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Bochner formulas, functional inequalities and generalized Ricci flow

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A B S T R A C T

As a consequence of the Bochner formula for the Bismut connection acting on gradients, we show sharp universal Poincaré and log-Sobolev inequalities along solutions to generalized Ricci flow. Using the two-form potential we define a twisted connection on spacetime which determines an adapted Brownian motion on the frame bundle, yielding an adapted Malliavin gradient on path space. We show a Bochner formula for this operator, leading to characterizations of generalized Ricci flow in terms of universal Poincaré and log-Sobolev type inequalities for the associated Malliavin gradient and Ornstein-Uhlenbeck operator.

1. Introduction

In the analysis of Ricci flow, the classic Bochner formula for gradients plays a key role. This basic formula underlies gradient estimates for solutions to the heat equation along Ricci flow, and yields functional inequalities such as Wasserstein distance monotonicity [13], and universal Poincaré and log-Sobolev inequalities [9]. Furthermore, these

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functional inequalities can be used to characterize supersolutions to Ricci flow [7,13]. Later, through a broad extension of the Bochner formula to functions on path space, Haslhofer-Naber gave a characterization of solutions to Ricci flow [7] in terms of universal functional inequalities. In this paper we extend this circle of ideas to the setting of *generalized Ricci flow*. A one-parameter family of metrics and two-forms (g_t, b_t) is a solution of generalized Ricci flow [15] if

$$\frac{\partial}{\partial t} g = -2\operatorname{Rc} + \tfrac{1}{2} H^2, \qquad \frac{\partial}{\partial t} b = -d_g^* H, \qquad H = H_0 + db,$$

where $dH_0 = 0$ and

$$H^2(X,Y) = \langle X \lrcorner H, Y \lrcorner H \rangle .$$

At times we will refer equivalently to the associated pair (g_t, H_t) as a solution to generalized Ricci flow. It is natural to express this equation using the curvature of the unique metric connection with torsion H, referred to as a *Bismut connection*. If we let D denote the Levi-Civita connection, the relevant Bismut connection is then

$$\nabla := D + \tfrac{1}{2} g^{-1} H, \qquad \operatorname{Rc}^{\nabla} = \operatorname{Rc} - \tfrac{1}{4} H^2 - \tfrac{1}{2} d_g^* H.$$

It follows that the generalized Ricci flow can be expressed as

$$\frac{\partial}{\partial t}(g - b) = -2\operatorname{Rc}^{\nabla},$$

where $\operatorname{Rc}^{\nabla}$ is the Ricci tensor of the Bismut connection. The flow equation arises naturally as renormalization group flow [14], and arises naturally from considerations in complex geometry [17–19] and generalized geometry [4,16]. We refer to [5] for further background on generalized Ricci flow.

As solutions to generalized Ricci flow are supersolutions to Ricci flow, the results on Ricci flow supersolutions mentioned above immediately apply without changes. However, by using the explicit geometric structure of generalized Ricci flow we obtain sharper results. First we show universal Poincaré and log-Sobolev inequalities along solutions to generalized Ricci flow, extending the result of [9]. It is possible to use these inequalities to give characterizations of supersolutions to generalized Ricci flow, although we do not carry this out here. To state the result we record some notation: given (M^n, g_t, H_t) a generalized Ricci flow on $M \times [0,T]$, for $(x_0, 0) \in M \times [0,T]$ let $(s,y) \mapsto p_{T,s}(x_0,y)$ denote the conjugate heat kernel (see Definition 2.3), and let

$$d\nu_s^{x_0} = p_{T,s}(x_0,y)\, dV_{g(s)}.$$

Throughout we adopt the convention that by a solution to generalized Ricci flow we mean a smooth solution where each time slice is complete with bounded geometry.

Theorem 1.1. *Let (M^n, g_t, H_t) be a solution to generalized Ricci flow defined on $M \times [0,T]$. Fix $x_0 \in M$ and $s \in [0,T)$. Then:*

(1) For all $\phi \in C_0^\infty(M)$ with $\int \phi d\nu_s^{x_0} = 0$, one has

$$\int \phi^2 d\nu_s^{x_0} \leqslant 2(T-s) \int |\nabla \phi|^2 \, d\nu_s^{x_0},$$

with equality if and only if either $\phi \equiv 0$, or $(M, g_t, H_t) \cong (M', g_t', H_t') \times (\mathbb{R}, dz^2, 0)$ for all $t \in [s,T]$ with $z(x_0) = 0$ and $\phi(x) = \lambda z$ for some constant $\lambda \in \mathbb{R}^*$.

(2) For all $\phi \in C_0^\infty(M)$ with $\int \phi^2 d\nu_s^{x_0} = 1$, one has

$$\int \phi^2 \log \phi^2 d\nu_s^{x_0} \leqslant 4(T-s) \int |\nabla \phi|^2 \, d\nu_s^{x_0},$$

with equality if and only if either $\phi \equiv 1$, or $(M, g_t, H_t) \cong (M', g_t', H_t') \times (\mathbb{R}, dz^2, 0)$ for all $t \in [s,T]$ with $z(x_0) = 0$ and $\phi(x) = \exp(\lambda z - 2\lambda^2(T-s))$ for some constant $\lambda \in \mathbb{R}^*$.

Going further, we will show a generalization of the infinite-dimensional Bochner formula for the Malliavin gradient on path space along Ricci flow as in [1,7,11]. The starting point of these constructions is to define a connection on the frame bundle of the spacetime associated to a time-dependent Riemannian manifold, originally employed in Hamilton's proof of the Harnack inequality for Ricci flow [6]. It turns out that it is possible to incorporate the two-form potential b_t into this construction in a way that fits very naturally with the generalized Ricci flow equation. For a family (M^n, g_t, b_t) defined for $t \in I$, we define a connection ∇ on $\pi^* TM \to M \times I$ which extends the given action of ∇ via

$$\nabla_t Y = \partial_t Y + \tfrac{1}{2} \partial_t \left(g_t - b_t \right) (Y, \cdot)^{\sharp_{g_t}}.$$

This operator admits a key Bochner formula, which is central to our constructions. In particular, given $(g_t, H_t = H_0 + db_t)$ a general one-parameter family, and u a solution of the time-dependent heat equation, one has that (Proposition 3.3)

$$\nabla_t \operatorname{grad}_{g_t} u = \Delta \operatorname{grad}_{g_t} u - \left(\mathrm{Rc}^{\nabla} + \tfrac{1}{2} \partial_t \left(g_t - b_t \right) \right) \left(\operatorname{grad}_{g_t} u, \cdot \right)^{\sharp_{g_t}}.$$

Thus, along a solution to generalized Ricci flow, the gradient of a solution to the heat equation itself satisfies a pure heat equation using the adapted derivative ∇. The main goal is to give an extension of the Bochner formula above to path space. In §4 we use the connection ∇ on spacetime defined above together with the antidevelopment map to give the Eels-Elworthy-Malliavin construction of Brownian motion in this setting. This in turn gives a notion of parallel gradient for martingales. We then prove a formula on the evolution of parallel gradients of martingales which generalizes the Bochner identity above:

$$d(\nabla^{\perp}_{\sigma}F_{\tau}) = \langle \nabla^{\perp}_{\tau}\nabla^{\perp}_{\sigma}F_{\tau},\, dW_{\tau}\rangle + (\mathrm{Rc}^{\nabla} + \frac{1}{2}\partial_t(g-b))_{\tau}(\nabla^{\perp}_{\tau}F_{\tau})\mathbb{1}_{[\sigma,T]}(\tau)\, d\tau$$
$$+ \nabla^{\perp}_{\sigma}F_{\sigma}\delta_{\sigma}(\tau)\, d\tau.$$

This is a generalization of the Bochner formula described above (cf. Corollary 4.22), which occurs as the case where F is a one-point cylinder function.

The path-space Bochner formula above can be used to give many equivalent characterizations of generalized Ricci flow. First, in Theorem 5.1 we give equivalent characterizations in terms of Bochner inequalities on path space. Next, in Theorem 5.2 we show equivalence with universal estimates on the norm and square norm of gradients of martingales. We note that the adapted geometry on path space also determines an Ornstein-Uhlenbeck operator by composing the Malliavin gradient with its adjoint. We show equivalence with universal Poincaré and log-Sobolev inequalities for this operator on path space, extending the inequalities of Theorem 1.1. The precise definitions of the objects in the theorem below appear in §4.

Theorem 1.2. *For an evolving family of manifolds $(M, g_t, H_t)_{t\in[0,T]}$, the following are equivalent:*

(1) The generalized Ricci flow

$$\partial_t(g-b) = -2\,\mathrm{Rc}^{\nabla}$$

is satisfied.

(2) For any $0 \leqslant \sigma \leqslant T' \leqslant T$ and any $F \in \mathfrak{C}_{T'}$, we have the estimate

$$\mathbb{E}_{(x,T')}[|\nabla^{\perp}_{\sigma}F_{\sigma}|^2] + 2\int_0^{T'} \mathbb{E}_{(x,T')}[|\nabla^{\perp}_{\tau}\nabla^{\perp}_{\sigma}F_{\tau}|^2]\, d\tau \leqslant \mathbb{E}_{(x,T')}[|\nabla^{\perp}_{\sigma}F|^2]$$

for all $x \in M$.

(3) For any $0 \leqslant \tau_1 \leqslant \tau_2 \leqslant T' \leqslant T$ the Ornstein-Uhlenbeck operator $\mathcal{L}_{(\tau_1,\tau_2)}$ on parabolic path space $L^2(P_{T'}\mathcal{M})$ satisfies the Poincaré inequality

$$\mathbb{E}_{(x,T')}[(F_{\tau_2} - F_{\tau_1})^2] \leqslant 2\mathbb{E}_{(x,T')}[F\,\mathcal{L}_{(\tau_1,\tau_2)}F]$$

for all $x \in M$.

(4) For any $0 \leqslant \tau_1 \leqslant \tau_2 \leqslant T' \leqslant T$ the Ornstein-Uhlenbeck operator $\mathcal{L}_{(\tau_1,\tau_2)}$ on parabolic path space $L^2(P_{T'}\mathcal{M})$ satisfies the log-Sobolev inequality

$$\mathbb{E}_{(x,T')}[(F^2)_{\tau_2}\log((F^2)_{\tau_2}) - (F^2)_{\tau_1}\log((F^2)_{\tau_1})] \leqslant 4\mathbb{E}_{(x,T')}[F\,\mathcal{L}_{(\tau_1,\tau_2)}F]$$

for all $x \in M$.

Moreover, if one of the conditions (1)-(4) is satisfied, we have:

(3a) For any $0 \leqslant T' \leqslant T$, $F \in \mathfrak{C}_{T'}$, we have the Poincaré Hessian estimate

$$\mathbb{E}_{(x,T')}[(F - \mathbb{E}_{(x,T')}[F])^2] + 4 \int_0^{T'} \int_0^{T'} \mathbb{E}_{(x,T')}[|\nabla_\tau^\perp \nabla_\sigma^\perp F_\tau|^2] \, d\sigma \, d\tau \leqslant 2 \int_0^{T'} \mathbb{E}_{(x,T')}[|\nabla_\sigma^\perp F|^2] \, d\sigma$$

for all $x \in M$.

(4a) For any $0 \leqslant T' \leqslant T$, $F \in \mathfrak{C}_{T'}$, we have the log-Sobolev Hessian estimate

$$\mathbb{E}_{(x,T')}[F^2 \log(F^2)] - \mathbb{E}_{(x,T')}[F^2] \log(\mathbb{E}_{(x,T')}[F^2]) + 2 \int_0^{T'} \int_0^{T'} \mathbb{E}_{(x,T')}[(F^2)_\tau |\nabla_\tau^\perp \nabla_\sigma^\perp \log((F^2)_\tau)|^2] \, d\sigma \, d\tau \leqslant 4 \int_0^{T'} \mathbb{E}_{(x,T')}[|\nabla_\sigma^\perp F|^2] \, d\sigma$$

for all $x \in M$.

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2. Universal Poincaré and log-Sobolev inequalities along generalized Ricci flow

2.1. Conventions

Before we begin we explicitly clarify our notational conventions and some elementary facts. Given data (M^n, g, H) of a smooth Riemannian manifold and closed three-form H, we let D denote the Levi-Civita connection of g and $\nabla = D + \frac{1}{2} g^{-1} H$ denote the Bismut connection, as explained in the introduction. We almost exclusively work with ∇, although in some proofs D makes an appearance. The connection ∇ induces connections on all tensor bundles, and furthermore we define a Laplace operator

$$\Delta = \operatorname{tr}_g \nabla\nabla$$

A fundamental point is that the Laplacian acting on functions is the same as the usual Levi-Civita Laplacian, although importantly this is no longer the case for the Laplacian acting on other tensor bundles, in particular acting on 1-forms and vector fields. Furthermore, we will typically deal with a one-parameter family $(g_t, H_t = H_0 + db_t)$ of Riemannian metrics and closed three-forms. Often we will simply describe this as (g_t, b_t), with the background fixed choice of H_0 not stated.

2.2. Heat operators

Definition 2.1. The heat operator and conjugate heat operator along a solution to generalized Ricci flow are defined by

$$\begin{aligned}\square &:= \partial_t - \Delta \\ \square^* &:= -\partial_t - \Delta + R - \frac{1}{4}|H|^2 .\end{aligned}$$

Lemma 2.2. *Let $[t_1, t_2] \subset [0, T]$. Let $u, v \colon M \times [t_1, t_2] \to \mathbb{R}$ be smooth functions with compact support in M. Then*

$$\int_{t_1}^{t_2} \int_M (\square u)v - (\square^* v)u \, dV \, dt = \left[\int_M uv \, dV \right]_{t_1}^{t_2} .$$

Definition 2.3. For $x, y \in M$ and $s < t \in [0, T]$, we let $p_{t,s}(x, y)$ denote the heat kernel based at (s, y), i.e. the unique minimal positive solution to the equations

$$\begin{aligned}&\square_{t,x} p_{t,s}(x, y) = 0, \\ &\lim_{t \downarrow s} p_{t,s}(x, y) = \delta_y(x).\end{aligned}$$

Observe that by the duality in Lemma 2.2, the heat kernel $p_{t,s}(x, y)$ equivalently solves the conjugate heat equation based at (t, x) in (s, y), i.e.

$$\begin{aligned}&\square^*_{s,y} p_{t,s}(x, y) = 0 \\ &\lim_{s \uparrow t} p_{t,s}(x, y) = \delta_x(y).\end{aligned}$$

Consequently, $p_{t,s}(x, y)$ is mass-preserving in y with respect to $dV_{g(s)}$ and

$$\int_M p_{t,s}(x, y) \, dV_{g(s)}(y) = 1$$

for all $s < t$ and $x \in M$. Moreover, the uniqueness implies the propagator property

$$p_{t,r}(x,z) = \int_M p_{t,s}(x,y) p_{s,r}(y,z)\, dm_s(y)$$

for all $r < s < t$ and $x, z \in M$. With this we can define the heat flow and the conjugate heat flow of a function $u \in C_0^\infty(M)$.

Definition 2.4. Let $u, v \in C_0^\infty(M)$ and $\mu \in \mathcal{P}(M)$. For $s \leqslant t \in [0,T]$, let $(t,x) \mapsto P_{t,s}u(x)$ denote the heat flow, i.e.

$$(P_{t,s}u)(x) = \int_M p_{t,s}(x,y) u(y)\, dV_{g(s)}(y).$$

For $t \geqslant s \in [0,T]$, let $(s,y) \mapsto P^*_{t,s}v(y)$ denote the conjugate heat flow, i.e.

$$(P^*_{t,s}v)(y) = \int_M p_{t,s}(x,y) v(x)\, dV_{g(t)}(x).$$

In other words, $(t,x) \mapsto P_{t,s}u(x)$ solves the (forward) heat equation from time s with initial condition u to time t, whereas $(s,y) \mapsto P^*_{t,s}v(y)$ solves the (backward) conjugate heat equation from time t with terminal condition v to time s. Lastly we record a useful identity for the heat flow $P_{t,s}$.

Lemma 2.5. *Let $t \in [0,T]$. For any family of smooth functions U_s parametrized by $s \in (0,t)$,*

$$\frac{d}{ds} P_{t,s} U_s = P_{t,s} \Box_s U_s.$$

Proof. By the definition of $P_{t,s}$ we have for every $u \in C_0^\infty(M)$

$$\frac{d}{ds} P_{t,s} u = -P_{t,s} \Delta_{g(s)} u.$$

The claim follows then from the Leibniz rule. □

2.3. The parabolic Bochner formula

A fundamental observation for Ricci flow equation is the gradient bound for solutions to the time-dependent heat equation. In fact, this observation extends to supersolutions of the Ricci flow, and thus automatically applies to solutions of generalized Ricci flow. The lemma below records the parabolic Bochner formulas for spacetime functions along a solution to generalized Ricci flow, which interestingly is expressed naturally in terms of the Hessian with respect to the Bismut connection.

