrium surfaces where multiple dining clubs can exist simultaneously along with non-group members as a result of group taxation (food sharing), cheating (freeloading), and cheating detection.

The remainder of this paper is organized as follows: In Section II, we analyse an evolutionary model of KPRP with a dining club. We study resource distribution through taxation and cheating in Section III. Cheating is modelled in an evolutionary context in Section IV. KPRP with multiple dynamic clubs is studied numerically in Section V. Finally, in Section VI we present conclusions and future directions.

II. MATHEMATICAL ANALYSIS

We first study KPRP with a single dining club. Let g be the size of the dining club and let n be the size of the population not in the dining club. The total population is given by $N = g + n$. We compute the probability that an individual eats. Consider an individual in the dining club. It is possible k individuals from the non-club members will choose the same restaurant. The probability that k agents choose this restaurant, while the remaining non-club agents choose a different restaurant is

$$\left(\frac{n+g-1}{n+g}\right)^{n-k}\left(\frac{1}{n+g}\right)^{k}.$$

Because each agent at the restaurant in question has an equal chance to eat, we must multiply this by $(k+1)^{-1}$ to obtain the probability of eating. Summing over all possible collisions between non-group members and a group member yields the probability that an individual in the dining club eats with probability

$$p_g(n,g) = \sum_{k=0}^{n}\binom{n}{k}\left(\frac{n+g-1}{n+g}\right)^{n-k}\left(\frac{1}{n+g}\right)^{k}\frac{1}{k+1},$$

We now compute the probability that a non-club member eats. Suppose k individuals from the non-club group choose the same restaurant as this non-club individual. Assuming no dining club member also chooses this restaurant, then $k+1$ individuals arrive at the common restaurant and the individual eats with probability,

$$\frac{n}{n+g}\frac{1}{k+1}\left(\frac{n+g-1}{n+g}\right)^{n-k-1}\left(\frac{1}{n+g}\right)^{k},$$

where the factor $n/(n+g)$ gives the probability that the individual does not choose the same restaurant as a dining club member. On the other hand, if the individual chooses a restaurant that has been chosen by a dining club member, then the probability that the individual eats is

$$\frac{g}{n+g}\frac{1}{k+2}\left(\frac{n+g-1}{n+g}\right)^{n-k-1}\left(\frac{1}{n+g}\right)^{k}.$$

Here the factor of $g/(n+g)$ gives the probability of collision with a dining club member and $(k+2)^{-1}$ appears because dining club member is also competing for the restaurant. Summing over all the ways these k individuals can be chosen from the $n-1$ other non-club members gives

$$p_n(n,g) = \sum_{k=0}^{n-1}\binom{n-1}{k}\left(\frac{n}{n+g}\frac{1}{k+1}+\frac{g}{n+g}\frac{1}{k+2}\right)\left(\frac{n+g-1}{n+g}\right)^{n-k-1}\left(\frac{1}{n+g}\right)^{k},$$

the probability that an arbitrary non dining club member eats.

If we assume $g = \alpha n$ and sum over k, then we can rewrite $p_g(n,g)$ in closed form as

$$p_g(n,\alpha) = \frac{\left(1-\frac{1}{\alpha n+n}\right)^{n}\left((\alpha+1)n\left(\left(\frac{1}{\alpha n+n-1}+1\right)^{n}-1\right)+1\right)}{n+1}.$$

Likewise, $p_n(n,g)$ can be written as

$$p_n(n,\alpha) = \frac{\left(1-\frac{1}{\alpha n+n}\right)^n}{n+1}\left\{\alpha^2 n+\alpha n-\alpha-n-1-\left[(\alpha+1)((\alpha-1)n-1)\left(\frac{1}{\alpha n+n-1}+1\right)^n\right]\right\}.$$

If we compute the limit as $n\to\infty$, this yields the asymptotic probabilities

$$p_g(\alpha) = \lim_{n\to\infty} p_g(n,\alpha) = \left(1-e^{-\frac{1}{\alpha+1}}\right)(\alpha+1), \tag{1}$$

and

$$p_n(\alpha) = \lim_{n\to\infty} p_n(n,\alpha) = -\alpha^2+e^{-\frac{1}{\alpha+1}}\left(\alpha^2+\alpha-1\right)+1. \tag{2}$$

For the remainder of this section and the next, we assume an infinite population. While it was easier to work with $g=\alpha n$ for the previous computation, for further analysis it is simpler to express g as a fraction of the total population. Let

$$\beta = \frac{g}{n+g} = \frac{\alpha}{1+\alpha}.$$

Substituting

$$\alpha = \frac{\beta}{1-\beta}. \tag{3}$$

into Eqs. (1) and (2) yields the simplified forms,

$$p_g(\beta) = \frac{1-e^{\beta-1}}{1-\beta} \quad \text{and}$$
$$p_n(\beta) = \frac{-2e\beta-e^{\beta}((\beta-3)\beta+1)+e}{e(\beta-1)^2}.$$

A simple plot shows that $p_g(\beta)\geq p_n(\beta)$ for all $\beta\in[0,1]$, see Fig. 1.

