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Scalar Curvature, Entropy, and Generalized Ricci Flow

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We derive a family of weighted scalar curvature monotonicity formulas for generalized Ricci flow, involving an auxiliary dilaton field evolving by a certain reaction–diffusion equation motivated by renormalization group flow. These scalar curvature monotonicities are dual to a new family of Perelman-type energy and entropy monotonicity formulas by coupling to a solution of the associated weighted conjugate heat equation. In the setting of Ricci flow, we further obtain a new family of convex Nash entropies and pseudolocality principles.

1 Introduction

A dominant theme in the analysis of Ricci flow is the understanding of curvature positivity conditions preserved by the flow [13, 14, 22, 23, 25]. Most fundamental among these, as observed by Hamilton in his original paper [22], is the preservation of a lower bound on the scalar curvature. This bound is essential for detailed analyses of heat kernels, ancient solutions, and singularity formation of Ricci flow (cf. e.g., [11, 16, 18, 26, 47–49]). A second dominant theme is the key role played by self-similar solutions of the flow, that is, Ricci solitons, which partly indicate the subtle interplay between Ricci flow and the diffeomorphism group. Such solutions, and the interaction between Ricci flow and the diffeomorphism group, lie at the foundation of various key estimates for Ricci flow, such as Hamilton's Harnack estimate [24], and Perelman's energy, entropy, reduced volume functionals, and differential Harnack estimate [32]. These various tools combine to reveal the structure of singular sets of Ricci flow [7–9, 32], leading to deep

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topological applications [10, 33]. In this paper, we extend the fundamental circle of ideas around scalar curvature monotonicity, Harnack estimates, and heat kernels to *generalized Ricci flow*. In the process, we obtain several new estimates for Ricci flow, in particular a new family of convex Nash entropies, and pseudolocality estimates

Given a smooth manifold M, a one-parameter family of metrics g_t and closed three-forms H_t is a solution of generalized Ricci flow if $H_t = H_0 + db_t$ and

$$\frac{\partial}{\partial t} g = -2\,\mathrm{Rc} + \tfrac{1}{2} H^2,$$
$$\frac{\partial}{\partial t} b = -d_g^* H,$$

where $H^2(X,Y) = \langle i_X H, i_Y H\rangle$, with i_X denoting the interior product and the inner product taken with respect to the time-dependent metric. This equation arises independently in mathematical physics [31, 34], complex geometry [40, 42], and generalized geometry [19, 36, 41], and we refer to [20] for further background. Some global existence and convergence results can be found in [2, 27, 37, 39]. Note that $H \equiv 0$ is preserved by the flow (cf. [20] Proposition 4.20), and the metric then solves Ricci flow. Thus, in the remainder of this paper, many results are for generalized Ricci flow, with the attendant results for Ricci flow occurring as a special case.

1.1 Scalar curvature monotonicity

Given a metric g, closed three-form H, and smooth function f, let

$$\mathrm{Rc}^{H,f} = \mathrm{Rc} - \tfrac{1}{4} H^2 + \nabla^2 f - \tfrac{1}{2}\left(d_g^* H + i_{\nabla f} H\right), \qquad R^{H,f} := R - \tfrac{1}{12}\,|H|^2 + 2\Delta f - |\nabla f|^2.$$

The tensor $\mathrm{Rc}^{H,f}$ reduces to the Ricci tensor of the Bismut connection with torsion H when $f = 0$, and in general can be motivated by extending ideas from Bakry–Emery [5] to the Laplacian of the Bismut connection acting on one-forms. The scalar curvature $R^{H,f}$ arises in the Lichnerowicz-type formula for the cubic Dirac operator of Bismut [12] in the case $f = 0$. The general case occurs when computing this formula using a weighted volume form (cf. [6, 32]).

Perelman's energy and entropy monotonicity formulas can be interpreted as differential inequalities for the weighted scalar curvature R^f. The key point is to allow the weight $u = e^{-f}$ to evolve by the conjugate heat equation. In this circle of ideas, the function $u = e^{-f}$ is the "dilaton" in physics terminology. Interestingly, in mathematical physics literature, a different equation is suggested for the dilaton in the RG flow of the H-twisted nonlinear sigma model [34]. Specifically, given (g_t, H_t) a solution of generalized Ricci flow, we let $\square = \frac{\partial}{\partial t} - \Delta$ denote the forward heat operator, and fix ϕ a solution to the associated *dilaton flow*,

$$\square \phi = \tfrac{1}{6} |H|^2.$$

With the setup above, we then obtain the following evolution equation for the generalized scalar curvature:

Proposition 1.1. (cf. Proposition 2.4) Given (M^n, g_t, H_t, ϕ_t) a solution to generalized Ricci flow, one has

$$\square R^{H,\phi} = 2 \left| \mathrm{Rc}^{H,\phi} \right|^2. \tag{1.1}$$

Remark 1.2. In the case of Ricci flow, the dilaton flow is simply the forward heat flow. Here, the weighted scalar curvature monotonicity appears in ([17] Chapter 7 Lemma 6.88). There, monotonicity formulas are shown for a one-parameter family of dilaton flows interpolating between the forward heat flow and the conjugate heat equation, which incidentally also extend to generalized Ricci flow.

Proposition 1.1 implies that a lower bound on $R^{H,\phi}$ is preserved on a compact manifold. Also, by applying the strong maximum principle, we obtain a rigidity result.

Corollary 1.3. (cf. Corollary 2.6) Let (M, g, H, ϕ) satisfy $R^{H,\phi} \geqslant 0$. If M is compact, then either

1. The triple (g, H, ϕ) defines a generalized Ricci soliton and $R^{H,\phi} \equiv 0$
2. The manifold M admits a triple $(\overline{g}, \overline{H}, \overline{\phi})$ such that $R^{\overline{H},\overline{\phi}} > 0$ everywhere.

Remark 1.4. The scalar curvature evolution equation of Proposition 1.1 and attendant corollaries can be generalized in several ways. In particular, noting that all quantities involved ultimately only depend on df, one may replace df with a general one-form α evolving by the operator $\square \alpha = \frac{1}{6} d |H|^2$, and obtain a monotone curvature quantity (cf. §2.2). Furthermore, the tensor H may be replaced with a formal linear combination of differential forms of all degrees, and with an appropriately weighted scalar curvature and dilaton flow, one again obtains a monotone curvature quantity (cf. §2.3). One special case of this is the extended Ricci flow system of List [29], coupling to an exact 1-form.

Another special case is the Ricci–Yang–Mills flow ([35, 39, 45, 46]), a coupling of the Ricci and Yang–Mills flows which in the case of abelian structure group corresponds to the case that H is a two-form, specifically the principal curvature.

1.2 Entropy formulas and Harnack estimates

Further structure is revealed when we treat $e^{-\phi}$ as a volume density, mirroring the role played by $u = e^{-f}$ in Perelman's work. Perelman's idea underlying his energy and entropy formulas is to let $u = e^{-f}$ be a solution of the conjugate heat equation $\square^* u = 0$, where

$$\square^* = -\frac{\partial}{\partial t} - \Delta + R.$$

Given this, one obtains the differential equation

$$\square^* \left(R^f u\right) = -2\left|\mathrm{Rc}^f\right|^2 u. \tag{1.2}$$

This is the pointwise computation underlying the monotonicity of Perelman's $\mathcal{F}$-functional, and including a further weighting of u by a time scale yields Perelman's entropy density monotonicity. There is a curious duality between equations (1.1) (in the setting of Ricci flow) and (1.2). On the one hand, the weighted scalar curvature is a supersolution to a forward heat equation when the weight satisfies the forward heat equation. On the other hand, after coupling to a solution of the conjugate heat equation, the weighted scalar curvature is a subsolution to the conjugate heat equation. We next clarify and deepen this apparent linkage, generalizing the circle of ideas around scalar curvature, entropy formulas, and conjugate heat kernels.

We return to the setting of generalized Ricci flow, where the conjugate heat operator takes the form

$$\square^* = -\frac{\partial}{\partial t} - \Delta + R - \tfrac{1}{4}\left|H\right|^2.$$

The fundamental formula ([20] Chapter 6) underlying the energy monotonicity that generalizes (1.2) is then

$$\square^* \left(R^{H,f} u\right) = -2\left|\mathrm{Rc}^{H,f}\right|^2 u. \tag{1.3}$$

When integrated against $e^{-f}dV_g$, this pointwise formula yields the gradient flow interpretation for generalized Ricci flow. As natural as this differential equation may seem, it is difficult to exploit in part because the conjugate heat operator itself is difficult to control. In the setting of Ricci flow, the a priori lower bound on the scalar curvature controls the reaction term in the conjugate heat operator, and this plays a key role in many applications.