Lemma 2.6. *Let (M^n, g_t, H_t) denote a solution to generalized Ricci flow. Then the following hold:*

(1) Given u a spacetime function,

$$\square \tfrac{1}{2} |\nabla u|^2 = -|\nabla \nabla u|^2 + \langle \nabla u, \nabla \square u \rangle .$$

(2) Fix u a spacetime function and $\varphi : \mathbb{R} \to \mathbb{R}$. Setting $U = \varphi(u)$, we have

$$\square U = \varphi' \square u - \varphi'' |\nabla u|^2 .$$

(3) Fix u a spacetime function and $\psi : \mathbb{R} \to \mathbb{R}$. Setting $U = \psi(u) |\nabla u|^2$ we have

$$\begin{aligned}\square U = & -2\psi(u) \left(|\nabla \nabla u|^2 + \langle \nabla u, \nabla \square u \rangle \right) - 4\psi'(u) \left\langle \nabla^2 u, \nabla u \otimes \nabla u \right\rangle \\ & - \psi''(u) |\nabla u|^4 + \psi'(u) \square u.\end{aligned}$$

Proof. For item (1) we apply the usual Bochner formula to obtain

$$\begin{aligned}\square \tfrac{1}{2} |\nabla u|^2 = & \langle \nabla \Delta u, \nabla u \rangle + \left\langle \mathrm{Rc} - \tfrac{1}{4} H^2, \nabla u \otimes \nabla u \right\rangle - \left|\nabla^2 u\right|^2 - \langle \Delta \nabla u, \nabla u \rangle \\ = & -\left| D^2 u \right|^2 - \tfrac{1}{4} |\nabla u \lrcorner H|^2 + \langle \nabla u, \nabla \square u \rangle .\end{aligned}$$

We furthermore observe that, since

$$\nabla = D + \tfrac{1}{2} H g^{-1},$$

it follows that

$$\nabla \nabla u = DDu + \frac{1}{2} \nabla u \lrcorner H.$$

And hence, since DDu is symmetric and $\nabla u \lrcorner H$ is skew-symmetric, it follows that

$$|\nabla \nabla u|^2 = |DDu|^2 + \tfrac{1}{4} |\nabla u \lrcorner H|^2 ,$$

yielding item (1). For item (2) we compute

$$\square \varphi(u) = \varphi'(u) \frac{\partial u}{\partial t} - \operatorname{div} (\varphi'(u) \nabla u) = \varphi'(u) \square u - \varphi''(u) |\nabla u|^2 .$$

For item (3) we compute

$$\begin{aligned}\square U &= \psi'(u) \frac{\partial u}{\partial t} |\nabla u|^2 + \psi(u) \frac{\partial}{\partial t} |\nabla u|^2 - \operatorname{div} \left(\psi'(u) \nabla u |\nabla u|^2 + \psi(u) \nabla |\nabla u|^2 \right) \\ &= \psi'(u) \square u + \psi(u) \square |\nabla u|^2 - \psi''(u) |\nabla u|^4 - 4\psi'(u) \left\langle \nabla^2 u, \nabla u \otimes \nabla u \right\rangle\end{aligned}$$

$$= -2\psi(u)\left(|\nabla\nabla u|^2 + \langle\nabla u, \nabla\Box u\rangle\right) - 4\psi'(u)\left\langle\nabla^2 u, \nabla u\otimes\nabla u\right\rangle$$
$$- \psi''(u)\,|\nabla u|^4 + \psi'(u)\Box u. \quad \Box$$

Using this lemma we give two useful intertwining relations of the heat flow, in particular generalizing the L^2-gradient estimate in the sense of Bakry-Émery.

Proposition 2.7. *Let (M^n, g_t, H_t) denote a solution to generalized Ricci flow.*

(1) For $u \in C_0^\infty(M)$ it holds

$$|\nabla P_{t,s}u|^2_{g(t)} = P_{t,s}(|\nabla u|^2_{g(s)}) - 2\int_s^t P_{t,r}\left(|\nabla\nabla P_{r,s}u|^2_{g(r)}\right) dr.$$

(2) For $u \in C_0^\infty(M)$ with $u \geqslant 0$ it holds

$$\frac{|\nabla P_{t,s}u|^2_{g(t)}}{P_{t,s}u} = P_{t,s}\left(\frac{|\nabla u|^2_{g(s)}}{u}\right) - 2\int_s^t P_{t,r}\left(P_{r,s}u\,|\nabla\nabla\log P_{r,s}u|^2_{g(r)}\right) dr.$$

Proof. By Lemma 2.5 and Lemma 2.6 (3) with $\psi = 1$ we have

$$\int_s^t \frac{d}{dr}P_{t,r}(|\nabla P_{r,s}u|^2_{g(r)})dr = \int_s^t P_{t,r}\Box_r\,|\nabla P_{r,s}u|^2_{g(r)}\,dr - 2\int_s^t P_{t,r}\left|\nabla^2 P_{r,s}u\right|^2_{g(r)} dr.$$

On the other hand, by the fundamental theorem of calculus

$$\int_s^t \frac{d}{dr}P_{t,r}(|\nabla P_{r,s}u|^2_{g(r)})dr = |\nabla P_{t,s}u|^2_{g(t)} - P_{t,s}(|\nabla u|^2_{g(s)}).$$

Combining the two last equations yields item (1).

In order to show item (2) we derive with Lemma 2.5 and Lemma 2.6 (3) with the choice $\psi(u) = \frac{1}{u}$,

$$\int_s^t \frac{d}{dr}P_{t,r}\left(\frac{|\nabla P_{r,s}u|^2_{g(r)}}{P_{r,s}u}\right)dr = \int_s^t P_{t,r}\Box_r\left(\frac{|\nabla P_{r,s}u|^2_{g(r)}}{P_{r,s}u}\right)dr$$
$$= -2\int_s^t P_{t,r}\left(\frac{|\nabla\nabla P_{r,s}u|^2_{g(r)}}{P_{r,s}u} - 2\frac{\left\langle\nabla^2 P_{r,s}u, \nabla P_{r,s}u\otimes\nabla P_{r,s}u\right\rangle}{(P_{r,s}u)^2} + \frac{|\nabla P_{r,s}u|^4_{g(r)}}{(P_{r,s}u)^3}\right)dr$$
$$= -2\int_s^t P_{t,r}\left(P_{r,s}u\,|\nabla\nabla\log P_{r,s}u|^2_{g(r)}\right) dr.$$

Again, the fundamental theorem of calculus yields the claim. □

2.4. Proof of Theorem 1.1

We end this section with the proof of Theorem 1.1. We apply the intertwining relations from Proposition 2.7 for certain test functions. We first prove a splitting result for the Bismut connection.

Proposition 2.8. *Let (M^n, g, H) be a smooth Riemannian manifold with H a closed three-form. Suppose there exists a closed nonvanishing 1-form α such that $\nabla\alpha \equiv 0$. Then the universal cover $\pi : \widetilde{M} \to M$ splits as $\widetilde{M} = M' \times \mathbb{R}$, the metric splits $\pi^* g = g' + \pi^*\alpha \otimes \pi^*\alpha$, where g' is a metric on M', and lastly $\pi^* H = \pi^*_{M'} H'$, where H' is a closed three-form on M'.*

Proof. Using the definition of ∇ we observe that

$$0 \equiv \nabla\alpha = D\alpha + \tfrac{1}{2}\alpha^\sharp \lrcorner H.$$

Since α is closed we have that $D\alpha$ is symmetric, whilst the final term is skew-symmetric, thus the two terms on the right hand side above vanish individually. In particular α is parallel with respect to the Levi-Civita connection, and the metric splitting of the universal cover is a standard consequence of the de Rham decomposition theorem.

To show the splitting property of H we let z denote a coordinate on the $\mathbb{R}$-factor of $\widetilde{M}$, and let $A, B, C \in TM'$. Note that by construction $\pi^*\alpha$ is a nonzero multiple of $\frac{\partial}{\partial z}$ and thus $\frac{\partial}{\partial z} \lrcorner \pi^* H \equiv 0$. Using this and that $\pi^* H$ is closed we furthermore obtain

$$0 = d\pi^* H\left(\frac{\partial}{\partial z}, A, B, C\right) = D_{\frac{\partial}{\partial z}} \pi^* H(A, B, C).$$

Thus $\pi^* H$ is parallel along $\frac{\partial}{\partial z}$, and it follows that $\pi^* H = \pi^*_{M'} H'$, with $dH' = 0$, as claimed. □

Proof of Theorem 1.1 (1). Let $\varphi(x) = x^2$. Let $u \in C_0^\infty(M)$ and recall that

$$d\nu_s^{x_0} = p_{T,s}(x_0, \cdot)\, dV_{g(s)}.$$

We note

$$\begin{aligned} -\int_s^T \frac{d}{dt} P_{T,t}(\varphi(P_{t,s}u))(x_0)\, dt &= P_{T,s}(\varphi(P_{s,s}u))(x_0) - P_{T,T}(\varphi(P_{T,s}u))(x_0) \\ &= \int_M \varphi(u)(y) p_{T,s}(x_0, y)\, dV_{g(s)}(y) \end{aligned}$$

$$
\begin{aligned}
&- \varphi\left(\int_M u(y) p_{T,s}(x_0,y)\, dV_{g(s)}(y)\right) \\
&= \int_M u^2 \, d\nu_s^{x_0} - \left(\int_M u \, d\nu_s^{x_0}\right)^2 .
\end{aligned}
$$

But on the other hand using Lemma 2.5 and Lemma 2.6 (2) we obtain

$$
\begin{aligned}
-\int_s^T \frac{d}{dt} P_{T,t}(\varphi(P_{t,s}u))(x_0)\, dt &= -\int_s^T P_{T,t} \Box_t (\varphi(P_{t,s}u))(x_0)\, dt \\
&= 2\int_s^T P_{T,t}(|\nabla P_{t,s} u|^2_{g(t)})(x_0)\, dt.
\end{aligned}
$$

Applying Proposition 2.7 (1) to the last term we get

$$
\begin{aligned}
&-\int_s^T \frac{d}{dt} P_{T,t}(\varphi(P_{t,s}u))(x_0)\, dt \\
&= 2\int_s^T P_{T,t}\left[P_{t,s}(|\nabla u|^2_{g(s)})(x_0) - 2\int_s^t P_{t,r}(|\nabla\nabla P_{r,s}u|^2)(x_0)\, dr\right] dt
\end{aligned}
$$

Combining these equations yields

$$
\begin{aligned}
&\int_M u^2 \, d\nu_s^{x_0} - \left(\int_M u \, d\nu_s^{x_0}\right)^2 \\
&\quad = 2\int_s^T P_{T,t} P_{t,s} \left|\nabla u\right|^2_{g(s)}(x_0)\, dt - 4\int_s^T P_{T,t}\left[\int_s^t P_{t,r}\left(\left|\nabla\nabla P_{r,s}u\right|^2_{g(r)}(x_0)\right) dr\right] dt \\
&\quad = 2\int_s^T P_{T,s} \left|\nabla u\right|^2_{g(s)}(x_0) dt - 4\int_s^T P_{0,t}\left[\int_s^t P_{t,r}\left(\left|\nabla\nabla P_{r,s}u\right|^2_{g(r)}(x_0)\right) dr\right] dt \\
&\quad = 2(T-s)\int_M \left|\nabla u\right|^2_{g(s)}(y) p_{T,s}(x_0,y)\, dV_{g(s)}(y) \\
&\qquad - 4\int_s^T P_{T,t}\left[\int_s^t P_{t,r}\left(\left|\nabla\nabla P_{r,s}u\right|^2_{g(r)}(x_0)\right) dr\right] dt
\end{aligned}
$$

$$= 2(T-s)\int_M |\nabla u|^2_{g(s)}\, d\nu_s^{x_0} - 4\int_s^T P_{T,t}\left[\int_s^t P_{t,r}\left(|\nabla\nabla P_{r,s}u|^2_{g(r)}(x_0)\right) dr\right] dt\,.$$

This implies the Poincaré inequality of item (1).

Equality occurs if and only if

$$\int_s^T P_{T,t}\left[\int_s^t P_{t,r}\left(|\nabla\nabla P_{r,s}u|^2_{g(r)}(x_0)\right) dr\right] dt = 0.$$

Due to the maximum principle of $P_{t,r}$ and $P_{T,t}$ we have that $|\nabla\nabla P_{r,s}u|_{g(r)}$ needs to vanish at time $r = s$, i.e. $|\nabla\nabla u|_{g(s)} = 0$. It follows from Proposition 2.8 that the data at time s splits as claimed. From uniqueness of solutions to generalized Ricci flow in the class of solutions with bounded geometry (cf. [2] which extends to generalized Ricci flow), it follows that the solution splits for all times, as claimed. □

Proof of Theorem 1.1 (2). Let $\varphi(x) = x\log x$ and $u \in C_0^\infty(M)$ with $u \geqslant 0$. With this we obtain from Proposition 2.7 (2) that

$$\int_M u\log u\, d\nu_s^{x_0} - \int_M u\, d\nu_s^{x_0} \log\left(\int_M u\, d\nu_s^{x_0}\right) = \int_s^T P_{T,t}\left(\frac{|\nabla P_{t,s}u|^2_{g(t)}}{P_{t,s}u}\right)(x_0)\, dt.$$

Applying Proposition 2.7 to the right hand side we get

$$\begin{aligned}
&\int_M u\log u\, d\nu_s^{x_0} - \int_M u\, d\nu_s^{x_0} \log\left(\int_M u\, d\nu_s^{x_0}\right) \\
&\quad = \int_s^T P_{T,t}\left[P_{t,s}\left(\frac{|\nabla u|^2_{g(s)}}{u}\right) - 2\int_s^t P_{t,r}(P_{r,s}u\,|\nabla\nabla \log P_{r,s}u|^2)\, dr\right](x_0)\, dt \\
&\quad = (T-s)P_{T,s}\left(\frac{|\nabla u|^2_{g(s)}}{u}\right) - 2\int_s^T P_{T,t}\left[\int_s^t P_{t,r}(P_{r,s}u\,|\nabla\nabla \log P_{r,s}u|^2)\, dr\right](x_0)\, dt \\
&\quad = (T-s)\int_M \frac{|\nabla u|^2_{g(s)}}{u}\, d\nu_s^{x_0} - 2\int_s^T P_{T,t}\left[\int_s^t P_{t,r}(P_{r,s}u\,|\nabla\nabla \log P_{r,s}u|^2)\, dr\right](x_0)\, dt.
\end{aligned}$$

Then if $\int_M u\, d\nu_s^{x_0} = 1$ we set $\phi = \sqrt{u}$ and obtain item (2). The case of equality is treated the same as in item (1). □

3. Twisted parallel transport and frame bundle formalism

In this section we define a connection on a spacetime adapted to a solution of generalized Ricci flow. The key point is a Bochner formula for solutions to the time-dependent heat equation, Proposition 3.3, which lies at the heart of the path space constructions to follow. We also use this connection to recast the time-dependent geometry on the frame bundle, which is necessary for the construction of the adapted Brownian motion et al.

3.1. The twisted connection on spacetime

Let (M^n, g_t, b_t) be a one-parameter family of generalized metrics. Let $\mathcal{M} = M \times I$ for some time interval I. Let ∂_t denote the canonical vector field on I lifted to $\mathcal{M}$. We define a connection on the vector bundle $\pi^* TM \to \mathcal{M}$ which extends the action of ∇ via

$$\nabla_t Y = \partial_t Y + \tfrac{1}{2}\partial_t \left(g_t + b_t\right)(Y, \cdot)^{\sharp_{g_t}}.$$

This generalizes Hamilton's spacetime connection introduced in his derivation of the Harnack estimate [6]. The term involving the time derivative of g is natural to include as it renders the connection compatible with the time-dependent metric. In fact one is free to add the action of an arbitrary skew-symmetric two-form as well and still preserve this property (cf. Lemma 3.1). As it turns out, the precise term $\frac{1}{2}\partial_t b_t$ gives the connection ∇ particularly favorable properties in the case of a solution to generalized Ricci flow.

Lemma 3.1. *The spacetime connection ∇ is compatible with g.*

Proof. This follows from

$$\begin{aligned}
\frac{d}{dt}\left|Y\right|^2_{g_t} &= \partial_t g_t\left(Y, Y\right) + g_t\left(\partial_t Y, Y\right) + g_t\left(Y, \partial_t Y\right) \\
&= g_t\left(\partial_t Y + \tfrac{1}{2}\partial_t g_t(Y,\cdot)^{\sharp_{g_t}}, Y\right) + g_t\left(Y, \partial_t Y + \tfrac{1}{2}\partial_t g_t(Y,\cdot)^{\sharp_{g_t}}\right) \\
&= g_t\left(\partial_t Y + \tfrac{1}{2}\partial_t\left(g_t + b_t\right)(Y,\cdot)^{\sharp_{g_t}}, Y\right) + g_t\left(Y, \partial_t Y + \tfrac{1}{2}\partial_t\left(g_t + b_t\right)(Y,\cdot)^{\sharp_{g_t}}\right) \\
&= 2g_t(\nabla_t Y, Y),
\end{aligned}$$

where the third line follows since b is skew-symmetric. □

Lemma 3.2. *Given (M, g, H) as above one has*

$$\Delta\nabla u = \nabla\Delta u + \mathrm{Rc}^{\nabla}(\nabla u, \cdot).$$

Proof. We choose local coordinates and let Γ denote the connection coefficients of ∇. We then compute

$$(\nabla\nabla\nabla u)_{ijk} = \partial_i (\nabla\nabla u)_{jk} - \Gamma_{ij}^l (\nabla\nabla u)_{lk} - \Gamma_{ik}^l (\nabla\nabla u)_{jl}$$
$$= \partial_i \left(\partial_j\partial_k u - \Gamma_{jk}^p \partial_p u\right) - \Gamma_{ij}^l (\partial_l\partial_k u - \Gamma_{lk}^p\partial_p u) - \Gamma_{ik}^l \left(\partial_j\partial_l u - \Gamma_{jl}^p \partial_p u\right).$$

It follows that

$$\begin{aligned}
&(\nabla\nabla\nabla u)_{ijk} - (\nabla\nabla\nabla u)_{jik}\\
&\quad = \partial_i \left(\partial_j\partial_k u - \Gamma_{jk}^p \partial_p u\right) - \Gamma_{ij}^l (\partial_l\partial_k u - \Gamma_{lk}^p\partial_p u) - \Gamma_{ik}^l \left(\partial_j\partial_l u - \Gamma_{jl}^p \partial_p u\right)\\
&\qquad - \left(\partial_j (\partial_i\partial_k u - \Gamma_{ik}^p \partial_p u) - \Gamma_{ji}^l (\partial_l\partial_k u - \Gamma_{lk}^p\partial_p u) - \Gamma_{jk}^l (\partial_i\partial_l u - \Gamma_{il}^p \partial_p u)\right)\\
&\quad = \partial_p u \left(-\partial_i \Gamma_{jk}^p + \partial_j \Gamma_{ik}^p - \Gamma_{jk}^l\Gamma_{il}^p + \Gamma_{ik}^l \Gamma_{jl}^p\right) - H_{ij}^l (\nabla\nabla u)_{lk}\\
&\quad = -\left(R^\nabla\right)_{ijk}^p d_p u - H_{ij}^l (\nabla\nabla u)_{lk}.
\end{aligned}$$

Also we have

$$\nabla_j\nabla_k u - \nabla_k\nabla_j u = \left(\partial_j\partial_k u - \Gamma_{jk}^p\partial_p u\right) - \left(\partial_k\partial_j u - \Gamma_{kj}^p \partial_p u\right) = -H_{jk}^p \partial_p u.$$