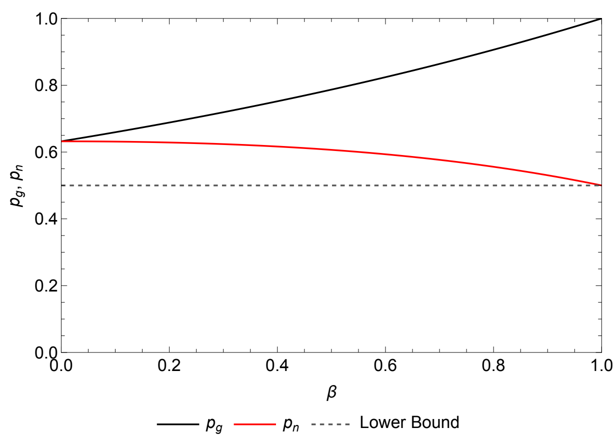


FIG. 1: A plot of $p_g(\beta)$ and $p_n(\beta)$ shows that it is always better for an individual to join the dining club than to remain independent.

In any realization of the KPRP, individuals will either eat or not eat a meal. Formally, an individual's meal size is either 0 or 1; i.e., the individual eats a complete meal or eats nothing. Since the KPRP describes a random process, let S_g be a random variable denoting the meal size for an individual in the dining club, and let S be a random variable denoting the meal size for a randomly chosen member of the entire population. These are both Bernoulli random variables and the probability of eating $p_g(\beta)$ is now easily seen as the expected (average) meal size $\langle S_g\rangle$. Using this

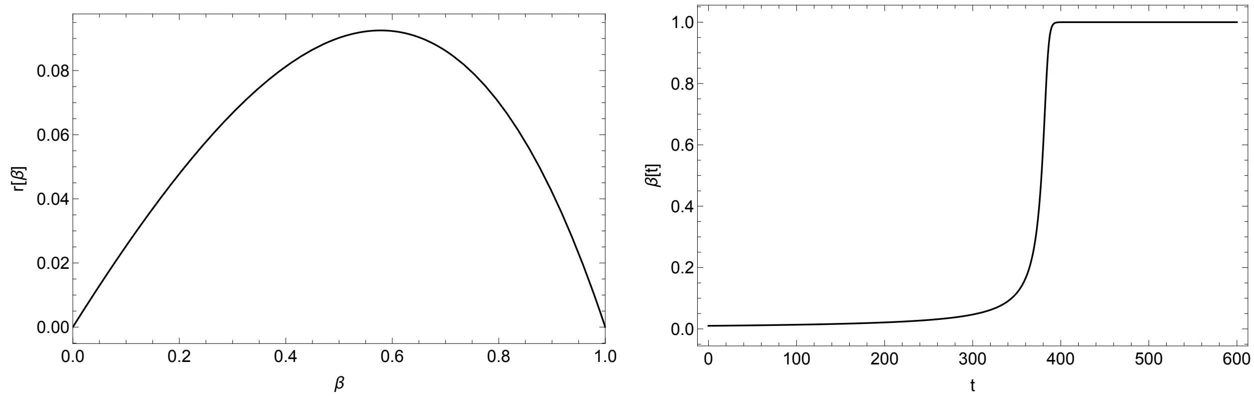


FIG. 2: (Left) The growth rate of $r(t)$ is an unimodal positive function with zeros at $\beta = 0$ and $\beta = 1$. (Right) The solution curve for $\beta(t)$ assuming $\beta(0) = 0.01$.

interpretation, and equating expected meal size with fitness, we assume that the rate of change of the proportion β is described by the replicator equation [30]

$$\dot{\beta} = \beta\left[p_g(\beta) - \bar{p}(\beta)\right] = \beta\left(\langle S_g\rangle - \langle S\rangle\right). \tag{4}$$

The population mean $\bar{p}(\beta) = \langle S\rangle$ can be computed as

$$\bar{p}(\beta) = \langle S\rangle = \frac{\alpha p_g(\alpha) + p_n(\alpha)}{1+\alpha},$$

and converted to an expression in β using Eqs. (1) to (3) as,

$$\langle S\rangle = \bar{p}(\beta) = e^{\beta-1}(\beta-1)+1.$$

Let

$$r(\beta) = p_g(\beta) - \bar{p}(\beta) = \langle S_g\rangle - \langle S\rangle = \frac{1-e^{\beta-1}}{1-\beta} - \left(e^{\beta-1}(\beta-1)+1\right),$$

be the growth rate of β. Then $r(0) = 0$ and we see that $\lim_{\beta\to 1} r(\beta) = 0$. That is, Eq. (4) has two fixed points. From Fig. 1, we must have $r(\beta) > 0$ for $0 < \beta < 1$. This is illustrated in Fig. 2 (left). It follows that $\beta(t)$ is described by a non-logistic sigmoid, as shown in Fig. 2 (right). We conclude that the decision to join the dining club is an evolutionarily stable strategy and the fixed point $\beta = 1$ is globally asymptotically stable while the fixed point $\beta = 0$ is asymptotically unstable.

III. SOCIAL SAFETY NETS AND DECEPTIVE FREE LOADING

Suppose the dining club imposes a *food tax* on its members at the rate $\kappa \in [0,1]$ so that if a diner is successful in obtaining food, then he reserves $\kappa \times 100\%$ of his meal to be shared with club members who choose a restaurant that is occupied by an independent individual. If we assume these resources are pooled and then shared equally, the expected meal size (normalized to the interval $[0,1]$) available for a club member who cannot obtain food on his own is given by

$$\tilde{p}_g(\beta) = \frac{g p_g(\beta)\kappa}{g - g p_g(\beta)} = \frac{p_g(\beta)\kappa}{1-p_g(\beta)}. \tag{5}$$

Note that sharing (for any value of κ) does not affect the expected meal size obtained by a group member, since we have the expected meal size

$$\langle S_g\rangle = (1-\kappa)p_g(\beta) + [1-p_g(\beta)]\frac{p_g(\beta)\kappa}{1-p_g(\beta)} = p_g(\beta). \tag{6}$$

We can construct a tax-rate that depends on β and ensures all participants in the dining club receive the same meal size. Setting $\tilde{p}_g(\beta) = 1-\kappa$ and solving, we obtain:

$$\kappa^* = 1 - p_g(\beta). \tag{7}$$

Thus, as β increases, the tax decreases. As a result of Eq. (6), the right-hand-side of Eq. (4) remains unchanged and the decision to join the dining club is still evolutionarily stable, even in the presence of sharing. That is $\beta = 1$ is still globally asymptotically stable.

