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# Global attractors for a novel nonlinear piezoelectric beam model with dynamic electromagnetic effects and viscoelastic memory

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**Abstract.** We consider a nonlinear coupled PDE model for a single piezoelectric beam retaining the electromagnetic effects and a long-range strain memory. Nonlinear source terms in both mechanical and electromagnetic equations and a viscous magnetic damping term in the electromagnetic equation are considered in the model. The mathematical analysis of this model is particularly needed for certain class of fully dynamic piezoelectric materials demonstrating a viscoelastic memory or creep. With an injection of magnetic damping, the structure of the dynamical system associated with the solutions of this system allows using the quasi-stability theory in order to obtain the existence of global and exponential attractors.

**Mathematics Subject Classification.** 93D20, 93D15, 74F15, 35Q60, 35Q74.

**Keywords.** Quasi-stability, Viscoelastic memory, Exponential attractors, Global attractors, Nonlinear piezoelectric beam, PVDF sensors, Creep.

## 1. Introduction

In this paper, a single-layer piezoelectric beam retaining the long-range viscoelastic memory (creep) and fully dynamic electromagnetic effects is considered. Modeling creep is simply considering that the stress and electric field at any instant may depend on both the instantaneous and the complete history of strains [50]. Following the modeling assumptions in [33,50] and denoting $v = v(x,t)$ and $p = p(x,t)$ the longitudinal displacement and total electric charge at point $x$ and at time $t$, respectively, the following sets of equations in the time domain, the equations of motion are

$$\begin{aligned} \rho v_{tt} - \alpha v_{xx} + \gamma\beta p_{xx} + \int_0^\infty \lambda(s) v_{xx}(t-s)ds + f_1(v,p) = h_1(x) \quad \text{in} \quad (0,L)\times(0,\infty), \\ \mu p_{tt} - \beta p_{xx} + \gamma\beta v_{xx} + g(p_t) + f_2(v,p) = h_2(x) \quad \text{in} \quad (0,L)\times(0,\infty) \end{aligned} \tag{1}$$

with initial conditions

$$\begin{aligned} &v(x,0) = v_0(x),\ v_t(x,0) = v_1(x),\ p(x,0) = p_0(x),\ p_t(0,x) = p_1(x), \quad x\in(0,L), \\ &v(x,-t) = v_2(x,t), \quad (x,t)\in(0,L)\times(0,\infty) \end{aligned} \tag{2}$$

where $f_1(v,p), f_2(v,p)$ represent internal forcing terms (sources), $h_1(x), h_2(x)$ represent external forces, and $g(p_t)$ denotes the distributed current damping. Moreover, $\lambda(t)$ is a relaxation or memory kernel, $v_0$, $v_1$, $v_2$, $p_0$, and $p_1$ are functions that belong to appropriate spaces, and $\alpha, \rho, \gamma, \beta, \mu$ are positive material constants with

$$\alpha := \alpha_1 + \gamma^2\beta, \quad \alpha_1 > 0, \tag{3}$$

and $\alpha_1$ satisfies

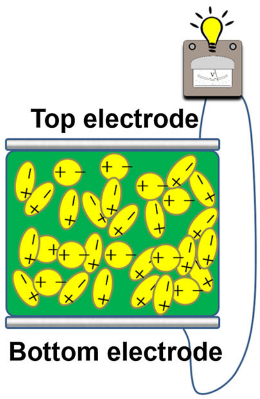
$$k_1 := \alpha_1 - \int_0^\infty \lambda(s)ds > 0. \tag{4}$$

Piezoelectric beams are multi-functional smart materials to develop electric displacement that is directly proportional to an applied mechanical stress; see Fig. 1. Operating a piezoelectric beam as an actuator requires an electrical input (voltage, current, or charge). One of the main components of the electrical input is the drive frequency which determines how fast a piezoelectric beam vibrates or changes its state. Periodic (regularly repeating) and arbitrary signals can be used to drive a piezoelectric beam, which corresponds to continuous control of vibrational modes. Due to their small size, flexibility, and high power density, they have become more and more promising in industrial applications such as from implantable biomedical devices to PVDF sensors [5,13,25,27,44,47].

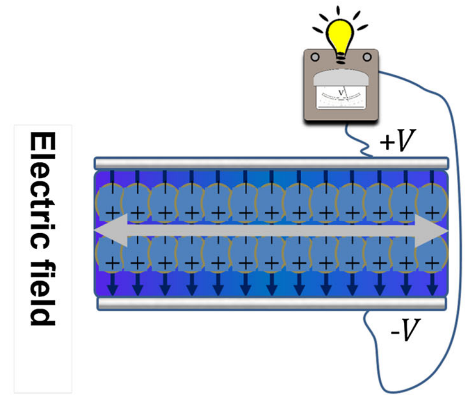
Two common piezoelectric materials are polymers (polyvinylidene fluoride, PVDF) and ceramics (lead zirconate titanate, PZT) [48]. Synthetic PVDF polymers are widely used to construct various types of sensors since having piezoelectric properties, and its flexibility allows applications of such sensors where brittle and rigid ceramic materials are not suitable under various loadings and temperatures [49]. The research on PVDF polymers reveals that the generic description of piezoelectricity has some shortcomings [50], and thus, classical continuum mechanics fails to predict the interactive effects of creep for PVDF polymers. In fact, it is observed that PVDF polymers tend to exhibit accelerated creep rates under superimposed static and cyclic loads. The results indicate that the linear viscoelastic theory describes a time response of PVDF polymers at the applied stress in both longitudinal and transverse directions [50]. Therefore, the piezoelectric materials, used as sensors or energy harvesters, should be characterized adequately by the long-memory dynamic modeling [12].

From the perspective of modeling of mechanical vibrations and electromagnetic effects on a piezoelectric beam during the motion, the existing literature predominantly uses the electrostatic/quasi-static approach due to the Maxwell's equations; see [26,33,48] and the references therein. Therefore, the amount of magnetic energy produced/stored is completely discarded [51]. However, these effects can be minor or major in certain applications [52, Chap. 8], [53]. In fact, the fully dynamic electromagnetic effects, unlike the electrostatic case, may have a dramatic effect on the boundary observability/controllability of certain class of single-layer or multilayer piezoelectric systems, and observability/controllability results are sensitive to material parameters if there is only one boundary controller applied to the piezoelectric layer, i.e., see [33,36–38]. In fact, unlike the existing literature, two boundary controllers are necessary for the exact controllability of mechanical and electromagnetic variables [41].

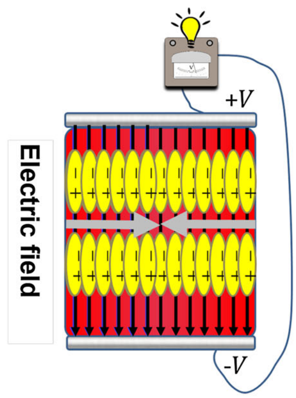
At this point, it is fair to mention about the research done to rigorously analyze wave equation-type nonlinear PDE models like (1)–(2). For example, there is a large literature study for a single wave equation modeling viscoelastic beams with a memory term added in a distributed fashion and other physics effects such as damping, delay, nonlinear source, and external force terms. It is worth mentioning the pioneer work of [11] where the asymptotic behavior of solutions of the viscoelastic equations (of memory type) is investigated. Among several works which deal with viscoelastic equations long-time memory, we refer the reader to a large class of papers on viscoelastic beams [6,28,32], Timoshenko beams [29], Berger plates [39], and the references therein. Memory effects on the overall well-posedness and stability of coupled systems of wave equations in the same domain can be considered via a boundary dissipation with the addition of the memory term in one of the equations [30], or a distributed damping term together with the memory term in one of the equations, i.e., [4,46]. There is also a large literature study on the transmission problems with a memory term, i.e., see [7] and the references therein. In the case of no damping, a lack of uniform stability result is shown since the memory term is not strong enough to exponentially stabilize dynamics both in the case of two elastic membranes [2] and in the case of full set of Maxwell's equations [43].



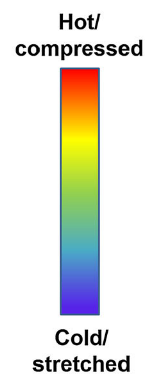

**(A)** A piezoelectric beam is an elastic beam with electrodes at the top and bottom surfaces. Electrodes are connected to an external electric circuit. As voltage is applied to its electrodes, charges separate and line up in the vertical direction.




**(B)** Positive voltages are applied and the beam actively stretches. Conversely, forced stretches produce positive voltages.




**(C)** Negative voltages are applied and the beam actively compressed. Conversely, forced compressions produce negative voltages.




**(D)** The color mapping indicates the voltage distribution.

FIG. 1. Piezoelectric materials can be used as actuators and sensors especially in vibration control and energy harvesting, respectively

The use of the quasi-stability theory is crucial for the long-time behavior analysis of (1)–(2) as a decomposition of the difference dynamics into a stable component and compact component. Note that the existence of a finite-dimensional attractor with optimal regularity and fractal exponential attractor is achieved by the quasi-stability theory; see [8–10] for pioneer work and the references therein. Inspired by the ample amount of results, similar approaches have been recently adopted by the authors to rigorously analyze the long-time memory behavior for piezoelectric beams with delay term [18], boundary memory terms [17], and thermal effects [22]. For other relevant work on wave systems, refer to [16,18–21,23,40,45] and references therein.

