Auditing for Core Stability in Participatory Budgeting

Kamesh Munagala[1(✉)], Yiheng Shen[1], and Kangning Wang[2]

[1] Duke University, Durham, NC 27708-0129, USA
{kamesh,ys341}@cs.duke.edu

[2] Simons Institute for the Theory of Computing, Berkeley, CA 94720-2190, USA
kangning@berkeley.edu

Abstract. We consider the *participatory budgeting* problem where each of n voters specifies additive utilities over m candidate projects with given sizes, and the goal is to choose a subset of projects (i.e., a *committee*) with total size at most k. Participatory budgeting mathematically generalizes multiwinner elections, and both have received great attention in computational social choice recently. A well-studied notion of group fairness in this setting is *core stability*: Each voter is assigned an "entitlement" of $\frac{k}{n}$, so that a subset S of voters can pay for a committee of size at most $|S| \cdot \frac{k}{n}$. A given committee is in the core if no subset of voters can pay for another committee that provides each of them strictly larger utility. This provides proportional representation to all voters in a strong sense. In this paper, we study the following auditing question: Given a committee computed by some preference aggregation method, how close is it to the core? Concretely, how much does the entitlement of each voter need to be scaled down by, so that the core property subsequently holds? As our main contribution, we present computational hardness results for this problem, as well as a logarithmic approximation algorithm via linear program rounding. We show that our analysis is tight against the linear programming bound. Additionally, we consider two related notions of group fairness that have similar audit properties. The first is *Lindahl priceability*, which audits the closeness of a committee to a market clearing solution. We show that this is related to the linear programming relaxation of auditing the core, leading to efficient exact and approximation algorithms for auditing. The second is a novel weakening of the core that we term the *sub-core*, and we present computational results for auditing this notion as well.

Keywords: Computational social choice · Participatory budgeting · Core stability

Supported by NSF grant CCF-2113798.

K. Wang—This work was done while Kangning Wang was at Duke University.

K. A. Hansen et al. (Eds.): WINE 2022, LNCS 13778, pp. 292–310, 2022.
https://doi.org/10.1007/978-3-031-22832-2_17

1 Introduction

The *participatory budgeting* problem [1,6,12,19,24] is motivated by real-world elections where voters decide which projects a city should fund subject to a budget constraint on the total cost of these projects. In this problem, there are m candidate projects forming a set C, and n voters. Each candidate j is associated with a size/cost s_j.

The *multiwinner election* problem [2,9,14,18,36] is commonly seen in practice, and has received significant research attention recently. Mathematically, it is a specialization of the participatory budgeting problem, where each candidate is of the same unit size.

In both settings, our goal is to pick a subset $T \subseteq C$ of candidates – which we call a *committee* – with total size at most a given value k, that is, $\sum_{j \in T} s_j \leq k$. Each voter i has a utility function $U_i(T)$ over subsets $T \subseteq C$ of candidates. In this paper, we assume the utility functions $\{U_i\}$ are *additive* across candidates. For some of our results, we also look at the more restricted case of multiwinner elections with *approval* (i.e. 0/1-additive) utilities: Each candidate is of unit size; each voter i "approves" a subset $A_i \subseteq C$ of candidates, and for any committee T, the utility function of voter i is simply $U_i(T) = |T \cap A_i|$, the number of approved candidates in the committee. We call this the Approval Election setting.

Core Stability. In both multiwinner elections and participatory budgeting, the methods used to aggregate preferences of voters are typically very simple, for instance, choosing the candidates who receive the most approval votes. This leads to a tension of such rules with *fairness* of the resulting outcome in terms of proportional representation of minority opinions, and a social planner may want to quantify this tension for any given election.

A notion of fairness in this context, which has been studied for over a century, is that of *core stability* [17,20,29,35,36]. This captures a strong notion of proportional representation. Given a committee W of size k, think of k as a budget, and split it equally among the n voters, so that each voter is entitled to a budget of $\frac{k}{n}$. For any subset $S \subseteq [n]$ of voters, their total entitlement is $|S| \cdot \frac{k}{n}$. If there is another committee T of size at most the entitlement $|S| \cdot \frac{k}{n}$, such that each voter $i \in S$ strictly prefers T to W, *i.e.*, $U_i(T) > U_i(W)$ for all $i \in S$, then these voters would have a justified complaint with W. A committee W where no subset $S \subseteq [n]$ of voters have a justified complaint is termed *core stable*.