Combining these we then have

$$\begin{aligned}
\nabla_i \Delta u &= g^{jk} \nabla_i\nabla_j\nabla_k u\\
&= g^{jk}\left(\nabla_j\nabla_i\nabla_k u - (R^\nabla)_{ijk}^p d_p u - H_{ij}^l (\nabla\nabla u)_{lk}\right)\\
&= g^{jk}\left(\nabla_j\nabla_k\nabla_i u - \nabla_j (H_{ik}^p d_p u) - (R^\nabla)_{ijk}^p d_p u - H_{ij}^l (\nabla\nabla u)_{lk}\right)\\
&= \Delta \nabla_i u - (\mathrm{Rc}^\nabla)_i^p d_p u - g^{jk}\left(\nabla_j (H_{ik}^p d_p u) + H_{ij}^l (\nabla\nabla u)_{lk}\right).
\end{aligned}$$

Then we simplify

$$\begin{aligned}
&g^{jk}\left(\nabla_j\left(H_{ik}^p d_p u\right) + H_{ij}^l (\nabla\nabla u)_{lk}\right)\\
&\quad = g^{jk}\left(D_j H_{ik}^p - \tfrac{1}{2} H_{ji}^l H_{lk}^p - \tfrac{1}{2} H_{jk}^l H_{il}^p + \tfrac{1}{2} H_{jl}^p H_{ik}^l\right) d_p u + g^{jk} H_{ik}^p \nabla_j\nabla_p u + g^{jk} H_{ij}^l \nabla_l \nabla_k u\\
&\quad = (d_g^* H_i^p) d_p u,
\end{aligned}$$

where the last line follows using the skew-symmetry of H. This finally yields, after rearranging,

$$\begin{aligned}
\Delta\nabla_i u &= \nabla_i \Delta u + (\mathrm{Rc}^\nabla)_i^p d_p u + (d_g^* H)_i^p d_p u\\
&= \nabla_i\Delta u + (\mathrm{Rc} - \tfrac{1}{4} H^2 - \tfrac{1}{2} d_g^* H)_i^p d_p u + (d_g^* H)_i^p d_p u\\
&= \nabla_i \Delta u + g^{pq}\, \mathrm{Rc}_{qi}^\nabla\, d_p u,
\end{aligned}$$

as claimed. □

Proposition 3.3. *Let (M^n, g_t, b_t) denote a time dependent family as above, and suppose u solves $\square u = 0$. Then*

$$\nabla_t \operatorname{grad}_{g_t} u = \Delta \operatorname{grad}_{g_t} u - \left(\mathrm{Rc}^{\nabla} + \tfrac{1}{2}\partial_t\left(g_t - b_t\right)\right)\left(\operatorname{grad}_{g_t} u, \cdot\right)^{\sharp_{g_t}}.$$

Proof. Using the definitions above and Lemma 3.2 we have

$$\begin{aligned}\nabla_t \operatorname{grad}_{g_t} u &= \partial_t\left(g_t^{-1} du\right) + \tfrac{1}{2}\partial_t\left(g_t + b_t\right)\left(\operatorname{grad}_{g_t} u, \cdot\right)^{\sharp_{g_t}} \\ &= \operatorname{grad}_{g_t} \Delta_{g_t} u - \partial_t g_t\left(\operatorname{grad}_{g_t} u, \cdot\right)^{\sharp_{g_t}} + \tfrac{1}{2}\partial_t\left(g_t + b_t\right)\left(\operatorname{grad}_{g_t} u, \cdot\right)^{\sharp_{g_t}} \\ &= \Delta \operatorname{grad}_{g_t} u - \left(\mathrm{Rc}^{\nabla} + \tfrac{1}{2}\partial_t(g_t - b_t)\right)\left(\operatorname{grad}_{g_t} u, \cdot\right)^{\sharp_{g_t}}. \quad \square\end{aligned}$$

3.2. Frame bundle formalism

We next use the spacetime connection to define an adapted geometry on the frame bundle. We refer to [12] for general background. Let (M^n, g_t, b_t) be a time-dependent family as above. Let $\mathcal{M} = M \times I$ for some time interval I. We define an O_n-bundle $\pi : \mathcal{F} \to \mathcal{M}$, where the fibers $\mathcal{F}_{(x,t)}$ are orthogonal maps $u : \mathbb{R}^n \to (T_x M, g_t)$. Given a curve γ_t in $\mathcal{M}$, the horizontal lift is a curve $u_t \in \mathcal{F}$ such that:

$$\pi \circ u_t = \gamma_t, \qquad \nabla_{\dot\gamma}(u_t v) = 0 \quad \text{for all } v \in \mathbb{R}^n.$$

By general theory, it follows that if we fix a point $u_0 \in \pi^{-1}\gamma_0$, there exists a unique horizontal lift u_t with initial condition u_0. Furthermore, given $aX + b\partial_t \in T_{(x,t)}\mathcal{M}$, and $u \in \mathcal{F}_{(x,t)}$, there exists a unique horizontal lift $aX^* + b\partial_t^*$ which satisfies

$$\pi_*(aX^* + b\partial_t^*) = aX + b\partial_t.$$

Here X^* is the usual horizontal lift of X with respect to ∇, and ∂_t^* the lift of ∂_t along the path which is constant in space.

The frame bundle $\mathcal{F}$ comes equipped with certain canonical vector fields as well. First, as above we have ∂_t^* which is the horizontal lift of ∂_t. Furthermore we define horizontal vector fields $\{E_i\}_{i=1}^n$ via

$$E_i(u) = (ue_i)^*,$$

where $\{e_i\}_{i=1}^n$ is the standard basis for $\mathbb{R}^n$. We furthermore define vertical vector fields

$$V_{ij}(u) = \left.\frac{d}{ds}\right|_{s=0}(u\exp(sA_{ij})), \qquad (A_{ij})_{kl} = (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}).$$

We will perform some computations below in local coordinates. To that end, given coordinates $\{x^i, t\}$ on $\mathcal{M}$ we canonically associate coordinates (x^i, t, e_j^k) on $\mathcal{F}$, where the functions e_j^k are defined by

$$ue_j = e_j^k \frac{\partial}{\partial x^k}.$$

We furthermore recall an identification between contravariant tensor fields on $\mathcal{M}$ and equivariant functions on $\mathcal{F}$. Given a smooth function $f : \mathcal{M} \to \mathbb{R}$ we set

$$\widetilde{f} = f \circ \pi.$$

Furthermore, given $\alpha = \alpha_i dx^i \in T^*\mathcal{M}$, we obtain $\widetilde{\alpha} : \mathcal{F} \to \mathbb{R}^n$ via

$$\widetilde{\alpha}_i(u) = \alpha_{\pi(u)}(ue_i).$$

This identification extends in an obvious way to any contravariant tensor.

Lemma 3.4. *With the setup above, one has:*

$$\widetilde{Xf} = X^* \widetilde{f}, \qquad \widetilde{\partial_t f} = \partial_t^* \widetilde{f}.$$

Proposition 3.5. *Given local coordinates $\{x^i, t\}$, we can express*

$$\begin{aligned} E_i &= e_i^k \left(\frac{\partial}{\partial x^k} - e_j^l \Gamma_{kl}^m \frac{\partial}{\partial e_j^m} \right) \\ V_{ij} &= e_j^k \frac{\partial}{\partial e_i^k} - e_i^k \frac{\partial}{\partial e_j^k} \\ \partial_t^* &= \partial_t - \tfrac{1}{2} \left(\widetilde{\partial_t g_{ik}} - \widetilde{\partial_t b_{ik}} \right) e_k^j \frac{\partial}{\partial e_i^j}. \end{aligned}$$

Proof. The first two items are already shown in [6]. To show the final item we first fix a frame $u_0 \in \mathcal{F}$, set $\pi(u_0) = (x_0, t_0)$, then define a curve $\gamma(t) = (x_0, t_0 + t)$, and let u_t denote the horizontal lift of γ_t. Recall that by construction, the vector field $u_t e_i$ is parallel, thus we can compute using the definition of the spacetime connection,

$$\begin{aligned} 0 = \nabla_{\frac{\partial}{\partial t}} (u_t e_i) &= \nabla_{\frac{\partial}{\partial t}} \left(e_i^j \frac{\partial}{\partial x^j} \right) \\ &= \frac{d}{dt} e_i^j \frac{\partial}{\partial x^j} + e_i^j \nabla_{\frac{\partial}{\partial t}} \frac{\partial}{\partial x^j} \\ &= \left(\frac{d}{dt} e_i^j + \tfrac{1}{2} \left(\widetilde{\partial_t g_{ik}} - \widetilde{\partial_t b_{ik}} \right) e_k^j \right) \frac{\partial}{\partial x^j}. \end{aligned}$$

It follows that

$$\partial_t^*|_{u_0} = \left. \frac{d}{dt} \right|_{t=0} \left(x_0, t_0 + t, e_i^j(t) \right)$$

$$
\begin{aligned}
&= \partial_t + \frac{d}{dt}\bigg|_{t=0} e_i^j(t)\frac{\partial}{\partial e_i^j} \\
&= \partial_t - \tfrac{1}{2}\left(\widetilde{\partial_t g}_{ik} - \widetilde{\partial_t b}_{ik}\right) e_k^j \frac{\partial}{\partial e_i^j},
\end{aligned}
$$

as claimed. □

Proposition 3.6. *With the setup above, for a contravariant tensor T one has*

$$
\widetilde{\nabla_X T} = X^* \widetilde{T}, \qquad \widetilde{\nabla_t T} = \partial_t^* \widetilde{T}
$$

$$
\nabla^2 T(ue_i, ue_j) = E_i E_j \widetilde{T}, \qquad \widetilde{\Delta T} = \sum_{i=1}^n E_i E_i \widetilde{T}.
$$

Proof. We prove these identities for $T = \alpha \in T^*$, as the general case is analogous. Fix a vector field X, local coordinates $\{x^i, t\}$, and express $X = X^i \frac{\partial}{\partial x^i}$. To simplify notation we let Γ denote the Christoffel symbols of ∇. We first express

$$
(\nabla_X \alpha)_i = X^j \left(\nabla_{\frac{\partial}{\partial x^j}} \alpha\right)_i = X^j \left(\frac{\partial \alpha_i}{\partial x^j} - \Gamma_{ji}^k \alpha_k\right).
$$

On the other hand we can express

$$
\left(\frac{\partial}{\partial x^i}\right)^* = \frac{\partial}{\partial x^i} - \Gamma_{il}^k e_j^l \frac{\partial}{\partial e_j^k}
$$

It follows that

$$
\begin{aligned}
X^* \widetilde{\alpha}_i &= X^m \left(\frac{\partial}{\partial x^m} - \Gamma_{ml}^k e_j^l \frac{\partial}{\partial e_j^k}\right)\left(\alpha_k e_i^k\right) \\
&= X^m \left(\frac{\partial \alpha_k}{\partial x^m} e_i^k - \Gamma_{ml}^k \alpha_k e_i^l\right) \\
&= X^m \left(\frac{\partial \alpha_k}{\partial x^m} - \Gamma_{mk}^l \alpha_l\right) e_i^k \\
&= \left(\widetilde{\nabla_X \alpha}\right)_i,
\end{aligned}
$$

as claimed. For the third claim we first compute

$$
\begin{aligned}
E_i E_j \widetilde{\alpha}_k &= \left(e_i^s\left(\frac{\partial}{\partial x^s} - e_a^t \Gamma_{st}^m \frac{\partial}{\partial e_a^m}\right)\right)\left(e_j^p\left(\frac{\partial}{\partial x^p} - e_b^d \Gamma_{pd}^q \frac{\partial}{\partial e_b^q}\right)\right)\left(e_k^r \alpha_r\right) \\
&= \left(e_i^s\left(\frac{\partial}{\partial x^s} - e_a^t \Gamma_{st}^m \frac{\partial}{\partial e_a^m}\right)\right)\left(e_j^p\left(e_k^r \frac{\partial \alpha_r}{\partial x^p} - e_k^d \Gamma_{pd}^r \alpha_r\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&= e_i^s \left(e_j^p \left(e_k^r \frac{\partial^2 \alpha_r}{\partial x^p \partial x^s} - e_k^d \frac{\partial \Gamma^r_{pd}}{\partial x^s} \alpha_r - e_k^d \Gamma^r_{pd} \frac{\partial \alpha_r}{\partial x^s} \right) \right) \\
&\quad - e_i^s e_j^t \Gamma^p_{st} \left(e_k^r \frac{\partial \alpha_r}{\partial x^p} - e_k^l \Gamma^r_{pl} \alpha_r \right) - e_i^s e_k^t \Gamma^r_{st} e_j^p \frac{\partial \alpha_r}{\partial x^p} + e_i^s e_k^t \Gamma^d_{st} e_j^p \Gamma^r_{pd} \alpha_r
\end{aligned}
$$

On the other hand we have

$$
\begin{aligned}
\left(\nabla^2 \alpha\right)_{ijk} &= \partial_i (\nabla \alpha)_{jk} - \Gamma^l_{ij} (\nabla \alpha)_{lk} - \Gamma^l_{ik} (\nabla \alpha)_{jl} \\
&= \partial_i \left(\partial_j \alpha_k - \Gamma^p_{jk} \alpha_p \right) - \Gamma^l_{ij} \left(\partial_l \alpha_k - \Gamma^p_{lk} \alpha_p \right) - \Gamma^l_{ik} \left(\partial_j \alpha_l - \Gamma^p_{jl} \alpha_p \right) \\
&= \alpha_{k,ij} - \Gamma^p_{jk,i} \alpha_p - \Gamma^p_{jk} \alpha_{p,i} - \Gamma^l_{ij} \alpha_{k,l} + \Gamma^l_{ij} \Gamma^p_{lk} \alpha_p - \Gamma^l_{ik} \alpha_{l,j} + \Gamma^l_{ik} \Gamma^p_{jl} \alpha_p.
\end{aligned}
$$

Comparing these two formulas gives the result. The final claim follows by tracing the third formula over an orthonormal basis. □

4. A Bochner formula on path space

4.1. Brownian motion and Ito's Lemma

Let $(M^n, g_t, b_t)_{t \in [0,T]}$ be a time-dependent family with spacetime connection as above. In what follows we give the Eels-Elworthy-Malliavin construction of Brownian motion adapted to our setting. This is a further generalization of the construction of [1,7]. Let $(x, T') \in \mathcal{M}$. It will be convenient to work with the backward time $\tau := T' - t$ and the convention that $\partial^*_\tau = -\partial^*_t$. Let us start with a smooth curve $\gamma_\tau = (x_\tau, T' - \tau)$ in $\mathcal{M}$ with $x_0 = x$ and denote by u_τ its horizontal lift. The anti-development $(w_\tau)_\tau \subset \mathbb{R}^n$ is the given as the solution of the ordinary differential equation

$$
\dot{u}_\tau = \partial^*_\tau + E_i(u_\tau) \dot{w}^i_\tau, \qquad w_0 = 0,
$$

which exists along γ by general theory. This equivalent formulation of parallel transport motivates the following stochastic differential equation

$$
\begin{aligned}
dU_\tau &= \partial^*_\tau \, d\tau + E_i(U_\tau) \circ dW^i_\tau, \\
U_0 &= u.
\end{aligned}
\tag{4.1}
$$

Here, (W_τ) is a Brownian motion on $\mathbb{R}^n$ and u is an initial frame at (x, T'). We use the convention that (W_τ) has $\Delta^{\mathbb{R}^n}$ as generator instead of $\frac{1}{2}\Delta^{\mathbb{R}^n}$, i.e. the covariation satisfies $dW^i_\tau dW^j_\tau = 2\delta_{ij} d\tau$ and $\circ$ refers to the Stratonovich integration. In this section we establish existence and uniqueness of (4.1) as well as a version of Ito's lemma.

Proposition 4.1. *The stochastic differential equation* (4.1) *has a unique continuous solution* $(U_\tau)_{\tau \in [0,T']}$ *and satisfies* $\pi_2(U_\tau) = T' - \tau$. *Furthermore, given any* C^2 *function* $\widetilde{f} : \mathcal{F} \to \mathbb{R}$ *we have*

$$d\widetilde{f}(U_\tau) = E_i\widetilde{f}(U_\tau)\,dW_\tau^i + \partial_\tau^*\widetilde{f}(U_\tau)\,d\tau + E_iE_i\widetilde{f}(U_\tau)\,d\tau. \tag{4.2}$$

Proof. We adapt the corresponding argument from [7]. First, we may embed the manifold $\mathcal{F}$ into $\mathbb{R}^N$ for some N. Then U_τ satisfies (4.1) if and only if the coordinate functions U_τ^a satisfy

$$dU_\tau^a = (\partial_\tau^*)^a\,d\tau + E_i^a(U_\tau)\circ dW_\tau^i$$

for all $a = 1,\dots,N$, see [10, Prop 1.2.7]. Since each vector field E_i is smooth and bounded since each time slice has bounded geometry, it follows from the standard theory for SDEs on Euclidean space that there is a unique solution on $[0,T']$, cf. [10, Theorem 1.1.8] and that this solution actually stays in $\mathcal{F}$, see [10, Theorem 1.2.9].