Suppose a proportion $\phi \in [0,1]$ of the independent population *that does not eat* can deceptively pose as club members, thereby sharing in the communally available food. In the presence of a food tax, the resulting decision to join the dining club now becomes a public goods problem. Then the expected meal size to anyone receiving shared food is given by

$$\tilde{p}_g(\beta) = \frac{\kappa g p_g(\beta)}{n[1-p_n(\beta)]\phi + [1-p_g(\beta)]g} = \frac{\alpha\kappa p_g(\beta)}{\phi[1-p_n(\beta)] + \alpha[1-p_g(\beta)]},$$

where α is defined in terms of β in Eq. (3). Let S_n be the random variable denoting the expected meal size for an independent member of the population. Then as a function of κ and ϕ,

$$\langle S_g \rangle = (1-\kappa)p_g(\beta) + [1-p_g(\beta)]\frac{\alpha\kappa p_g(\beta)}{\alpha[1-p_g(\beta)] + [1-p_n(\beta)]\phi} \text{ and} \tag{8}$$

$$\langle S_n \rangle = p_n(\beta) + [1-p_n(\beta)]\phi\frac{\alpha\kappa p_g(\beta)}{\alpha[1-p_g(\beta)] + [1-p_n(\beta)]\phi}. \tag{9}$$

It is possible but unwieldy to compute $r(\beta,\phi) = \langle S_g \rangle - \langle S \rangle$ using the expected meal size with deception rate ϕ and group size β. Plotting sample curves for $r(\beta,\phi)$ shows that the growth rate now changes sign at some value $\beta(\phi)$; see Fig. 3 (left). As a consequence of this, the replicator equation for β is given by

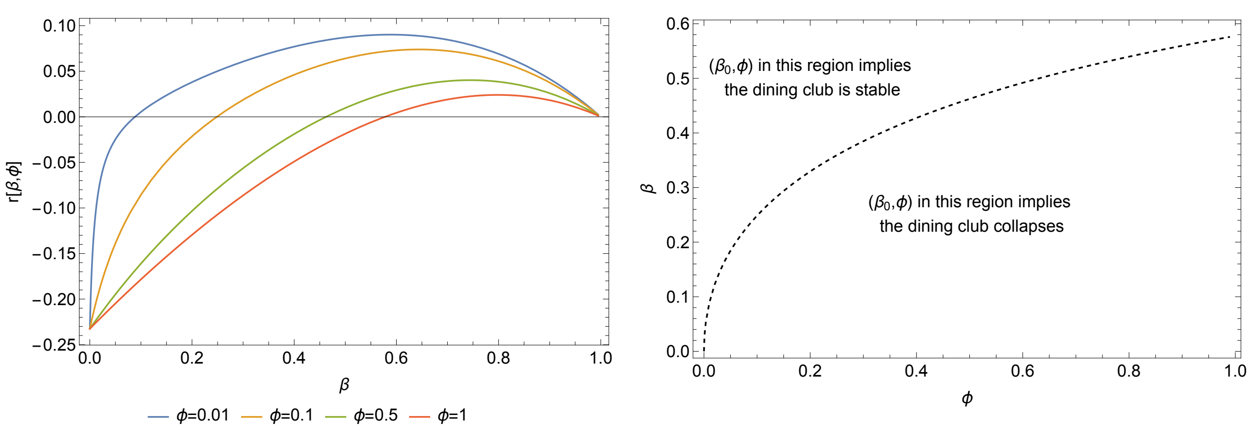


FIG. 3: (Left) The rate function $r(\beta,\phi)$ for varying values of ϕ shows that $r(t)$ changes sign as a function of β. (Right) The solution curve for β^* as a function of ϕ so that $r(\beta^*,\phi) = 0$.

$$\dot{\beta} = \beta\left(\langle S_g \rangle - \langle S \rangle\right).$$

These dynamics exhibit a new unstable equilibrium point, illustrating a bifurcation in parameter ϕ with numerically computed bifurcation diagram shown in Fig. 3 (right). An example solution flow (for various initial conditions) is shown in Fig. 4. We can compute $\beta^* \approx 0.577$ for $\phi = 1$. This is particularly interesting because we have essentially constructed a public goods problem in which joining the dining club enforces a taxation rate of $\kappa = 1 - p_g(\beta)$ on the members, who are then guaranteed (the public good of) a meal each day. The presence of freeloaders destabilizes the group formation process, but does not guarantee that a group cannot form. Since $\beta^*(\phi)$ is monotonically increasing, it follows that if ϕ grows slowly enough so that at any time $\beta(t) > \beta^*[\phi(t)]$, then the dining club will grow to include the entire population. If $\beta(t) < \beta^*[\phi(t)]$, then the dining club collapses. We impose an evolutionary dynamic on the freeloaders in the next section to study this effect.