The core has a "fair taxation" interpretation [23,29]. The quantity $\frac{k}{n}$ can be thought of as the tax contribution of a voter, and a committee in the core has the property that no sub-group of voters could have spent their share of tax money in a way that *all* of them were better off. It subsumes notions of fairness such as Pareto-optimality, proportionality, and various forms of justified representation [3,4,22] that have been extensively studied in multiwinner election and fairness literature. Note that the core is *oblivious* to how the demographic slices are defined – it attempts to be fair to *all* subsets of voters. This is a desirable feature in practice, since demographic slices are often not known upfront, and there could be hidden sub-groups that can only be inferred from voter preferences.

Approximate Stability. The core is a very appealing group fairness notion; however, even in very simple settings, the core could be empty [20]. This motivates approximation, where the entitlement $\frac{k}{n}$ of each voter is scaled by a factor of θ.

Definition 1. *For $\theta \leq 1$, a committee W of size at most k lies in the θ-*approximate core *if for all $S \subseteq [n]$, there is no* deviating committee *T with size at most $\theta \cdot |S| \cdot \frac{k}{n}$, such that for all $i \in S$, we have $U_i(T) > U_i(W)$.*

It is known [25] that a $\frac{1}{32}$-approximate core solution always exists for very general utility functions of the voters.

Auditing for Approximate Stability. Though the existence of approximate core solutions is a strong positive result, the algorithms for finding such solutions are often complex. Indeed, even in settings where the core is known to be always non-empty, for instance when candidates can be chosen fractionally [23], the non-emptiness is an existence result that needs an expensive fixed point computation. On the other hand, in practice, what are implemented are typically the simplest and most explainable social choice methods such as Single Transferable Vote (STV). Therefore, from the perspective of a societal decision maker, such as a civic body running a participatory budgeting election, it becomes important to answer the following *auditing* question for any given election:

> Given a committee W of size at most k found by some implemented preference aggregation method, how close is it to being core stable, i.e., what is the smallest value of θ_c such that W *does not* lie in the θ_c-approximate core for that instance?

Note that if a committee W lies in the core, then $\theta_c > 1$, else $\theta_c \leq 1$. Such an auditing question is useful even if the decision maker themselves is not sensitive for fairness because it allows for review of implemented decision rules via a third party or government agency. Further, the set of deviating voters that correspond to the θ_c-approximation yield a demographic that are unhappy with the current outcome, and this can be analyzed further by policy makers.

We term the above question as the *core auditing* problem. In this paper, we study the computational complexity of core auditing. In that process, we define both stronger and weaker notions of fairness and audit these notions as well.

1.1 Our Results

Hardness and Approximation Algorithm. We show in Sect. 3 that for Approval Elections, the value of θ_c in the core auditing problem is NP-Hard to approximate to a factor better than $1 + \frac{1}{e} > 1.367$. We further show that this APX-Hardness persists even when voters are allowed to choose a fractional deviating committee. We also show that the problem remains NP-Hard when each voter approves a constant number of candidates, and each candidate is approved by a constant number of voters. These results significantly strengthen the NP-Hardness result presented in [11].

On the positive side, in Sect. 4, we design an $O(\min(\log m, \log n))$ approximation algorithm for the value θ_c, where m and n are the number of candidates and voters respectively. We do this via linear program rounding. Our program (and indeed, our auditing question itself) is an interesting generalization of the densest subgraph problem [15], where the goal is to choose a subgraph with maximum average degree. Given a graph, treat voters as edges and candidates as vertices that are approved by the incident edges; further assume any voter needs utility 2 (that is, both end-points) in a feasible deviation. Then, the value of θ_c is precisely the density of the densest subgraph (to scaling). We combine ideas from the rounding for densest subgraph (where the rounding produces the integer optimum without approximation) with that from maximum coverage to design our rounding scheme. We further show that our linear program has an integrality gap of $\Omega(\min(\log m, \log n))$, showing that we cannot do any better against an LP lower bound. Our proof in Sect. 4 applies to the Approval Election setting. We extend this to general candidate sizes and arbitrary additive utilities via knapsack cover inequalities in Sect. 5, leading to an $O(\min(\log m, \log n))$ approximation factor. In the full paper [31], we show that both our hardness and approximation results easily extend to settings where candidates can be fractionally chosen in the committees.