For the system (1)–(2) without the nonlinear damping term $g(p_t)$, i.e.,

$$\rho v_{tt} - \alpha v_{xx} + \gamma\beta p_{xx} + \int_0^\infty \lambda(s) v_{xx}(t-s)ds + f_1(v,p) = h_1(x) \quad \text{in} \quad (0,L)\times(0,\infty), \tag{5}$$

$$\mu p_{tt} - \beta p_{xx} + \gamma\beta v_{xx} + f_2(v,p) = h_2(x) \quad \text{in} \quad (0,L)\times(0,\infty), \tag{6}$$

with the same set of boundary conditions

$$v(0,t) = v_x(L,t) = p(0,t) = p_x(L,t) = 0, \quad t \geq 0, \tag{7}$$

the memory term is not strong enough to exponentially stabilize a coupled system of nonlinear equations of this sort. Indeed, the system (5)–(7) is not even a gradient system. For the system (5)–(7) to be exponentially stable, it is necessary to add more dissipation to the system. Adding the nonlinear damping term $g(p_t)$ to the electromagnetic equation is physical and viable, and it can be implemented easily through the circuit attached to the electrodes. Indeed, it is the electric current injected through the electrodes.

The interaction between the memory term $\int_0^\infty \lambda(s) v_{xx}(t-s)ds$ and the damping term $g(p_t)$ in (1)–(2) with the consideration of natural boundary conditions for the clamped-free beam generates mathematical hurdles that are difficult to overcome by the Lyapunov's approaches. This same difficulty is not present for the coupled-wave systems involving two damping terms $g_1(v_t)$ and $g_2(p_t)$, i.e., [46], or Timoshenko beams [34], where many authors consider the well-known equalizing relationship $\frac{\kappa}{\rho_1} = \frac{b}{\rho_2}$ (something not preferred in applications) to overcome the difficulty imposed by the Lyapunov's approach.

The novelty of the work here can be summarized as the following:

(i) The results obtained here are novel for fully dynamic and non-compactly coupled piezoelectric beam systems.
(ii) The system (1)–(2) deals with the interaction between the memory effect and the nonlinear damping, which makes it challenging in using the Lyapunov approach.
(iii) Since the system (1) is accompanied by the natural clamped-free boundary conditions (2), it is rather a difficult task to construct the quasi-stability result.
(iv) More importantly, there is no restriction for the speeds of wave propagations $\frac{\alpha}{\rho}$ and $\frac{\beta}{\mu}$ in (1), corresponding to mechanical and electromagnetic vibrations, respectively.

The outline of the paper is as follows: In Sect. 2, assumptions and notations together with the functional analytic setup are proposed. In Sect. 3, the Cauchy problem is formulated with energy of solutions. In Sect. 4, existence, uniqueness, and continuous dependence of global solutions are discussed. In Sect. 5, the existence of a compact global attractor is proved. Finally, in Sect. 6, the existence of an exponential attractor is proved.

## 2. Notations and assumptions

In this work, $L^\varrho(0,L)$, $1 \leq \varrho < \infty$ denote the Lebesgue spaces of measurable functions on $(0,L)$ whose $\varrho$th power is integrable and endowed with the norm $\|\cdot\|_\varrho$. In particular, when $\varrho = 2$, we use the notation $\|\cdot\|_2$ and $\langle\cdot,\cdot\rangle$ the norm and inner product in $L^2(0,L)$, respectively. $L^\infty(0,L)$ represents the space of measurable functions that are essentially bounded on $(0,L)$, endowed with the norm $\|\cdot\|_\infty$. Moreover, $H^n(0,L)$ $(n = 1,2)$ endowed with the norm $\|\cdot\|_{H^n}$ denotes the Sobolev spaces whose elements are functions in $L^2(0,L)$ such that the weak derivative of $r$th order with $r \leq n$ belongs to $L^2(0,L)$. We also define the following Sobolev space:

$$H^1_*(0,L) = \Big\{ u \in H^1(0,L) : u(0) = 0 \Big\}.$$

Since $u(0) = 0$, the Poincaré's inequality holds

$$\|u\|_2 \leq c_p \|u_x\|_2, \quad \forall\, u \in H^1_*(0, L), \tag{8}$$

and therefore, $\|u\|_{H^1_*(0,L)} := \|u_x\|_2$ is an equivalent norm in $H^1_*(0, L)$.

**Assumption 2.1.** Assume that the following set of hypotheses on the memory (relaxation) kernel $\lambda$ holds:

$$\lambda \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+), \quad \lambda(s) \geq 0 \quad \text{and} \quad \lambda'(s) \leq 0, \quad \forall s \in \mathbb{R}^+, \tag{9}$$

$$\lambda'(s) + \delta_1 \lambda(s) \leq 0 \quad \text{for some} \quad \delta_1 > 0, \ \ \forall s \in \mathbb{R}^+, \quad \lambda_0 := \int_0^\infty \lambda(s) ds = \lambda(0) > 0. \tag{10}$$

Besides, the requirement that $\lambda$ has a unitary mass translates into

$$\int_0^\infty s\lambda(s) ds = 1. \tag{11}$$

Let $\lambda$ be a memory kernel satisfying the assumptions (9)–(11). Now, we consider the weighted Hilbert space

$$\mathcal{M} := L^2(\mathbb{R}^+; H^1_*(0, L)) = \left\{ u : \mathbb{R}^+ \to H^1_*(0, L); \ \int_0^\infty \lambda(s) \|u_x(s)\|^2 ds < \infty \right\} \tag{12}$$

for which the inner product and the norm (induced by the inner product) are given, respectively, by

$$\langle u, v \rangle_{\mathcal{M}} := \int_0^\infty \lambda(s) \langle u_x(s), v_x(s) \rangle \, ds,$$

$$\|u\|^2_{\mathcal{M}} = \int_0^\infty \lambda(s) \|u_x(s)\|_2^2 ds \tag{13}$$

for all $u, v \in \mathcal{M}$. Now, we define the linear operator $\mathcal{T}$ on $\mathcal{M}$ by

$$\mathcal{T} u := -u_s, \qquad u \in \mathscr{D}(\mathcal{T}) \tag{14}$$

where

$$\mathscr{D}(\mathcal{T}) := \left\{ u \in \mathcal{M} \ \middle| \ u_s \in \mathcal{M}, \ u(0) = 0 \right\}.$$

For every $u \in \mathscr{D}(\mathcal{T})$, the nonnegative functional

$$\Gamma(u) := -\int_0^\infty \lambda'(s) \|u_x(s)\|_2^2 \, ds$$

is well defined, and the following identity holds:

$$2\langle \mathcal{T} u, u \rangle_{\mathcal{M}} = -\Gamma(u). \tag{15}$$

Moreover, following the assumption (10) on $\lambda$, we deduce the inequality

$$\delta_1 \|u\|^2_{\mathcal{M}} \leq \Gamma(u), \tag{16}$$

which is crucial for the rest of the paper.

Using the operator $\mathcal{T}$ above and the relative history of $v$ defined by $\eta(t,s) = v(t) - v(t-s)$, introduced first by Dafermos [11], the system (1)–(2) can be rewritten in the following equivalent form

$$\rho v_{tt} - \tilde{k} v_{xx} + \gamma\beta p_{xx} - \int_0^\infty \lambda(s)\eta_{xx}(s)ds + f_1(v,p) = h_1 \quad \text{in} \quad (0,L)\times(0,\infty), \tag{17}$$

$$\mu p_{tt} - \beta p_{xx} + \gamma\beta v_{xx} + g(p_t) + f_2(v,p) = h_2 \quad \text{in} \quad (0,L)\times(0,\infty), \tag{18}$$

$$\eta_t - \mathcal{T}\eta - \ v_t = 0 \quad \text{in} \quad (0,L)\times(0,\infty)\times(0,\infty) \tag{19}$$

with boundary conditions

$$v(0,t) = v_x(L,t) = p(0,t) = p_x(L,t) = \eta(0,s,t) = \eta_x(L,s,t) = 0, \quad t \geq 0, \quad s \geq 0 \tag{20}$$

and initial conditions

$$\begin{aligned} &v(x,0) = v_0(x),\ v_t(x,0) = v_1(x),\ p(x,0) = p_0(x),\ p_t(0,x) = p_1(x), \quad x \in (0,L),\\ &\eta(x,s,0) = \eta_0(x,s), \quad (x,s) \in (0,L)\times(0,\infty). \end{aligned} \tag{21}$$

Here,

$$\tilde{k} = k_1 + \gamma^2\beta. \tag{22}$$

Now, consider the Hilbert space

$$\mathcal{H} = H^1_*(0,L) \times H^1_*(0,L) \times L^2(0,L) \times L^2(0,L) \times \mathcal{M} \tag{23}$$

with the following inner product:

$$(U,\tilde{U})_{\mathcal{H}} = \rho\langle\phi,\tilde{\phi}\rangle + \mu\langle\varphi,\tilde{\varphi}\rangle + k_1\langle v_x,\tilde{v}_x\rangle + \beta\langle\gamma v_x - p_x, \gamma\tilde{v}_x - \tilde{p}_x\rangle + \langle\eta,\tilde{\eta}\rangle_{\mathcal{M}}$$

where $U = (v,p,\phi,\varphi,\eta), \tilde{U} = (\tilde{v},\tilde{p},\tilde{\phi},\tilde{\varphi},\tilde{\eta}) \in \mathcal{H}$. The corresponding norm is then given by

$$\|U\|^2_{\mathcal{H}} = \rho\|\phi\|_2^2 + \mu\|\varphi\|_2^2 + k_1\|v_x\|_2^2 + \beta\|\gamma v_x - p_x\|_2^2 + \|\eta\|^2_{\mathcal{M}}.$$

Observe that there exists a constant $\kappa_0 > 0$ such that

$$\|v_x\|_2^2 + \|p_x\|_2^2 \leq \kappa_0\big(k_1\|v_x\|_2^2 + \beta\|\gamma v_x - p_x\|_2^2\big). \tag{24}$$

Indeed, noting that

$$\|p_x\|_2^2 = \|\gamma v_x - p_x - \gamma v_x\|_2^2 \leq 2\|\gamma v_x - p_x\|_2^2 + 2\gamma^2\|v_x\|_2^2,$$

we have

$$\|v_x\|_2^2 + \|p_x\|_2^2 \leq (2\gamma^2+1)\|v_x\|_2^2 + 2\|\gamma v_x - p_x\|_2^2.$$

Therefore, (24) holds with $\kappa_0 = \max\{(2\gamma^2+1)k_1^{-1}, 2\beta^{-1}\}$. Combining (8) and (24), there exists a constant $d_0 > 0$ such that

$$\|v\|_2^2 + \|p\|_2^2 \leq d_0\left(k_1\|v_x\|_2^2 + \beta\|\gamma v_x - p_x\|_2^2\right). \tag{25}$$

**Assumption 2.2.** The following are assumed for the external forces and source terms:

(i) The external forces $h_1, h_2 \in L^2(0,L)$.