In order to show (4.2) we convert the Stratonovich integral in (4.1) into an Ito integral by dropping the $\circ$ and adding one half times the covariation of $E_i(U_\tau)$ and W_τ:

$$dU_\tau^a = (\partial_\tau^*)^a\,d\tau + E_i^a(U_\tau)\,dW_\tau^i + \frac{1}{2}\,dE_i^a(U_\tau)\,dW_\tau^i.$$

For the covariation term we compute, using Ito's lemma in Euclidean space,

$$dE_i^a(U_\tau)\,dW_\tau^i = \frac{\partial}{\partial x^b}E_i^a(U_\tau)\,dU_\tau^b\,dW_\tau^i = 2\frac{\partial}{\partial x^b}E_i^a(U_\tau)E_i^b(U_\tau)\,d\tau.$$

Here, we also used the fact that the covariation of a continuous process and a process of finite variation vanishes. Now, let $\widetilde{f}\colon \mathcal{F}\to\mathbb{R}$ be a C^2 function. Then, by Ito's lemma in Euclidean space,

$$\begin{aligned} d\widetilde{f}(U_\tau) &= \frac{\partial}{\partial x^a}\widetilde{f}(U_\tau)\,dU_\tau^a + \frac{1}{2}\frac{\partial^2}{\partial x^a\partial x^b}\widetilde{f}(U_\tau)\,dU_\tau^a\,dU_\tau^b \\ &= \frac{\partial}{\partial x^a}\widetilde{f}(U_\tau)(\partial_\tau^*)^a\,d\tau + \frac{\partial}{\partial x^a}\widetilde{f}(U_\tau)E_i^a(U_\tau)\,dW_\tau^i \\ &\quad + \frac{\partial}{\partial x^a}\widetilde{f}(U_\tau)\frac{\partial}{\partial x^b}E_i^a(U_\tau)E_i^b(U_\tau)\,d\tau + \frac{\partial^2}{\partial x^a\partial x^b}\widetilde{f}(U_\tau)E_i^a(U_\tau)E_i^b(U_\tau)\,d\tau. \end{aligned}$$

Finally, since

$$\begin{aligned} &\frac{\partial}{\partial x^a}\widetilde{f}(U_\tau)(\partial_\tau^*)^a = \partial_\tau^*\widetilde{f}(U_\tau) \\ &\frac{\partial}{\partial x^a}\widetilde{f}(U_\tau)E_i^a(U_\tau) = E_i(U_\tau)\widetilde{f}(U_\tau) \\ &\frac{\partial}{\partial x^a}\widetilde{f}(U_\tau)\frac{\partial}{\partial x^b}E_i^a(U_\tau)E_i^b(U_\tau) + \frac{\partial^2}{\partial x^a\partial x^b}\widetilde{f}(U_\tau)E_i^a(U_\tau)E_i^b(U_\tau) = E_i(U_\tau)E_i(U_\tau)\widetilde{f}(U_\tau), \end{aligned}$$

we obtain (4.2):

$$d\widetilde{f}(U_\tau) = \partial_\tau^*\widetilde{f}(U_\tau)\,d\tau + E_i(U_\tau)\widetilde{f}(U_\tau)\,dW_\tau^i + E_i(U_\tau)E_i(U_\tau)\widetilde{f}(U_\tau)\,d\tau.$$

Lastly, note that with the choice $\widetilde{f} = \pi_2$ we get $d\widetilde{f}(U_\tau) = -\,d\tau$. Furthermore setting $\pi_2(U_0) = T'$ we get $\pi_2(U_\tau) = T' - \tau$. □

Let $P_0\mathbb{R}^n$ denote the Euclidean path space based at the origin, i.e. the space of all continuous curves $\{w_\tau | w_0 = 0\}_{\tau\in[0,T']} \subset \mathbb{R}^n$. We denote by Γ_0 the Wiener measure on $P_0\mathbb{R}^n$. The path space has a canonical filtration $\Sigma_\tau^{\mathbb{R}^n}$ generated by the evaluation maps $\{e_\sigma\colon P_0\mathbb{R}^n \to \mathbb{R}^n | e_\sigma(w) = w_\sigma, \sigma \leqslant \tau\}$. With the help of (4.1) we can transfer the notion of Wiener measure to the path space over $\mathcal{F}$ and $\mathcal{M}$.

Definition 4.2. Let $P_u\mathcal{F}$ and $P_{(x,T')}\mathcal{M}$ be the space of continuous curves, $\{u_\tau | u_0 = u, \pi_2(u_\tau) = T' - \tau\}_{\tau\in[0,T']} \subset \mathcal{F}$ and $\{\gamma_\tau = (x_\tau, T' - \tau) | \gamma_0 = (x, T')\}_{\tau\in[0,T']}$ respectively.

It will be convenient from time to time to work with the *total path space* $P_{T'}\mathcal{M} = \bigcup_{x\in M} P_{(x,T')}\mathcal{M}$.

Definition 4.3. Let $U\colon P_0\mathbb{R}^n \to P_u\mathcal{F}$ solve (4.1) and let $\Pi\colon P_u\mathcal{F} \to P_{(x,T')}\mathcal{M}$ defined by $\Pi(U)_\tau = \pi(U_\tau)$.

(1) We call $\Gamma_u := U_*(\Gamma_0)$ and $\Gamma_{(x,T')} := \Pi_*\Gamma_u$ the *Wiener measures* of horizontal Brownian motion on $\mathcal{F}$ and Brownian motion on space-time $\mathcal{M}$ respectively.
(2) The filtrations on $P_u\mathcal{F}$ and $P_{(x,T')}\mathcal{M}$ are given by $\Sigma_\tau^{\mathcal{M}} := (\Pi \circ U)_*\Sigma_\tau^{\mathbb{R}^n}$ and $\Sigma_\tau^{\mathcal{F}} := U_*\Sigma_\tau^{\mathbb{R}^n}$.
(3) We call $\pi(U_\tau) = (X_\tau, T' - \tau)$ *Brownian motion* on $\mathcal{M}$ based at $\pi(u) = (x, T')$.
(4) We call the family of isometries $\{S_\tau := U_0U_\tau^{-1}\colon (T_{X_\tau}M, g_{T'-\tau}) \to (T_xM, g_{T'})\}$ *stochastic parallel transport* along the Brownian curve X_τ.

Proposition 4.4. *Let w on $\mathcal{M}$ be a solution to the heat equation*

$$\Box w = 0, \quad w|_s = f,$$

where $f \in C^\infty(M)$ and $s \in [0, T']$. Then

$$w(x,T) = \mathbb{E}_{(x,T')}[f(X_{T'-s})].$$

Proof. We consider the lift $\widetilde{w}(U_\tau)$ and obtain by (4.2)

$$d\widetilde{w}(U_\tau) = E_i\widetilde{w}(U_\tau)\, dW_\tau^i + \partial_\tau^*\widetilde{w}(U_\tau)\, d\tau + E_iE_i\widetilde{w}(U_\tau)\, d\tau.$$

Since w solves the heat equation, by virtue of Lemma 3.4 and Proposition 3.6 the last two terms vanish. Integrating on $(0, T' - s)$ we get

$$\widetilde{w}(U_{T'-s}) - \widetilde{w}(U_0) = \int_0^{T'-s} E_i\widetilde{w}(U_\tau)\, dW_\tau^i. \tag{4.3}$$

Taking expectations, and since the Ito integral of an adapted process is a martingale, we have

$$\mathbb{E}_{(x,T')}[f(X_{T'-s})] - w(x,T') = \mathbb{E}[\widetilde{w}(U_{T'-s}) - \widetilde{w}(U_0)] = 0.$$

Here we used that $\widetilde{w}(U_0) = w(x,T')$ and $\widetilde{w}(U_{T'-s}) = w(X_{T'-s}, s) = f(X_{T'-s})$. □

A further corollary is that the Wiener measure can be characterized by the heat kernels.

Corollary 4.5. *Let $0 \leqslant \tau_1 < \tau_2 < \ldots < \tau_k \leqslant T'$ be a partition and $A_1, \ldots, A_k \subset M^n$ Borel sets. Then it holds*

$$\begin{aligned}&\mathbb{P}[X_{\tau_j} \in A_j, j = 1, \ldots, k] \\ &= \int\limits_{A_k} \cdots \int\limits_{A_1} p_{T',T'-\tau_1}(x, y_1) \cdots p_{T'-\tau_{k-1},T'-\tau_k}(y_{k-1}, y_k)\, dV_{g(T'-\tau_1)}(y_1) \cdots dV_{g(T'-\tau_k)}(y_k).\end{aligned}$$

4.2. Feynman-Kac formula

Proposition 4.6. *Let $s \in [0, T']$, $A_t \in \mathrm{End}(TM)$ and Y a vector valued solution of the heat equation with potential, $\nabla_t Y = \Delta_{g_t} Y + A_t Y$, with $Y|_s = Z \in C_0^\infty(TM)$, then*

$$Y(x,T') = \mathbb{E}_{(x,T')}[R_{T'-s} S_{T'-s} Z(X_{T'-s})], \tag{4.4}$$

where $R_\tau = R_\tau(\gamma)\colon T_xM \to T_xM$ is the solution of the ODE $\frac{d}{d\tau} R_\tau = R_\tau S_\tau A_{T'-\tau} S_\tau^{-1}$ with $R_0 = \mathrm{id}$.

Proof. Let $\widetilde{Y}\colon \mathcal{F} \to \mathbb{R}^n$, $\widetilde{Y}(U) = u^{-1} Y_{\pi u}$. Applying the Ito formula (4.2), we obtain

$$\begin{aligned}d\widetilde{Y}(U_\tau) &= E_i \widetilde{Y}(U_\tau)\, dW_\tau^i + \partial_\tau^* \widetilde{Y}(U_\tau)\, d\tau + E_i E_i \widetilde{Y}(U_\tau)\, d\tau \\ &= E_i \widetilde{Y}(U_\tau)\, dW_\tau^i - A_{T'-\tau} \widetilde{Y}(U_\tau)\, d\tau,\end{aligned}$$

where we used Lemma 3.4 and Proposition 3.6. Let $\widetilde{R}_\tau\colon \mathbb{R}^n \to \mathbb{R}^n$ be the solution of the ODE $\frac{d}{d\tau} \widetilde{R}_\tau \widetilde{A}_{T'-\tau}$ with $R_0 = \mathrm{id}$. Then

$$d(\widetilde{R}_\tau \widetilde{Y}(U_\tau)) = \widetilde{R}_\tau H_i \widetilde{Y}(U_\tau)\, dW_\tau^i. \tag{4.5}$$

Integrating on $[0, T'-s]$ and taking expectations, we obtain

$$\widetilde{Y}(u) = \mathbb{E}_u[\widetilde{R}_{T'-s} \widetilde{Y}(U_{T'-s})].$$

Finally, we compute

$$
\begin{aligned}
Y(x,T') = u\widetilde{Y}(u) &= \mathbb{E}_u[U_0\widetilde{R}_{T'-s}U_0^{-1}U_0U_{T'-s}^{-1}U_{T'-s}\widetilde{Y}(U_{T'-s})] \\
&= \mathbb{E}_u[\widetilde{R}_{T'-s}\widetilde{S}_{T'-s}Z(X_{T'-s})],
\end{aligned}
$$

since $U_{T'-s}\widetilde{Y}(U_{T'-s}) = Y(X_{T'-s}, s) = Z(X_{T'-s})$ and $R_\tau = U_0\widetilde{R}_\tau U_0^{-1}$. Indeed, the last equality holds since

$$
\begin{aligned}
\frac{d}{d\tau}(U_0\widetilde{R}_\tau U_0^{-1}) = U_0\widetilde{R}_\tau\widetilde{A}_{T'-\tau} &= U_0\widetilde{R}_\tau U_0^{-1}U_0U_\tau^{-1}\widetilde{A}_{T'-\tau}U_\tau^{-1}U_\tau U_0^{-1} \\
&= U_0\widetilde{R}_\tau U_0^{-1}S_\tau A_{T'-\tau}S_\tau^{-1},
\end{aligned}
$$

which shows that R_τ and $U_0\widetilde{R}_\tau U_0^{-1}$ solve the same ODE, and thus must be equal. □

4.3. Induced martingales and parallel gradients

Definition 4.7. Let $F\colon P_{T'}\mathcal{M} \to \mathbb{R}$ be integrable. Then, we define the induced martingale as

$$F_\tau(\gamma) := \mathbb{E}_{(x,T')}[F|\Sigma_\tau](\gamma).$$

Note that then F_τ satisfies the martingale property, i.e. for all $\sigma \leqslant \tau$

$$\mathbb{E}_{(x,T')}[F_\tau|\Sigma_\sigma] = F_\sigma,$$

by the definition of conditional expectation and that $\Sigma_\sigma \subset \Sigma_\tau$.

The next results concern the induced martingale of an integrable function F. Note that integrability is not a big restriction, since each uniformly integrable martingale can be represented as the induced martingale of an integrable function. Explicitly, by standard results the induced martingale satisfies the following:

Proposition 4.8. *Let $F\colon P_{T'}\mathcal{M} \to \mathbb{R}$ be integrable. Then, for almost every Brownian curve $\{\gamma_\tau\}_{\tau\in[0,T']}$ we have for the induced martingale*

$$F_\tau(\gamma) := \mathbb{E}_{(x,T')}[F|\Sigma_\tau](\gamma) = \int\limits_{P_{\gamma_\tau}\mathcal{M}} F(\gamma|_{[0,\tau]} * \gamma')\, d\Gamma_{\gamma_\tau}(\gamma'),$$

where we integrate over all γ' in the based path space $P_{\gamma_\tau}\mathcal{M}$ and $*$ denotes the concatenation of the two curves $\gamma|_{[0,T']}$ and γ'.

The analysis to follows exploits a nice set of nice test functions on path space, namely cylinder functions:

Definition 4.9. Given $\boldsymbol{\tau} = \{\tau_j\}_{j=1}^k$ be a partition of $[0, T']$ we define evaluation maps

$$e_{\boldsymbol{\tau}} \colon P_{T'}\mathcal{M} \to M^k, \qquad e_{\boldsymbol{\tau}}(\gamma) = (\pi_1\gamma_{\tau_1}, \pi_1\gamma_{\tau_2}, \ldots, \pi_1\gamma_{\tau_k}).$$

Given a partition τ and a smooth compactly supported function $f \colon M^k \to \mathbb{R}$ we obtain a *cylinder function*

$$F \colon P_{T'}\mathcal{M} \to \mathbb{R}, \qquad F(\gamma) = f(e_{\boldsymbol{\tau}}(\gamma)).$$

The space of all cylinder functions is denoted $\mathfrak{C}_{T'}$.

Definition 4.10. Let $F \in \mathfrak{C}_{T'}$ and fix $\gamma \in P_{T'}\mathcal{M}$. Given V a vector field along γ we let $\xi^\varepsilon = (x_\tau^\varepsilon, T' - \tau)_{\tau \in [0,T']}$ denote a one-parameter family of curves such that $\xi^0 = \gamma$ and $\frac{\partial}{\partial \varepsilon}\big|_{\varepsilon=0} x_\tau^\varepsilon = V_\tau$. Then

$$D_V F := \frac{\partial}{\partial \varepsilon}\bigg|_{\varepsilon=0} f(e_{\boldsymbol{\tau}}(\xi^\varepsilon)).$$

In our setting we will only use a special class of vector fields V. In particular, let $\mathcal{H}$ denote the Hilbert space of H^1-curves $(h_\tau)_{\tau \geqslant 0}$ in $(T_xM, g_{T'})$ with $h_0 = 0$ equipped with the inner product

$$\langle h_1, h_2 \rangle_{\mathcal{H}} = \int_0^{T'} \left\langle \dot{h}_1, \dot{h}_2 \right\rangle_{(T_xM, g_{T'})} d\tau.$$

Given $(h_\tau)_{\tau \geqslant 0} \in \mathcal{H}$ we let $V_\tau(\gamma) = S_\tau^{-1}(\gamma) h_\tau$.

This derivative operator admits a key integration by parts formula, cf. [3], [7, Theorem A.1]. In the statement below, for $(h_\tau)_{\tau \geqslant 0} \in \mathcal{H}$ we set

$$\langle h_\tau, dW_\tau \rangle = (U_0^{-1} h_\tau)_i \, dW_\tau^i,$$

noting that this inner product is independent of the initial choice of frame U_0. The theorem is proved in an appendix (§6).

Theorem 4.11. *Let $F, G \in \mathfrak{C}_{T'}$, let $(h_\tau)_{\tau \geqslant 0} \in \mathcal{H}$ and write $V = (S_\tau^{-1} h_\tau)_{\tau \geqslant 0}$. Then*

$$D_V^* G := -D_V G + \frac{1}{2} G \int_0^{T'} \left\langle \frac{d}{d\tau} h_\tau - S_\tau (\mathrm{Rc}^\nabla + \frac{1}{2}\partial_t (g - b))_{T-\tau}^\dagger S_\tau^{-1} h_\tau, \, dW_\tau \right\rangle \quad (4.6)$$

satisfies $\mathbb{E}_{(x,T')}[D_V F G] = \mathbb{E}_{(x,T')}[F D_V^* G]$.

Definition 4.12. Let $\sigma \in [0, T']$ and let $F \in \mathfrak{C}_{T'}$. The *parallel gradient* $\nabla^{\perp}_{\sigma} F \colon P_{(x,T')}\mathcal{M} \to (T_x M, g_{T'})$ is defined as

$$D_{V^\sigma} F(\gamma) = \left\langle \nabla^{\perp}_{\sigma} F(\gamma), v \right\rangle_{(T_x M, g_{T'})},$$

where $V^{\sigma}_{\tau} = S^{-1}_{\tau} v \mathbb{1}_{[\sigma,T']}(\tau)$ and $v \in (T_x M, g_{T'})$. Explicitly, if we have the representation $F = f \circ e_{\boldsymbol{\tau}}$, $\boldsymbol{\tau} = \{\tau_j\}_{j=1}^k$, it follows that

$$\nabla^{\perp}_{\sigma} F(\gamma) = \sum_{\tau_j \geqslant \sigma} S_{\tau_j} \operatorname{grad}^{(j)}_{g_{T'-\tau_j}} f(\pi_1 \gamma_{\tau_1}, \ldots, \pi_1 \gamma_{\tau_k}). \tag{4.7}$$

Definition 4.13. Given $F \in \mathfrak{C}_{T'}$, its *Malliavin derivative* $\nabla^{\mathcal{H}} F \colon P_{(x,T')}\mathcal{M} \to \mathcal{H}$ is defined as

$$D_V F(\gamma) = \left\langle \nabla^{\mathcal{H}} F, h \right\rangle_{\mathcal{H}},$$

for every $h \in \mathcal{H}$ and $V = (S^{-1}_{\tau} h_{\tau})_{\tau \geqslant 0}$. It follows that the parallel gradient is the time derivative of the Malliavin gradient $\frac{d}{d\tau}(\nabla^{\mathcal{H}} F)_{\tau} = \nabla^{\perp}_{\tau} F$ and furthermore

$$|\nabla^{\mathcal{H}} F|^2_{\mathcal{H}} = \int_0^{T'} |\nabla^{\perp}_{\tau} F|^2 \, d\tau.$$

Definition 4.14. Given the setup above and $0 \leqslant \tau_1 \leqslant \tau_2 \leqslant T'$, we define the *Ornstein-Uhlenbeck operator*

$$\mathcal{L}_{(\tau_1,\tau_2)} := \int_{\tau_1}^{\tau_2} {\nabla^{\perp}_{\tau}}^* \nabla^{\perp}_{\tau} \, d\tau.$$

Remark 4.15. Our discussion above and proofs below work exclusively with cylinder functions. Due to the integration by parts formula (4.6) the Malliavin gradient is closable and can be extended to a closed unbounded operator from $L^2(P_{(x,T')}\mathcal{M})$ to $L^2(P_{(x,T')}\mathcal{M}, \mathcal{H})$ with $\mathfrak{C}_{T'}$ being a dense subset of the domain (cf. [10] Section 8). The definitions of all derivative operators considered here can be similarly extended.