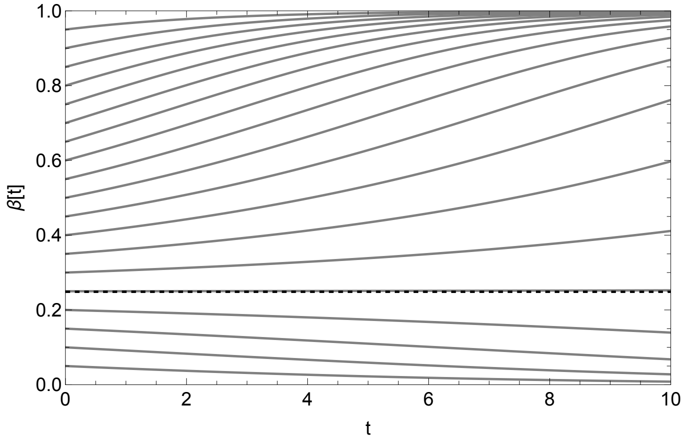


FIG. 4: Here $\phi = 0.1$ and we show the instability of the interior fixed point. With $\beta(0) > \beta^*$ all members of the population are eventually driven to join the dining club. If $\beta(0) < \beta^*$, the dining club fails as a result of freeloading.

IV. EVOLVING FREELOADERS

If we divide the population into three groups, dining club members (g), non-dining club freeloaders (f) and non-dining club non-freeloaders (h), we can construct an evolutionary dynamic for the freeloaders. Let χ be the proportion of the population that is not in the dining club and will freeload (cheating) and $\eta = 1 - \beta - \chi$ to be the proportion of the population that is not in the dining club and not freeloading (honest). Then the population of freeloaders is $\chi(n + \alpha n)$. The expected meal size to any agent accepting communal food is then

$$\frac{\kappa g p_g(\beta)}{g[1 - p_g(\beta)] + [1 - p_n(\beta)]\chi(n + \alpha n)} = \frac{\kappa \alpha p_g(\beta)}{\alpha[1 - p_g(\beta)] + [1 - p_n(\beta)]\chi(1 + \alpha)} = \frac{\kappa \alpha p_g(\beta)}{\alpha[1 - p_g(\beta)] + [1 - p_n(\beta)]\chi(1 - \beta)^{-1}}. \quad (10)$$

Let S_g be as before, and let S_f be the random variable denoting the meal size for an individual in the freeloading group and S_h be the random variable denoting meal size for an individual from the non-freeloading non-dining club group. It follows from Eqs. (8) to (10) that

$$\langle S_g \rangle = (1 - \kappa)p_g(\beta) + [1 - p_g(\beta)]\frac{\alpha \kappa p_g(\beta)}{\alpha[1 - p_g(\beta)] + [1 - p_n(\beta)]\chi(1 - \beta)^{-1}},$$
$$\langle S_f \rangle = p_n(\beta) + [1 - p_n(\beta)]\frac{\alpha \kappa p_g(\beta)}{\alpha[1 - p_g(\beta)] + [1 - p_n(\beta)]\chi(1 - \beta)^{-1}}, \text{ and}$$
$$\langle S_h \rangle = p_n(\beta).$$

Here, we have replaced ϕ with its definition in terms of χ and β. Employing the same reasoning we used to obtain Eq. (4), we can construct replicator equations for proportions β, χ and η.

The population mean meal size is

$$\langle S \rangle = \chi \langle S_f \rangle + \beta \langle S_g \rangle + \eta \langle S_h \rangle .$$

The dynamics of η (the non-freeloading, non-dining club group) are extraneous, and we can focus on the two-dimensional system

$$\dot{\beta} = \beta \left(\langle S_g \rangle - \langle S \rangle \right)$$
$$\dot{\chi} = \chi \left(\langle S_f \rangle - \langle S \rangle \right),$$

which do not depend on the value of η. Fig. 5 shows the dynamics of this evolutionary system. It is straightforward to compute that when $\beta = 0$, then $\langle S_g \rangle - \langle S \rangle = \langle S_f \rangle - \langle S \rangle = 0$ for all values of $\chi \in [0, 1]$. Thus, the dynamics freeze on the left boundary of the simplex

$$\Delta_2 = \left\{ (\beta, \chi) \in \mathbb{R}^2 : \beta + \chi \leq 1,\ \beta \geq 0,\ \chi \geq 0 \right\}.$$

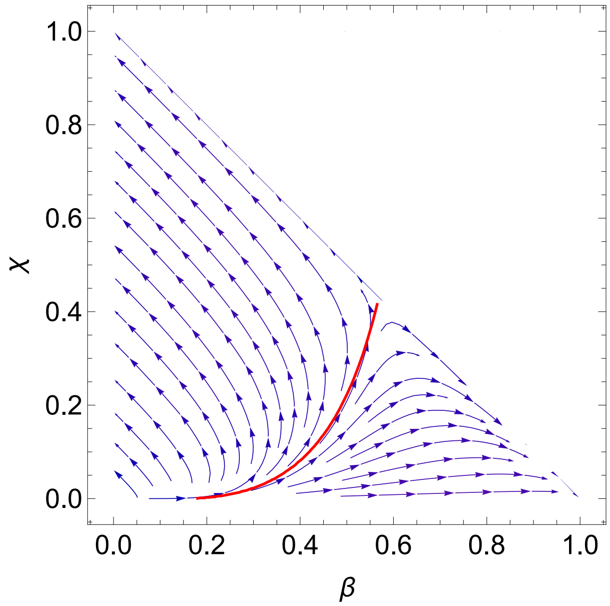


FIG. 5: A phase portrait of the two-dimensional system showing the dynamics of (β, χ). The red curve shows a numerically computed boundary between the basin of attraction of $(\beta, \chi) = (1, 0)$ and $(\beta, \chi) = (0, 1)$.