(ii) There exists a function $F \in C^2(\mathbb{R}^2)$ such that

$$\nabla F = (f_1, f_2). \tag{26}$$

(iii) There exist $q \geq 1$ and $C > 0$ such that

$$|\nabla f_i(v,p)| \leq C\left(1 + |v|^{q-1} + |p|^{q-1}\right), \quad i = 1,2. \tag{27}$$

(iv) There exist constants $d \geq 0$, $m_F > 0$ with

$$0 \leq d < \frac{1}{2d_0} \tag{28}$$

such that

$$F(v,p) \geq -d\left(|v|^2 + |p|^2\right) - m_F. \tag{29}$$

Moreover,

$$\nabla F(v,p)\cdot(v,p) - F(v,p) \geq -d\left(|v|^2 + |p|^2\right) - m_F. \tag{30}$$

**Remark 2.3.** Assumption 2.2, with few variations, can be found in several works such as [3,14,15,31,35,42]. An example of a function satisfying this assumption is

$$F(v,p) = |v+p|^4 - |v+p|^2 + |vp|^2. \tag{31}$$

**Assumption 2.4.** Consider an increasing function $g \in C^1(\mathbb{R})$ with $g(0) = 0$. In addition, assume that there exist constants $m, M > 0$ such that

$$m \leq g'(s) \leq M \quad \forall\, s \in \mathbb{R}. \tag{32}$$

By the mean value theorem and (32), the following monotonicity property is obtained:

$$m|u-v|^2 \leq (g(u)-g(v))(u-v) \leq M|u-v|^2, \quad \forall\, u,v \in \mathbb{R}. \tag{33}$$

## 3. Cauchy's problem

Let $U(t) = (v(t), p(t), v_t(t), p_t(t), \eta(t))$. The system (17)-(19) (with boundary and initial conditions) can be rewritten as an abstract initial value problem (Cauchy's problem) in $\mathcal{H}$:

$$\begin{cases} U_t(t) + (\mathcal{A}_1 + \mathcal{A}_2)U(t) + \mathcal{F}(U(t)) = 0,\, t > 0, \\ U(0) = U_0, \end{cases} \tag{34}$$

where $U_t = \frac{dU}{dt}$ and $U_0 = (v_0, p_0, v_1, p_1, \eta_0)$, and the operators $\mathcal{A}_1 : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$, $\mathcal{A}_2 : \mathcal{H} \to \mathcal{H}$ and $\mathcal{B} : \mathcal{H} \to \mathcal{H}$ are defined by

$$\mathcal{A}_1 \begin{pmatrix} v \\ p \\ \phi \\ \varphi \\ \eta \end{pmatrix} = \begin{pmatrix} -\phi \\ -\varphi \\ \rho^{-1}(-\tilde{k}v_{xx} + \gamma\beta p_{xx} - \int_0^\infty \lambda(s)\eta_{xx}(s)ds) \\ \mu^{-1}(-\beta p_{xx} + \gamma\beta v_{xx}) \\ -\mathcal{T}\eta - \phi \end{pmatrix}, \quad \mathcal{A}_2 \begin{pmatrix} v \\ p \\ \phi \\ \varphi \\ \eta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \mu^{-1}g(\varphi) \\ 0 \end{pmatrix}, \tag{35}$$

with

$$\begin{aligned} D(\mathcal{A}_1) :=& \Big\{(v,p,\phi,\varphi,\eta) \in \mathcal{H};\ v,p \in H^2(0,L),\ \phi,\varphi \in H^1_*(0,L),\ \eta \in D(\mathcal{T}), \\ & v_x(L) = p_x(L) = 0,\ \ \tilde{k}v_{xx} + \int_0^\infty \lambda(s)\eta_{xx}(s)ds \in L^2(0,L)\Big\}, \end{aligned} \tag{36}$$

$$\mathcal{F} \begin{pmatrix} v \\ p \\ \phi \\ \varphi \\ \eta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \rho^{-1}(f_1(v,p) - h_1) \\ \mu^{-1}(f_2(v,p) - h_2) \\ 0 \end{pmatrix}. \tag{37}$$

**Definition 3.1.** A *strong solution* to (34) on $[0,T)$ is a continuous function $U:[0,T)\to\mathcal{H}$ such that $U(0)=U_0$, $U$ is differentiable a.e. on $[0,T)$ and Lipschitz continuous, $U(t)\in D(\mathcal{A})$ for any $t\in[0,T)$.

**Definition 3.2.** A *generalized solution* to (34) on $[0,T]$ with $T>0$ is a function $U\in C([0,T],\mathcal{H})$ with $U(0)=U_0$ for which there exists a sequence of strong solutions $(U_n)\in C([0,T],\mathcal{H})$ of

$$\frac{d}{dt}U_n+(\mathcal{A}_1+\mathcal{A}_2)U_n+\mathcal{F}(U_n)=f_n,\quad n=1,2,\dots \tag{38}$$

with $U_n\to U$ in $C([0,T],\mathcal{H})$ and $f_n\to 0$ on $L^1(0,T;\mathcal{H})$. A function $U\in C([0,T);\mathcal{H})$ with $0<T\le\infty$ is a generalized solution to (34) on $[0,T)$ if $U$ is a generalized solution to (34) on $[0,T']$ for any $0<T'<T$.

### 3.1. Energy of solutions

Given a solution $U(t)=(v(t),p(t),p_t(t),v_t(t),\eta(t))$ to (34) on $[0,T)$, define the *energy* $E(t)$ of $U(t)$ by

$$\begin{aligned}E(t)&:=\frac{\rho}{2}\|v_t(t)\|_2^2+\frac{\mu}{2}\|p_t(t)\|_2^2+\frac{k_1}{2}\|v_x(t)\|_2^2+\frac{\beta}{2}\|\gamma v_x(t)-p_x(t)\|_2^2+\frac{1}{2}\|\eta(t)\|_{\mathcal{M}}^2\\&=\frac{1}{2}\|U(t)\|_{\mathcal{H}}^2,\end{aligned} \tag{39}$$

and define the *modified energy* $\mathscr{E}(t)$ of $U(t)$ by

$$\mathscr{E}(t):=E(t)+\int_0^L F(v(t),p(t))dx-\langle h_1,v(t)\rangle-\langle h_2,p(t)\rangle. \tag{40}$$

**Lemma 3.3.** *The modified energy (40) associated with solution $U(t)=(v(t),p(t),v_t(t),p_t(t),\eta(t))$ of (34) on $[0,T)$ is non-increasing. Moreover, there exist constants $\chi_0,C_F>0$ such that*

$$\chi_0\|U(t)\|_{\mathcal{H}}^2\le\mathscr{E}(0)+C_F,\quad t\in[0,T). \tag{41}$$

*In other words, every solution $U(t)$ of (34) always remains inside a closed ball whose radius depends on $U(0)$.*

*Proof.* Suppose $U(t)$ is a strong solution. Multiply (17) by $v_t$, (18) by $p_t$, and (19) by $-k(s)\eta_{xx}$. Now, integrating by parts the first two equations in the variable $x$ over $[0,L]$, and the third one over $[0,\infty)\times[0,L]$ with respect $x$ and $s$, yields

$$\frac{d}{dt}\mathscr{E}(t)=-\langle g(p_t),p_t\rangle-\frac{1}{2}\Gamma(\eta)\le 0. \tag{42}$$

Therefore, $\mathscr{E}(t)$ is non-increasing, and in particular

$$\mathscr{E}(t)\le\mathscr{E}(0),\quad t\in[0,T). \tag{43}$$

Next, (29) is integrated over $[0,L]$ with respect to $x$ and (25) is applied to obtain

$$\begin{aligned}\int_0^L F(v,p)dx&\ge -d(\|v\|_2^2+\|p\|_2^2)-Lm_F\ge -dd_0\left(k_1\|v_x\|_2^2+\beta\|\gamma v_x-p_x\|_2^2\right)-Lm_F\\&\ge -dd_0\|U(t)\|_{\mathcal{H}}^2-Lm_F.\end{aligned} \tag{44}$$

Now, (44) together with (28) leads to

$$\mathscr{E}(t)\ge\left(\frac{1}{2}-dd_0\right)\|U(t)\|_{\mathcal{H}}^2-Lm_F-\int_0^L(h_1v+h_2p)\,dx.$$

By letting

$$\chi_0 = \frac{1}{4}\big(1 - 2dd_0\big) > 0, \tag{45}$$

and using the estimate

$$\int_0^L (h_1 v + h_2 p)\,\mathrm{d}x \leq \frac{\chi_0}{d_0}\left(\|v\|_2^2 + \|p\|_2^2\right) + \frac{d_0}{4\chi_0}\left(\|h_1\|_2^2 + \|h_2\|_2^2\right),$$

the inequality in (41) is obtained with

$$C_F = Lm_F + \frac{d_0}{4\chi_0}\left(\|h_1\|_2^2 + \|h_2\|_2^2\right).$$

Finally, by a density argument, (41) is valid for every generalized solution on $[0, T']$ for any $0 < T' < T$. □

## 4. Well-posedness

The following results are needed to prove thee global well-posedness of (34).