We get a better behaved heat kernel by including a further weight given by a solution to the dilaton flow. Indeed, the conjugate of the heat operator, taken with respect to the measure $dm = e^{-\phi}dV_g$, is

$$\square_\phi^* = -\frac{\partial}{\partial t} - \Delta + 2\nabla\phi + R^{H,\phi}.$$

We then obtain a generalization of (1.3):

Proposition 1.5. (cf. Proposition 3.3) Let (M, g_t, H_t, ϕ_t) denote a solution to generalized Ricci flow, and suppose $u = e^{-f}$ is a solution of $\square_\phi^* u = 0$. Then

$$\square_\phi^* \left(R^{H,f+\phi} u\right) = -2\left|\mathrm{Rc}^{H,f+\phi}\right|^2 u.$$

Note that in this formula there are naturally two functional degrees of freedom given by a solution to the dilaton flow, and then a solution to the weighted conjugate heat equation. This furthermore yields a generalization of the gradient flow property of generalized Ricci flow (cf. Proposition 3.7).

By including further weighting of u by a time scale we also obtain a generalization of Perelman's shrinker entropy monotonicity and differential Harnack estimate, parameterized by the choices of f and ϕ. In particular, recall Perelman's entropy density

$$W^{H,F} = \tau R^{H,F} + F - n,$$

To obtain a monotone quantity under generalized Ricci flow, we must include a further functional parameter that measures the concentration of H, namely a solution to the *conjugate dilaton flow*. In particular, we let ψ denote a solution of

$$\square_\phi^*(\psi u) = -\tfrac{1}{6}\,|H|^2 u.$$

We then obtain:

Proposition 1.6. (cf. Proposition 3.12) Let (M, g_t, H_t, ϕ_t) denote a solution to generalized Ricci flow, and suppose $u = (4\pi\tau)^{-\frac{n}{2}} e^{-f}$ is a solution of $\square_\phi^* u = 0$, where $\tau = T - t$ for some fixed T. Furthermore, suppose ψ is a solution of the conjugate dilaton flow. Then

$$\square_\phi^* \left[(W^{H,f+\phi} + \psi) u \right] = -2\tau \left| \mathrm{Rc}^{H,f+\phi} - \frac{1}{2\tau} g \right|^2 u.$$

This yields that if u approaches a weighted Dirac delta at some forward time then

$$(W^{H,f+\phi} + \psi) u \leqslant 0,$$

generalizing Perelman's Harnack estimate (cf. Corollary 3.14). Due to the presence of the conjugate dilaton flow solution, finding further geometric applications requires a more detailed understanding of the torsion H.

1.3 Further applications to Ricci flow

To conclude, we observe some formal extensions of some key results in the analysis of Ricci flow to the setting of weighted scalar curvature. First we note an extension of the definition of the Nash and Perelman entropies, adapted to weighted scalar curvature. In Proposition 4.2, we show convexity of the Nash entropy, extending fundamental observations in [26]. Going further, recall a key application of Perelman's differential Harnack estimate is the pseudolocality estimate [32], which roughly says that almost Euclidean regions will regularize for a short time. The strength of this result is that "almost Euclidean," is measured in a very weak sense, namely by a lower scalar curvature bound and an almost-Euclidean isoperimetric inequality. Based on the generalized entropy monotonicity formulas above, we give an extension of this result, involving the weighted scalar curvature and isoperimetric inequality (Theorem 1.7). The proof follows Perelman's original proof until the final stages, where the entropy integrand is manipulated to exploit the weighted scalar curvature bound. In the end, we require a technical result relating the weighted isoperimetric inequality to the weighted log-Sobolev inequality (Theorem 4.7), proved using the method of Steiner symmetrization.

Theorem 1.7. (cf. Theorem 4.8) For every $\alpha > 0$, there exist $\delta, \epsilon > 0$ satisfying the following: suppose we have a smooth pointed Ricci flow solution $(M, (p_0, 0), g_t)$ defined for $t \in [0, (\epsilon r_0)^2]$, such that each time slice is complete. Suppose that there exists $\phi_0 \in C_0^\infty(M)$ such that

1. $R^{\phi_0}(p,0) \geqslant -r_0^{-2}$ for any $p \in B_0(p_0, r_0)$,
2. $\phi_0(p,0) \geqslant -\delta$ for any $p \in B_0(p_0, r_0)$,
3. The ϕ_0-weighted isoperimetric constant of $B_0(p_0, r_0)$ satisfies $I_n^{\phi_0} \geqslant (1-\delta)c_n$, where c_n denotes the Euclidean isoperimetric constant.

Then $|\mathrm{Rm}|\,(p,t) < \alpha t^{-1} + (\epsilon r_0)^{-2}$ whenever $0 < t \leqslant (\epsilon r_0)^2$ and $d_t(p, p_0) \leqslant \epsilon r_0$.

2 Weighted Scalar Curvature Monotonicity Formulas

In this section, we show several monotonicity formulas for weighted scalar curvatures and certain further generalizations. First, we prove Proposition 1.1 of the introduction, and derive a rigidity result using the strong maximum principle. Then we extend to a more general setting where df is replaced by an arbitrary 1-form, and then to the setting where the metric flow is coupled to the heat flow for differential forms of arbitrary degree.

2.1 Weighted scalar curvature monotonicity and rigidity results

To begin, we formalize some definitions from the introduction.

Definition 2.1. Given a smooth manifold M, a triple (g, H, ϕ) of a Riemannian metric, closed three-form H, and function ϕ determine a *twisted Bakry–Emery curvature*

$$\mathrm{Rc}^{H,\phi} = \mathrm{Rc} - \tfrac{1}{4}H^2 + \nabla^2\phi - \tfrac{1}{2}\left(d_g^* H + i_{\nabla\phi}H\right).$$

These data also determine a *generalized scalar curvature*

$$R^{H,\phi} := R - \tfrac{1}{12}\,|H|^2 + 2\Delta\phi - |\nabla\phi|^2.$$

Definition 2.2. Given (M, g_t, H_t) a solution to generalized Ricci flow, a one-parameter family ϕ_t satisfies the *dilaton flow* if

$$\Box\phi = \tfrac{1}{6}\,|H|^2.$$

Remark 2.3. We separate the terminology of a solution to the dilaton flow associated to a solution of generalized Ricci flow to emphasize that the two flows are decoupled, and for instance the initial data for ϕ are arbitrary. On the other hand, for convenience, we will also refer to a triple (g_t, H_t, ϕ_t) as a solution of generalized Ricci flow, where a particular solution ϕ to the dilaton flow has been selected.

Proposition 2.4. (cf. Proposition 1.1) Let (M^n, g_t, H_t, ϕ_t) be a solution to generalized Ricci flow. Then

$$\square R^{H,\phi} = 2\left|\mathrm{Rc}^{H,\phi}\right|^2.$$

Proof. We compute the heat operator acting on each term of $R^{H,\phi}$ separately. First, a standard computation (cf. [20] Lemma 5.11) yields

$$\square R = -\tfrac{1}{2}\Delta|H|^2 + \tfrac{1}{2}\operatorname{div}\operatorname{div} H^2 + 2\left\langle \mathrm{Rc}, \mathrm{Rc} - \tfrac{1}{4}H^2\right\rangle.$$

Also, we have

$$\square\left(-\tfrac{1}{12}|H|^2\right) = \left\langle \tfrac{1}{8}H^2 - \tfrac{1}{2}\mathrm{Rc}, H^2\right\rangle - \tfrac{1}{6}\left\langle \Delta_d H, H\right\rangle + \tfrac{1}{12}\Delta|H|^2.$$

Next, we have

$$\begin{aligned}\square\Delta\phi &= \left\langle 2\,\mathrm{Rc} - \tfrac{1}{2}H^2, \nabla^2\phi\right\rangle + \left\langle -\tfrac{1}{2}\operatorname{div} H^2 + \tfrac{1}{4}d|H|^2, d\phi\right\rangle \\ &\quad + \Delta\left(\Delta\phi + \tfrac{1}{6}|H|^2\right) - \Delta\Delta\phi \\ &= \left\langle 2\,\mathrm{Rc} - \tfrac{1}{2}H^2, \nabla^2\phi\right\rangle + \left\langle -\tfrac{1}{2}\operatorname{div} H^2 + \tfrac{1}{4}d|H|^2, d\phi\right\rangle + \tfrac{1}{6}\Delta|H|^2.\end{aligned}$$

Furthermore, using the Bochner formula

$$\begin{aligned}\square|\nabla\phi|^2 &= 2\left\langle \nabla\left(\Delta\phi + \tfrac{1}{6}|H|^2\right), \nabla\phi\right\rangle + \left\langle 2\,\mathrm{Rc} - \tfrac{1}{2}H^2, d\phi\otimes d\phi\right\rangle - \Delta|\nabla\phi|^2 \\ &= -2\left|\nabla^2\phi\right|^2 + \tfrac{1}{3}\left\langle \nabla|H|^2, \nabla\phi\right\rangle - \tfrac{1}{2}\left\langle H^2, d\phi\otimes d\phi\right\rangle.\end{aligned}$$