4.4. Martingale representation theorem

Proposition 4.16. *Let $F \in \mathfrak{C}_{T'}$ and let F_τ be the induced martingale. Then F_τ solves*

$$dF_\tau = \left\langle \nabla^{\perp}_{\tau} F_\tau, dW_\tau \right\rangle, \qquad F|_{\tau=0} = F_0.$$

Remark 4.17. This result shows that martingales are the natural generalization of the (backward) heat-flow to the path space $P\mathcal{M}$. Indeed, let $F_\tau(\gamma) = f_\tau(\pi_1\gamma_\tau)$ for some smooth function $f\colon \mathcal{M} \to \mathbb{R}$. Then Proposition 4.16 together with (4.2) yield

$$0 = df_\tau(\pi_1\gamma_\tau) - \langle \nabla f_\tau(\pi_1\gamma_\tau), dW_\tau\rangle = (\partial_\tau + \Delta_{g_{T-\tau}})f_\tau(\pi_1\gamma_\tau),$$

which means that f_τ solves the backward heat equation.

Proof. Let $F(\gamma) = f(\pi_1\gamma_{\tau_1}, \dots, \pi_1\gamma_{\tau_k})$, where $f\colon M^k \to \mathbb{R}$ is a smooth compactly supported function. Then, for $\tau \in (\tau_l, \tau_{l+1})$, by Corollary 4.5 and Proposition 4.8 we have

$$\begin{aligned}
F_\tau(\gamma) &= \int\limits_{P_{\gamma_\tau}\mathcal{M}} F(\gamma|_{[0,\tau]} * \gamma')\, d\Gamma_{\gamma_\tau}(\gamma') \\
&= \int\limits_{P_{\gamma_\tau}\mathcal{M}} f(\pi_1\gamma_{\tau_1}, \dots, \pi_1\gamma_{\tau_l}, \pi_1\gamma_{\tau_{l+1}-\tau}, \dots, \pi_1\gamma_{\tau_k-\tau})\, d\Gamma_{\gamma_\tau}(\gamma') \\
&= \int\limits_{M^{k-l}} f(X_{\tau_1}, \dots, X_{\tau_l}, y_{l+1}, \dots, y_k) p_{T'-\tau, T'-\tau_{l+1}}(X_\tau, y_{l+1}) \dots \\
&\qquad p_{T'-\tau_{k-1}, T'-\tau_k}(y_{k-1}, y_k)\, dV_{g_{T'-\tau_{l+1}}}(y_{l+1}) \dots dV_{g_{T'-\tau_k}}(y_k) \\
&=: f_\tau(X_{\tau_1}, \dots, X_{\tau_l}, X_\tau).
\end{aligned}$$

Note that for $(x_1, \dots, x_l)$ fixed, $(x, \tau) \mapsto f_\tau(x_1, \dots, x_l, x)$ solves $(\partial_\tau + \Delta^{(l+1)})f_\tau = 0$, where $\Delta^{(l+1)}$ acts on the last entry.

Let $\widetilde{f}_\tau = f_\tau \circ \otimes_1^{l+1}\pi_1 \circ \otimes_1^{l+1}\pi$ and $\widetilde{F}_\tau = F_\tau \circ \Pi$. Then $\widetilde{F}_\tau(U) = \widetilde{f}_\tau(U_{\tau_1}, \dots, U_{\tau_l}, U_\tau)$. According to (4.2) we have then

$$\begin{aligned}
d\widetilde{F}_\tau(U) &= d\widetilde{f}_\tau(U_{\tau_1}, \dots, U_{\tau_l}, U_\tau) \\
&= (\partial_\tau^*\widetilde{f}_\tau(U_{\tau_1}, \dots, U_{\tau_l}, U_\tau) + E_i^{(l+1)}E_i^{(l+1)}\widetilde{f}_\tau(U_{\tau_1}, \dots, U_{\tau_l}, U_\tau)\, d\tau \\
&\qquad + \left\langle E_i^{(l+1)}\widetilde{f}_\tau(U_{\tau_1}, \dots, U_{\tau_l}, U_\tau),\, dW_\tau^i\right\rangle.
\end{aligned}$$

Note that due to Proposition 3.6 we have $\partial_\tau^*\widetilde{f}_\tau + E_i^{(l+1)}E_i^{(l+1)}\widetilde{f}_\tau = 0$. Next, we compute

$$\begin{aligned}
E_i^{(l+1)}\widetilde{f}_\tau(U_{\tau_1}, \dots, U_{\tau_l}, U_\tau) &= (U_\tau e_i)^*\widetilde{f}_\tau(U_{\tau_1}, \dots, U_{\tau_l}, U_\tau) \\
&= (U_\tau e_i) f_\tau(X_{\tau_1}, \dots, X_{\tau_l}, X_\tau) \\
&= \left\langle U_\tau e_i, \operatorname{grad}^{(l+1)}_{g_{T'-\tau}} f_\tau(X_{\tau_1}, \dots, X_{\tau_l}, X_\tau)\right\rangle_{(T_{X_\tau}M, g_{T'-\tau})} \\
&= \left\langle S_\tau U_\tau e_i, S_\tau \operatorname{grad}^{(l+1)}_{g_{T'-\tau}} f_\tau(X_{\tau_1}, \dots, X_{\tau_l}, X_\tau)\right\rangle_{(T_xM, g_{T'})} \\
&= \left\langle U_0 e_i, \nabla_\tau^\perp F_\tau(\gamma)\right\rangle_{(T_xM, g_{T'})},
\end{aligned}$$

where we used Lemma 3.4 in the second line and (4.7) in the last line. All in all we find

$$dF_\tau(\gamma) = d\widetilde{F}_\tau(U) = \left\langle U_0 e_i, \nabla_\tau^\perp F_\tau(\gamma)\right\rangle_{(T_x M, g_{T'})} dW_\tau^i = \left\langle \nabla_\tau^\perp F_\tau(\gamma), dW_\tau \right\rangle,$$

which was the claim. □

Corollary 4.18. *Let $F \in \mathfrak{C}_{T'}$. Then the quadratic variation $[F,F]_\tau$ of the induced martingale F_τ satisfies*

$$d[F,F]_\tau = 2|\nabla_\tau F_\tau|^2 \, d\tau.$$

4.5. Evolution of the parallel gradient

In the next result we give is about the evolution of the parallel gradient.

Theorem 4.19. *Let $F \in \mathfrak{C}_{T'}$ and let $\sigma \geqslant 0$ be fixed. Then the parallel gradient of the induced martingale $\nabla_\sigma^\perp F_\tau$ satisfies*

$$\begin{aligned} d(\nabla_\sigma^\perp F_\tau) = {} & \left\langle \nabla_\tau^\perp \nabla_\sigma^\perp F_\tau, dW_\tau \right\rangle + (\mathrm{Rc}^\nabla + \frac{1}{2}\partial_t(g-b))_\tau (\nabla_\tau^\perp F_\tau) \mathbb{1}_{[\sigma,T']}(\tau)\, d\tau \\ & + \nabla_\sigma^\perp F_\sigma \delta_\sigma(\tau)\, d\tau, \end{aligned}$$

where $\left\langle \mathrm{Rc}^\nabla + \frac{1}{2}\partial_t(g-b))_\tau(v), w \right\rangle_{T_x M, g_{T'}} = (\mathrm{Rc}_{g_t}^\nabla + \frac{1}{2}\partial_t(g-b))|_{t=T'-\tau}(S_\tau^{-1}v, S_\tau^{-1}w).$

Proof. Since F_τ is Σ_τ-measurable, i.e. it depends only on times smaller than τ, we have that $\nabla_\sigma^\perp F_\tau = 0$ as soon as $\tau < \sigma$. At $\sigma = \tau$ we have a jump discontinuity, which is expressed in the δ-notation $\nabla_\sigma^\perp F_\sigma \delta_\sigma(\tau)$. For $\tau > \sigma$ we aim to show the evolution

$$d(\nabla_\sigma^\perp F_\tau) = \left\langle \nabla_\tau^\perp \nabla_\sigma^\perp F_\tau, dW_\tau \right\rangle + (\mathrm{Rc}^\nabla + \frac{1}{2}\partial_t(g-b))_\tau (\nabla_\tau^\perp F_\tau)\, d\tau. \tag{4.8}$$

Let $F(\gamma) = f(\pi_1\gamma_{\tau_1}, \ldots \pi_1\gamma_{\tau_k})$ be a cylinder function. Let $\tau \in (\tau_l, \tau_{l+1})$, then $F_\tau(\gamma) = f_\tau(X_{\tau_1}, \ldots, X_{\tau_k}, X_\tau)$ as in the proof of Proposition 4.16 and by virtue of (4.7)

$$\begin{aligned} \nabla_\sigma^\perp F_\tau(\gamma) = {} & \sum_{\tau_j \geqslant \sigma} S_{\tau_j} \operatorname{grad}^{(j)}_{g_{T'-\tau_j}} f_\tau(X_{\tau_1}, \ldots, X_{\tau_k}, X_\tau) \\ & + S_\tau \operatorname{grad}^{(l+1)}_{g_{T'-\tau}} f_\tau(X_{\tau_1}, \ldots, X_{\tau_k}, X_\tau). \end{aligned}$$

Consider $G_i(U) := \left\langle U_0 e_i, \nabla_\sigma^\perp F_\tau(\Pi U)\right\rangle$. Then

$$\begin{aligned} G_i(U) := {} & \left\langle U_0 e_i, \nabla_\sigma^\perp F_\tau(\Pi U)\right\rangle_{T_x M, g_{T'}} \\ = {} & \sum_{\tau_j \geqslant \sigma} \left\langle U_{\tau_j} e_i, \operatorname{grad}^{(j)}_{g_{T'-\tau_j}} f_\tau(X_{\tau_1}, \ldots, X_{\tau_k}, X_\tau)\right\rangle_{T_{X_{\tau_j}} M, g_{T'-\tau_j}} \end{aligned}$$

$$+ \left\langle U_\tau e_i, \operatorname{grad}^{(l+1)}_{g_{T'-\tau}} f_\tau(X_{\tau_1}, \ldots, X_{\tau_k}, X_\tau) \right\rangle_{T_{X_\tau}M, g_{T'-\tau}}$$

$$= \sum_{\tau_j \geqslant \sigma} E_i^{(j)} \widetilde{f}_\tau(U_{\tau_1}, \ldots, U_{\tau_l}, U_\tau) + E_i^{(l+1)} \widetilde{f}_\tau(U_{\tau_1}, \ldots, U_{\tau_l}, U_\tau),$$

where we used Lemma 3.4 in the third line. Then with (4.2) we find

$$\begin{aligned}
dG_i(U) = &\sum_{\tau_j \geqslant \sigma} \left(\partial_\tau^* E_i^{(j)} \widetilde{f}_\tau(U_\tau) + E_m^{(l+1)} E_m^{(l+1)} E_i^{(j)} \widetilde{f}_\tau(U_{\tau_1}, \ldots, U_{\tau_l}, U_\tau) \right) d\tau \\
&+ E_m^{(l+1)} E_i^{(j)} \widetilde{f}_\tau(U_\tau)\, dW_\tau^m \\
&+ \left(\partial_\tau^* E_i^{(l+1)} \widetilde{f}_\tau(U_\tau) + E_m^{(l+1)} E_m^{(l+1)} E_i^{(l+1)} \widetilde{f}_\tau(U_{\tau_1}, \ldots, U_{\tau_l}, U_\tau) \right) d\tau \\
&+ E_m^{(l+1)} E_i^{(l+1)} \widetilde{f}_\tau(U_\tau)\, dW_\tau^m \\
= &\sum_{\tau_j \geqslant \sigma} E_i^{(j)} (\partial_\tau^* + E_m^{(l+1)} E_m^{(l+1)}) \widetilde{f}_\tau(U_\tau)\, d\tau + E_m^{(l+1)} E_i^{(j)} \widetilde{f}_\tau(U_\tau)\, dW_\tau^m \\
&+ E_i^{(l+1)} (\partial_\tau^* + E_m^{(l+1)} E_m^{(l+1)}) \widetilde{f}_\tau(U_\tau)\, d\tau + E_m^{(l+1)} E_i^{(l+1)} \widetilde{f}_\tau(U_\tau)\, dW_\tau^m \\
&+ [\partial_\tau^* + E_m^{(l+1)} E_m^{(l+1)}, E_i] \widetilde{f}_\tau(U_\tau)\, d\tau.
\end{aligned}$$

Recall that $(\partial_\tau^* + E_m^{(l+1)} E_m^{(l+1)}) \widetilde{f}_\tau = 0$ due to Proposition 3.6. Furthermore, using Propositions 3.3 and 3.6 we deduce

$$[\partial_\tau^* + E_m^{(l+1)} E_m^{(l+1)}, E_i] \widetilde{f}_\tau = (\widetilde{\mathrm{Rc}^\nabla} + \frac{1}{2} \widetilde{\partial_t (g-b)})_{im} E_m^{(l+1)} \widetilde{f}_\tau.$$

All in all this gives us

$$\begin{aligned}
dG_i(U) = &\sum_{\tau_j \geqslant \sigma} E_m^{(l+1)} E_i^{(j)} \widetilde{f}_\tau(U_\tau)\, dW_\tau^m + E_m^{(l+1)} E_i^{(l+1)} \widetilde{f}_\tau(U_\tau)\, dW_\tau^m \\
&+ (\widetilde{\mathrm{Rc}^\nabla} + \frac{1}{2} \widetilde{\partial_t (g-b)})_{im} E_m^{(l+1)} \widetilde{f}_\tau
\end{aligned}$$

Projecting down yields

$$\begin{aligned}
&\sum_{\tau_j \geqslant \sigma} E_m^{(l+1)} E_i^{(j)} \widetilde{f}_\tau(U_\tau)\, dW_\tau^m + E_m^{(l+1)} E_i^{(l+1)} \widetilde{f}_\tau(U_\tau)\, dW_\tau^m \\
&= \left\langle \sum_{\tau_j \geqslant \sigma} (S_\tau \otimes S_{\tau_j}) \nabla^{(l+1)} \nabla^{(j)} f_\tau + (S_\tau \otimes S_\tau) \nabla^{(l+1)} \nabla^{(l+1)} f_\tau, dW_\tau \otimes U_0 e_i \right\rangle \\
&= \left\langle \nabla_\tau^\perp \nabla_\sigma^\perp F_\tau(\gamma), dW_\tau \otimes U_0 e_i \right\rangle,
\end{aligned}$$

and

$$(\widetilde{\mathrm{Rc}^{\nabla}}+\frac{1}{2}\partial_t\widetilde{(g-b)})_{im}E_m^{(l+1)}\widetilde{f}_\tau(U_\tau)\,d\tau=\left\langle(\mathrm{Rc}^{\nabla}+\frac{1}{2}\partial_t(g-b))_\tau(\nabla_\tau^{\perp}F_\tau)\,d\tau,U_0e_i\right\rangle,$$

giving the result. □

Lemma 4.20. *Let $F\in\mathfrak{C}_{T'}$ and $\tau,\sigma\geqslant 0$ fixed. Then*

$$\nabla_\tau^{\perp}|\nabla_\sigma^{\perp}F|^2=2\left\langle\nabla_\tau^{\perp}\nabla_\sigma^{\perp}F,\nabla_\sigma^{\perp}F\right\rangle.$$

Proof. Let $F(\gamma)=f(\pi_1\gamma_{\tau_1},\cdots,\pi_1\gamma_{\tau_k})$. Then, as in the proof of Theorem 4.19

$$\left\langle\nabla_\sigma^{\perp}F(\gamma),U_0e_a\right\rangle=\sum_{\tau_j\geqslant\sigma}E_a^{(j)}\widetilde{f}(U_{\tau_1},\ldots,U_{\tau_k}).$$

Hence

$$\begin{aligned}\left\langle\nabla_\tau^{\perp}|\nabla_\sigma^{\perp}F|^2,U_0e_b\right\rangle&=\sum_{\tau_k\geqslant\tau}E_b^{(k)}\sum_{a=1}^{n}(\sum_{\tau_j\geqslant\sigma}E_a^{(j)}\widetilde{f})^2\\&=2\sum_{a=1}^{n}\sum_{\tau_k\geqslant\tau}\sum_{\tau_j\geqslant\sigma}E_b^{(k)}E_a^{(j)}\widetilde{f}(\sum_{\tau_j\geqslant\sigma}E_a^{(j)}\widetilde{f}).\end{aligned}$$

Projecting down and using Proposition 3.6 yields the claim. □

Corollary 4.21. *Let $F\in\mathfrak{C}_{T'}$ and $\sigma\geqslant 0$ fixed. Then $\nabla_\sigma^{\perp}F_\tau\colon P_{(x,T')}\mathcal{M}\to(T_xM,g_{T'})$ satisfies*

(1) the quadratic Bochner identity

$$\begin{aligned}d(|\nabla_\sigma^{\perp}F_\tau|^2)=&\left\langle\nabla_\tau^{\perp}|\nabla_\sigma^{\perp}F_\tau|^2,dW_\tau\right\rangle+2(\mathrm{Rc}^{\nabla}+\frac{1}{2}\partial(g-b))_\tau(\nabla_\tau^{\perp}F_\tau,\nabla_\sigma^{\perp}F_\tau)\,d\tau\\&+2|\nabla_\tau^{\perp}\nabla_\sigma^{\perp}F_\tau|^2\,d\tau+|\nabla_\tau^{\perp}F_\tau|^2\delta_\sigma(\tau)\,d\tau,\end{aligned}$$

(2) and the linear Bochner identity

$$\begin{aligned}d|\nabla_\sigma^{\perp}F_\tau|=&\left\langle\nabla_\tau^{\perp}|\nabla_\sigma^{\perp}F_\tau|,dW_\tau\right\rangle+\frac{|\nabla_\tau^{\perp}\nabla_\sigma^{\perp}F_\tau|^2-|\nabla_\tau^{\perp}|\nabla_\sigma^{\perp}F_\tau||^2}{|\nabla_\sigma^{\perp}F_\tau|}\,d\tau\\&+\frac{1}{|\nabla_\sigma^{\perp}F_\tau|}(\mathrm{Rc}^{\nabla}+\frac{1}{2}\partial_t(g-b))_\tau(\nabla_\tau^{\perp}F_\tau,\nabla_\sigma^{\perp}F_\tau)\,d\tau+|\nabla_\tau^{\perp}F_\tau|\delta_\sigma(\tau)\,d\tau.\end{aligned}$$

Here, we denote $(\mathrm{Rc}^{\nabla}+\frac{1}{2}\partial(g-b))_\tau(v,w)=(\mathrm{Rc}_{g_t}^{\nabla}+\frac{1}{2}\partial_t(g-b))|_{t=T'-\tau}(S_\tau^{-1}v,S_\tau^{-1}w)$.