There is a single hyperbolic saddle on the boundary of Δ_2 that can be numerically computed as $(\beta, \chi) \approx (0.578, 0.422)$. The two boundary equilibria $(\beta, \chi) = (1, 0)$ and $(\beta, \chi) = (0, 1)$ are both locally asymptotically stable. Thus, the long-run population behaviour is dependent on the initial conditions. We can numerically construct a curve of initial conditions showing this dichotomous behaviour. This is shown in Fig. 6 and as the red curve in Fig. 5. As β_0

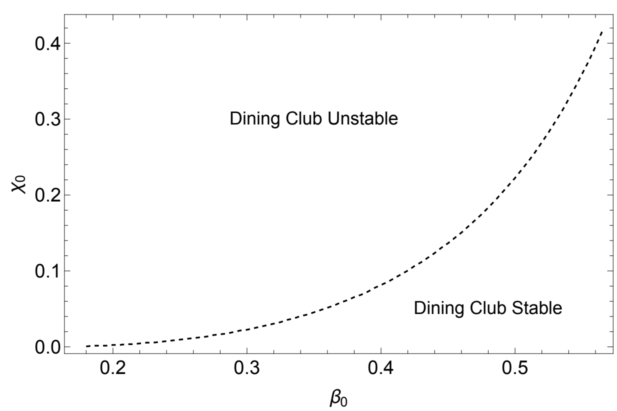


FIG. 6: (Right) Numerically computed curve showing the boundary between the stable and unstable dining club strategy for varying initial conditions.

approaches $\beta^* \approx 0.578$ corresponding to equilibrium point for $\phi = 1$, the curve stops because χ_0 would need to lie outside the simplex to cause the dining club to collapse. It is interesting to note that the phase portrait illustrates trajectories in which both β and χ are increasing up to a point, followed by either the collapse of the dining club (while χ continues to increase) or the collapse of the freeloading group, as all population members join the dining club (and β continues to increase).

V. NUMERICAL RESULTS ON MULTIPLE DINING CLUBS

We now consider KPRP with two dining clubs. We model three groups of agents $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{F}$, denoting the populations in dining club one and dining club two and the population in neither club, respectively. We estimate

$\langle S_{g_1} \rangle$, $\langle S_{g_2} \rangle$ and $\langle S_f \rangle$ using Monte Carlo simulation. This Monte Carlo simulation is then embedded into a larger dynamic process for updating the groups.

In the Monte Carlo simulation, the free agent group acts normally, choosing a restaurant randomly. The members of the dining clubs also choose restaurants randomly, but with the constraint that no two agents in a dining club may choose the same restaurant. Since we are studying this system numerically, we introduce two kinds of taxation policies:

1. Policy I: We assume a given tax rate κ with no redistribution; i.e., the tax goes to maintain the dining club in some form.

2. Policy II: Agents within the dining club are taxed at a rate κ given by Eq. (7), and food is redistributed to club members who did not initially eat (and possibly to freeloaders). Freeloaders are the free agents who did not get food that day. They will randomly choose a dining club to eat in if they do not get food on a given day with a probability 1. That is, we assume $\phi = 1$. We also introduce a probability ρ that such cheaters will be caught. If a freeloader is caught, their food is not redistributed and becomes waste. So from the stand point of the restaurant, the wasted food is still food served, but instead of it benefiting agents it is discarded because the freeloader is caught.

In the dynamic model that follows, we refer to the process of simulating groups eating over several days by the function `MonteCarlo`$(\mathcal{F}, \mathcal{G}_1, \mathcal{G}_2, \kappa, \rho)$. The system dynamics of our simulation are then described by the following steps:

1: **Input:** $\mathcal{F}$, $\mathcal{G}_1$, $\mathcal{G}_2$.
2: **while** There is at least one agent in each group **do**
3:    Compute $(\langle S_{g_1} \rangle, \langle S_{g_2} \rangle, \langle S_f \rangle) =$ `MonteCarlo`$(\mathcal{F}, \mathcal{G}_1, \mathcal{G}_2, \kappa, \phi)$.
4:    Set $\mathcal{P} = \mathcal{F} \cup \mathcal{G}_1 \cup \mathcal{G}_2$.
5:    Choose two agents i and j at random from $\mathcal{P}$.
6:    Let `Group`(i) (resp. `Group`(j)) be the group to which i (resp. j) belongs.
7:    Let p_i (resp. p_j) be the probability that i (resp. j) eats.
8:    **if** $p_i > p_j$ **then**
9:       Move j to `Group`(i)
10:    **else if** $p_j > p_i$ **then**
11:       Move i to `Group`(j)
12:    **end if**
13:    Remove i and j from $\mathcal{P}$.
14:    **if** $|\mathcal{P}| > 1$ **then**
15:       **goto** 5
16:    **else**
17:       **goto** 3
18:    **end if**
19: **end while**

It is clear in the dynamics simulated by this model, there are three equilibria corresponding to the cases when all agents are in $\mathcal{F}$ or $\mathcal{G}_1$ or $\mathcal{G}_2$. Example trajectories produced by the simulation are shown in Fig. 7 for three initial conditions. Let $\langle S_{g_1} \rangle$, $\langle S_{g_2} \rangle$ and $\langle S_f \rangle$ be the mean meal size of an individual in the three groups. We can show empirically that in a no-tax situation, the expected meal size (probability of eating) approaches 1, just as it does in the single dining club case. See Fig. 2 (right). To generate Fig. 8, we initialized each dining club with 100 members and 100 non-club members. We used the Monte Carlo algorithm to simulate 300 trajectories for the group sizes and recorded the empirical probability of eating (expected meal size) over time. In Fig. 8, we show the expected meal size (with 95% confidence interval) as a function of time for the instances when Dining Club 1 (or 2) was dominant as well as the population mean meal size. Just as in the single dining club case, restaurant utilization (mean meal size) approaches 1. We explore the effect tax rate and initial condition has on long-run behavior of the system in the subsequent experimental results.