Combining the above formulas and using the definition of $R^{H,\phi}$ yields

$$\begin{aligned}\square R^{H,\phi} &= 2\left\langle \mathrm{Rc}, \mathrm{Rc} - \tfrac{1}{4}H^2\right\rangle + \left\langle \tfrac{1}{8}H^2 - \tfrac{1}{2}\mathrm{Rc}, H^2\right\rangle + 2\left\langle 2\,\mathrm{Rc} - \tfrac{1}{2}H^2, \nabla^2\phi\right\rangle + 2\left|\nabla^2\phi\right|^2 \\ &\quad + \left\langle -\operatorname{div} H^2 + \tfrac{1}{2}d|H|^2, d\phi\right\rangle - \tfrac{1}{3}\left\langle \nabla|H|^2, \nabla\phi\right\rangle + \tfrac{1}{2}\left\langle H^2, d\phi\otimes d\phi\right\rangle \\ &\quad - \tfrac{1}{12}\Delta|H|^2 + \tfrac{1}{2}\operatorname{div}\operatorname{div} H^2 - \tfrac{1}{6}\left\langle \Delta_d H, H\right\rangle.\end{aligned} \tag{2.1}$$

We further recall the identity (cf. [20] Lemma 3.19)

$$\operatorname{div} H^2 = \tfrac{1}{6}\nabla|H|^2 - \left\langle d_g^* H, H\right\rangle,$$

where $\left\langle d_g^* H, H \right\rangle = (d_g^* H)^{kl} H_{ikl}$. This has the further consequence

$$\operatorname{div} \operatorname{div} H^2 = \tfrac{1}{6} \Delta \left| H \right|^2 + \tfrac{1}{3} \left\langle \Delta_d H, H \right\rangle + \left| d_g^* H \right|^2.$$

Plugging these into (2.1) yields

$$\Box R^{H,\phi} = 2 \left| \mathrm{Rc} - \tfrac{1}{4} H^2 + \nabla^2 \phi \right|^2 + \tfrac{1}{2} \left| d_g^* H + i_{\nabla \phi} H \right|^2 = 2 \left| \mathrm{Rc}^{H,\phi} \right|^2,$$

as claimed. ■

Corollary 2.5. Let (M^n, g_t, H_t, ϕ_t) be a solution to generalized Ricci flow on a compact manifold. Then, for any smooth existence time t, one has

$$\inf_{M \times \{t\}} R^{H,\phi} \geqslant \inf_{M \times \{0\}} R^{H,\phi}.$$

Proof. This follows from the maximum principle applied to Proposition 2.4 ■

Corollary 2.6. (cf. Corollary 1.3) Let (M, g, H, ϕ) satisfy $R^{H,\phi} \geqslant 0$. If M is compact, then either

1. The triple (g, H, ϕ) defines a generalized Ricci soliton and $R^{H,\phi} \equiv 0$
2. The manifold M admits a triple $(\overline{g}, \overline{H}, \overline{\phi})$ such that $R^{\overline{H},\overline{\phi}} > 0$ everywhere.

Proof. This follows from the strong maximum principle applied to the evolution equation of Proposition 2.4. ■

2.2 One-form scalar curvature monotonicity

In the context of generalized geometry, the dilaton ϕ, or more accurately its differential $d\phi$, plays the role of a *divergence operator* [19, 20], which is necessary to define the generalized Ricci tensor. A natural class of divergence operators are defined by a 1-form which need not even be closed. Next, we extend the results above to this more general setting.

Definition 2.7. Given a smooth manifold M, a triple (g, H, α) of a Riemannian metric, closed three-form H, and one-form α determine a *generalized scalar curvature*

$$R^{H,\alpha} = R - \tfrac{1}{12} \left| H \right|^2 - 2 d^* \alpha - \left| \alpha \right|^2.$$

Definition 2.8. Given (M, g_t, H_t) solution to generalized Ricci flow, a one-parameter family α_t satisfies the *dilaton flow* if

$$\frac{\partial}{\partial t}\alpha = \Delta_d \alpha + \tfrac{1}{6} d\,|H|^2. \tag{2.2}$$

Proposition 2.9. Let $(M^n, g_t, H_t, \alpha_t)$ be a solution to generalized Ricci flow. Then

$$\square R^{H,\alpha} = 2\left|\mathrm{Rc} - \tfrac{1}{4}H^2 + L_{\frac{1}{2}\alpha^\sharp} g\right|^2 + \tfrac{1}{2}|d\alpha|^2 + \tfrac{1}{2}\left|d_g^* H + i_{\alpha^\sharp} H\right|^2.$$

Proof. The proof is nearly identical to that of Proposition 2.4. We first note

$$\begin{aligned}\square d^*\alpha &= \left\langle 2\,\mathrm{Rc} - \tfrac{1}{2}H^2, \nabla\alpha\right\rangle + \left\langle -\tfrac{1}{2}\operatorname{div} H^2 + \tfrac{1}{4} d\,|H|^2, \alpha\right\rangle \\ &\quad + d^*\left(\Delta_d \alpha + \tfrac{1}{6} d\,|H|^2\right) - \Delta d^*\alpha \\ &= \left\langle 2\,\mathrm{Rc} - \tfrac{1}{2}H^2, \nabla\alpha\right\rangle + \left\langle -\tfrac{1}{2}\operatorname{div} H^2 + \tfrac{1}{4} d\,|H|^2, \alpha\right\rangle + \tfrac{1}{6}\Delta\,|H|^2.\end{aligned}$$

Furthermore, using the Bochner formula

$$\begin{aligned}\square\,|\alpha|^2 &= 2\left\langle \Delta_d\alpha + \tfrac{1}{6} d\,|H|^2, \alpha\right\rangle + \left\langle 2\,\mathrm{Rc} - \tfrac{1}{2}H^2, \alpha\otimes\alpha\right\rangle - \Delta\,|\alpha|^2 \\ &= -2\,|\nabla\alpha|^2 + \tfrac{1}{3}\left\langle d\,|H|^2, \alpha\right\rangle - \tfrac{1}{2}\left\langle H^2, \alpha\otimes\alpha\right\rangle.\end{aligned}$$

Using these and arguing as in Proposition 2.4, we obtain

$$\begin{aligned}\square R^{H,\alpha} &= 2\left|\mathrm{Rc} - \tfrac{1}{4}H^2 + \nabla\alpha\right|^2 + \tfrac{1}{2}\left|d_g^* H + i_{\alpha^\sharp} H\right|^2 \\ &= 2\left|\mathrm{Rc} - \tfrac{1}{4}H^2 + L_{\frac{1}{2}\alpha^\sharp} g\right|^2 + \tfrac{1}{2}|d\alpha|^2 + \tfrac{1}{2}\left|d_g^* H + i_{\alpha^\sharp} H\right|^2,\end{aligned}$$

as claimed. ∎

Remark 2.10. An a priori lower bound for $R^{H,\alpha}$ as in Corollary 2.5 and a rigidity result as in Corollary 1.3 are immediate consequences.

2.3 Further generalizations of Ricci flow

The results of this paper extend to a more general class of geometric evolution equations coupled to differential forms. In particular, let $H = \bigoplus_{k=1}^n H_k$ denote a closed section of

$\Lambda^* T^* M$, where the subscript indicates the degree of the differential form, and consider the system of equations

$$\frac{\partial}{\partial t} g = -2\,\mathrm{Rc} + \tfrac{1}{2} H^2, \qquad \frac{\partial}{\partial t} H = \Delta_d H, \tag{2.3}$$

where as before $H^2(X, Y) = \langle i_X H, i_Y H \rangle$. We note that the constant factor of $\frac{1}{2}$ on the term H^2 may seem arbitrary, but as the coupled partial differential equation for H is linear, this constant can be tuned to any positive value, independently for each value of k if desired. This system of equations obeys the same fundamental regularity of properties as generalized Ricci flow, and has an interpretation as Ricci flow on more general Courant algebroids (cf. [20]). For this setup, we define the generalized scalar curvature

$$R^{H,\phi} = R - \tfrac{1}{4} \sum_{k=1}^{n} \tfrac{1}{k} \left| H_k \right|^2 + 2\Delta\phi - |\nabla\phi|^2.$$

Also, we attach a dilaton flow of the form

$$\Box \phi = \tfrac{1}{4} \sum_{k=1}^{n} \tfrac{k-1}{k} \left| H_k \right|^2. \tag{2.4}$$

Proposition 2.11. Suppose (g_t, H_t) is a solution of (2.3) and ϕ_t is a solution of (2.4). Then

$$\Box R^{H,\phi} = 2 \left| \mathrm{Rc} - \tfrac{1}{4} H^2 + \nabla^2 \phi \right|^2 + \tfrac{1}{2} \left| d_g^* H + i_{\nabla\phi} H \right|^2.$$

Proof. For a closed differential form H_k of degree k, one has the Bianchi identities

$$\begin{aligned} \operatorname{div} H_k^2 &= -\langle d^* H_k, H_k \rangle + \frac{1}{2k} d \left| H_k \right|^2 \\ \operatorname{div} \operatorname{div} H_k^2 &= \tfrac{1}{2k} \Delta \left| H_k \right|^2 + \tfrac{1}{k} \langle \Delta_d H, H \rangle + \left| d_g^* H_k \right|^2. \end{aligned}$$

Using these, a straightforward modification of the proof of Proposition 2.4 gives the result. ■

Remark 2.12. An a priori lower bound for $R^{H,\phi}$ as in Corollary 2.5 and a rigidity result as in Corollary 1.3 are immediate consequences.