Proof. As in the previous proof, it is enough to consider the case $\sigma<\tau$. By Ito's Lemma and Theorem 4.19 we have

$$\begin{aligned} d(|\nabla^{\perp}_{\sigma} F_\tau|^2) =& 2\left\langle \nabla^{\perp}_{\sigma} F_\tau, d(\nabla^{\perp}_{\sigma} F_\tau)\right\rangle + d[\nabla^{\perp}_{\sigma} F_\tau, \nabla^{\perp}_{\sigma} F_\tau] \\ =& 2\left\langle \nabla^{\perp}_{\sigma} F_\tau, \left\langle \nabla^{\perp}_{\tau}\nabla^{\perp}_{\sigma} F_\tau,\, dW_\tau\right\rangle + (\mathrm{Rc}^{\nabla} + \frac{1}{2}\partial_t(g-b))_\tau(\nabla^{\perp}_{\tau} F_\tau)\, d\tau \right\rangle \\ &+ 2|\nabla^{\perp}_{\tau}\nabla^{\perp}_{\sigma} F_\tau|^2\, d\tau. \end{aligned}$$

Noticing that $2\left\langle \nabla^{\perp}_{\sigma} F_\tau, \left\langle \nabla^{\perp}_{\tau}\nabla^{\perp}_{\sigma} F_\tau,\, dW_\tau\right\rangle\right\rangle = \left\langle \nabla^{\perp}_{\tau}|\nabla^{\perp}_{\sigma} F_\tau|^2, dW_\tau\right\rangle$ due to Lemma 4.20, this proves the quadratic Bochner identity.

In order to show the linear Bochner identity, we use the Ito-Tanaka-Meyer formula for the convex function $|\cdot|\colon \mathbb{R}^n \to \mathbb{R}$, cf. [8]. Let us note that there is no local time at the origin, since we assume dimension > 1. $\square$

Corollary 4.22. *The generalized Bochner formula on $P\mathcal{M}$ reduces to*

$$\frac{1}{2}(\partial_\tau + \Delta_{g_{T'-\tau}})|\nabla f_\tau|^2 = |\nabla\nabla f_\tau|^2 + (\mathrm{Rc}^{\nabla} + \frac{1}{2}\partial_t(g-b))|_{t=T'-\tau}(\nabla f_\tau, \nabla f_\tau),$$

where $f_\tau = P_{T'-\tau,T'-\tau_1} f$, $f\colon M \to \mathbb{R}$ is a smooth function, $0 < \tau_1 < T'$ is fixed, and $\tau < \tau_1$.

Proof. Define

$$F_\tau(\gamma) = \begin{cases} P_{T'-\tau,T'-\tau_1} f(\pi_1\gamma_\tau) & \text{if } \tau < \tau_1 \\ f(\pi_1\gamma_{\tau_1}) & \text{if } \tau \geqslant \tau_1. \end{cases}$$

It follows from Proposition 4.8 that this defines a martingale on $P\mathcal{M}$. Moreover,

$$|\nabla^{\perp}_{0} F_\tau|(\gamma) = |\nabla^{\perp}_{\tau} F_\tau|(\gamma) = |\nabla f_\tau|(\pi_1\gamma_\tau)$$

and

$$|\nabla^{\perp}_{0}\nabla^{\perp}_{\tau} F_\tau|(\gamma) = |\nabla\nabla f_\tau|(\pi_1\gamma_\tau)$$

By virtue of Corollary 4.21 we have

$$\begin{aligned} d(|\nabla f_\tau|^2) - \left\langle \nabla^{\perp}_{\tau}|\nabla f_\tau|^2, dW_\tau\right\rangle = 2|\nabla\nabla f_\tau|^2\, d\tau + 2(\mathrm{Rc}^{\nabla} \\ + \frac{1}{2}\partial_t(g-b))|_{t=T'-\tau}(\nabla f_\tau, \nabla f_\tau)\, d\tau, \end{aligned}$$

with all quantities evaluated at $\pi_1\gamma_\tau$. Then by Ito's formula (4.2), eventually lifting everything on the frame bundle, we find for the left hand side

$$d(|\nabla f_\tau|^2) - \left\langle \nabla^{\perp}_{\tau}|\nabla f_\tau|^2, dW_\tau\right\rangle = (\partial_\tau + \Delta_{g_{T'-\tau}})|\nabla f_\tau|^2\, d\tau.$$

All in all we obtain

$$(\partial_\tau + \Delta_{g_{T'-\tau}})|\nabla f_\tau|^2\, d\tau = 2|\nabla\nabla f_\tau|^2\, d\tau + 2(\mathrm{Rc}^\nabla + \frac{1}{2}\partial_t(g-b))|_{t=T'-\tau}(\nabla f_\tau, \nabla f_\tau)\, d\tau,$$

which holds at $\pi_1\gamma_\tau$ for Γ_x-a.e. curves γ, which means by the definition of the Wiener measure Γ_x it holds for a.e. $y \in M$. By smoothness of f we obtain the claim for every $y \in M$. □

Lemma 4.23. *Let $F \in \mathfrak{C}_{T'}$ be nonnegative and $\tau, \sigma \geqslant 0$ fixed. Then*

$$\nabla^\perp_\sigma \nabla^\perp_\tau \log F = F^{-1}\nabla^\perp_\sigma\nabla^\perp_\tau F - F^{-2}\nabla^\perp_\tau F \otimes \nabla^\perp_\sigma F.$$

Proof. The proof follows by computing on the frame bundle as in the proof of Lemma 4.20. □

Corollary 4.24. *Let $F \in \mathfrak{C}_{T'}$ be nonnegative and let F_τ be the induced martingale. Then $X_\tau := F_\tau^{-1}|\nabla^{\mathcal{H}} F_\tau|^2 - F_\tau \log F_\tau$ satisfies*

$$\begin{aligned} dX_\tau = {}& \langle \nabla^\perp_\tau X_\tau, dW_\tau \rangle + 2F_\tau \left(\int_0^{T'} |\nabla^\perp_\tau \nabla^\perp_\sigma \log F_\tau|^2\, d\sigma \right) d\tau \\ & + 2F_\tau^{-1} \left(\int_0^{T'} (\mathrm{Rc}^\nabla + \frac{1}{2}\partial_t(g-b))_\tau(\nabla^\perp_\sigma F_\tau, \nabla^\perp_\tau F_\tau)\, d\sigma \right) d\tau. \end{aligned}$$

Proof. Note that

$$d(F_\tau \log F_\tau) = \langle \nabla^\perp_\tau (F_\tau \log F_\tau), dW_\tau \rangle + F_\tau^{-1}|\nabla^\perp_\tau F_\tau|^2\, d\tau \tag{4.9}$$

due to Proposition 4.16 and the standard Ito formula. For the other term we compute

$$d(F_\tau^{-1}|\nabla^{\mathcal{H}} F_\tau|^2) = F_\tau^{-1} d|\nabla^{\mathcal{H}} F_\tau|^2 + |\nabla^{\mathcal{H}} F_\tau|^2 dF_\tau^{-1} + d[F_\tau^{-1}, |\nabla^{\mathcal{H}} F_\tau|^2].$$

Noticing that

$$\begin{aligned} d|\nabla^{\mathcal{H}} F_\tau|^2 = {}& \langle \nabla^\perp_\tau |\nabla^{\mathcal{H}} F_\tau|^2, dW_\tau \rangle \\ & + 2\int_0^{T'} \left((\mathrm{Rc}^\nabla + \frac{1}{2}\partial_t(g-b))_\tau(\nabla_\tau F_\tau, \nabla^\perp_\sigma F_\tau) + |\nabla^\perp_\tau \nabla^\perp_\sigma F_\tau|^2 \right) d\sigma\, d\tau \\ & + |\nabla^\perp_\tau F_\tau|^2\, d\tau \end{aligned}$$

due to Corollary 4.21 and that

$$dF_\tau^{-1} = \langle \nabla^\perp_\tau (F_\tau^{-1}), dW_\tau \rangle + 2F_\tau^{-3}|\nabla^\perp_\tau F_\tau|^2\, d\tau$$

we compute using Lemma 4.20 and Lemma 4.23

$$\begin{aligned} d(F_\tau^{-1}|\nabla^{\mathcal{H}} F_\tau|^2) = & \langle \nabla_\tau^\perp (F_\tau^{-1}|\nabla^{\mathcal{H}} F_\tau|^2), dW_\tau \rangle + 2F_\tau \left(\int_0^{T'} |\nabla_\tau^\perp \nabla_\sigma \log F_\tau|^2 \, d\sigma \right) d\tau \\ & + 2F_\tau^{-1} \left(\int_0^{T'} (\mathrm{Rc}^\nabla + \frac{1}{2}\partial_t (g-b))_\tau (\nabla_\sigma F_\tau, \nabla_\tau F_\tau) \, d\sigma \right) d\tau \\ & + F_\tau^{-1} |\nabla_\tau^\perp F_\tau|^2 \, d\tau. \end{aligned} \tag{4.10}$$

All in all, combining (4.9) with (4.10) we get

$$\begin{aligned} dX_\tau = & \langle \nabla_\tau^\perp X_\tau, dW_\tau \rangle + 2F_\tau \left(\int_0^{T'} |\nabla_\tau^\perp \nabla_\sigma^\perp \log F_\tau|^2 \, d\sigma \right) d\tau \\ & + 2F_\tau^{-1} \left(\int_0^{T'} (\mathrm{Rc}^\nabla + \frac{1}{2}\partial_t (g-b))_\tau (\nabla_\sigma^\perp F_\tau, \nabla_\tau^\perp F_\tau) \, d\sigma \right) d\tau. \quad \square \end{aligned}$$

5. Characterizations of generalized Ricci flow

Theorem 5.1. *For an evolving family of manifolds $(M, g_t, H_t)_{t\in[0,T]}$, the following are equivalent:*

(1) The generalized Ricci flow is satisfied:

$$\partial_t (g-b) = -2\,\mathrm{Rc}^\nabla .$$

(2) Let $0 \leqslant \sigma \leqslant T' \leqslant T$ and $F \in \mathfrak{C}_{T'}$. Then the induced martingales satisfy the Bochner inequality

$$d|\nabla_\sigma^\perp F_\tau|^2 \geqslant \langle \nabla_\tau^\perp |\nabla_\sigma^\perp F_\tau|^2, dW_\tau \rangle + 2|\nabla_\tau^\perp \nabla_\sigma^\perp F_\tau|^2 \, d\tau + |\nabla_\sigma^\perp F_\sigma|^2 \delta_\sigma(\tau) \, d\tau.$$

(3) Let $0 \leqslant \sigma \leqslant T' \leqslant T$ and $F \in \mathfrak{C}_{T'}$. Then the induced martingales satisfy the weak Bochner inequality

$$d|\nabla_\sigma^\perp F_\tau|^2 \geqslant \langle \nabla_\tau^\perp |\nabla_\sigma^\perp F_\tau|^2, dW_\tau \rangle + |\nabla_\sigma^\perp F_\sigma|^2 \delta_\sigma(\tau) \, d\tau.$$

(4) Let $0 \leqslant \sigma \leqslant T' \leqslant T$ and $F \in \mathfrak{C}_{T'}$. Then the induced martingales satisfy the linear Bochner inequality

$$d|\nabla_\sigma^\perp F_\tau| \geqslant \langle \nabla_\tau^\perp |\nabla_\sigma^\perp F_\tau|, dW_\tau \rangle + |\nabla_\sigma^\perp F_\sigma| \delta_\sigma(\tau) \, d\tau.$$

(5) Let $0 \leqslant \sigma \leqslant T' \leqslant T$ and $F \in \mathfrak{C}_{T'}$. Then the induced martingales satisfy

$$\tau \mapsto |\nabla_\sigma^\perp F_\tau| \text{ is a submartingale.}$$

Proof. (1) ⇒ (2): Let $\partial_t(g-b) = -2\,\mathrm{Rc}^\nabla$. Then the claim directly follows from Corollary 4.21.

(2) ⇒ (3): This follows from omitting the Hessian part.

(2) ⇒ (4): Assume $\tau > \sigma$. Note that due to Corollary 4.21, we have that

$$\begin{aligned} d|\nabla_\sigma^\perp F_\tau| = &\left\langle \nabla_\tau^\perp |\nabla_\sigma^\perp F_\tau|, dW_\tau \right\rangle + \frac{|\nabla_\tau^\perp \nabla_\sigma^\perp F_\tau|^2 - |\nabla_\tau^\perp|\nabla_\sigma^\perp F_\tau||^2}{|\nabla_\sigma^\perp F_\tau|}\, d\tau \\ &+ \frac{1}{|\nabla_\sigma^\perp F_\tau|}(\mathrm{Rc}^\nabla + \frac{1}{2}\partial_t(g-b))_\tau(\nabla_\tau^\perp F_\tau, \nabla_\sigma^\perp F_\tau)\, d\tau. \end{aligned} \tag{5.1}$$

Now, comparing (5.1) with (2) by standard Ito's lemma and using that $|\nabla_\tau^\perp|\nabla_\sigma^\perp F_\tau||^2 \leqslant |\nabla_\tau^\perp \nabla_\sigma^\perp F_\tau|^2$ we directly find that

$$d|\nabla_\sigma^\perp F_\tau| \geqslant \left\langle \nabla_\tau^\perp |\nabla_\sigma^\perp F_\tau|, dW_\tau \right\rangle.$$

Together with $\nabla_\sigma^\perp F_\tau = 0$ for $\tau < \sigma$, this yields (4).

(4) ⇒ (3): This follows directly by applying Ito's formula and (4).

(4) ⇔ (5): Clearly, (4) implies (5). For (5) ⇒ (4), note that (5) implies that the absolutely continuous part in (5.1) must be nonnegative, which deduces (4).

(3) ⇒ (1): Let $F\colon P_{(x,T')}\mathcal{M} \to \mathbb{R}$. By Corollary 4.21 we know that

$$\tau \mapsto |\nabla_0^\perp F_\tau|^2 - \int_0^\tau (\mathrm{Rc}^\nabla + \frac{1}{2}\partial_t(g-b))_\rho(\nabla_\rho^\perp F_\rho, \nabla_0^\perp F_\rho) + 2|\nabla_\rho^\perp \nabla_0^\perp F_\rho|^2\, d\rho$$

is a martingale and consequently

$$\mathbb{E}_{(x,T')}\left[|\nabla_0^\perp F_\varepsilon|^2 - \int_0^\varepsilon (\mathrm{Rc}^\nabla + \frac{1}{2}\partial(g-b))_\tau(\nabla_\tau^\perp F_\tau, \nabla_0^\perp F_\tau) + 2|\nabla_\tau^\perp \nabla_0^\perp F_\tau|^2\, d\rho \right] = |\nabla_0^\perp F_0|^2.$$

By virtue of (3) $\tau \mapsto |\nabla_0^\perp F_\tau|^2$ is a submartingale and in particular

$$\mathbb{E}_{(x,T')}[|\nabla_0^\perp F_\varepsilon|^2] \geqslant |\nabla_0^\perp F_0|^2.$$

Together this implies that

$$\mathbb{E}_{(x,T')}\left[\int_0^\varepsilon (\mathrm{Rc}^\nabla + \frac{1}{2}\partial(g-b))_\tau(\nabla_\tau^\perp F_\tau, \nabla_0^\perp F_\tau) + 2|\nabla_\tau^\perp \nabla_0^\perp F_\tau|^2\, d\tau \right] \geqslant 0. \tag{5.2}$$

Now we consider two choices of cylinder function for F. Let the first one be $f_1\colon M \to \mathbb{R}$ with $f_1(x) = 0$, $\nabla f_1(x) = v$, and $\nabla^2 f_1(x) = 0$, where $v \in (T_xM, g_{T'})$. Then define $F\colon P_{(x,T')}\mathcal{M} \to \mathbb{R}$ by $F(\gamma) = f_1(\pi_1(\gamma_\varepsilon))$. For $\tau \leqslant \varepsilon$ we have

$$\nabla_0^\perp F_\tau = \nabla_\tau^\perp F_\tau = S_\tau \nabla P_{T'-\tau,T'-\varepsilon} f_1(\pi_1(\gamma_\tau)), \quad |\nabla_\tau^\perp \nabla_0^\perp F_\tau| = |\nabla^2 P_{T'-\tau,T'-\varepsilon} f_1(\pi_1(\gamma_\tau))|$$

and thus $\nabla_\tau^\perp F_\tau = v + o(\varepsilon)$ and $|\nabla_\tau^\perp \nabla_0^\perp F_\tau| = o(\varepsilon)$.