In a situation with adaptive taxation (see Eq. (7)), the two dining clubs can stabilize in size (when started with identical memberships) and the result is a probability of eating that is higher than $1 - e^{-1}$ but not exactly 1, as the two groups compete for common resources. This is illustrated in Fig. 9.

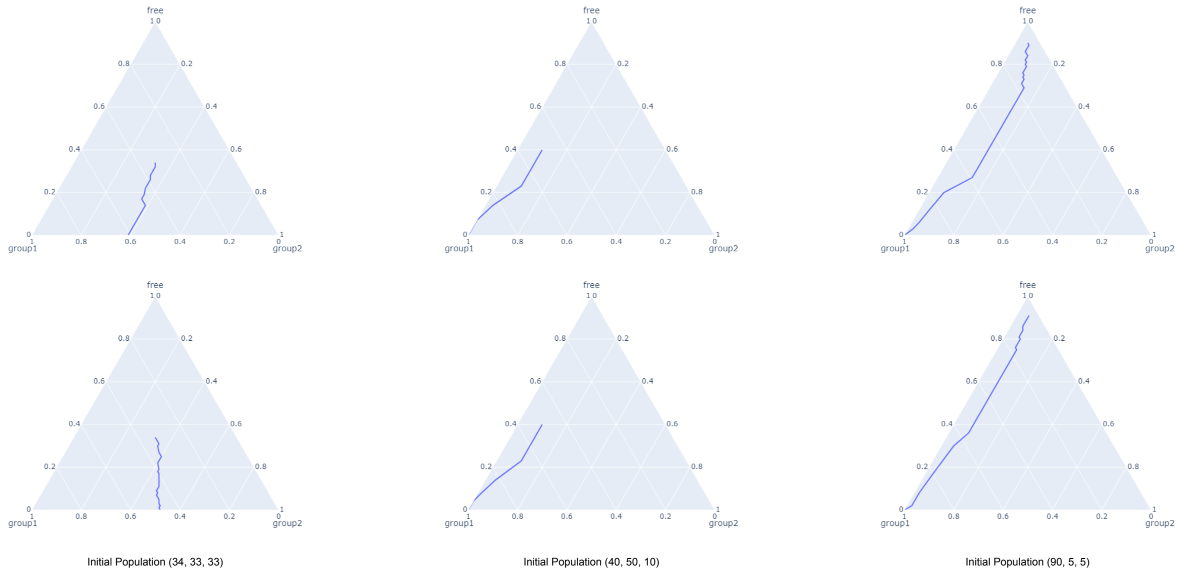


FIG. 7: (Left) Two example trajectories starting with 34 non-group members, 33 members in group 1 and 33 members in group 2. (Middle) Two example trajectories starting with 40 non-group members, 50 members in group 1 and 10 members in group 2. (Right) Two example trajectories starting with 90 non-group members, 5 members in group 1 and 5 members in group 2. When the two dining clubs start with the same number of individuals, the population will randomly evolve so that one club is dominant (assuming no taxation and food redistribution)

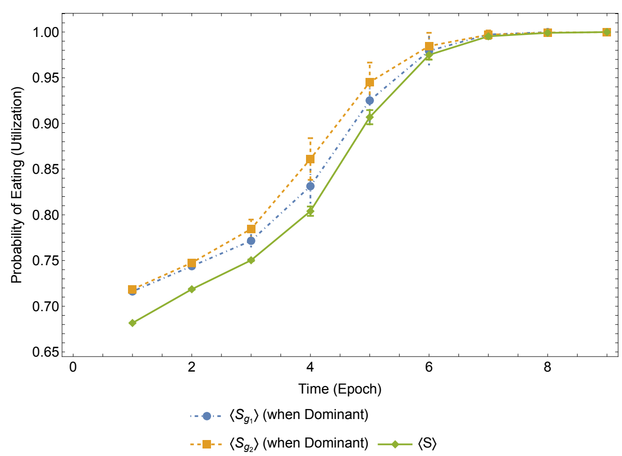


FIG. 8: In a Monte Carlo simulation, there is a 50% chance that either of the dining clubs will become the dominant group. Here, we show that the dominant group always approaches a probability of 1 of eating and hence so does the entire population.