It is an interesting question to close the gap between our hardness result (constant factor) and our approximation ratio. The difficulty lies with density problems in general, where hardness of approximation results have been hard to come by; see for instance, the k-densest subgraph problem [27].

Lindahl Priceability. A closely related notion of fairness, considered in [23,29,32,33] is that of committees that can be supported by market clearing prices. The notion of *Lindahl equilibrium* is a pricing scheme that strengthens the core, meaning that if the former exists, it lies in the core. In this scheme, each voter i is assigned price p_{ij} for candidate j, and these prices are such that for any candidate, the total price is equal to its size. If a voter buys their optimal set of candidates subject to the total price paid being at most their entitlement, k/n, then all voters choose the *same* committee. This is therefore a market clearing notion with per-voter prices such that the optimal voter action given these prices and equal entitlements results in a common committee being chosen. If committees could be chosen fractionally, it is known via a fixed point argument that the Lindahl equilibrium always exists [23]. However, these need not exist when considering integer committees.

In this paper, we consider an integer version of this concept that we term *Lindahl priceability.* We show that this notion implies the core. As with the core, in Sect. 6, we define the approximation factor θ_ℓ to which a given committee satisfies Lindahl priceability, via scaling the entitlement k/n of each voter by that factor. We show via LP duality not only that the quantity θ_ℓ can be audited in polynomial time for approval elections, but also that this computation coincides with the LP relaxation to the core auditing program. This results in a novel and somewhat surprising connection between the Lindahl priceability and the core for Approval elections, where the approximation factor θ_ℓ for

Lindahl priceability is found via the LP relaxation to the program that computes the approximation factor θ_c for the core. Further, our approach easily extends to show computational results for general utilities and sizes.

Our notion is related to the *cost efficient Lindahl equilibrium* proposed recently in [33] for APPROVAL ELECTIONS. However, there is a crucial difference: While they translate the fractional Lindahl equilibrium to the integer case, we translate the gradient optimality conditions implied by the fractional equilibrium to the integer case. To illustrate that our definition is different, note that while there are simple instances of APPROVAL ELECTIONS on which the former notion does not exist, we do not know such an instance for our definition.

Weak Priceability and Sub-core. In Sect. 7, we finally connect our work to another notion of priceability first studied in [34]. This notion is a relaxation of Lindahl priceability for APPROVAL ELECTIONS, where voters cannot greedily augment the current committee given the prices and their entitlement. We term this "weak priceability" and use this to define a new relaxation of the core, termed *sub-core*, which only allows voters to deviate and gain utility from super-sets. We show that weakly priceable committees lie in the sub-core. Further, though the sub-core appears like a weak notion of fairness, we show that it remains NP-HARD to audit. We finally present an $O(\min(\log m, \log n))$ approximation to the auditing question using same techniques as for auditing the core.

In practice, committees found by social choice rules are likely to be much better approximations to the sub-core compared to the core. Hence, it is desirable to show a practitioner closeness to weaker notions of fairness such as the sub-core in addition to closeness to the core.

Omitted Proofs. For lack of space, all omitted proofs are in the full paper [31].

1.2 Related Work

Proportionality in Social Choice. The earliest work that considers proportional representation dates back to the late 1800's [17], and several voting rules attempting to achieve it, such as PAV [36] and Phragmén [10] rules also date back to then. There has been resurgence of interest in axiomatizing proportionality [3,4,8,14,22,30] partly driven by real-world applications of such elections to areas such as participatory budgeting [1,6,24], and partly due to local bodies and countries implementing rules such as ranked choice voting that attempt to achieve proportionality, in their elections. These advances have made auditing fairness notions such as closeness to the core and weaker group fairness notions imperative in these settings.