The second choice is $f_2\colon M \times M \to \mathbb{R}$ with $f_2(x,x) = 0$, $\nabla^{(1)} f_2(x,x) = 2v$, $\nabla^{(2)} f_2(x,x) = -v$, and $\nabla^2 f_2(x,x) = 0$. Let $F(\gamma) = f_2(\pi_1(\gamma_0), \pi_1(\gamma(\varepsilon)))$. Then, for $\tau \leqslant \varepsilon$,

$$\begin{aligned}
F_\tau(\gamma) &= P^{(2)}_{T'-\tau,T'-\varepsilon} f_2(x, \pi_1\gamma_\tau), \\
\nabla_0^\perp F_\tau &= \nabla^{(1)} P^{(2)}_{T'-\tau,T'-\varepsilon} f_2(x, \pi_1\gamma_\tau) + S_\tau \nabla^{(2)} P^{(2)}_{T'-\tau,T'-\varepsilon} f_2(x, \pi_1\gamma_\tau) = v + o(\varepsilon), \\
\nabla_\tau^\perp F_\tau &= S_\tau \nabla^{(2)} P^{(2)}_{T'-\tau,T'-\varepsilon} f_2(x, \pi_1\gamma_\tau) = -v + o(\varepsilon), \\
|\nabla_\tau^\perp \nabla_0^\perp F_\tau| &\leqslant |\nabla^{(2)}\nabla^{(1)} P^{(2)}_{T'-\tau,T'-\varepsilon} f_2(x, \pi_1\gamma_\tau)| + |\nabla^{(2)}\nabla^{(2)} P^{(2)}_{T'-\tau,T'-\varepsilon} f_2(x, \pi_1\gamma_\tau)| = o(\varepsilon).
\end{aligned}$$

Inserting both choices into (5.2) we get

$$(\mathrm{Rc}^\nabla + \frac{1}{2}\partial(g-b))_\varepsilon(v,v) = (\mathrm{Rc}^\nabla + \frac{1}{2}\partial_t(g-b))|_{t=T'-\varepsilon}(S_\varepsilon^{-1}v, S_\varepsilon^{-1}v) = o(\varepsilon).$$

Letting $\varepsilon \to 0$ we get $(\mathrm{Rc}^\nabla + \frac{1}{2}\partial_t(g-b))|_{t=T'} = 0$. □

Theorem 5.2. *For an evolving family of manifolds $(M, g_t, H_t)_{t\in[0,T]}$, the following are equivalent:*

(1) The generalized Ricci flow is satisfied:

$$\partial_t(g-b) = -2\,\mathrm{Rc}^\nabla.$$

(2) Let $0 \leqslant \sigma \leqslant T' \leqslant T$ and $F \in \mathfrak{C}_{T'}$. Then the induced martingales satisfy the gradient estimate

$$|\nabla_\sigma^\perp F_{\tau_1}| \leqslant \mathbb{E}_{(x,T')}[|\nabla_\sigma^\perp F_{\tau_2}||\Sigma_{\tau_1}]$$

for all $0 \leqslant \tau_1 \leqslant \tau_2 \leqslant T'$ and $x \in M$.

(3) Let $0 \leqslant \sigma \leqslant T' \leqslant T$ and $F \in \mathfrak{C}_{T'}$. Then the induced martingales satisfy the gradient estimate

$$|\nabla_\sigma^\perp F_{\tau_1}|^2 \leqslant \mathbb{E}_{(x,T')}[|\nabla_\sigma^\perp F_{\tau_2}|^2|\Sigma_{\tau_1}]$$

for all $0 \leqslant \tau_1 \leqslant \tau_2 \leqslant T'$ and $x \in M$.

(4) Let $0 \leqslant \sigma \leqslant T' \leqslant T$ *and* $F \in \mathfrak{C}_{T'}$*. Then we have the gradient estimate*

$$|\nabla_x \mathbb{E}_{(x,T')}[F]|^2 \leqslant \mathbb{E}_{(x,T')}[|\nabla_0^{\perp} F|^2]$$

for all $x \in M$*.*

(5) For any $T' \in [0,T]$*,* $F \in \mathfrak{C}_{T'}$*, the induced martingales satisfy the quadratic variation estimate*

$$\mathbb{E}_{(x,T')}\left[\frac{d[F,F]_\tau}{d\tau}\right] \leqslant 2\mathbb{E}_{(x,T')}[|\nabla_\tau^{\perp} F|^2],$$

for all $\tau \in [0,T']$ *and* $x \in M$*.*

Proof. (1) ⇒ (2) ⇒ (3): (2) immediately follows from (5.1) and integrating in τ and taking expectations. Claim (3) follows then from (2) by Cauchy-Schwarz:

$$|\nabla_\sigma^{\perp} F_{\tau_1}|^2 \leqslant \left(\mathbb{E}_{(x,T')}\left[|\nabla_\sigma^{\perp} F_{\tau_2}||\Sigma_{\tau_1}\right]\right)^2 \leqslant \mathbb{E}_{(x,T')}\left[|\nabla_\sigma^{\perp} F_{\tau_2}|^2|\Sigma_{\tau_1}\right]$$

for all $\tau_2 \geqslant \tau_1$.

(3) ⇒ (5): Note that according to Theorem 4.19 $d[F,F]_\tau = 2|\nabla_\tau^{\perp} F_\tau|^2\, d\tau$ and hence (3) yields

$$\begin{aligned}\mathbb{E}_{(x,T')}\left[\frac{d[F,F]_\tau}{d\tau}\right] &= 2\mathbb{E}_{(x,T')}[|\nabla_\tau^{\perp} F_\tau|^2] \leqslant 2\mathbb{E}_{(x,T')}\left[\mathbb{E}_{(x,T')}[|\nabla_\tau^{\perp} F|^2|\Sigma_\tau]\right] \\ &= 2\mathbb{E}_{(x,T')}[|\nabla_\tau^{\perp} F|^2].\end{aligned}$$

(5) ⇒ (4): This follows by recalling from Corollary 4.18 that $\frac{d}{d\tau}[F,F]_\tau = 2|\nabla_\tau^{\perp} F_\tau|^2$ and applying (5).

(4) ⇒ (1): This follows by choosing 1-point and 2-point cylinder functions similarly as in the proof of the implication Theorem 5.1 (3) ⇒ (1). □

Theorem 1.2. *For an evolving family of manifolds* $(M, g_t, H_t)_{t\in[0,T]}$*, the following are equivalent:*

(1) The generalized Ricci flow is satisfied:

$$\partial_t(g-b) = -2\,\mathrm{Rc}^{\nabla}.$$

(2) For any $0 \leqslant \sigma \leqslant T' \leqslant T$ *and any* $F \in \mathfrak{C}_{T'}$*, we have the estimate*

$$\mathbb{E}_{(x,T')}[|\nabla_\sigma^{\perp} F_\sigma|^2] + 2\int_0^{T'} \mathbb{E}_{(x,T')}[|\nabla_\tau^{\perp}\nabla_\sigma^{\perp} F_\tau|^2]\, d\tau \leqslant \mathbb{E}_{(x,T')}[|\nabla_\sigma^{\perp} F|^2]$$

for all $x \in M$*.*

(3) For any $0 \leqslant \tau_1 \leqslant \tau_2 \leqslant T' \leqslant T$ the Ornstein-Uhlenbeck operator $\mathcal{L}_{(\tau_1,\tau_2)}$ on parabolic path space $L^2(P_{T'}\mathcal{M})$ satisfies the Poincaré inequality

$$\mathbb{E}_{(x,T')}[(F_{\tau_2} - F_{\tau_1})^2] \leqslant 2\mathbb{E}_{(x,T')}[F\,\mathcal{L}_{(\tau_1,\tau_2)}F]$$

for all $x \in M$.

(4) For any $0 \leqslant \tau_1 \leqslant \tau_2 \leqslant T' \leqslant T$ the Ornstein-Uhlenbeck operator $\mathcal{L}_{(\tau_1,\tau_2)}$ on parabolic path space $L^2(P_{T'}\mathcal{M})$ satisfies the log-Sobolev inequality

$$\mathbb{E}_{(x,T')}[(F^2)_{\tau_2}\log((F^2)_{\tau_2}) - (F^2)_{\tau_1}\log((F^2)_{\tau_1})] \leqslant 4\mathbb{E}_{(x,T')}[F\,\mathcal{L}_{(\tau_1,\tau_2)}F]$$

for all $x \in M$.

Moreover, if one of the conditions (1)-(4) is satisfied, we have:

(3a) For any $0 \leqslant T' \leqslant T$, $F \in \mathfrak{C}_{T'}$, we have the Poincaré Hessian estimate

$$\mathbb{E}_{(x,T')}[(F - \mathbb{E}_{(x,T')}[F])^2] + 4\int_0^{T'}\int_0^{T'} \mathbb{E}_{(x,T')}[|\nabla_\tau^\perp \nabla_\sigma^\perp F_\tau|^2]\,d\sigma\,d\tau$$
$$\leqslant 2\int_0^{T'} \mathbb{E}_{(x,T')}[|\nabla_\sigma^\perp F|^2]\,d\sigma$$

for all $x \in M$.

(4a) For any $0 \leqslant T' \leqslant T$, $F \in \mathfrak{C}_{T'}$, we have the log-Sobolev Hessian estimate

$$\mathbb{E}_{(x,T')}[F^2\log(F^2)] - \mathbb{E}_{(x,T')}[F^2]\log(\mathbb{E}_{(x,T')}[F^2])$$
$$+ 2\int_0^{T'}\int_0^{T'} \mathbb{E}_{(x,T')}[(F^2)_\tau|\nabla_\tau^\perp\nabla_\sigma^\perp \log((F^2)_\tau)|^2]\,d\sigma\,d\tau \leqslant 4\int_0^{T'} \mathbb{E}_{(x,T')}[|\nabla_\sigma^\perp F|^2]\,d\sigma$$

for all $x \in M$.

Proof. (1) $\Rightarrow$ (2): Assertion (2) follows directly from integrating Theorem 5.1 (2) and taking expectations.

(2) $\Rightarrow$ (3): Using Ito's isometry and Theorem 5.2 yields

$$\mathbb{E}_{(x,T')}[(F_{\tau_2} - F_{\tau_1})^2] = 2\mathbb{E}_{(x,T')}[\int_{\tau_1}^{\tau_2} |\nabla_\sigma^\perp F_\sigma|^2\,d\sigma]$$
$$\leqslant 2\mathbb{E}_{(x,T')}[\int_{\tau_1}^{\tau_2} |\nabla_\sigma^\perp F|^2\,d\sigma] = 2\mathbb{E}_{(x,T')}[F\,\mathcal{L}_{(\tau_1,\tau_2)}F],$$

where we used Theorem 5.2 (2) in the inequality.

(3) ⇒ (1): Dividing (3) by $\tau_2 - \tau_1$ and letting $\tau_2 - \tau_1 \to 0$ we find

$$\mathbb{E}_{(x,T')}\left[\frac{d[F,F]_\tau}{d\tau}\right] \leqslant 2\mathbb{E}_{(x,T')}[|\nabla^\perp_\tau F|^2],$$

which is Theorem 5.2 (5).

(1) ⇒ (4): Take $G = F^2$. Then, by (4.9)

$$\begin{aligned}
&\mathbb{E}_{(x,T')}[(F^2)_{\tau_2}\log(F^2)_{\tau_2} - (F^2)_{\tau_1}\log(F^2)_{\tau_1}] \\
&= \mathbb{E}_{(x,T')}\left[\int_{\tau_1}^{\tau_2} G_\tau^{-1}|\nabla^\perp_\tau G_\tau|^2\, d\tau\right] \\
&\leqslant \mathbb{E}_{(x,T')}\left[\int_{\tau_1}^{\tau_2} G_\tau^{-1}\mathbb{E}_{(x,T')}\left[|\nabla^\perp_\tau G_{T'}||\Sigma_\tau\right]^2\, d\tau\right] \\
&\leqslant 4\mathbb{E}_{(x,T')}\left[\int_{\tau_1}^{\tau_2} |\nabla^\perp_\tau F|^2\, d\tau\right] \\
&= 4\mathbb{E}_{(x,T')}[F\mathcal{L}_{(\tau_1,\tau_2)}F],
\end{aligned}$$

where we used Theorem 5.2 (2) in the second step and Cauchy-Schwarz in the third.

(4) ⇒ (3): We apply (4) to $F^2 = 1 + \varepsilon G$ and obtain by Taylor approximation

$$\frac{1}{2}\mathbb{E}_{(x,T')}[\varepsilon^2 G^2_{\tau_2} - \varepsilon^2 G^2_{\tau_1}] \leqslant \varepsilon^2\mathbb{E}_{(x,T')}[G\mathcal{L}_{(\tau_1,\tau_2)}G] + o(\varepsilon^2).$$

Dividing by ε^2 and letting $\varepsilon \to 0$ we obtain

$$\frac{1}{2}\mathbb{E}_{(x,T')}[G^2_{\tau_2} - G^2_{\tau_1}] \leqslant \mathbb{E}_{(x,T')}[G\mathcal{L}_{(\tau_1,\tau_2)}G].$$

Noticing that $\mathbb{E}_{(x,T')}[G^2_{\tau_2} - G^2_{\tau_1}] = \mathbb{E}_{(x,T')}[(G_{\tau_2} - G_{\tau_1})^2]$ proves the claim.

This proves the equivalence of (1)-(4). Next we show the remaining implications.

(2) ⇒ (3a): Apply Proposition 4.16 and Ito's isometry and integrate (2) on $(0, T')$.

(1) ⇒ (4a): Let $G = F^2$ and consider $X_\tau = G_\tau^{-1}|\nabla^{\mathcal{H}} G_\tau|^2 - G_\tau \log(G_\tau)$. Note that according to Corollary 4.24, we have that

$$\begin{aligned}
dX_\tau = \;&\left\langle \nabla^\perp_\tau X_\tau, dW_\tau\right\rangle + 2G_\tau\left(\int_0^{T'} |\nabla^\perp_\tau\nabla^\perp_\sigma \log(G_\tau)|^2\, d\sigma\right)\, d\tau \\
&+ 2G_\tau^{-1}\left(\int_0^{T'} (\mathrm{Rc}^\nabla + \frac{1}{2}\partial_t(g-b))_\tau(\nabla^\perp_\tau G_\tau, \nabla^\perp_\sigma G_\tau)\, d\sigma\right)\, d\tau
\end{aligned}$$

$$\geqslant \left\langle \nabla^{\perp}_{\tau} X_\tau, dW_\tau \right\rangle + 2G_\tau \left(\int_0^{T'} |\nabla^{\perp}_{\tau} \nabla^{\perp}_{\sigma} \log(G_\tau)|^2 \, d\sigma \right) d\tau,$$

where we used (1) in the last equation. Integration and taking expectations yields

$$\mathbb{E}_{(x,T')}[X'_T] - \mathbb{E}_{(x,T')}[X_0] \geqslant 2\mathbb{E}_{(x,T')} \left[\int_0^{T'} G_\tau \left(\int_0^{T'} |\nabla^{\perp}_{\tau} \nabla^{\perp}_{\sigma} \log(G_\tau)|^2 \, d\sigma \right) d\tau \right],$$

and evaluating the expectations on the left hand side

$$\begin{aligned} \mathbb{E}_{(x,T')}[X_0] &= -\mathbb{E}_{(x,T')}[F^2] \log(\mathbb{E}_{(x,T')}[F^2]) \\ \mathbb{E}_{(x,T')}[X_{T'}] &= 4\mathbb{E}_{(x,T')}[|\nabla^{\mathcal{H}} F|^2] - \mathbb{E}_{(x,T')}[F^2 \log(F^2)]. \end{aligned}$$

Putting everything together yields

$$\begin{aligned} &4\mathbb{E}_{(x,T')}[|\nabla^{\mathcal{H}} F|^2] - \mathbb{E}_{(x,T')}[F^2 \log(F^2)] + \mathbb{E}_{(x,T')}[F^2] \log(\mathbb{E}_{(x,T')}[F^2]) \\ &\geqslant 2 \int_0^{T'} \int_0^{T'} \mathbb{E}_{(x,T')} \left[(F^2)_\tau |\nabla^{\perp}_{\tau} \nabla^{\perp}_{\sigma} \log((F^2)_\tau)|^2 \right] d\sigma \, d\tau, \end{aligned}$$

which is (4a). □

6. Appendix: Integration by parts

Proof of Theorem 4.11. Since D_V satisfies the product rule, it is enough to show that

$$\mathbb{E}_{(x,T')}[D_V F] = \frac{1}{2}\mathbb{E}_{(x,T')} \left[F \int_0^{T'} \left\langle \frac{d}{d\tau} h_\tau - S_\tau (\mathrm{Rc}^{\nabla} + \frac{1}{2}\partial_t (g-b))^{\dagger}_{T'-\tau} S_\tau^{-1} h_\tau, \, dW_\tau \right\rangle \right]$$

for all $F \in \mathfrak{C}_{T'}$. We prove this by induction on the order k of the cylinder function F.

$k = 1$: Let $F(\gamma) = f(x_\sigma)$ and let $s = T' - \sigma$. Since $w(x,t) = P_{t,s} f(x)$ satisfies the heat equation, the gradient satisfies

$$\nabla_t \operatorname{grad}_{g_t} w = \Delta_{g_t} w - (\mathrm{Rc}^{\nabla} + \frac{1}{2}\partial_t (g_t - b_t))(\operatorname{grad}_{g_t} w, \cdot)^{\#_{g_t}}$$

by Proposition 3.3. By the Feynman-Kac formula (Proposition 4.6) we have

$$\operatorname{grad}_{g_{T'}} w(x, T') = \mathbb{E}_{(x,T')}[R_\sigma \, S_\sigma \operatorname{grad}_{g_s} f(X_\sigma)], \tag{6.1}$$

where $R_\tau = R_\tau(\gamma)\colon (T_xM, g_{T'}) \to (T_xM, g_{T'})$ solves the ODE $\frac{d}{d\tau}R_\tau = R_\tau\, S_\tau\, (\mathrm{Rc}^\nabla + \frac{1}{2}\partial_t(g-b))_{T'-\tau}S_\tau^{-1}$ with $R_0 = \mathrm{id}$, and where we view $(\mathrm{Rc}^\nabla + \frac{1}{2}\partial_t(g-b))_{T'-\tau}$ as endomorphism of TM using the metric $g_{T'-\tau}$. Note also by (4.3)

$$f(X_\sigma) = w(x,T') + \int_0^\sigma E_i\widetilde{w}(U_\tau)\, dW_\tau^i, \tag{6.2}$$

where $\widetilde{w}$ is the invariant lift.

Let $(z_\tau)_{\tau\in[0,T']} \in \mathcal{H}$. Then by (6.2) and Ito's isometry

$$\begin{aligned}\mathbb{E}_{(x,T')}\left[f(X_\sigma)\int_0^\sigma \left\langle R_\tau^\dagger \dot z_\tau, dW_\tau\right\rangle\right] &= \mathbb{E}_{(x,T')}\left[\int_0^\sigma E_i\widetilde{w}(U_\tau)\, dW_\tau^i \int_0^\sigma \left\langle R_\tau^\dagger \dot z_\tau, dW_\tau\right\rangle\right] \\ &= 2\mathbb{E}_{(x,T')}\left[\int_0^\sigma \left\langle \nabla^E\widetilde{w}(U_\tau), U_0^{-1}R_\tau^\dagger \dot z_\tau\right\rangle\, d\tau\right] \\ &= 2\mathbb{E}_{(x,T')}\left[\int_0^\sigma \left\langle R_\tau U_0\nabla^E\widetilde{w}(U_\tau), \dot z_\tau\right\rangle_{g'_T}\, d\tau\right],\end{aligned}$$

where $R_\tau^\dagger$ is the transpose of R_τ.