**Lemma 4.1.** *The operator $\mathcal{A}_1$ is m-accretive.*

*Proof.* By a standard computation,

$$\langle \mathcal{A}_1 U, U\rangle_{\mathcal{H}} = -\frac{1}{2}\int_0^\infty \lambda'(s)\|\eta_x(s)\|_2^2 ds \geq 0, \quad \forall U = (v, p, \phi, \varphi, \eta) \in D(\mathcal{A}_1). \tag{46}$$

Additionally, $R(I + \mathcal{A}_1) = \mathcal{H}$. Letting $U^* = (v^*, p^*, \phi^*, \varphi^*, \eta^*) \in \mathcal{H}$, it is aimed to obtain $U = (v, p, \phi, \varphi, \eta) \in D(\mathcal{A}_1)$ such that

$$(I + \mathcal{A}_1)U = U^*. \tag{47}$$

Note that (47) is equivalent to the following system:

$$v - \phi = v^*, \tag{48}$$

$$p - \varphi = p^*, \tag{49}$$

$$\rho\phi - \tilde{k}v_{xx} + \gamma\beta p_{xx} - \int_0^\infty \lambda(s)\eta_{xx}(s)ds = \rho\phi^*, \tag{50}$$

$$\mu\varphi - \beta p_{xx} + \gamma\beta v_{xx} = \mu\varphi^*, \tag{51}$$

$$\eta - \mathcal{T}\eta - \phi = \eta^*. \tag{52}$$

Now, observe that

$$\eta(s) = (1 - e^{-s})\phi + e^{-s}\int_0^s e^{\tau}\eta^*(\tau)d\tau \tag{53}$$

satisfies (52) with $\eta(0) = 0$, and the second term on the right side in (53)

$$s \in \mathbb{R}^+ \mapsto e^{-s}\int_0^s e^{\tau}\eta^*(\tau)d\tau \tag{54}$$

belongs to $\mathcal{M}$. This can be seen by changing the order within the integral

$$\begin{aligned}
&\int_0^\infty \lambda(s)\left\|e^{-s}\int_0^s e^\tau \eta_x^*(\tau)d\tau\right\|_2^2 ds = \int_0^\infty \lambda(s)e^{-2s}\int_0^L \left(\int_0^s e^\tau \eta_x^*(\tau)d\tau\right)^2 dxds\\
&\quad\le \int_0^\infty \lambda(s)e^{-2s}\int_0^L \left(\int_0^s e^\tau d\tau\right)\left(\int_0^s e^\tau |\eta_x^*(\tau)|^2 d\tau dx\right) ds\\
&\quad\le \int_0^\infty \lambda(s)e^{-s}\int_0^s e^\tau \|\eta_x^*(\tau)\|_2^2 d\tau ds = \int_0^\infty\int_\tau^\infty \lambda(s)e^{-s}e^\tau\|\eta_x^*(\tau)\|_2^2 ds d\tau\\
&\quad= \int_0^\infty e^{2\tau}\|\eta_x^*(\tau)\|_2^2\left(\int_\tau^\infty \lambda(s)e^{-2s}ds\right)d\tau \le \int_0^\infty \lambda(\tau)\|\eta_x^*(\tau)\|_2^2 d\tau < \infty.
\end{aligned}\tag{55}$$

Therefore, $\phi \in H_*^1(0,L)$ implies $\eta(s) \in H_*^1(0,L)$ for $s > 0$ and $\eta \in \mathcal{M}$,

$$\eta_s(s) = e^{-s}\phi - e^{-s}\int_0^s e^\tau \eta^*(\tau)d\tau + \eta^*(s) \in \mathcal{M} \tag{56}$$

and thus $\eta \in D(\mathcal{T})$. By using (48) and (53) in ((50),(49), (51)),

$$\begin{aligned}
\rho v - (\tilde{k}_0 + \gamma^2\beta)v_{xx} + \gamma\beta p_{xx} &= \vartheta^*,\\
\mu p - \beta p_{xx} + \gamma\beta v_{xx} &= \mu\varphi^* + \mu p^*
\end{aligned}\tag{57}$$

where (by (4))

$$\tilde{k}_0 := \alpha_1 - \int_0^\infty \lambda(s)e^{-s}ds > 0 \tag{58}$$

and

$$\vartheta^* := \rho\phi^* + \rho v^* + \int_0^\infty \lambda(s)e^{-s}\int_0^s e^\tau \eta_{xx}^*(\tau)d\tau ds - \int_0^\infty \lambda(s)(1-e^{-s})ds v_{xx}^*. \tag{59}$$

Note that $\vartheta^* \in H_*^{-1}(0,L)$. Since $v_x^* \in L^2(0,L)$ and for $w \in H_*^1(0,L)$ with $\|w_x\| \le 1$, a calculation similar to (55) (see [24] for more details) and from Holder's inequality, the following is obtained:

$$\begin{aligned}
&\left|\left\langle \int_0^\infty \lambda(s)e^{-s}\int_0^s e^\tau \eta_{xx}^*(\tau)d\tau ds, w\right\rangle\right| = \left|\left\langle \int_0^\infty \lambda(s)e^{-s}\int_0^s e^\tau \eta_x^*(\tau)d\tau ds, w_x\right\rangle\right|\\
&\quad\le \int_0^\infty \lambda(s)e^{-s}\int_0^s e^\tau \|\eta_x^*(\tau)\| d\tau ds \le\ k_0^{1/2}\|\eta^*\|_{\mathcal{M}},
\end{aligned}\tag{60}$$

and analogously,

$$\left|\left\langle \int_0^\infty \lambda(s)(1-e^{-s})ds v_{xx}^*, w\right\rangle\right| \le k_0\|v_x^*\|_2. \tag{61}$$

Now, for solving the system (57), we use a standard variational approach in order to obtain a bilinear functional $\mathcal{B} : (H^1_*(0,L) \times H^1_*(0,L))^2 \to \mathbb{R}$ defined by

$$\mathcal{B}\Big((v,p),(\widetilde{v},\widetilde{p})\Big) := \rho\langle v,\widetilde{v}\rangle + (\tilde{k}_0+\gamma^2\beta)\langle v_x,\widetilde{v}_x\rangle + \mu\langle p,\widetilde{p}\rangle + \beta\langle \gamma v_x - p_x, \gamma\widetilde{v}_x - \widetilde{p}_x\rangle. \tag{62}$$

It is not difficult to verify that $\mathcal{B}$ is continuous and coercive. Therefore, it follows from the Lax–Milgram's theorem that the system (57) has a unique solution $(v,p)$. Now, (48) and (49) implies $\phi,\varphi \in H^1_*(0,L)$, and (53) implies $\eta \in \mathcal{M}$, and $(57)_2$ implies $-p+\gamma v \in H^2(0,L)$. This together with (50) yields $\tilde{k}v_{xx} + \int\limits_0^\infty \lambda(s)\eta_{xx}(s)ds \in L^2(0,L)$, and therefore, $v,p \in H^2(0,L)$. Hence, $(v,p,\phi,\varphi,\eta) \in D(\mathcal{A}_1)$ and the result follows from [54][Lemma 2.2.3]. □

**Lemma 4.2.** *The operator $\mathcal{A}_2$ is accretive and Lipschitz continuous.*

*Proof.* It follows from (35) that for any $U = (v,p,\phi,\varphi,\eta)$ and $\widetilde{U} = (\widetilde{v},\widetilde{p},\widetilde{\phi},\widetilde{\varphi},\widetilde{\eta})$ in $\mathcal{H}$

$$\langle \mathcal{A}_2(U) - \mathcal{A}_2(\widetilde{U}), U - \widetilde{U}\rangle_{\mathcal{H}} = \langle g(\varphi) - g(\tilde{\varphi}), \varphi - \tilde{\varphi}\rangle \geq 0. \tag{63}$$

Therefore, $\mathcal{A}_2$ is accretive. Considering (33) leads to

$$\|\mathcal{A}_2(U) - \mathcal{A}_2(\widetilde{U})\|_{\mathcal{H}} \leq C\|U - \widetilde{U}\|_{\mathcal{H}} \tag{64}$$

for some positive constant $C$ independent of $U$ and $\widetilde{U}$. The proof is now complete. □

**Lemma 4.3.** *The operator $\mathcal{A} = \mathcal{A}_1 + \mathcal{A}_2$ is m-accretive.*

*Proof.* Since $\mathcal{A}_1$ is m-accretive and $\mathcal{A}_2$ is accretive and Lipschitz continuous, it follows from [10][Proposition 2.2.3] that $\mathcal{A}$ is m-accretive. □