Notions of Approximate Core. In addition to the notion of approximation presented in Definition 1, a different notion allows deviating voters to use their entire entitlement, but requires them to extract at least a factor $\theta > 1$ larger utility on deviation. Under this notion, it is shown in [34] that a classic voting rule called

PAV [36] achieves a 2-approximation. This result was generalized to show a constant approximate core for arbitrary submodular utility functions and general candidate sizes in [32]. An analogous result for clustering was presented in [16]. Our work directly shows that this notion of approximation can be audited in a *bicriteria* fashion as follows: If the given committee is a c-approximation without violating entitlements, we can determine if it is a c-approximation had entitlements been violated by a factor of $O(\log m)$. It is an interesting open question to remove the bicriteria nature of this result.

Auditing for Fairness. The question of auditing has become salient given the increasing democratization of societal decision making, for instance via processes like participatory budgeting. In the context of social choice, there are natural properties that are easy to achieve algorithmically but hard to audit. For instance, checking if an arbitrary outcome is Pareto-optimal is computationally hard [5], while achieving it via some algorithm is easy. We take a further step in this direction by studying the approximate audit of arguably the strongest possible group fairness notion, the core, as well as related fairness properties.

Going beyond social choice, the notion of auditing for group fairness has gained prevalence in machine learning. Here, the "voters" are data points, and the "committee" is a classifier. We wish to audit if the classifier provides comparable accuracy for various demographic slices. The work of [26] formulates and presents algorithms for this problem.

2 Mathematical Program for $\boldsymbol{\theta_c}$

For most of this paper, we consider the Approval Election setting. Recall that in this setting, each voter i "approves" a set $A_i \subseteq C$ of unit-sized candidates, and its utility for a committee $T \subseteq C$ is simply $U_i(T) = |A_i \cap T|$. Our hardness results hold even for this simple setting, while our approximation algorithms hold for general additive utilities and sizes (see Sect. 5).

We first present a mathematical program that computes θ_c given a committee $W \subseteq C$ of size at most k, as in Definition 1. In this program, there is a variable $z_i \in \{0, 1\}$ that captures whether voter i deviates, and a variable $x_j \in \{0, 1\}$ that captures whether candidate j is present in the deviating committee. If this is a feasible deviation, then the utility of each voter for which $z_i = 1$ must strictly increase, which means $\forall i \in [n]$, $\sum_{j \in A_i} x_j \geq (U_i(W) + 1) \cdot z_i$.

Next, let $R = \frac{n}{k}$. Then, the budget available to the deviating voters is $\frac{1}{R} \sum_i z_i$, while the size of the committee to which they deviate is $\sum_{j \in C} x_j$. This means the entitlement k/n of each voter must be scaled by a factor of $R \cdot \frac{\sum_{j \in C} x_j}{\sum_i z_i}$, so that the voters with $z_i = 1$ do not have enough entitlement to pay for this deviating committee. Since the goal is to have no deviations at all, the value θ_c is simply the solution to the following mathematical program:

$$\text{Minimize } R \cdot \frac{\sum_{j \in C} x_j}{\sum_i z_i}, \text{ s.t.}$$
$$\forall i \in [n], \sum_{j \in C \cap A_i} x_j \geq z_i \cdot (U_i(W) + 1);$$
$$\forall i \in [n], \ \forall j \in [m], \ x_j, z_i \in \{0, 1\}.$$

The above program attempts to maximize the ratio of the number of constraints satisfied via setting z_i to 1, to the number of x_j variables set to 1.

3 Hardness of Auditing the Core

As mentioned before, all hardness results in this section apply to the APPROVAL ELECTION setting, where the utilities are binary, and candidate sizes are unit. We first show that the core auditing problem, that is, the problem of computing θ_c for a given committee W, is NP-HARD even in a "constant degree" setting. This strengthens an NP-HARDNESS result for the core in [11].

Theorem 1. *Deciding whether a committee W does not lie in the core (that is, deciding whether its $\theta_c \leq 1$) is* NP-HARD *when each voter approves at most 6 candidates (that is, $|A_i| \leq 6$ for all voters $i \in [n]$), and each candidate lies in at most 2 of the sets A_i.*

We now show that the core auditing problem is in fact APX-HARD.

Theorem 2. *For any constant $\gamma > 0$, approximating θ_c to within a factor of $1 + \frac{1}{e} - \gamma$ is* NP-HARD.