A. Simulation Results

For each simulation, we divide 100 agents into $\mathcal{F}$, $\mathcal{G}_1$ and $\mathcal{G}_2$. To construct an approximation for the basins of attraction for the three equilibrium populations, we ran the simulation using 1,000 replications and simulated each possible (discrete) starting population size for $|\mathcal{F}|$, $|\mathcal{G}_1|$ and $|\mathcal{G}_2|$.

a. Tax Policy I: We explore the effect of varying κ (the tax rate) from 0.05 to 0.15. Both clubs use the same tax rate. However, they begin with different proportions of the population. To manage simulation time, we executed the while loop at most, 10,000 times. If all players had not joined a single community by then, we declared this a failed run, suggesting slow convergence from this initial condition. The outcome of almost all experiments resulted in a dominant group (either free agents or dinning clubs) being formed. This is illustrated in Fig. 10. Let β_1 and β_2 be the proportion of the population in dining clubs one and two, respectively, and let $\nu = 1 - \beta_1 - \beta_2$ be the free

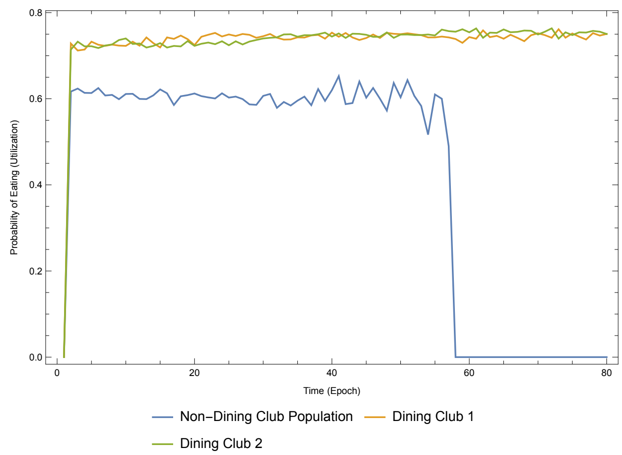


FIG. 9: An example run in which the non-club member population goes to zero and the two clubs maintain relatively stable populations over time. (Initial group sizes are $|\mathcal{F}| = 30$, $|\mathcal{G}_1 = 35|$, $|\mathcal{G}_2| = 35$. Final groups size at the end of the simulation are $|\mathcal{G}_1 = 49|$, $|\mathcal{G}_2| = 51$.

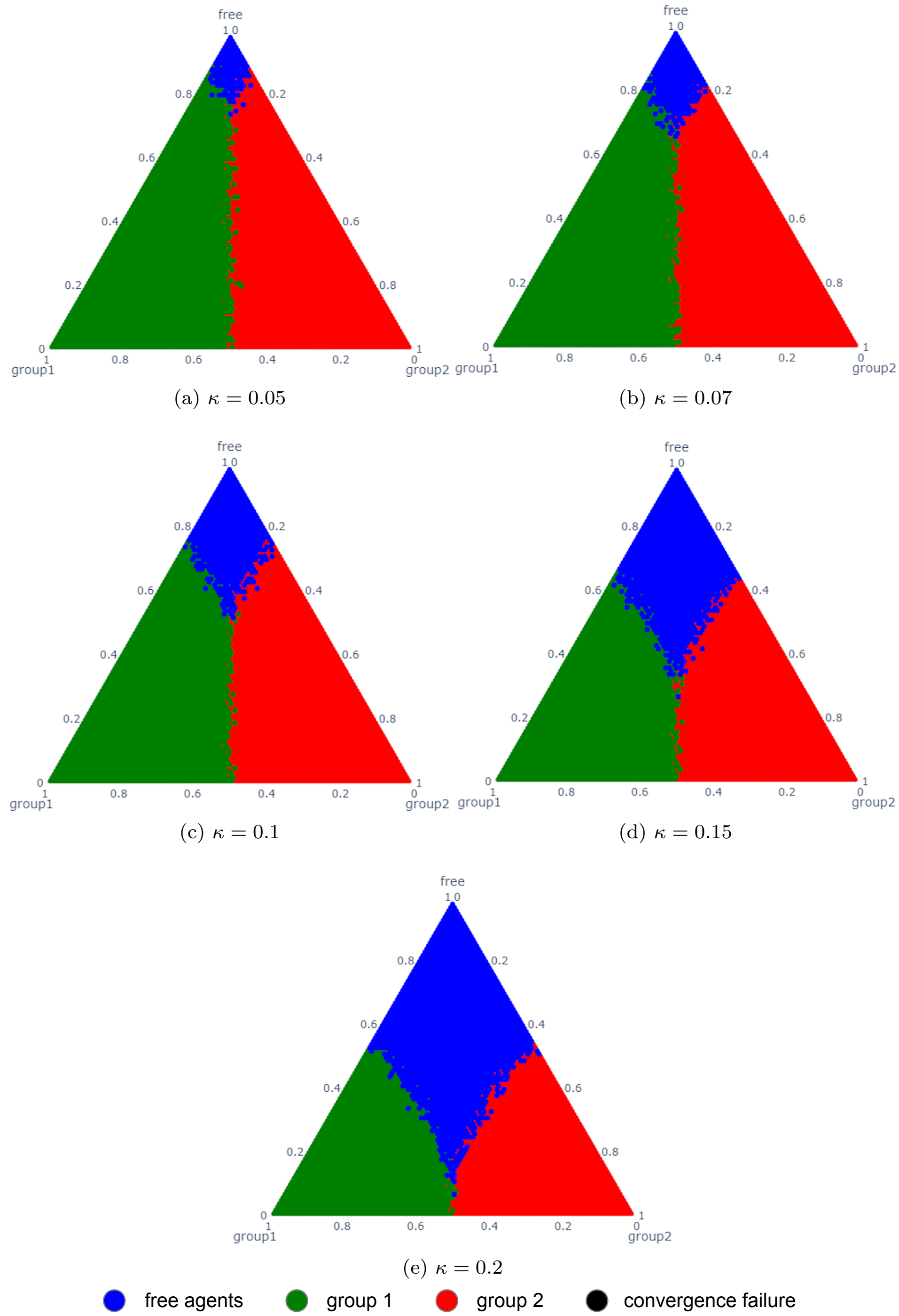
group proportion. Then the dynamics can be projected to the two-dimensional unit simplex Δ_2 embedded in $\mathbb{R}^3$ with coordinates (β_1, β_2, η). When the simulation converges, we can determine the ω-limit set of trajectories leaving (near) an initial condition $(\beta_1^0, \beta_2^0, \eta^0)$. Fig. 10 shows that the size of the tax rate κ is correlated with the size of the basin of attraction for the free agent group. The dynamics roughly partition the simplex into three basins of attraction, with the basins of attraction for the two dining clubs exhibiting symmetry as expected. On the boundaries of these regions, we expect unstable coexistence of multiple groups would be possible. This is qualitatively similar to the unstable fixed point identified in Fig. 4.