**Lemma 4.4.** *The operator $\mathcal{F}$ is locally Lipschitz.*

*Proof.* Let $U = (v,p,\phi,\varphi,\eta)$, $\widetilde{U} = (\widetilde{v},\widetilde{p},\widetilde{\phi},\widetilde{\varphi},\widetilde{\eta})$ in $\mathcal{H}$ and $K > 0$ such that

$$\|U\|_{\mathcal{H}}, \|\widetilde{U}\|_{\mathcal{H}} \leq K. \tag{65}$$

It follows from (37) that

$$\|\mathcal{F}(U) - \mathcal{F}(\widetilde{U})\|^2_{\mathcal{H}} = \rho^{-1}\int\limits_0^L |f_1(v,p) - f_1(\widetilde{v},\widetilde{p})|^2 dx + \mu^{-1}\int\limits_0^L |f_2(v,p) - f_2(\widetilde{v},\widetilde{p})|^2 dx. \tag{66}$$

By (27) and the mean value theorem, there exists $\theta \in (0,1)$ such that

$$\begin{aligned} |f_i(v,p) - f_i(\tilde{v},\tilde{p})|^2 &= |\nabla f_i(\theta(v,p) + (1-\theta)(\tilde{v},\tilde{p}))|^2 |(v,p) - (\tilde{v},\tilde{p})|^2 \\ &\leq C\left(|v|^{q-1} + |\tilde{v}|^{q-1} + |p|^{q-1} + |\tilde{p}|^{q-1} + 1\right)^2 \left(|v-\tilde{v}|^2 + |p-\tilde{p}|^2\right). \end{aligned} \tag{67}$$

Therefore, there exists a constant $C_K > 0$ such that

$$\int\limits_0^L |f_i(v,p) - f_i(\tilde{v},\tilde{p})|^2 \,\mathrm{d}x \leq C_R\|U - \tilde{U}\|^2_{\mathcal{H}}, \quad i = 1,2. \tag{68}$$

Finally, substituting (68) into (66), we conclude that there exists $\tilde{C}_R > 0$ such that

$$\|\mathcal{F}(U) - \mathcal{F}(\tilde{U})\|_{\mathcal{H}} \leq \tilde{C}_R\|U - \tilde{U}\|_{\mathcal{H}}.$$

This proves that $\mathcal{F}$ is locally Lipschitz continuous. □

**Theorem 4.5.** *(***Existence of Global Solution***) Consider the Cauchy's problem (34).*

***a.*** *If $U_0 \in \mathcal{H}$, (34) has a unique global generalized solution.*

***b.*** *If $U_0 \in D(\mathcal{A})$ where $\mathcal{A} = \mathcal{A}_1 + \mathcal{A}_2$, the generalized solution obtained in (a) is strong solution.*

***c.*** *If $U^1(t)$ and $U^2(t)$ are two solutions to (34), there exists a positive constant $C_0 = C_0(U^1(0), U^2(0))$ such that for every $T > 0$*

$$\|U^1(t) - U^2(t)\|_{\mathcal{H}} \le e^{C_0 t}\|U^1(0) - U^2(0)\|_{\mathcal{H}}, \quad 0 \le t \le T. \tag{69}$$

*Proof.* **(a) and (b):** Since $\mathcal{A}$ is m-accretive, (34) is a local Lipschitz disturbance for

$$\begin{cases} U_t(t) + \mathcal{A}U(t) = 0, \quad t > 0, \\ U(0) = U_0. \end{cases} \tag{70}$$

It follows from [10][Theorem 2.3.8] that there exist $T_{\max} > 0$ such that if $U_0 \in \mathcal{H}$, (34) has a unique generalized solution on $[0, T_{\max})$, and if $U_0 \in D(\mathcal{A})$, (34) has a unique strong solution on $[0, T_{\max})$. Let $U(t)$ be a solution of (34). By (41), we have

$$\lim_{t \to T_{\max}^-} \|U(t)\|_{\mathcal{H}} < \infty. \tag{71}$$

Therefore, we conclude that $T_{\max} = \infty$ following from Theorem 2.3.8 in [10], and hence, the solution is global.

**(c):** By letting $U^1(t) = (v^1, p^1, v_t^1, p_t^1, \eta^1)$ and $U^2(t) = (v^2, p^2, v_t^2, p_t^2, \eta^2)$ be strong solutions of (34), $U(t) = U^1(t) - U^2(t) = (v, p, v_t, p_t, \eta)$ is a solution of

$$\begin{aligned} \rho v_{tt} - \tilde{k} v_{xx} + \gamma\beta p_{xx} - \int_0^\infty \lambda(s)\eta_{xx}(s)ds &= -\left(f_1(v^1, p^1) - f_1(v^2, p^2)\right) \\ \mu p_{tt} - \beta p_{xx} + \gamma\beta v_{xx} &= -\left(g(p_t^1) - g(p_t^2)\right) - \left(f_2(v^1, p^1) - f_2(v^2, p^2)\right) \\ \eta_t + \eta_s &= v_t. \end{aligned} \tag{72}$$

Now, multiply $(72)_1$ by $v_t$, $(72)_2$ by $p_t$, $(72)_3$ by $\lambda(s)\eta_{xx}$, and integrate by parts the two first over $[0, L]$ with respect to $x$ and the third over $[0, L] \times [0, \infty]$ with respect to $x$ and $s$ to obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|U(t)\|_{\mathcal{H}}^2 &\le -m\|p_t\|_2^2 + \int_0^\infty \lambda'(s)\|\eta_x\|_2^2 ds - \int_0^L (f_1(v^1, p^1) - f_1(v^2, p^2))v_t dx \\ &- \int_0^L (f_2(v^1, p^1) - f_2(v^2, p^2))p_t dx \le -\int_0^L (f_1(v^1, p^1) - f_1(v^2, p^2))v_t dx \\ &- \int_0^L (f_2(v^1, p^1) - f_2(v^2, p^2))p_t dx. \end{aligned} \tag{73}$$

where (32) is taken into account. Next, apply the Young's inequality to have

$$\begin{aligned} \int_0^L (f_1(v^1, p^1) - f_1(v^2, p^2))v_t dx &\le \frac{1}{2}\int_0^L |f_1(v^1, p^1) - f_1(v^2, p^2)|^2 dx + \frac{1}{2}\|v_t\|_2^2, \\ \int_0^L (f_2(v^1, p^1) - f_2(v^2, p^2))p_t dx &\le \frac{1}{2}\int_0^L |f_2(v^1, p^1) - f_2(v^2, p^2)|^2 dx + \frac{1}{2}\|p_t\|_2^2. \end{aligned} \tag{74}$$

Following the traces of a calculation similar to the one in Lemma 4.4, a positive constant $C_0$ depending on $U^1(0)$ and on $U^2(0)$ is obtained such that for $i = 1, 2$ :

$$\frac{1}{2}\int_0^L |f_1(v^1, p^1) - f_1(v^2, p^2)|^2 dx \le C_0 \|U(t)\|^2_{\mathcal{H}}. \tag{75}$$

Combining (73), (74), and (75) leads to

$$\frac{d}{dt}\|U(t)\|^2_{\mathcal{H}} \le C_0 \|U(t)\|^2_{\mathcal{H}} \tag{76}$$

where $C_0$ represents a generic positive constant of $U^1(0)$ and $U^2(0)$. Finally, (69) holds true on $[0, T]$ for any $T > 0$. Using a density argument, (69) holds for every generalized solution. □

**Remark 4.6.** Once the existence of a global solution for (34) is established, represent $U(t, U_0)$ the solution of (34) for all $U_0 \in \mathcal{H}$, and, this gives rise to a family of operators $S(t)$, $t \in \mathbb{R}$ of $\mathcal{H}$ itself defined by

$$S(t)U_0 = U(t, U_0). \tag{77}$$

From Theorem 4.5, $t \to S(t)U_0$ is Lipschitz continuous for any $U_0 \in \mathcal{H}$, and from (69), $S(t)$ is continuous for any $t \ge 0$ with $S(0) = I$ where $I$ is the identity operator on $\mathcal{H}$. It is not difficult to show that $S(t)$ defined in (77) satisfies the semigroup property. Therefore, the pair $(\mathcal{H}, S(t))$ is a dynamical system with $S(t)$ being a $C_0$-semigroup. Henceforth, the properties of this dynamical system shall be studied in order to obtain global and exponential attractors.

## 5. Global attractor

### 5.1. Quasi-stability

The purpose of this subsection is to establish the quasi-stability for the dynamical system $(\mathcal{H}, S(t))$ on every bounded subset of $\mathcal{H}$ which is positively invariant. It is proved that the dynamical system satisfies the stabilizability estimate [10, Section 7.9] for such sets.