3 Weighted Energy and Entropy Formulas

In [32], Perelman introduced differential inequalities for the weighted scalar curvature, coupled to a solution of the conjugate heat equation. These key estimates complement the a priori scalar curvature lower bound, and underpin the proofs of κ-noncollapsing and pseudolocality for Ricci flow. In this section, complementary to our a priori lower bound for the generalized scalar curvature, we generalize these estimates to the case of generalized Ricci flow, by further coupling to a solution of the dilaton flow. The key point is to treat the auxiliary solution to the dilaton flow as a shift in Perelman's dilaton f, in particular constructing f as a solution to the ϕ-*weighted* conjugate heat equation. This leads to a family of differential inequalities with now *two* auxiliary functional parameters ϕ and f.

3.1 Weighted conjugate heat operators

In this subsection, we define the weighted conjugate heat operator, then show some elementary properties of this equation and its relation to the classic conjugate heat equation.

Definition 3.1. Let (M, g_t, H_t, ϕ_t) denote a solution to generalized Ricci flow. Define the *conjugate heat operator*

$$\square^* = -\frac{\partial}{\partial t} - \Delta + R - \tfrac{1}{4}\left|H\right|^2.$$

Also, we define the *weighted conjugate heat operator*

$$\square^*_\phi = -\frac{\partial}{\partial t} - \Delta + 2\nabla\phi + R^{H,\phi}.$$

Lemma 3.2. Let (M, g_t, H_t, ϕ_t) denote a solution to generalized Ricci flow. Given u_t, v_t smooth functions, we have

1. $\frac{d}{dt}\int uv\,\mathrm{d}V_g = \int_M (v\square u - u\square^* v)\,\mathrm{d}V_g$.
2. $\frac{d}{dt}\int uve^{-\phi}\mathrm{d}V_g = \int_M \left(v\square u - u\square^*_\phi v\right)e^{-\phi}\mathrm{d}V_g$.
3. A solution to $\square^* u = 0$ preserves mass against dV_g, that is, $\frac{d}{dt}\int_M u\mathrm{d}V_g = 0$.
4. A solution to $\square^*_\phi u = 0$ preserves mass against $e^{-\phi}dV_g$, that is, $\frac{d}{dt}\int_M ue^{-\phi}\mathrm{d}V_g = 0$.
5. One has $\square^*(ue^{-\phi}) = \left(\square^*_\phi u\right)e^{-\phi}$.

Proof. Items (1) and (3) are elementary consequences of the generalized Ricci flow equations. To show item (2), we compute

$$\begin{aligned}
&\frac{d}{dt}\int_M uve^{-\phi}\mathrm{d}V_g\\
&= \int_M \left[v\frac{\partial}{\partial t}u + u\frac{\partial}{\partial t}v + uv\left(-R + \tfrac{1}{4}|H|^2 - \Delta\phi - \tfrac{1}{6}|H|^2\right)\right]e^{-\phi}\mathrm{d}V_g\\
&= \int_M \left[v\Box u + u\left(e^{\phi}\Delta(e^{-\phi}v) + \frac{\partial}{\partial t}v + v(-R + \tfrac{1}{12}|H|^2 - \Delta\phi)\right)\right]e^{-\phi}\mathrm{d}V_g\\
&= \int_M \left[v\Box u + u\left(\frac{\partial}{\partial t}v + \Delta v - 2\langle\nabla v,\nabla\phi\rangle + v\left(-R + \tfrac{1}{12}|H|^2 - 2\Delta\phi + |\nabla\phi|^2\right)\right)\right]e^{-\phi}\mathrm{d}V_g\\
&= \int_M \left[v\Box u + u\left(\frac{\partial}{\partial t} + \Delta - 2\nabla\phi - R^{H,\phi}\right)v\right]e^{-\phi}\mathrm{d}V_g\\
&= \int_M \left[v\Box u - u\Box_\phi^* v\right]e^{-\phi}\mathrm{d}V_g,
\end{aligned}$$

as claimed. Item (4) is an elementary consequence of item (2). To show item (5), we directly compute

$$\begin{aligned}
\Box^*\left(ue^{-\phi}\right) &= (\Box^* u)e^{-\phi} - 2\left\langle\nabla u,\nabla e^{-\phi}\right\rangle - u\left(\frac{\partial}{\partial t} + \Delta\right)e^{-\phi}\\
&= \left[\left(-\frac{\partial}{\partial t} - \Delta + 2\nabla\phi + R - \tfrac{1}{4}|H|^2\right)u\right]e^{-\phi} + u\left(\frac{\partial}{\partial t}\phi + \Delta\phi - |\nabla\phi|^2\right)e^{-\phi}\\
&= \left[\left(-\frac{\partial}{\partial t} - \Delta + 2\nabla\phi + R - \tfrac{1}{12}|H|^2 + 2\Delta\phi - |\nabla\phi|^2\right)u\right]e^{-\phi}\\
&= \left(\Box_\phi^* u\right)e^{-\phi},
\end{aligned}$$

as claimed. ■

3.2 Energy density monotonicity

Proposition 3.3. (cf. Proposition 1.5) Let (M, g_t, H_t, ϕ_t) denote a solution to generalized Ricci flow, and suppose $u = e^{-f}$ is a solution of $\Box_\phi^* u = 0$. Then

$$\Box_\phi^*\left(R^{H,f+\phi}u\right) = -2\left|\mathrm{Rc}^{H,f+\phi}\right|^2 u.$$

Proof. In [20] Theorem 6.12, it is shown that if $v = e^{-F}$ is a solution of the conjugate heat equation $\square^* v = 0$, then

$$\square^* \left(R^{H,F} v\right) = -2 \left|\mathrm{Rc}^{H,F}\right| v.$$

It follows from Lemma 3.2 item (5) that $v = ue^{-\phi} = e^{-f-\phi}$ is a solution of $\square^* v = 0$. Furthermore, again using Lemma 3.2 item (5), we compute

$$\left(\square^*_{\phi} \left(R^{f+\phi} u\right)\right) e^{-\phi} = \square^* \left(R^{f+\phi} v\right) = -2 \left|\mathrm{Rc}^{H,f+\phi}\right| ue^{-\phi},$$

giving the claim. ■

Corollary 3.4. Let (M^n, g_t, H_t, ϕ_t) denote a solution to generalized Ricci flow. Suppose $u = e^{-f}$ is a solution of $\square^*_{\phi} u = 0$. Then

$$\sup_{M\times\{t\}} R^{H,f+\phi} \geqslant \sup_{M\times\{0\}} R^{H,f+\phi}.$$

Proof. It follows from Proposition 1.5 that $R^{f+\phi}$ is a subsolution of a backwards heat-type equation; therefore, by the maximum principle, its supremum is nonincreasing as a function of $-t$, therefore nondecreasing as a function of t. ■

3.3 Gradient property revisited and steady solitons

The results in the previous subsection underpin a generalization of the gradient flow interpretation of generalized Ricci flow. To begin, we recall the usual gradient flow interpretation. Define

$$\mathcal{F}(g,H,f) = \int_M \left(|\nabla f|^2 + R - \tfrac{1}{12}|H|^2\right) e^{-f} \mathrm{d}V_g,$$

$$\lambda(g,H) = \inf_{\{f \,|\, \int_M e^{-f} dV_g = 1\}} \mathcal{F}(g,H,f).$$

Following [32], it was shown in [31] that generalized Ricci flow is the gradient flow of λ. Furthermore, there is a general monotonicity formula for $\mathcal{F}$ once $u = e^{-f}$ is imposed to solve the conjugate heat equation along the flow. By explicitly including the dilaton shift ϕ, we get an infinite dimensional family of eigenvalues λ for which generalized Ricci flow is the gradient flow, following Proposition 1.5.

Definition 3.5. Given (M^n, g, H, ϕ), we define

$$\lambda(g, H, \phi) = \inf_{\{f \mid \int_M e^{-f} e^{-\phi} dV_g = 1\}} \mathcal{F}(g, H, f + \phi).$$

Lemma 3.6. The quantity $\lambda(g, H, \phi)$ is the lowest eigenvalue of the operator

$$\mathcal{L} = -4\Delta + 4\nabla\phi + R^{H,\phi}.$$

Proof. Integrating by parts, we can express, for $w = e^{-\frac{f}{2}}$,

$$\mathcal{F}(g, H, f + \phi) = \int_M \left(4 |\nabla w|^2 + R^{H,\phi} w^2\right) e^{-\phi} \mathrm{d}V_g,$$

and the result follows from a standard argument ■

Proposition 3.7. Generalized Ricci flow is the gradient flow of $\lambda(g, H, \phi)$.