We will reduce from the maximum set coverage problem on regular instances.

Lemma 1 (Regular Maximum Coverage [21]). *The universe contains qd elements. There are ξ sets, each with d elements. It is* NP-HARD *to distinguish between the following two cases:*

1. *"YES" instances: There exist q sets that cover the universe.*
2. *"NO" instances: No collection of q sets can cover $(1 - 1/e + \varepsilon) \cdot qd$ elements.*

Proof (Proof of Theorem 2). For each instance of the regular Max Covering Problem, there are qd elements and ξ sets. We construct the following instance for auditing the core:

- There are ξ main candidates. Each candidate corresponds to a set. There are $\frac{1}{e}(q-1)qd$ dummy candidates.
- There are two group of voters. The first group contains $\frac{1}{e} \cdot qd$ voters. They each approve $q - 1$ disjoint dummy candidates, and all the main candidates.

- The second group contains qd voters. Each of these voters corresponds to an element of the covering instance. She approves the main candidates whose corresponding set contains her corresponding element. Therefore, there are $(1 + 1/e)qd$ voters. Add dummy voters who do not approve any candidates, so that the total number of voters is $n = q(q-1)d^2$.
- The budget for committee selection is $k = (q-1)qd$. The current committee W contains all the dummy candidates. All voters in the first group have utility $q-1$ while all voters in the second group have utility 0 in W.
- Note that each voter is assigned a budget of $\frac{1}{R} = \frac{k}{n} = \frac{(q-1)qd}{q(q-1)d^2} = \frac{1}{d}$.

If the maximum coverage instance is a "YES" instance, choose as the deviating committee the q main candidates whose corresponding sets cover the universe. We call a voter "satisfied" if her utility has strictly increased compared to the current committee W. From the program in Sect. 2, θ_c is $R = d$ times the minimum ratio of the total number of selected candidates to the number of satisfied voters. Since we have selected q candidates, the voters in the first group receive utility q and are therefore satisfied. Moreover, since the chosen candidates' corresponding sets cover the universe of qd elements, the voters in the second group receive utility at least one, and are therefore satisfied. Therefore,

$$\theta_c \leq R \cdot \frac{q}{qd \cdot (1 + 1/e)} = \frac{1}{1 + 1/e}.$$

Suppose the maximum coverage instance is a "NO" instance. We will show that $\theta_c \geq 1 - o(1)$. First suppose a deviating committee is composed of $s < q$ main candidates. These candidates can cover at most ds voters from the second group. For the first group, they provide utility s to each voter. If t of these voters are satisfied, we must have chosen $(q-s)t$ dummy candidates. This means the scaling factor needed is at least

$$R \cdot \frac{s + (q-s)t}{ds + t} = \frac{s + (q-s)t}{s + t/d} > 1.$$

If the number of main candidates in the deviating committee is at least q, the voters in the first group are all satisfied and we don't need to choose dummy candidates. Consider an arbitrary q-candidate subset of these selected candidates. All voters in the first group are satisfied by these candidates, since they receive utility q from them. Since the coverage instance is a "NO" instance, no more than $(1 - 1/e + \varepsilon) \cdot kd$ voters in the second group are satisfied by this subset. Suppose there are r remaining candidates in the deviation, each candidate can only increase the number of satisfied voters by at most $r \cdot d$. Therefore,

$$\theta_c \geq R \cdot \frac{q + r}{r \cdot d + (1 - 1/e + \varepsilon) \cdot q \cdot d + (1/e) \cdot q \cdot d} = \frac{R}{d} \cdot \frac{q + r}{r \cdot d + q \cdot d \cdot (1 + \varepsilon)} \geq \frac{1}{1 + \varepsilon}.$$

Since the gap of θ_c between the constructed auditing instance from "YES" instances and from "NO" instances is at least $\frac{1+1/e}{1+\varepsilon}$, approximating θ_c to within this factor is NP-Hard.