**Lemma 5.1.** *Suppose that Assumptions 2.1, 2.2, and 2.4 hold. Let $B$ be a bounded positively invariant set in $\mathcal{H}$, and let $S(t)U^i = (v^i, p^i, v^i_t, p^i_t, \eta^i)$ be the weak solutions of (17)–(21) with initial conditions $U^i \in B$, $i = 1, 2$. Then, there exist constants $\mu_0, \gamma_0, C'_B > 0$ such that*

$$E(t) \le \gamma_0 e^{-\mu_0 t} E(0) + C'_B \int_0^t e^{-\mu_0(t-s)}(\|v(s)\|^2_{2q} + \|p(s)\|^2_{2q})\,\mathrm{d}s \tag{78}$$

*where $v = v^1 - v^2$ and $p = p^1 - p^2$.*

*Proof.* Note that $u = v^1 - v^2$, $p = p^1 - p^2$ and $\eta = \eta^1 - \eta^2$ solve the system

$$\begin{cases} \rho v_{tt} - \tilde{k} v_{xx} + \gamma\beta p_{xx} - \int\limits_0^\infty \lambda(s)\eta_{xx}(s)ds = -F_1(v, p) \quad \text{in} \quad (0, L) \times (0, \infty), \\ \mu p_{tt} - \beta p_{xx} + \gamma\beta v_{xx} + G(p_t) = -F_2(v, p) \quad \text{in} \quad (0, L) \times (0, \infty), \\ \eta_t - \mathcal{T}\eta_s - \ v_t = 0 \quad \text{in} \quad (0, L) \times (0, \infty) \times (0, \infty) \end{cases} \tag{79}$$

with boundary conditions

$$v(0) = v_x(L) = p(0) = p_x(L) = 0$$

and initial conditions

$$\big(v(0), p(0), v_t(0), p_t(0)\big) = U^1 - U^2 \tag{80}$$

where

$$G(p_t)=g(p_t^1)-g(p_t^2),\quad F_i(v,p)=f_i(v^1,p^1)-f_i(v^2,p^2),\quad i=1,2.$$

**Step 1.** It is claimed that there exists a constant $C>0$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t)\le-\frac{m}{2}\|p_t\|_2^2-\frac{1}{2}\Gamma(\eta)+C_B\big(\|v\|_{2q}^2+\|p\|_{2q}^2\big)+\rho\varepsilon_1\|v_t\|_2^2,\quad\forall\,\varepsilon_1>0. \tag{81}$$

By multiplying the first equation in (79) by $v_t$, the second one by $p_t$, the third one by $\lambda(s)\eta_{xx}$ respectively, and integrating by parts the first and second equations over $[0,L]$ with respect to $x$ and the third one over $[0,L]\times[0,\infty)$ with respect $x$ and $s$, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E(t)&=-\big\langle G(p_t),\,p_t\big\rangle+\big\langle\mathcal{T}\eta,\ \eta\big\rangle_{\mathcal{M}}-\big\langle F_1(v,p),\,v_t\big\rangle-\big\langle F_2(v,p),\,p_t\big\rangle\\&\le-m\|p_t\|_2^2-\frac{1}{2}\Gamma(\eta)-\big\langle F_1(v,p),\,v_t\big\rangle-\big\langle F_2(v,p),\,p_t\big\rangle.\end{aligned}\tag{82}$$

By (27), Hölder's inequality and Young's inequalities, and Lemma 3.3, there exists a constant $C_B>0$ such that

$$\begin{aligned}\big|\big\langle F_1(v,p),\,v_t\big\rangle\big|&\le C\left(\|v^1\|_{2q}^{q-1}+\|v^2\|_{2q}^{q-1}+\|p^1\|_{2q}^{q-1}+\|p^2\|_{2q}^{q-1}+1\right)\big(\|v\|_{2q}+\|p\|_{2q}\big)\|v_t\|_2\\&\le C_B\big(\|v\|_{2q}+\|p\|_{2q}\big)\|v_t\|_2\le C_B\big(\|v\|_{2q}^2+\|p\|_{2q}^2\big)+\rho\varepsilon_1\|v_t\|_2^2,\quad\forall\,\varepsilon_1>0.\end{aligned}\tag{83}$$

In a similar fashion, it follows that

$$\begin{aligned}\big|\big\langle F_2(v,p),\,p_t\big\rangle\big|&\le C\left(\|v^1\|_{2q}^{q-1}+\|v^2\|_{2q}^{q-1}+\|p^1\|_{2q}^{q-1}+\|p^2\|_{2q}^{q-1}+1\right)\big(\|v\|_{2q}+\|p\|_{2q}\big)\|p_t\|_2\\&\le C_B\big(\|v\|_{2q}+\|p\|_{2q}\big)\|p_t\|_2\le C_B\big(\|v\|_{2q}^2+\|p\|_{2q}^2\big)+\frac{m}{2}\|p_t\|_2^2,\quad m>0.\end{aligned}\tag{84}$$

Substituting the estimates (83) and (84) in (82) leads to (81).
**Step 2.** Multiplying the equation $(79)_1$ by $v$ and integrating over $[0,L]$ leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{F}(t)-\rho\|v_t\|_2^2+k_1\|v_x\|_2^2+\gamma\beta\big\langle\gamma v_x-p_x,\,v_x\big\rangle+\int_0^\infty\lambda(s)\big\langle\eta_x(s),\,v_x\big\rangle ds+\big\langle F_1(v,p),\,v\big\rangle=0 \tag{85}$$

where

$$\mathscr{F}(t):=\rho\big\langle v_t,\,v\big\rangle. \tag{86}$$

Additionally,

$$\int_0^\infty\lambda(s)\big\langle\eta_x(s),v_x\big\rangle ds\le C\|v_x\|_2\|\eta\|_{\mathcal{M}}\le\frac{k_1}{4}\|v_x\|_2^2+C\Gamma(\eta). \tag{87}$$

Using Young's inequality in (85) leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{F}(t)\le-\frac{k_1}{2}\|v_x\|_2^2+\rho\|v_t\|_2^2+C\|\gamma v_x-p_x\|_2^2+C\Gamma(\eta)+\big\langle F_1(v,p),\,v\big\rangle. \tag{88}$$

By (27), Hölder's inequality and the embedding $L^{2q}(0,L)\hookrightarrow L^2(0,L)$, it follows that

$$\begin{aligned}\big|\big\langle F_1(v,p),\,v\big\rangle\big|&\le C\left(\|v^1\|_{2q}^{q-1}+\|v^2\|_{2q}^{q-1}+\|p^1\|_{2q}^{q-1}+\|p^2\|_{2q}^{q-1}+1\right)\big(\|v\|_{2q}+\|p\|_{2q}\big)\|v\|_2\\&\le C_B\big(\|v\|_{2q}+\|p\|_{2q}\big)\|v\|_{2q}\le C_B\big(\|v\|_{2q}^2+\|p\|_{2q}^2\big).\end{aligned}\tag{89}$$

Next, substitute the estimates (89) in (88) to get

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{F}(t)\le-\frac{k_1}{2}\|v_x\|_2^2+\rho\|v_t\|_2^2+C\|\gamma v_x-p_x\|_2^2+C\Gamma(\eta)+C_B\big(\|v\|_{2q}^2+\|p\|_{2q}^2\big). \tag{90}$$

**Step 3.** Multiplying the equation $(79)_1$ by $(\gamma v - p)$ and integrating over $[0,L]$ leads to

$$\rho\big\langle v_{tt},\, \gamma v - p\big\rangle + k_1\big\langle v_x,\, \gamma v_x - p_x\big\rangle + \gamma\beta\|\gamma v_x - p_x\|_2^2 + \int\limits_0^\infty \lambda(s)\big\langle \eta_x(s), \gamma v_x - p_x\big\rangle ds$$
$$+ \big\langle F_1(v,p),\, \gamma v - p\big\rangle = 0.$$

Now, define

$$\mathscr{G}(t) := -\rho\big\langle v_t,\, \gamma v - p\big\rangle. \tag{91}$$

By $v_{tt}(\gamma v - p) = \frac{\partial}{\partial t}[v_t(\gamma v - p)] - v_t(\gamma v - p)_t$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{G}(t) = &-\rho\gamma\|v_t\|_2^2 + \rho\big\langle v_t,\, p_t\big\rangle + k_1\big\langle v_x,\, \gamma v_x - p_x\big\rangle + \gamma\beta\|\gamma v_x - p_x\|_2^2\\ &+ \int\limits_0^\infty \lambda(s)\big\langle \eta_x(s),\, \gamma v_x - p_x\big\rangle ds + \big\langle F_1(v,p),\, \gamma v - p\big\rangle = 0.\end{aligned} \tag{92}$$

Next, utilize the Young's inequality to obtain

$$\int\limits_0^\infty \lambda(s)\big\langle \eta_x(s),\, \gamma v_x - p_x\big\rangle ds \le C\|\gamma v_x - p_x\|_2^2 + C\Gamma(\eta), \quad \big|\big\langle F_1(v,p),\, \gamma v - p\big\rangle\big| \le C_B(\|v\|_{2q}^2 + \|p\|_{2q}^2),$$

and

$$k_1\big\langle v_x,\, \gamma v_x - p_x\big\rangle \le k_1\varepsilon_2\|v_x\|_2^2 + C\|\gamma v_x - p_x\|_2^2, \quad \forall\, \varepsilon_2 > 0,$$

Therefore, for all $\varepsilon_2 > 0$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{G}(t) = -\frac{\rho\gamma}{2}\|v_t\|_2^2 + C\|p_t\|_2^2 + k_1\varepsilon_2\|v_x\|_2^2 + C\|\gamma v_x - p_x\|_2^2 + C\Gamma(\eta) + C_B(\|v\|_{2q}^2 + \|p\|_{2q}^2). \tag{93}$$

**Step 4.** Multiply the equation $(79)_2$ by $(\gamma v - p)$ and integrate over $[0,L]$ to have

$$\mu\big\langle p_{tt},\, \gamma v - p\big\rangle - \beta\|\gamma v_x - p_x\|_2^2 + \big\langle G(p_t),\, \gamma v - p\big\rangle + \big\langle F_2(v,p),\, \gamma v - p\big\rangle = 0. \tag{94}$$