Critical points for λ are steady generalized Ricci solitons, and recent constructions and classification results for these objects have appeared in [4, 38, 43, 44], including examples on compact manifolds. The next proposition shows that, on such a steady soliton, the dilaton flow (suitably normalized) converges to the relevant soliton function f.

Proposition 3.8. Suppose (M^n, g, H, f) is a steady generalized Ricci soliton, that is, $\mathrm{Rc}^{H,f} \equiv 0$, and let $\lambda = \lambda(g, H)$. Given $\phi_0 \in C^\infty(M)$ a smooth function, the solution to the gauge-fixed normalized dilaton flow

$$(\square + \nabla f)\, \phi = \tfrac{1}{6} |H|^2 - \lambda$$

with initial condition ϕ_0 exists on $[0, \infty)$ and $\lim_{t\to\infty} \phi_t = f + c$ for some $c \in \mathbb{R}$.

Proof. We recall the basic identities for a soliton:

$$R - \tfrac{1}{4} |H|^2 + \Delta f = 0,$$
$$R^{H,f} = \lambda.$$

The first follows by tracing $\mathrm{Rc}^{H,f} = 0$ and the second follows by first observing that R^f is constant by a Bianchi identity (cf. [20] Proposition 4.33), then observing by integration against f, suitably normalized, that this constant must be $\lambda(g,H)$. Using these identities, we compute

$$\begin{aligned}(\square+\nabla f)(\phi-f) &= \tfrac{1}{6}|H|^2-\lambda+\Delta f-|\nabla f|^2\\ &= \tfrac{1}{6}|H|^2-\left(R-\tfrac{1}{12}|H|^2+2\Delta f-|\nabla f|^2\right)+\Delta f-|\nabla f|^2\\ &= -R+\tfrac{1}{4}|H|^2-\Delta f\\ &= 0.\end{aligned}$$

As $\square+\nabla f$ is a strictly parabolic operator with no constant term, it follows from standard results that ϕ_t exists for all time, and moreover that $\phi-f$ approaches a constant. ■

3.4 Entropy density monotonicity

Definition 3.9. Given a smooth manifold M, a triple (g,H,F) of a Riemannian metric, closed three-form H, function F and $\tau>0$ determine a *generalized entropy density*

$$W^{H,F}=\tau R^{H,F}+F-n.$$

In the setting of Ricci flow, the quantity W^F is Perelman's entropy density, which satisfies a key monotonicity property. To obtain a monotone entropy quantity for generalized Ricci flow, we require a further functional parameter, which can be used to measure the concentration of H at a given point.

Definition 3.10. Let (M^n,g_t,H_t,ϕ_t) denote a solution to generalized Ricci flow, and suppose u is a solution of $\square^*_\phi u=0$. A function ψ is a solution of the *conjugate dilaton flow* if

$$\square^*_\phi(\psi u)=-\tfrac{1}{6}|H|^2u.$$

The lemma below shows that the flow is well-posed for arbitrary smooth terminal data.

Lemma 3.11. Given (M^n,g_t,H_t,ϕ_t) a solution of generalized Ricci flow, suppose $u=e^{-f}$ is a positive solution of $\square^*_\phi u=0$. Then ψ is a solution of the conjugate dilaton flow if and only if

$$\left(-\frac{\partial}{\partial t}-\Delta+2\nabla(f+\phi)\right)\psi=-\tfrac{1}{6}|H|^2.$$

Proof. Given a smooth function ψ, we formally compute

$$\begin{aligned}\square_\phi^*(\psi u) &= \left(\left(-\frac{\partial}{\partial t} - \Delta + 2\nabla\phi\right)\psi\right)u + \psi\square_\phi^* u - 2\langle\nabla\psi, \nabla u\rangle \\ &= \left(\left(-\frac{\partial}{\partial t} - \Delta + 2\nabla(f+\phi)\right)\psi\right)u.\end{aligned}$$

Using that u is positive, the result follows. ■

Proposition 3.12. (cf. Proposition 1.6) Let (M, g_t, H_t, ϕ_t) denote a solution to generalized Ricci flow, and suppose $u = (4\pi\tau)^{-\frac{n}{2}}e^{-f}$ is a solution of $\square_\phi^* u = 0$, where $\tau = T - t$ for some fixed T. Furthermore, suppose ψ is a solution of the conjugate dilaton flow. Then

$$\square_\phi^*\left[(W^{H,f+\phi} + \psi)u\right] = -2\tau\left|\mathrm{Rc}^{H,f+\phi} - \frac{1}{2\tau}g\right|^2 u.$$

Proof. First observe, using $\square_\phi^* u = 0$,

$$\begin{aligned}0 = \square_\phi^* u &= \left(-\frac{\partial}{\partial t} - \Delta + 2\nabla\phi + R^{H,\phi}\right)\left((4\pi\tau)^{-\frac{n}{2}}e^{-f}\right) \\ &= \left(-\frac{n}{2\tau} + \frac{\partial f}{\partial t} + \Delta f - |\nabla f|^2 - 2\langle\nabla\phi, \nabla f\rangle + R^{H,\phi}\right)\left((4\pi\tau)^{-\frac{n}{2}}e^{-f}\right).\end{aligned} \tag{3.1}$$

Now we compute using Proposition 1.5,

$$\begin{aligned}&\left(-\frac{\partial}{\partial t} - \Delta + 2\nabla\phi\right)W^{H,f+\phi} \\ &\quad= R^{H,f+\phi} + \tau\left(-\frac{\partial}{\partial t} - \Delta + 2\nabla\phi\right)R^{H,f+\phi} + \left(-\frac{\partial}{\partial t} - \Delta + 2\nabla\phi\right)(f+\phi) \\ &\quad= \left(2\Delta(f+\phi) - |\nabla f+\phi|^2 + R - \tfrac{1}{12}|H|^2\right) \\ &\qquad + \tau\left(-2\left|\mathrm{Rc} - \tfrac{1}{4}H^2 + \nabla^2(f+\phi)\right|^2 - \tfrac{1}{2}\left|d_g^* H + i_{\nabla(f+\phi)}H\right|^2 - 2\left\langle\nabla R^{H,f+\phi}, \nabla f\right\rangle\right) \\ &\qquad + \left(-\frac{n}{2\tau} - |\nabla f|^2 + 2\Delta\phi - |\nabla\phi|^2 + R - \tfrac{1}{12}|H|^2\right) + \left(-2\Delta\phi + 2|\nabla\phi|^2 - \tfrac{1}{6}|H|^2\right) \\ &\quad= -2\tau\left|\mathrm{Rc} - \tfrac{1}{4}H^2 + \nabla^2(f+\phi) - \frac{1}{2\tau}g\right|^2 - \tfrac{\tau}{2}\left|d_g^* H + i_{\nabla(f+\phi)}H\right|^2 + \tfrac{1}{6}|H|^2 \\ &\qquad - 2\left\langle\nabla(\tau R^{f+\phi}), \nabla f\right\rangle - 2|\nabla f|^2 - 2\langle\nabla f, \nabla\phi\rangle \\ &\quad= -2\tau\left|\mathrm{Rc}^{H,f+\phi} - \frac{1}{2\tau}g\right|^2 + \tfrac{1}{6}|H|^2 - 2\left\langle\nabla W^{H,f+\phi}, \nabla f\right\rangle.\end{aligned}$$

We then obtain

$$
\begin{aligned}
\square_\phi^* \left(W^{H,f+\phi} u \right) &= \left(-\frac{\partial}{\partial t} - \Delta + 2\nabla\phi + R^{H,\phi} \right) (W^{H,f+\phi} u) \\
&= \left(\left(-\frac{\partial}{\partial t} - \Delta + 2\nabla\phi \right) W^{H,f+\phi} \right) u + W^{H,f+\phi} \square_\phi^* u - 2 \left\langle \nabla W^{H,f+\phi}, \nabla u \right\rangle \\
&= -2\tau \left| \mathrm{Rc}^{H,f+\phi} - \frac{1}{2\tau} g \right|^2 u + \tfrac{1}{6} \left| H \right|^2 u,
\end{aligned}
$$

and the result follows. ■

3.5 Perelman's Harnack estimate

To prove the generalization of Perelman's Harnack estimate, we require some technical estimates for the conjugate heat kernel along solutions to generalized Ricci flow, which follow from straightforward modifications of results in [15] (cf. also [30, 47]). We fix (M^n, g_t, H_t, ϕ_t) a solution of generalized Ricci flow on $M \times [0, T]$, and let $Z(x, t, y, s)$ denote the fundamental solution of $\square u = 0$. We fix $p \in M$ and define $u(x, t) = Z(p, T, x, t)$. It follows that u is a solution of the conjugate heat equation $\square^* u = 0$. We define f by $u = (4\pi\tau)^{-\frac{n}{2}} e^{-f}$ as above, where $\tau = T - t$. Furthermore, we fix $0 < t_0 < T$ and let $h_{t_0} \geqslant 0$ denote a smooth compactly supported function, and let h_t denote the solution to $\square h = 0$ on $[t_0, T]$ with initial condition h_{t_0}. We then claim the following fundamental characteristics for the entropy density $W^{H,f}$.