Let $N_\tau := R_\tau U_0 \nabla^E\widetilde{w}(U_\tau) = R_\tau S_\tau\, \mathrm{grad}_{g_{T'-\tau}}\, w(X_\tau, T'-\tau)$. Integration by parts gives

$$\begin{aligned}\mathbb{E}_{(x,T')}\left[\int_0^\sigma \left\langle N_\tau, \dot z_\tau\right\rangle_{g_{T'}}\, d\tau\right] &= \mathbb{E}_{(x,T')}\left[\left\langle z_\sigma, N_\sigma\right\rangle_{g_{T'}} - \int_0^\sigma \left\langle z_\tau, dN_\tau\right\rangle_{g_{T'}}\, d\tau\right] \\ &= \mathbb{E}_{(x,T')}\left[\left\langle z_\sigma, N_\sigma\right\rangle_{g_{T'}}\right],\end{aligned}$$

where we used in the last step that N_τ is a martingale, cf. equation (4.5). With this we obtain

$$\mathbb{E}_{(x,T')}\left[f(X_\sigma)\int_0^{T'} \left\langle R_\tau^\dagger \dot z_\tau, dW_\tau\right\rangle\right] = 2\mathbb{E}_{(x,T')}\left[\left\langle R_\sigma^\dagger z_\sigma, S_\sigma\, \mathrm{grad}_{g_\sigma} f(X_\sigma)\right\rangle_{g_{T'}}\right],$$

where we used that $\mathbb{E}_{(x,T')}\left[f(X_\sigma)\int_\sigma^{T'}\left\langle R_\tau^\dagger \dot z_\tau, dW_\tau\right\rangle\right] = 0$. Finally, let $h_\tau = R_\tau^\dagger z_\tau$. Then

$$R_\tau^\dagger \dot z_\tau = \dot h_\tau - S_\tau(\mathrm{Rc}^\nabla + \frac{1}{2}\partial_t(g-b))_{T'-\tau}^\dagger S_\tau^{-1}h_\tau,$$

which is the claim.

Now we prove the inductive step, assuming the result for $k-1$-point cylinder functions. Let $F(\gamma) = f(x_{\sigma_1}, \dots, x_{\sigma_k})$ and let $s_i = T' - \sigma_i$. Then

$$\mathbb{E}_{(x,T')}[D_V F] = \sum_{j=1}^{k} \mathbb{E}_{(x,T')} \left\langle h_{\sigma_j}, S_{\sigma_j} \operatorname{grad}^{(j)}_{g_{s_j}} f(X_{\sigma_1}, \dots, X_{\sigma_k}) \right\rangle_{g_{T'}}. \tag{6.3}$$

We define a function

$$\alpha(x_1, \dots, x_{k-1}) := \mathbb{E}_{(x_{k-1}, s_{k-1})} f(x_1, \dots, x_{k-1}, X'_{\sigma_k - \sigma_{k-1}}),$$

where X' is based at x_{k-1}. Then, for $j = 1, \dots, k-2$, we have

$$\operatorname{grad}^{(j)}_{g_{s_j}} \alpha(x_1, \dots, x_{k-1}) = \mathbb{E}_{(x_{k-1}, s_{k-1})} \operatorname{grad}^{(j)}_{g_{s_j}} f(x_1, \dots, x_{k-1}, X'_{\sigma_k - \sigma_{k-1}}). \tag{6.4}$$

For $j = k-1$ we have by the product rule and (6.1)

$$\begin{aligned} \operatorname{grad}^{(k-1)}_{g_{s_{k-1}}} \alpha(x_1, \dots, x_{k-1}) &= \mathbb{E}_{(x_{k-1}, s_{k-1})} \operatorname{grad}^{(k-1)}_{g_{s_{k-1}}} f(x_1, \dots, x_{k-1}, X'_{\sigma_k - \sigma_{k-1}}) \\ &+ \mathbb{E}_{(x_{k-1}, s_{k-1})}[R'_{\sigma_k - \sigma_{k-1}} S'_{\sigma_k - \sigma_{k-1}} \operatorname{grad}_{g_{s_k}} f(x_1, \dots, x_{k-1}, X'_{\sigma_k - \sigma_{k-1}})], \end{aligned} \tag{6.5}$$

where $R'_\tau = R'_\tau(\gamma)\colon (T_{x_{k-1}}M, g_{s_{k-1}}) \to (T_{x_{k-1}}M, g_{s_{k-1}})$ solves the ODE $\frac{d}{d\tau} R'_\tau = R'_\tau S'_\tau (\mathrm{Rc}^\nabla + \frac{1}{2}\partial_t(g-b))_{s_{k-1}-\tau} S'^{-1}_\tau$ with $R_0 = \mathrm{id}$. Now, let $\Gamma\colon P_{(x,T')}\mathcal{M} \to \mathbb{R}$ be the $(k-1)$-point cylinder function induced by α

$$G(\gamma) = \alpha(x_{\sigma_1}, \cdots, x_{\sigma_{k-1}}).$$

Then, by (6.3), (6.4), and (6.5), and by the law of total expectation

$$\begin{aligned} \mathbb{E}_{(x,T')}[D_V F] &= \mathbb{E}_{(x,T')}[D_V G] + \mathbb{E}_{(x,T')} \left[\left\langle h_{\sigma_k}, S_{\sigma_k} \operatorname{grad}^{(k)}_{g_{s_k}} f(X_{\sigma_1}, \dots, X_{\sigma_k}) \right\rangle_{g_{T'}} \right] \\ &- \mathbb{E}_{(x,T')} \Bigg[\mathbb{E}_{(X_{\sigma_{k-1}}, s_{k-1})} \\ &\times \left[\left\langle h_{\sigma_{k-1}}, S_{\sigma_{k-1}} R'_{\sigma_k - \sigma_{k-1}} S'_{\sigma_k - \sigma_{k-1}} \operatorname{grad}_{g_{s_k}} f(X_{\sigma_1}, \dots, X_{\sigma_{k-1}}, X'_{\sigma_k - \sigma_{k-1}}) \right\rangle_{g_{T'}} \right] \Bigg] \end{aligned} \tag{6.6}$$

By induction, the claim holds for $k-1$ and thus

$$\mathbb{E}_{(x,T')}[D_V G] = \frac{1}{2} \mathbb{E}_{(x,T')} \left[G \int_0^{s_{k-1}} \left\langle \frac{d}{d\tau} h_\tau - S_\tau (\mathrm{Rc}^\nabla + \frac{1}{2}\partial_t(g-b))^\dagger_{T'-\tau} S_\tau^{-1} h_\tau,\, dW_\tau \right\rangle \right]. \tag{6.7}$$

The second term on the right hand side of (6.6) we decompose into

$$
\begin{aligned}
&\mathbb{E}_{(x,T')}\left[\left\langle h_{\sigma_k}, S_{\sigma_k}\,\mathrm{grad}^{(k)}_{g_{s_k}} f(X_{\sigma_1},\dots,X_{\sigma_k})\right\rangle_{g_{T'}}\right] \\
&\quad= \mathbb{E}_{(x,T')}\left[\left\langle h_{\sigma_k}-h_{\sigma_{k-1}}, S_{\sigma_k}\,\mathrm{grad}^{(k)}_{g_{s_k}} f(X_{\sigma_1},\dots,X_{\sigma_k})\right\rangle_{g_{T'}}\right] \\
&\qquad+ \mathbb{E}_{(x,T')}\left[\left\langle h_{\sigma_{k-1}}, S_{\sigma_k}\,\mathrm{grad}^{(k)}_{g_{s_k}} f(X_{\sigma_1},\dots,X_{\sigma_k})\right\rangle_{g_{T'}}\right].
\end{aligned}
\tag{6.8}
$$

Then, for the first term in (6.8) the Markov property at $(X_{\sigma_{k-1}}, s_{k-1})$ together with the induction hypothesis for one-point cylinder functions implies

$$
\begin{aligned}
&\mathbb{E}_{(x,T')}\left[\left\langle h_{\sigma_k}-h_{\sigma_{k-1}}, S_{\sigma_k}\,\mathrm{grad}^{(k)}_{g_{s_k}} f(X_{\sigma_1},\dots,X_{\sigma_k})\right\rangle_{g_{T'}}\right] \\
&= \mathbb{E}_{(x,T')}\mathbb{E}_{(X_{\sigma_{k-1}},s_{k-1})} \\
&\quad\times\left[\left\langle S^{-1}_{\sigma_{k-1}}(h_{\sigma_k}-h_{\sigma_{k-1}}), S'_{\sigma_k-\sigma_{k-1}}\,\mathrm{grad}^{(k)}_{g_{s_k}} f(X_{\sigma_1},\dots,X_{\sigma_{k-1}},X'_{\sigma_k-\sigma_{k-1}})\right\rangle_{g_{s_{k-1}}}\right] \\
&= \frac{1}{2}\mathbb{E}_{(x,T')}\left[F\int_{s_{k-1}}^{s_k}\left\langle \frac{d}{d\tau}h_\tau - S_\tau(\mathrm{Rc}^\nabla+\frac{1}{2}\partial_t(g-b))^\dagger S_\tau^{-1}(h_\tau-h_{\sigma_{k-1}}), dW_\tau\right\rangle\right].
\end{aligned}
\tag{6.9}
$$

Similarly for the other term in (6.8)

$$
\begin{aligned}
&\mathbb{E}_{(x,T')}\left[\left\langle h_{\sigma_{k-1}}, S_{\sigma_k}\,\mathrm{grad}^{(k)}_{g_{s_k}} f(X_{\sigma_1},\dots,X_{\sigma_k})\right\rangle_{g_{T'}}\right] \\
&= \mathbb{E}_{(x,T')}\mathbb{E}_{(X_{\sigma_{k-1}},s_{k-1})} \\
&\quad\times\left[\left\langle S^{-1}_{\sigma_{k-1}}h_{\sigma_{k-1}}, S'_{\sigma_k-\sigma_{k-1}}\,\mathrm{grad}^{(k)}_{g_{s_k}} f(X_{\sigma_1},\dots,X_{\sigma_{k-1}},X'_{\sigma_k-\sigma_{k-1}})\right\rangle_{g_{s_{k-1}}}\right],
\end{aligned}
$$

which we combine with the third term on the right hand side in (6.6) and obtain

$$
\begin{aligned}
&\mathbb{E}_{(x,T')}\mathbb{E}_{(X_{\sigma_{k-1}},s_{k-1})} \\
&\times\left[\left\langle S^{-1}_{\sigma_{k-1}}h_{\sigma_{k-1}}, S'_{\sigma_k-\sigma_{k-1}}\,\mathrm{grad}^{(k)}_{g_{s_k}} f(X_{\sigma_1},\dots,X_{\sigma_{k-1}},X'_{\sigma_k-\sigma_{k-1}})\right\rangle_{g_{s_{k-1}}}\right] \\
&- \mathbb{E}_{(x,T')}\mathbb{E}_{(X_{\sigma_{k-1}},s_{k-1})} \\
&\times\left[\left\langle h_{\sigma_{k-1}}, S_{\sigma_{k-1}}R'_{\sigma_k-\sigma_{k-1}}S'_{\sigma_k-\sigma_{k-1}}\,\mathrm{grad}_{g_{s_k}} f(X_{\sigma_1},\dots,X_{\sigma_{k-1}},X'_{\sigma_k-\sigma_{k-1}})\right\rangle_{g_{T'}}\right] \\
&= \mathbb{E}_{(x,T')}\mathbb{E}_{(X_{\sigma_{k-1}},s_{k-1})} \\
&\times\left[\left\langle (\mathrm{id}-R'_{\sigma_k-\sigma_{k-1}})^\dagger S^{-1}_{\sigma_{k-1}}h_{\sigma_{k-1}}, S'_{\sigma_k-\sigma_{k-1}}\,\mathrm{grad}_{g_{s_k}} f(X_{\sigma_1},\dots,X_{\sigma_{k-1}},X'_{\sigma_k-\sigma_{k-1}})\right\rangle_{g_{s_{k-1}}}\right]
\end{aligned}
$$

$$= -\frac{1}{2}\mathbb{E}_{(x,T')}\left[F\int_{s_{k-1}}^{s_k}\left\langle S_\tau(\mathrm{Rc}^\nabla + \frac{1}{2}\partial_t(g-b))^\dagger S_\tau^{-1}h_{\sigma_{k-1}}, dW_\tau\right\rangle\right]. \tag{6.10}$$

Here, we used the induction hypothesis for

$$w_\tau := (\mathrm{id} - R'^{\dagger}_{\tau-\sigma_{k-1}})S^{-1}_{\sigma_{k-1}}h_{\sigma_{k-1}}.$$

Then a simple calculation shows that

$$\begin{aligned}
&\frac{d}{d\tau}w_\tau - S'_{\tau-\sigma_{k-1}}(\mathrm{Rc}^\nabla + \frac{1}{2}\partial_t(g-b))^\dagger S'^{-1}_{\tau-\sigma_{k-1}}w_\tau \\
&= -S'_{\tau-\sigma_{k-1}}(\mathrm{Rc}^\nabla + \frac{1}{2}\partial_t(g-b))^\dagger S'^{-1}_{\tau-\sigma_{k-1}}S^{-1}_{\sigma_{k-1}}h_{\sigma_{k-1}},
\end{aligned}$$

which gives (6.10).

Adding (6.7), (6.9), (6.10), we obtain

$$\begin{aligned}
&\mathbb{E}_{(x,T')}[D_V F] \\
&= \frac{1}{2}\mathbb{E}_{(x,T')}\left[G\int_0^{s_{k-1}}\left\langle \frac{d}{d\tau}h_\tau - S_\tau(\mathrm{Rc}^\nabla + \frac{1}{2}\partial_t(g-b))^\dagger_{T'-\tau}S_\tau^{-1}h_\tau,\, dW_\tau\right\rangle\right] \\
&\quad + \frac{1}{2}\mathbb{E}_{(x,T')}\left[F\int_{s_{k-1}}^{s_k}\left\langle \frac{d}{d\tau}h_\tau - S_\tau(\mathrm{Rc}^\nabla + \frac{1}{2}\partial_t(g-b))^\dagger S_\tau^{-1}(h_\tau - h_{\sigma_{k-1}}), dW_\tau\right\rangle\right] \\
&\quad - \frac{1}{2}\mathbb{E}_{(x,T')}\left[F\int_{s_{k-1}}^{s_k}\left\langle S_\tau(\mathrm{Rc}^\nabla + \frac{1}{2}\partial_t(g-b))^\dagger S_\tau^{-1}h_{\sigma_{k-1}}, dW_\tau\right\rangle\right] \\
&= \frac{1}{2}\mathbb{E}_{(x,T')}\left[F\int_0^{T'}\left\langle \frac{d}{d\tau}h_\tau - S_\tau(\mathrm{Rc}^\nabla + \frac{1}{2}\partial_t(g-b))^\dagger_{T'-\tau}S_\tau^{-1}h_\tau,\, dW_\tau\right\rangle\right],
\end{aligned}$$

which proves the theorem. □

Data availability

No data was used for the research described in the article.

References

[1] Marc Arnaudon, Kolehe Abdoulaye Coulibaly, Anton Thalmaier, Brownian motion with respect to a metric depending on time: definition, existence and applications to Ricci flow, C. R. Math. Acad. Sci. Paris 346 (13–14) (2008) 773–778.

[2] Bing-Long Chen, Xi-Ping Zhu, Uniqueness of the Ricci flow on complete noncompact manifolds, J. Differ. Geom. 74 (1) (2006) 119–154.
[3] B. Driver, A Cameron-Martin type quasi-invariance theorem for Brownian motion on a compact Riemannian manifold, J. Funct. Anal. 110 (2) (1992) 272–376.
[4] M. Garcia-Fernandez, Ricci flow, Killing spinors, and T-duality in generalized geometry, Adv. Math. 350 (2019) 1059–1108.
[5] Mario Garcia-Fernandez, Jeffrey Streets, Generalized Ricci Flow, University Lecture Series, AMS, 2020.
[6] Richard S. Hamilton, The Harnack estimate for the Ricci flow, J. Differ. Geom. 37 (1) (1993) 225–243.
[7] Robert Haslhofer, Aaron Naber, Characterizations of the Ricci flow, J. Eur. Math. Soc. 20 (5) (2018) 1269–1302.
[8] Robert Haslhofer, Aaron Naber, Ricci curvature and Bochner formulas for martingales, Commun. Pure Appl. Math. 71 (6) (2018) 1074–1108.
[9] Hans-Joachim Hein, Aaron Naber, New logarithmic Sobolev inequalities and an ϵ-regularity theorem for the Ricci flow, Commun. Pure Appl. Math. 67 (9) (2014) 1543–1561.
[10] Elton P. Hsu, Stochastic Analysis on Manifolds, vol. 38, American Mathematical Society, 2002.
[11] Christopher Kennedy, A Bochner formula on path space for the Ricci flow, arXiv:1909.04193.
[12] Shoshichi Kobayashi, Katsumi Nomizu, Foundations of Differential Geometry, vol. I, Wiley Classics Library, John Wiley & Sons, Inc., New York, 1996, reprint of the 1963 original, A Wiley-Interscience Publication.
[13] Robert J. McCann, Peter M. Topping, Ricci flow, entropy and optimal transportation, Am. J. Math. 132 (3) (2010) 711–730.
[14] Joseph Polchinski, String Theory, vol. I: An Introduction to the Bosonic String, Cambridge Monographs on Mathematical Physics, Cambridge University Press, Cambridge, 2005, reprint of the 2003 edition.
[15] Jeffrey Streets, Regularity and expanding entropy for connection Ricci flow, J. Geom. Phys. 58 (7) (2008) 900–912.
[16] Jeffrey Streets, Generalized geometry, T-duality, and renormalization group flow, J. Geom. Phys. 114 (2017) 506–522.
[17] Jeffrey Streets, Pluriclosed flow and the geometrization of complex surfaces, in: Geometric Analysis, in: Progr. Math., vol. 333, Birkhäuser/Springer, Cham, 2020, pp. 471–510.
[18] Jeffrey Streets, Gang Tian, A parabolic flow of pluriclosed metrics, Int. Math. Res. Not. (16) (2010) 3101–3133.
[19] Jeffrey Streets, Gang Tian, Generalized Kähler geometry and the pluriclosed flow, Nucl. Phys. B 858 (2) (2012) 366–376.