Let

$$\mathscr{H}(t) := -\mu\big\langle p_t,\, \gamma v - p\big\rangle. \tag{95}$$

By $p_{tt}(\gamma v - p) = \frac{\partial}{\partial t}[p_t(\gamma v - p)] - p_t(\gamma v - p)_t$, and the Cauchy–Schwarz inequality, the following is immediate:

$$\frac{d}{dt}\mathscr{H}(t) \le -\mu\gamma\big\langle p_t,\, v_t\big\rangle + \mu\|p_t\|_2^2 - \beta\|\gamma v_x - p_x\|_2^2 + \big\langle G(p_t),\, \gamma v - p\big\rangle + \big\langle F_2(v,p),\, \gamma v - p\big\rangle. \tag{96}$$

Now, using the Young's and Poincaré's inequalities and the following estimates:

$$\big\langle G(p_t),\, \gamma v - p\big\rangle \le C\|p_t\|_2^2 + \frac{\beta}{2}\|\gamma v_x - p_x\|_2^2 \quad \text{and} \quad \big|\big\langle F_2(v,p),\, \gamma v - p\big\rangle\big| \le C_B(\|v\|_{2q}^2 + \|p\|_{2q}^2) \tag{97}$$

the following is obtained:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{H}(t) \le -\frac{\beta}{2}\|\gamma v_x - p_x\|_2^2 + \mu\varepsilon_3\|v_t\|_2^2 + C\|p_t\|_2^2 + C_B(\|v\|_{2q}^2 + \|p\|_{2q}^2), \quad \forall\, \varepsilon_3 > 0. \tag{98}$$

**Step 5.** Consider the functional

$$\mathscr{L}(t) := N_1 E(t) + \mathscr{F}(t) + N_2\mathscr{G}(t) + N_3\mathscr{H}(t).$$

By Young's and Poincaré's inequalities, the following holds true:

$$|\mathscr{L}(t) - N_1E(t)| \le |\mathscr{F}(t)| + N_2|\mathscr{G}(t)| + N_3|\mathscr{H}(t)| \le \tilde{\eta}E(t), \quad \tilde{\eta} > 0. \tag{99}$$

Consequently, for $N_1 > \tilde{\eta}$

$$\big(N_1 - \tilde{\eta}\big)E(t) \le \mathscr{L}(t) \le \big(N_1 + \tilde{\eta}\big)E(t), \quad \forall t \ge 0. \tag{100}$$

**Step 6.** By the estimates (81), (90), (93), and (98):

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{L}(t) \le &-\big(N_2\rho\gamma/2 - N_1\rho\varepsilon_1 - N_3\mu\varepsilon_3 - \rho\big)\|v_t\|_2^2 - \big(mN_1/2 - N_2C - N_3C\big)\|p_t\|_2^2\\ &- \big(k_1/2 - N_2k_1\varepsilon_2\big)\|v_x\|_2^2 - \big(N_3\beta/2 - C - N_2C\big)\|\gamma v_x - p_x\|_2^2\\ &- \big(N_1/2 - C - N_2C\big)\Gamma(\eta) + \big(1 + N_1 + N_2 + N_3\big)C_B\big(\|v\|_{2q}^2 + \|p\|_{2q}^2\big).\end{aligned}$$

Now, choose $\varepsilon_1 := 1/4N_1$, $\varepsilon_2 := 1/4N_2$, and $\varepsilon_3 := \rho/4\mu N_3$ to obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{L}(t) \le &-\big(N_2\gamma - 3\big)\frac{\rho}{2}\|v_t\|_2^2 - \big(mN_1/2 - N_2C - N_3C\big)\|p_t\|_2^2 - \frac{k_1}{4}\|v_x\|_2^2\\ &- \big(N_3\beta/2 - C - N_2C\big)\beta\|\gamma v_x - p_x\|_2^2 - \big(N_1/2 - C - N_2C\big)\Gamma(\eta)\\ &+ \big(1 + N_1 + N_2 + N_3\big)C_B\big(\|v\|_{2q}^2 + \|p\|_{2q}^2\big).\end{aligned}$$

Choosing further that $N_2 > 3/\gamma$, $N_3 > \frac{2C}{\beta}(1+N_2)$, and $N_1 > \max\big\{\frac{2}{m}C(N_2+N_3),\, 2C(1+N_2),\, \tilde{\eta}\big\}$, there exists $N_0 > 0$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{L}(t) \le -N_0E(t) + C_B\big(\|v\|_{2q}^2 + \|p\|_{2q}^2\big).$$

Next, the second inequality in (100) is utilized to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{L}(t) \le -\frac{N_0}{N_1 + \tilde{\eta}}\mathscr{L}(t) + C_B(\|v\|_{2q}^2 + \|p\|_{2q}^2). \tag{101}$$

Finally, the Gronwall's lemma is applied to (101) to get

$$\mathscr{L}(t) \le e^{-\frac{N_0}{N_1+\tilde{\eta}}t}\mathscr{L}(0) + C_B\int_0^t e^{-\frac{N_0}{N_1+\tilde{\eta}}(t-s)}(\|v(s)\|_{2q}^2 + \|p(s)\|_{2q}^2)\,\mathrm{d}s.$$

Choosing $\gamma_0 := \frac{N_1+\tilde{\eta}}{N_1-\tilde{\eta}} > 0$, $\mu_0 := \frac{N_0}{N_1+\tilde{\eta}} > 0$, $C'_B := \frac{C_B}{N_1-\tilde{\eta}} > 0$ and reusing (100), the following inequality holds true:

$$E(t) \le \gamma_0e^{-\mu_0t}E(0) + C'_B\int_0^t e^{-\mu_0(t-s)}(\|v(s)\|_{2q}^2 + \|p(s)\|_{2q}^2)\,\mathrm{d}s.$$

Hence, (78) is obtained. □

### 5.2. Gradient system

Recall that a dynamical system $(H, S(t))$ is gradient if it possesses a strict Lyapunov functional. That is, a functional $\Phi : H \to \mathbb{R}$ is a strict Lyapunov function for a system $(H, S(t))$ if

(i) the map $t \to \Phi(S(t)z)$ is non-increasing for each $z \in H$,
(ii) if $\Phi(S(t)z) = \Phi(z)$ for some $z \in H$ and for all $t$, then $z$ is a stationary point of $S(t)$, that is, $S(t)z = z$.

**Lemma 5.2.** *The dynamical system $(\mathcal{H}, S(t))$ corresponding to problem (17)–(21) is gradient. Moreover, there exists a Lyapunov functional $\Phi$ defined in $\mathcal{H}$ such that*

***a.*** *the Lyapunov functional $\Phi$ is bounded from above on any bounded subset of $\mathcal{H}$,*
***b.*** *the set $\Phi_R = \{U_0 \in \mathcal{H}\ ;\ \Phi(U_0) \le R\}$ is bounded in $\mathcal{H}$ for every $R > 0$.*

*Proof.* Consider the functional $\Phi$ on $\mathcal{H}$ defined by

$$\Phi(v,p,\phi,\varphi,\eta)=\frac{1}{2}\|(v,p,\phi,\varphi,\eta)\|_{\mathcal{H}}^2+\int_0^L F(v,p)dx-\int_0^L h_1 v dx-\int_0^L h_2 p dx. \tag{102}$$

Note that if $U(t)=(v(t),p(t),v_t(t),p_t(t),\eta(t))=S(t)U_0$ is a solution of (34), for any $U_0=(v_0,p_0,v_1,p_1,\eta_0)\in\mathcal{H}$, $t\mapsto\Phi(S(t)U_0)$ is non-increasing by Lemma 3.3. Moreover, if $\Phi(S(t)U_0)=\Phi(U_0)$ for any $t>0$, (42) leads to

$$-m\|p_t\|_2^2-\frac{1}{2}\Gamma(\eta)=0 \tag{103}$$

which implies that

$$p_t(t)=0,\quad t>0\quad\Longrightarrow\quad p(t)=p_0,\quad t>0. \tag{104}$$

On the other hand, by (103) and (16),

$$0=\Gamma(\eta)\leq-\delta_1\|\eta\|_{\mathcal{M}}^2.$$

Therefore, $\eta(t,s)=0$ for $t,s>0$. Noting that $\eta_t-\mathcal{T}\eta_s-\ v_t=0$, we can get

$$v_t(t)=0,\quad t>0\quad\Longrightarrow\quad v(t)=v_0,\quad t>0, \tag{105}$$

and consequently, $U_0=(v_0,p_0,0,0,0)\in\mathscr{N}$. This simply means that $\Phi$ is a strict Lyapunov function for $(\mathcal{H},S(t))$ on $\mathcal{H}$, and therefore, $(\mathcal{H},S(t))$ is gradient system.

From (42), $\Phi$ can be easily seen to be bounded from above on bounded subsets of $\mathcal{H}$. Let $U(t)$ be the mild solution (corresponding to $U_0$) to problem (17)-(21) such that $\Phi(U_0)\leq R$. Then, it is inferred from (41) that

$$\chi_0\|S(t)U_0\|_{\mathcal{H}}^2\leq\Phi(U_0)+C_F\leq R+C_F.$$

Hence, $\Phi_R$ is a bounded set of $\mathcal{H}$. □

**Lemma 5.3.** *The set* $\mathscr{N}=\{U=(v,p,0,0,0)\in\mathcal{H};-\tilde{k}v_{xx}+\gamma\beta p_{xx}+f_1(v,p)=h_1,\quad-\beta p_{xx}+\gamma\beta v_{xx}+f_2(v,p)=h_2\}$ *of stationary solutions is bounded in* $\mathcal{H}$.