Proposition 3.13. (cf. [15] Theorem 7.1) Given the setup above,

1. For any $t_0 < t < T$ one has that $hW^{H,f} \in L^1(M, g_t)$.
2. For any $t_0 < t_1 < t_2 < T$ one has

$$
\int_M hW^{H,f} \mathrm{d}V_{g_{t_1}} \leqslant \int_M hW^{H,f} \mathrm{d}V_{g_{t_2}} + \int_{t_1}^{t_2} \int_M \tfrac{1}{6} h \left| H \right|^2 u \mathrm{d}V_{g_t} \mathrm{d}t.
$$

3. One has

$$
\limsup_{t \to T^-} \int_M hW^{H,f} \mathrm{d}V_{g_t} \leqslant 0.
$$

Corollary 3.14. Let (M^n, g_t, H_t, ϕ_t) denote a solution to generalized Ricci flow on a compact manifold, defined for $t \in [0, T]$. Suppose $u = (4\pi(T-t))^{-\frac{n}{2}} e^{-f}$ approaches $e^{\phi}\delta_{x_0}$ as $t \to T$ and solves $\square_\phi^* u = 0$. Finally, let ψ denote the unique solution of the conjugate dilaton flow associated to u with $\psi_T \equiv 0$. Then, for any $0 \leqslant t_0 \leqslant T$, one has

$$\left(W^{H,f+\phi} + \psi\right) u \leqslant 0.$$

Proof. Suppose there exists a point (y_0, t_0) such that

$$V := \left(W^{H,f+\phi} + \psi\right) u(y_0, t_0) > 0.$$

Choose a smooth nonnnegative function h such that $h(y_0) = 1$ and $\sup_M h \leqslant 1$. Solve the forward heat equation $\square h = 0$ on $[t_0, T]$ with initial data h at time t_0. Note that by the maximum principle h remains nonnegative. We then compute using Lemma 3.2 and Proposition 3.12,

$$\frac{d}{dt}\int_M hVe^{-\phi}\mathrm{d}V_g = -\int_M h\square_\phi^* Ve^{-\phi}\mathrm{d}V_g \geqslant 0.$$

However, we obtain a contradiction since by construction $\int_M hVe^{-\phi}dV_{g_{t_0}} > 0$ and by Proposition 3.13 and the fact that $\psi_T \equiv 0$,

$$\lim_{t\to T}\int_M hVe^{-\phi}\mathrm{d}V_g = 0.$$

■

4 Weighted Nash Entropy and Pseudolocality Estimates

In this section, we give an extension of the Nash entropy convexity for Ricci flow to the setting of weighted scalar curvature. Then, we derive a pseudolocality principle for Ricci flow in terms of weighted scalar curvature generalizing Perelman's result [32]. The key new technical point is a weighted isoperimetric inequality and its relationship to a weighted log-Sobolev inequality.

4.1 Weighted Nash and Perelman Entropies

Definition 4.1. Given a smooth Riemannian manifold (M, g), $f, \phi \in C^\infty(M)$ and $\tau > 0$, let $d\nu = (4\pi\tau)^{-\frac{n}{2}}e^{-f}e^{-\phi}dV_g$, and define the *weighted Nash entropy* by

$$\mathcal{N}(g,\phi,f,\tau) := \int_M f\mathrm{d}\nu - \frac{n}{2}. \tag{4.1}$$

Furthermore, define the *weighted Perelman entropy* by

$$\mathcal{W}(g,\phi,f,\tau) := \int_M \left[\tau R^{f+\phi} + f - n\right] \mathrm{d}\nu. \tag{4.2}$$

Proposition 4.2. Let (M, g_t, ϕ_t) denote a solution to Ricci flow defined on $(-T, 0]$, and suppose u_t is a solution of the conjugate heat equation $\Box_\phi^* u = 0$, and define f via $u = (4\pi\tau)^{-\frac{n}{2}} e^{-f}$, where $\tau = -t$, and suppose the measure $d\nu$ has unit mass. Then

$$\frac{d}{d\tau}(\tau\mathcal{N}) = \mathcal{W}, \qquad \frac{d}{d\tau}\mathcal{W} = -2\tau \int_M \left|\mathrm{Rc}^{f+\phi} - \frac{1}{2\tau} g\right|^2 \mathrm{d}\nu.$$

Proof. Since $\Box_\phi^* u = 0$, an elementary computation shows that we have that

$$\frac{d}{d\tau} d\nu = \left(u^{-1}\Delta u - 2u^{-1}\langle\nabla\phi, \nabla u\rangle - \Delta\phi + |\nabla\phi|^2\right) d\nu.$$

Furthermore, using the evolution equation (3.1) satisfied by f and integration by parts, we compute

$$\begin{aligned}\frac{d}{d\tau}\int_M f \mathrm{d}\nu =& \int_M \left(\Delta f - |\nabla f|^2 - 2\langle\nabla\phi, \nabla f\rangle + R^\phi - \frac{n}{2\tau}\right) \mathrm{d}\nu \\ &+ \int_M f\left(u^{-1}\Delta u - 2u^{-1}\langle\nabla\phi, \nabla u\rangle - \Delta\phi + |\nabla\phi|^2\right) \mathrm{d}\nu \\ =& \int_M \left(R^{f+\phi} - \frac{n}{2\tau}\right) \mathrm{d}\nu.\end{aligned}$$

Thus, using that $\int_M d\nu \equiv 1$, we obtain

$$\frac{d}{d\tau}(\tau\mathcal{N}) = \int_M \left[\tau R^{f+\phi} + f - n\right] \mathrm{d}\nu = \mathcal{W},$$

as claimed. Furthermore, we note that we can express

$$\mathcal{W} = \int_M \left[\mathcal{W}^{f+\phi} - \phi\right] \mathrm{d}\nu.$$

Then, it follows by Lemma 3.2 and Proposition 3.12 with $\psi \equiv 0$ that

$$\begin{aligned}\frac{d}{d\tau}\mathcal{W} =& \frac{d}{d\tau}\int_M \left[\left(\mathcal{W}^{f+\phi} - \phi\right) u\right] e^{-\phi} \mathrm{d}V_g = \int_M \left[\Box_\phi^*(\mathcal{W}^{f+\phi} u) + u\Box\phi - \phi\Box_\phi^* u\right] e^{-\phi} \mathrm{d}V_g \\ =& -2\tau \int_M \left|\mathrm{Rc}^{f+\phi} - \frac{1}{2\tau} g\right|^2 \mathrm{d}\nu,\end{aligned}$$

as claimed. ■

4.2 Weighted log-Sobolev and isoperimetric inequalities

Definition 4.3. Given a Riemannian manifold (M^n, g) and $\phi \in C^\infty(M)$, we say that these data satisfy a *ϕ-weighted Sobolev inequality* if there exists $\Lambda = \Lambda(g, \phi) > -\infty$ such that for all $u = (2\pi)^{-\frac{n}{2}} e^{-f}$ such that $\int_M u e^{-\phi} \mathrm{d}V_g = 1$ one has

$$\int_M \left(\tfrac{1}{2} |\nabla f|^2 + f - n\right) u e^{-\phi} \mathrm{d}V_g \geqslant \Lambda. \tag{4.3}$$

Remark 4.4. Given $\psi \in W^{1,2}(M)$, define f by

$$\psi^2 = (2\pi)^{-\frac{n}{2}} e^{-f} \int_M \psi^2 e^{-\phi} \mathrm{d}V_g = u \int_M \psi^2 e^{-\phi} \mathrm{d}V_g.$$

It then follows that (4.3) is equivalent to

$$\int_M \left(2 |\nabla \psi|^2 - \psi^2 \log \psi^2\right) e^{-\phi} \mathrm{d}V_g + \log\left(\int_M \psi^2 e^{-\phi} \mathrm{d}V_g\right) \int_M \psi^2 e^{-\phi} \mathrm{d}V_g$$
$$\geqslant \left(\frac{n}{2} \log(2\pi) + n + \Lambda\right) \int_M \psi^2 e^{-\phi} \mathrm{d}V_g.$$

Definition 4.5. Fix (M, g) a Riemannian manifold and ϕ a smooth function. We say that (M, g) satisfies a *ϕ-weighted isoperimetric inequality* with constant I_n^ϕ if for all compact domains $\Omega \subset M$ with C^1 boundary one has

$$\left[\mathrm{Area}_{e^{-\phi} dA}(\partial \Omega)\right]^n \geqslant I_n^\phi \left[\mathrm{Vol}_{e^{-\phi} dV_g}(\Omega)\right]^{n-1}.$$

Remark 4.6. The ϕ-weighted isoperimetric inequality is scale-invariant for the metric g, whereas for a constant λ an elementary argument shows that $I_n^{\phi+\lambda} = e^{-\lambda} I_n^\phi$.