Hardness of Auditing Fractional Committees. One natural question is whether the above hardness stems from the integrality requirement on the committee (the x_j variables in the program in Sect. 2) or the voters (the z_i variables). In the full paper [31], we show that the auditing problem remains hard to approximate to constant factors even when the committees can be chosen *fractionally*. This corresponds to allowing the variables $\{x_j\}$ to be fractional in $[0, 1]$. This shows that the hardness of the problem stems mainly from insisting $\{z_i\}$ be integral. The proof of this result is similar to the previous proof.

4 A Logarithmic Approximation for Auditing the Core

Our main result in this section is the following theorem, which we prove for the Approval Election setting. The proof for general candidate sizes and general additive utilities is presented in Sect. 5.

Theorem 3. *Given a committee W of size at most k, its θ_c value can be computed within $O(\min(\log m, \log n))$ factor in polynomial time, where m, n are the total number of candidates and voters respectively.*

4.1 LP Relaxation

Given a committee W, we start with the mathematical program from Sect. 2 and relax the variables to be fractional. This yields the following program. To see that this is a relaxation, if $z_i = 0$ for some i, then the first constraint is trivially satisfied. On the other hand, if $z_i = 1$, then we can increase all y_{ij} so that $y_{ij} = x_j$, thereby recovering the constraint in the integer program from Sect. 2. Therefore, any solution to the integer program is a feasible solution to the program below.

$$\text{Minimize } R \cdot \frac{\sum_{j=1}^{m} x_j}{\sum_{i=1}^{n} z_i}, \text{ s.t.}$$
$$\forall i \in [n], \sum_{j \in A_i} y_{ij} \geq z_i \cdot (U_i(W) + 1);$$
$$\forall i \in [n], \ \forall j \in A_i, \ y_{ij} \leq x_j;$$
$$\forall i \in [n], \ \forall j \in A_i, \ y_{ij} \leq z_i;$$
$$\forall i \in [n], j \in [m], \ x_j, z_i, y_{ij} \geq 0.$$

This can be written as a LP if we omit the denominator from the objective and add the constraint $\sum_i z_i \geq 1$, and hence can be solved in polynomial time.

Denote $u_i = U_i(W)$. For the committee W, we further denote

$$\theta_p = R \cdot \frac{\sum_{j=1}^{m} x_j}{\sum_{i=1}^{n} z_i} \tag{1}$$

where the variables are set based on the optimal solution to the linear relaxation. Therefore, $\theta_p \leq \theta_c$.

4.2 Proof of Theorem 3

We will show that θ_p is an $O(\log m)$ approximation to θ_c. The proof of the $O(\log n)$ approximation is similar and presented in the full paper [31].

By scaling the LP, we can assume that $\max_i\{z_i\} = 1$. Therefore, all $y_{ij} \leq z_i \leq 1$ and $x_j = \min_{i:j\in A_i}\{y_{ij}\} \leq 1$: all the variables are in the range $[0, 1]$.

$O(\log m)$ *Approximation.* Given the fractional solution, we note that $y_{ij} = \min(x_j, z_i)$. We now construct an integral solution by the following steps:

1. Pick $\alpha \in [0, 1]$ uniformly at random. If $z_i > \alpha$, set $\hat{z}_i = 1$; else $\hat{z}_i = 0$.
2. Let $x'_j = \max\{\frac{1}{2m^2}, x_j\}$.
3. If $2x'_j > \alpha$, then set $\hat{x}_j = 1$; else set $\hat{x}_j = 1$ with probability $2x'_j/\alpha$. We round each $\hat{x}_j$ independently.
4. If $\hat{z}_i = 1$, check if $\sum_{j\in A_i} \min\{\hat{x}_j, \hat{z}_i\} \geq u_i$. If so, set $\hat{\hat{z}}_i = 1$; else set $\hat{\hat{z}}_i = 0$.

Suppose the largest z_i is $z_{i^*} = 1$, we have $\sum_{j\in A_{i^*}} y_{ij} \geq 1$. Therefore, for some j, $y_{i^*j} \geq 1/m$. Therefore $\sum_{j=1}^{m} x_j \geq \frac{1}{m}$. Since Step 2 increases $\sum_j x_j$ by at most $\frac{1}{2m}$, we have $\frac{\sum_j x'_j}{\sum_j x_j} \leq 3/2$.