b. Tax Policy II: In a second set of experiments, we allow freeloading but assume the freeloader may be caught with probability ρ. When this occurs, the food that the freeloader was supposed to is not redistributed and goes to waste. We let ρ vary between 0 and 1 and used Eq. (7) to set the tax policy. The cheating probability was fixed at $\phi = 1$. As before, both clubs shared the same tax rate and we executed the while loop at most, 10,000 times. If all players had not joined a single community by then, we declared this a failed run, suggesting slow convergence from this initial condition. Basins of attraction for various fixed points are shown in Fig. 11. It is interesting to notice that there are a substantial number of failed cases between the clubs. This suggests an area of slow dynamics and possibly the existence of a slow manifold. Constructing a mathematical model of this scenario is an area reserved for future work, since it is unclear exactly how the dynamics are changing in this region.

VI. CONCLUSIONS AND FUTURE DIRECTIONS

In this paper, we studied the Kolkata Paise Restaurant Problem (KPRP) with dining clubs. Agents in a dining club mutually agree to visit separate restaurants, thereby increasing the probability that they eat (obtain a resource). An evolutionary game model was formulated describing the choice to join the dining club. We showed that joining the dining club is an evolutionarily stable strategy, even when members are taxed (in food) and resources are distributed. When cheating was introduced to the non-dining club members, i.e. the non-dining club members could deceptively benefit from the communal food collected by the dining club, a new unstable fixed point appears. We analysed this bifurcation as well as the decision to cheat using the resulting replicator dynamic. Numeric experiments on two dining clubs show that the behaviour in this case is similar to the case with one dining club, but may exhibit richer dynamics.

There are several directions for future research. Studying the theoretical properties of two (or more) dining clubs is clearly of interest. Adding many groups (i.e., so that the number of groups is a proportion of the number of players) might lead to unexpected phenomena. Also, allowing groups to compete for membership (by varying tax rates) might create interesting dynamics. As part of this research, investigation of the dynamics on the boundary both in theory and through numeric simulation would be of interest. Exploring multiple dining clubs with a spatial component might also lead to interesting results. If we introduce a diffusion component to the evolutionay dynamic with multiple groups, we should expect to see travelling wave solutions spreading out from the group origin points, as in the Fisher-KPP equation [31–33]. However, how these waves interact when multiple equivalent clubs meet is unclear. In two dimensions, it may lead to fractal geometry, like that seen in compact Apollonian packing. In this



(a) $\kappa = 0.05$ (b) $\kappa = 0.07$ (c) $\kappa = 0.1$ (d) $\kappa = 0.15$ (e) $\kappa = 0.2$

FIG. 10: Basins of attraction for various tax rates are shown in a ternary plot. The different colours indicate where the model converges from the given starting point.

case, characterizing the resulting spatial structures would be of great interest. A final area of future research would be to investigate the effect of taxing cheaters who are caught, thus allowing them to eat, but discouraging them from cheating. Determining the impact on the basins of attraction in this case would be the primary research objective.

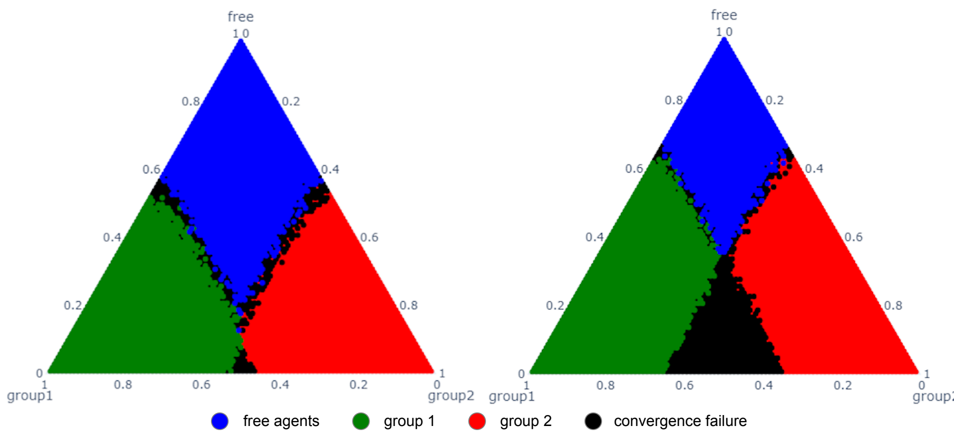


FIG. 11: (Left) We show the basins of attraction when the probability that a cheater is caught is set at 0.5. (Right) Basins of attraction when the probability that a cheater is caught is 1.

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