Theorem 4.7. Suppose (M^n, g) is a Riemannian manifold and fix ϕ a smooth function. Suppose $\overline{B}(x_0, R)$ is compact, and that $(B(x_0, R), g)$ satisfies a ϕ-weighted isoperimetric inequality with constant I_n^ϕ. Then, for any C^1 function ψ compactly supported in $B(x_0, R)$, one has

$$\int_M \left(2 |\nabla \psi|^2 - \psi^2 \log \psi^2\right) e^{-\phi} \mathrm{d}V_g + \log\left(\int_M \psi^2 e^{-\phi} \mathrm{d}V_g\right) \int_M \psi^2 e^{-\phi} \mathrm{d}V_g$$
$$\geqslant \left(\frac{n}{2} \log(2\pi) + n + \log\left(\frac{I_n^\phi}{c_n}\right)\right) \int_M \psi^2 e^{-\phi} \mathrm{d}V_g,$$

where c_n denotes the Euclidean isoperimetric constant. In particular, $(B(x_0,R),g)$ satisfies a ϕ-weighted Sobolev inequality with constant

$$\Lambda = \log\left(\frac{I_n^\phi}{c_n}\right). \tag{4.4}$$

Proof. For simplicity of notation, we can just set $M = B(x_0,R)$. Also, we perform a useful rescaling. If we set $\widetilde{g} = \left(\frac{c_n}{I_n^\phi}\right)^{\frac{2}{n}} g$, then the required inequality is equivalent to

$$\begin{aligned}\int_M \left(2\left(\frac{c_n}{I_n^\phi}\right)^{\frac{2}{n}} |\nabla\psi|^2 - \psi^2\log\psi^2\right) e^{-\phi}\mathrm{d}V_{\widetilde{g}} + \log\left(\int_M \psi^2 e^{-\phi}\mathrm{d}V_{\widetilde{g}}\right)\int_M \psi^2 e^{-\phi}\mathrm{d}V_{\widetilde{g}} \\ \geqslant \left(\frac{n}{2}\log(2\pi)+n\right)\int_M \psi^2 e^{-\phi}\mathrm{d}V_{\widetilde{g}}.\end{aligned} \tag{4.5}$$

We will show (4.5), dropping the tildes from the notation for simplicity. By an approximation argument, it suffices to consider the case $\psi \geqslant 0$. For $s > 0$, let $M_s = \{x \in M \mid \psi \geqslant s\}$. Let $\Gamma_s = \partial M_s$, and let

$$F(s) := \mathrm{Vol}_{e^{-\phi}dV_g}(M_s).$$

Choose $r_0 < \infty$ such that

$$\mathrm{Vol}_{e^{-\phi}dV_g}(\{\psi > 0\}) = \omega_n r_0^n = \mathrm{Vol}_{dV_{\mathbb{R}^n}}(B_{r_0}).$$

Choose the unique rotationally invariant function $h : \mathbb{R}^n \to \mathbb{R}$ such that

$$\mathrm{Vol}_{dV_{\mathbb{R}^n}}(\{h \geqslant s\}) = F(s),$$

and such that $h(x) = 0$ for $|x| \geqslant r_0$. Furthermore, set $M_s' = \{h \geqslant s\}$, and $\Gamma_s' = \partial M_s'$. We recall the statement of the co-area formula,

$$\int_M H\,|\nabla f|\,\mathrm{d}V_g = \int_{-\infty}^{\infty}\int_{\{f=s\}} H\mathrm{d}A\mathrm{d}s.$$

Applying this with $H = e^{-\phi}\,|\nabla\psi|^{-1}$ and $f = \psi$ on M_t yields

$$F(t) = \mathrm{Vol}_{e^{-\phi}dV_g}(M_t) = \int_t^\infty \int_{\{\psi=s\}} |\nabla\psi|^{-1} e^{-\phi}\mathrm{d}A\mathrm{d}s.$$

This implies that for almost all s one has

$$-\frac{dF}{ds}(s) = \int_{\{\psi=s\}} |\nabla\psi|^{-1} e^{-\phi} \mathrm{d}A.$$

Thus, for an arbitrary Lipschitz function λ, we have

$$-\int_0^\infty \lambda \frac{\mathrm{d}F}{\mathrm{d}s}\mathrm{d}s = \int_0^\infty \lambda \int_{\{\psi=s\}} |\nabla\psi|^{-1} e^{-\phi} \mathrm{d}A\mathrm{d}s.$$

On the other hand, applying the co-area formula with $H = \lambda(\psi)\,|\nabla\psi|^{-1}e^{-\phi}$ yields

$$\int_{\{\psi>0\}} \lambda(\psi)e^{-\phi}\mathrm{d}V_g = \int_0^\infty \lambda(s) \int_{\{\psi=s\}} |\nabla\psi|^{-1} e^{-\phi} \mathrm{d}A\mathrm{d}s.$$

Integrating by parts, we have

$$\int_0^\infty \frac{\mathrm{d}\lambda}{\mathrm{d}s} F\mathrm{d}s = \int_{\{\psi>0\}} \lambda(\psi)e^{-\phi}\mathrm{d}V_g.$$

By construction, we may argue similarly with the function $\lambda(h)$ on $\mathbb{R}^n$ to obtain

$$\int_{\{h>0\}} \lambda(h)\mathrm{d}V_{\mathbb{R}^n} = \int_0^\infty \frac{\mathrm{d}\lambda}{\mathrm{d}s} F(s) = \int_{\{\psi>0\}} \lambda(\psi)e^{-\phi}\mathrm{d}V_g.$$

Choosing $\lambda(s) = (\log s^2)s^2$ and $\lambda(s) = s^2$ thus implies

$$\int_M (\log\psi^2)\psi^2\mathrm{d}V_g = \int_{\mathbb{R}^n} (\log h^2)h^2\mathrm{d}V_{\mathbb{R}^n}, \qquad \int_M \psi^2\mathrm{d}V_g = \int_{\mathbb{R}^n} h^2\mathrm{d}V_{\mathbb{R}^n}. \tag{4.6}$$

This reduces the result to comparing to comparing the gradient terms. By the co-area formula, we have

$$\int_t^\infty \int_{\Gamma_s} |\nabla\psi|^{-1} e^{-\phi}\mathrm{d}A_g\mathrm{d}s = F(t) = \int_t^\infty \int_{\Gamma_s'} |\nabla h|^{-1}\mathrm{d}A_{\mathbb{R}^n}\mathrm{d}s.$$

Differentiating this yields

$$\int_{\Gamma_s} |\nabla\psi|^{-1} e^{-\phi}\mathrm{d}A = \int_{\Gamma_s'} |\nabla h|^{-1}\mathrm{d}A_{\mathbb{R}^n}.$$

Furthermore, note that by construction, one has

$$(\mathrm{Area}(\Gamma_s'))^n = c_n(\mathrm{Vol}(M_s'))^{n-1} = c_n(\mathrm{Vol}_{e^{-\phi}dV_g}(M_s))^{n-1}$$
$$\leqslant \frac{c_n}{I_n^\phi}(\mathrm{Area}_{e^{-\phi}dA}(\Gamma_s))^n.$$

Furthermore, using that h is rotationally symmetric, and applying Hölder's inequality, we get

$$\begin{aligned}\int_{\Gamma_s'} |\nabla h|\, \mathrm{d}A_{\mathbb{R}^n} \cdot \int_{\Gamma_s'} |\nabla h|^{-1} \mathrm{d}A_{\mathbb{R}^n} &= (\mathrm{Area}(\Gamma_s'))^2 \\ &\leqslant \left(\frac{c_n}{I_n^\phi}\right)^{\frac{2}{n}} (\mathrm{Area}_{e^{-\phi}dA}(\Gamma_s))^2 \\ &\leqslant \left(\frac{c_n}{I_n^\phi}\right)^{\frac{2}{n}} \int_{\Gamma_s} |\nabla\psi|\, e^{-\phi}\mathrm{d}A_g \cdot \int_{\Gamma_s} |\nabla\psi|^{-1} e^{-\phi}\mathrm{d}A_g.\end{aligned}$$

This then implies

$$\int_{\Gamma_s'} |\nabla h|\, \mathrm{d}A_{\mathbb{R}^n} \leqslant \left(\frac{c_n}{I_n^\phi}\right)^{\frac{2}{n}} \int_{\Gamma_s} |\nabla\psi|\, e^{-\phi}\mathrm{d}A_g.$$