*Proof.* By letting $U=(v,p,0,0,0)\in\mathscr{N}$, $U$ satisfies

$$\begin{aligned}&-\tilde{k}v_{xx}+\gamma\beta p_{xx}+f_1(v,p)=h_1,\\&-\beta p_{xx}+\gamma\beta v_{xx}+f_2(v,p)=h_2.\end{aligned} \tag{106}$$

Multiplying $(106)_1$ by $v$ and $(106)_2$ by $p$ and integrating by parts each one over $[0,L]$ leads to

$$k_1\|v_x\|_2^2+\beta\|\gamma v_x-p_x\|_2^2=-\int_0^L\nabla F(v,p)\cdot(v,p)dx+\int_0^1(h_1v+h_2p)dx. \tag{107}$$

where (26) is used. Considering (25), (29), and (30) yields

$$-\int_0^L\nabla F(v,p)\cdot(v,p)dx\leq 2dd_0\big(k_1\|v_x\|_2^2+\beta\|\gamma v_x-p_x\|_2^2\big)+2Lm_F, \tag{108}$$

and therefore, we obtain the following from (107), (108), and (45):

$$4\chi_0\big(k_1\|v_x\|_2^2+\beta\|\gamma v_x-p_x\|_2^2\big)\leq 2Lm_F+\int_0^L(h_1v+h_2p)\,\mathrm{d}x. \tag{109}$$

By Young's inequality and (25), it follows that

$$\int_0^L (h_1 v + h_2 p)\,\mathrm{d}x \le \frac{\chi_0}{d_0}\big(\|v\|_2^2 + \|p\|_2^2\big) + \frac{d_0}{4\chi_0}\big(\|h_1\|_2^2 + \|h_2\|_2^2\big)$$

$$\le \chi_0\big(k_1\|v_x\|_2^2 + \beta\|\gamma v_x - p_x\|_2^2\big) + \frac{d_0}{4\chi_0}\big(\|h_1\|_2^2 + \|h_2\|_2^2\big).$$

Finally, substituting the last estimate in (109) results in

$$3\chi_0\|U\|^2 \le 2Lm_F + \frac{d_0}{4\chi_0}\big(\|h_1\|_2^2 + \|h_2\|_2^2\big) \tag{110}$$

which shows that the set $\mathscr{N}$ is bounded in $\mathcal{H}$. $\square$

**Theorem 5.4.** *The dynamical system ($\mathcal{H}$,$S(t)$) possesses a compact global attractor $\mathfrak{A} = \mathscr{M}(\mathscr{N})$, where $\mathscr{M}(\mathscr{N})$ is the unstable manifold emanating from $\mathscr{N}$. Moreover, $\mathfrak{A}$ has finite fractal dimension.*

*Proof.* Since the system $(\mathcal{H}, S(t))$ is quasi-stable by Lemma 5.1, $(\mathcal{H}, S(t))$ is asymptotically smooth by Proposition 7.9.4 in [10]. Thus, noting Lemmas 5.2 and 5.2 and using Corollary 7.5.7 in [10], it is known that $(\mathcal{H}, S(t))$ has a compact global attractor given by $\mathfrak{A} = \mathscr{M}(\mathscr{N})$. Finally, the attractor $\mathfrak{A}$ has finite fractal dimension by Theorem 7.9.6 in [10]. $\square$

## 6. Regularity and exponential attractors

**Theorem 6.1.** *Every trajectory $\{(v(t), p(t), p_t(t), v_t(t), \eta^t(t))\}$ in $\mathfrak{A}$ has further regularity*

$$\|(v,p)\|_{(H^2(0,1)\cap H^1_*(0,L))^2} + \|(v_t,p_t)\|_{(H^1_*(0,L))^2} + \|(v_{tt},p_{tt})\|_{(L^2(0,L))^2} \le R, \tag{111}$$

*for some $R > 0$.*

*Proof.* Since the system $(\mathcal{H}, S(t))$ is quasi-stable on the attractor $\mathfrak{A}$, it follows from Theorem 7.9.8 in [10] that any complete trajectory $(v, p, p_t, v_t, \eta)$ in $\mathfrak{A}$ enjoys the following regularity properties:

$$v_t \in L^\infty(\mathbb{R}, H^1_*(0,L)) \cap C(\mathbb{R}, L^2(0,L)), \quad p_t \in L^\infty(\mathbb{R}, H^1_*(0,L)) \cap C(\mathbb{R}, L^2(0,L))$$

and

$$v_{tt} \in L^\infty(\mathbb{R}, L^2(0,L)), \quad p_{tt} \in L^\infty(\mathbb{R}, L^2(0,L)), \quad \eta_t \in L^\infty(\mathbb{R}, \mathcal{M}).$$

Noting that the nonlinear terms are continuous and using (17)-(18), w $v_{xx} \in L^\infty(\mathbb{R}, L^2(0,L))$ and $p_{xx} \in L^\infty(\mathbb{R}, L^2(0,L))$ are obtained. Hence, the proof is complete. $\square$

**Definition 6.2.** A compact set $\mathfrak{A}_{\exp} \subset \mathcal{H}$ is called an *exponential attractor* for $(\mathcal{H}, S(t))$ if

- $\mathfrak{A}_{\exp}$ is a positively invariant set, that is, $S(t)\mathfrak{A}_{\exp} \subset \mathfrak{A}_{\exp}$ for all $t \ge 0$;
- $\mathfrak{A}_{\exp}$ has finite fractal dimension in $\mathcal{H}$;
- $\mathfrak{A}_{\exp}$ attracts bounded sets of $\mathcal{H}$ at an exponential rate, that is, for any bounded set $D \subset \mathcal{H}$ there exist $t_D, C_D, \gamma_D > 0$ such that

$$\mathrm{dist}_{\mathcal{H}}(S(t)D, \mathfrak{A}_{\exp}) \le C_D e^{-\gamma_D(t-t_D)}, \quad \forall t \ge t_D,$$

  where $\mathrm{dist}_{\mathcal{H}}$ represents the Hausdorff semi-distance in $\mathcal{H}$. If there exists an exponential attractor only having finite dimension in some extended space $\widetilde{\mathcal{H}} \supseteq \mathcal{H}$ , then this exponentially attracting set is called generalized fractal exponential attractor.

The proof of the following result is based on Theorem 7.9.9 of [10]. Note that this theorem does not provide an estimate for the fractal dimension in extended phase space $\widetilde{\mathcal{H}}$.

**Theorem 6.3.** *The dynamical system $(\mathcal{H}, S(t))$ possesses a generalized exponential attractor $\mathfrak{A}_{exp} \subset \mathcal{H}$ with finite fractal dimension in extended space*

$$\mathcal{H}_{-1} = (H_*^{-1}(0,L))^2 \times (L^2(0,L))^2 \times \mathcal{M}^{-1} \supset \mathcal{H}, \tag{112}$$

*where $H_*^{-1}(0,L)$ is the dual space of $H_*^1(0,L)$ pivoted with respect to $L^2(0,L)$, and $\mathcal{M}^{-1}$ is the dual space of $\mathcal{M}$.*

*Proof.* Consider the set $\mathcal{B}_R$ given by

$$\mathcal{B}_R = \{U \in \mathcal{H};\ \Phi(U) \le R\}, \tag{113}$$

where $\Phi$ is the Lyapunov function defined by (102). For $R$ large enough, we have $\mathcal{B}_R$ absorbing positively invariant. Therefore, $(\mathcal{H}, S(t))$ is quasi-stable on $\mathcal{B}_R$. This way, there exists a positive constant $C_\mathcal{B}$ such that if $U_0 \in \mathcal{B}_R$ and $U(t) = (v(t), p(t), v_t(t), p_t(t), \eta(t)) = S(t)U_0$

$$\|(v(t), p(t), v_t(t), p_t(t), \eta(t))\|_\mathcal{H} \le C_\mathcal{B}. \tag{114}$$

From (17)–(19), we obtain

$$\|(v_t(t), p_t(t), v_{tt}(t), p_{tt}(t), \eta_t(t))\|_{\mathcal{H}_{-1}} \le \widetilde{C}_\mathcal{B} \tag{115}$$

where $\widetilde{C}_\mathcal{B}$ is a positive constant depending on $\mathcal{B}_R$. So, for any $t_1, t_2 \in [0,T]$ we have

$$\|S(t_1)U_0 - S(t_2)U_0\|_{\mathcal{H}_{-1}} = \left\| \int_{t_1}^{t_2} U_t(\tau) d\tau \right\|_{\mathcal{H}_{-1}} \le \int_{t_1}^{t_2} \|U_t(\tau)\|_{\mathcal{H}_{-1}} d\tau \le \widetilde{C}_\mathcal{B} |t_1 - t_1|. \tag{116}$$

This means that $t \mapsto S(t)U_0$ is Hölder continuous on $\mathcal{H}_{-1}$ for any $U_0 \in \mathcal{B}_R$. Therefore, the result follows from [10, Theorem 7.9.9]. □

**Remark 6.4.** Note that proving the existence of exponential attractors even for a single wave equation is a difficult task. Most of the techniques used in the literature are developed for the study of parabolic problems, e.g., see [1]. In particular, this difficulty is pronounced due to the lack of regularity of the memory component $\eta$ in (19). For this reason, the existence of the so-called generalized exponential attractors, defined in Definition 6.2, is proved in the similar fashion as discussed in [9,10]. It is crucial to point out that the major difference here is that the set $\mathfrak{A}_{\text{exp}}$ has a finite fractal dimension in an extended phase space $\widetilde{\mathcal{H}}$ containing $\mathcal{H}$. Hence, exponential attractors can be considered in weaker phase spaces.

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