The co-area formula then implies

$$\begin{aligned}\left(\frac{c_n}{I_n^\phi}\right)^{\frac{2}{n}} \int_M |\nabla\psi|^2 e^{-\phi}\mathrm{d}V_g &= \left(\frac{c_n}{I_n^\phi}\right)^{\frac{2}{n}} \int_0^\infty \int_{\Gamma_s} |\nabla\psi|\, e^{-\phi}\mathrm{d}A_g \mathrm{d}s \\ &\geqslant \int_0^\infty \int_{\Gamma_s'} |\nabla h|\, \mathrm{d}A_{\mathbb{R}^n} \\ &= \int_{\mathbb{R}^n} |\nabla h|^2 \mathrm{d}V_{\mathbb{R}^n}.\end{aligned} \tag{4.7}$$

Combining (4.6), (4.7) and applying the Euclidean logarithmic Sobolev inequality [21] yields

$$
\begin{aligned}
&\int_M \left(2\left(\frac{c_n}{I_n^\phi}\right)^{\frac{2}{n}} |\nabla\psi|^2 - \psi^2\log\psi^2\right) e^{-\phi}\mathrm{d}V_g + \log\left(\int_M \psi^2 e^{-\phi}\mathrm{d}V_g\right)\int_M \psi^2 e^{-\phi}\mathrm{d}V_g \\
&\quad - \left(\frac{n}{2}\log(2\pi)+n\right)\int_M \psi^2 e^{-\phi}\mathrm{d}V_g \\
&\geqslant \int_{\mathbb{R}^n}\left(2|\nabla h|^2 - h^2\log h^2\right)\mathrm{d}V_{\mathbb{R}^n} + \log\left(\int_{\mathbb{R}^n} h^2\mathrm{d}V_{\mathbb{R}^n}\right)\int_{\mathbb{R}^n} h^2\mathrm{d}V_{\mathbb{R}^n} \\
&\quad - \left(\frac{n}{2}\log(2\pi)+n\right)\int_{\mathbb{R}^n} h^2\mathrm{d}V_{\mathbb{R}^n} \\
&\geqslant 0,
\end{aligned}
$$

as required. ■

4.3 Weighted pseudolocality estimate

Theorem 4.8. (cf. Theorem 1.7) For every $\alpha > 0$, there exist $\delta, \epsilon > 0$ satisfying the following: suppose we have a smooth pointed Ricci flow solution $(M, (p_0, 0), g_t)$ defined for $t \in [0, (\epsilon r_0)^2]$, such that each time slice is complete. Suppose that there exists $\phi_0 \in C_0^\infty(M)$ such that

1. $R^{\phi_0}(p, 0) \geqslant -r_0^{-2}$ for any $p \in B_0(p_0, r_0)$,
2. $\phi_0(p, 0) \geqslant -\delta$ for any $p \in B_0(p_0, r_0)$,
3. The ϕ_0-weighted isoperimetric constant of $B_0(p_0, r_0)$ satisfies $I_n^{\phi_0} \geqslant (1-\delta)c_n$, where c_n denotes the Euclidean isoperimetric constant.

Then $|\mathrm{Rm}|(p, t) < \alpha t^{-1} + (\epsilon r_0)^{-2}$ whenever $0 < t \leqslant (\epsilon r_0)^2$ and $d_t(p, p_0) \leqslant \epsilon r_0$.

Remark 4.9. In the hypotheses above, there is both a lower bound for ϕ_0 in item (2) and an implicit upper bound for ϕ_0 in item (3) (cf. Remark 4.6).

Proof. We follow the proof in [28] and briefly indicate the initial phases. By scale invariance of the hypotheses and conclusion, it suffices to consider the case $r_0 = 1$ and also $\alpha < \frac{1}{100n}$. If the theorem is false, there exist sequences $\epsilon_k \to 0, \delta_k \to 0$ pointed Ricci flow solutions $(M_k, (p_{0,k}, 0), g^k)$ and functions ϕ_0^k, which satisfy the hypotheses of the theorem but for which there exists (p_k, t_k) with $0 < t_k \leqslant \epsilon_k$, $d(p_k, t_k) \leqslant \epsilon_k$, but $|\mathrm{Rm}|(p_k, t_k) \geqslant \alpha t_k^{-1} + \epsilon_k^{-2}$. Choose $A_k = \frac{1}{100n\epsilon_k}$, and employ ([28] Lemma 31.1) to obtain a new sequence $(\overline{p}_k, \overline{t}_k)$, which lie at the center of parabolic balls of a controlled size. We let $u_k = (4\pi(\overline{t}_k - t))^{-\frac{n}{2}} e^{-F_k}$ satisfy $\square^* u_k = 0$ and $\lim_{t \to \overline{t}_k^-} u(p,t) = \delta_{\overline{p}_k}(p)$. Furthermore, let $W^k = W^{F_k}$. It follows from ([28] Lemma 33.4) that there exists $\beta > 0$ so that for all sufficiently large k, there exists $\widetilde{t}_k \in [\overline{t}_k - \frac{1}{2}\alpha Q_k^{-1}, \overline{t}_k]$ with $\int_{B_k} W^k \mathrm{d}V_k \leqslant -\beta$, where $Q_k = |\mathrm{Rm}|\,(\overline{p}_k, \overline{t}_k)$, and $B_k = B_{\widetilde{t}_k}(\overline{p}_k, \sqrt{\overline{t}_k - \widetilde{t}_k})$.

The aim is to use this to contradict the weighted log-Sobolev inequality at the initial time. We drop the subscripts and work with a particular large k. One constructs a function h on spacetime from a modified distance function satisfying certain estimates we will use below (cf. [28] §34). Using Perelman's Harnack estimate and estimates for h, we produce a function $u = (4\pi\overline{t})^{-\frac{n}{2}} e^{-F}$ at time zero with mass arbitrarily close to 1, which satisfies (cf. [28] (34.6))

$$\beta(1 - A^{-2}) \leqslant -\int W^F h u \mathrm{d}V_g.$$

We drop the subscript on ϕ_0 and define f via $F = f + \phi$. We thus express

$$\begin{aligned}
\beta(1 - A^{-2}) &\leqslant \int_M \left[\left(-2\Delta\phi - 2\Delta f + |\nabla(f+\phi)|^2 - R\right)\overline{t} - (f+\phi) + n\right] h u \mathrm{d}V_g \\
&= \int_M \left[\left(-R^\phi - 2\Delta f + |\nabla f|^2 + 2\langle \nabla f, \nabla\phi\rangle\right)\overline{t} - (f+\phi) + n\right] h u \mathrm{d}V_g.
\end{aligned} \tag{4.8}$$

We set $\widetilde{f} = f - \log h$ and integrate by parts to obtain

$$\begin{aligned}
&\int_M \left(-2\Delta f + |\nabla f|^2 + 2\langle \nabla f, \nabla\phi\rangle\right) h e^{-f-\phi} \mathrm{d}V_g \\
&\quad = \int_M \left(2\left\langle \nabla f, \nabla(h e^{-f-\phi})\right\rangle + (|\nabla f|^2 + 2\langle \nabla f, \nabla\phi\rangle) h e^{-f-\phi}\right) \mathrm{d}V_g \\
&\quad = \int_M \left\langle \nabla f, 2h^{-1}\nabla h - \nabla f\right\rangle h e^{-f-\phi} \mathrm{d}V_g \\
&\quad = \int_M \left(-\left|\nabla\widetilde{f}\right|^2 + h^{-2}|\nabla h|^2\right) h e^{-f-\phi} \mathrm{d}V_g.
\end{aligned}$$

Furthermore, by hypothesis $-R^\phi h \leqslant 1$ and by construction $h^{-1}|\nabla h|^2 \leqslant \frac{10}{(10A\epsilon)^2}$, so that

$$\int_M \overline{t}\left(\frac{|\nabla h|^2}{h} - R^\phi h\right) u \mathrm{d}V_g \leqslant A^{-2} + \epsilon^2.$$

Also, by construction, we have

$$-\int_M uh\log h \leqslant cA^{-2}$$

for a controlled constant c. Finally, by hypothesis (2), we note that

$$\int_M -\phi hu \leqslant \delta.$$

Plugging the above observations into (4.8) yields

$$\beta(1-A^{-2})-(1+c)A^{-2}-\epsilon^2-\delta \leqslant \int_M \left(-\bar{t}\,|\nabla\widetilde{f}|^2-\widetilde{f}+n\right)(4\pi\bar{t})^{-\frac{n}{2}}e^{-\widetilde{f}}e^{-\phi}\mathrm{d}V_g.$$

For ϵ,δ sufficiently small, it follows then that

$$\tfrac{1}{2}\beta \leqslant \int_M \left(-\bar{t}\,|\nabla\widetilde{f}|^2-\widetilde{f}+n\right)(4\pi\bar{t})^{-\frac{n}{2}}e^{-\widetilde{f}}e^{-\phi}\mathrm{d}V_g.$$

For δ chosen sufficiently small, this contradicts hypothesis (3) using Theorem 4.7. ■

Remark 4.10. Theorem 1.7 implies via smoothing a diffeomorphism finiteness result for the class of manifolds with a weighted scalar curvature lower bound, almost nonnegative weight, volume upper bound, and almost-Euclidean weighted isoperimetric inequality (cf. [28] Theorem 37.1).

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