We first bound the expectation of $\hat{x}_j$. If $x'_j < 1/2$, since $x'_j \geq \frac{1}{2m^2}$, we have:

$$\mathbb{E}[\hat{x}_j] = \int_{\alpha=0}^{2x'_j} 1 \, \mathrm{d}\alpha + \int_{\alpha=2x'_j}^{1} 2x'_j/\alpha \, \mathrm{d}\alpha = 2x'_j + 2x'_j \cdot \ln\alpha \Big|_{2x'_j}^{1} \leq 2x'_j \cdot (1 + 2\ln m).$$

Therefore, we have

$$\mathbb{E}\Big[\sum_j \hat{x}_j\Big] \leq 2(1 + 2\ln m)\sum_j x'_j \leq 3(1 + 2\ln m)\sum_j x_j.$$

We now bound $\mathbb{E}\left[\sum_i \hat{\hat{z}}_i\right]$. Let $P_i \triangleq \{j \in A_i : 2x'_j < \alpha\}$, $Q_i \triangleq \{j \in A_i : 2x'_j \geq \alpha\}$. Since $\hat{x}_j = 1$ for $j \in Q_i$, conditioned on $\hat{z}_i = 1$, we have:

$$\Pr\left(\hat{\hat{z}}_i = 0\right) = \Pr\left(\sum_{j\in A_i} \min\{\hat{x}_j, \hat{z}_i\} < u_i + 1\right) = \Pr\left(\sum_{j\in P_i} \hat{x}_j < u_i + 1 - |Q_i|\right).$$

By the constraints in the optimization and since $y_{ij} = \min(x_j, z_i)$, we have

$$\sum_{j\in P_i} \min\{x_j, z_i\} + \sum_{j\in Q_i} \min\{x_j, z_i\} \geq z_i \cdot (u_i + 1).$$

Since the second term is capped by $z_i \cdot |Q_i|$, we have $\sum_{j\in P_i} x_j \geq z_i \cdot \big((u_i + 1) - |Q_i|\big)$. When $\hat{z}_i = 1$, we have $\alpha < z_i$, and thus

$$\mathbb{E}\left[\sum_{j\in P_i} \hat{x}_j\right] \geq 2 \cdot \left(\sum_{j\in P_i} x'_j\right)/\alpha \geq 2 \cdot \big((u_i + 1) - |Q_i|\big) \cdot z_i/\alpha \geq 2 \cdot \big((u_i + 1) - |Q_i|\big).$$

By Chernoff Bounds on the independent binary random variables $\{\hat{x}_j\}$, we have

$$\Pr\left(\sum_{j\in P_i} \hat{x}_j < u_i + 1 - |Q_i| \,\middle|\, \hat{z}_i = 1\right) < \left(\frac{e^{-1/2}}{\sqrt{1/2}}\right)^2 = 2/e.$$

Therefore, we have

$$\mathbb{E}\left[\sum_i \hat{\hat{z}}_i\right] = \sum_i \mathbb{E}\Big[\hat{z}_i \cdot \big(1 - \Pr(\hat{\hat{z}}_i = 0)\Big] \geq \sum_i \mathbb{E}\Big[\hat{z}_i \cdot (1 - \frac{2}{e})\Big] \geq (1 - \frac{2}{e}) \cdot \sum_i z_i.$$

Since $\{\hat{x}_j\}$ and $\{\hat{\hat{z}}_i\}$ form a valid solution to the program in Sect. 2, there exists a setting of these variables such that

$$\frac{\sum_j \hat{x}_j}{\sum_i \hat{\hat{z}}_i} \leq \frac{\mathbb{E}[\sum_j \hat{x}_j]}{\mathbb{E}[\sum_i \hat{\hat{z}}_i]} \leq \frac{3(1 + 2\ln m)}{1 - 2/e} \cdot \frac{\sum_j x_j}{\sum_i z_i} = \frac{3(1 + 2\ln m)}{1 - 2/e} \cdot \theta_p.$$

Therefore, we have $\theta_p \leq \theta_c \leq \frac{3(1+2\ln m)}{1-2/e} \cdot \theta_p$, completing the proof of the $O(\log m)$ approximation. The proof of the $O(\log n)$ approximation is in the full paper [31].

4.3 Integrality Gap Instance

In the full paper [31], we prove the following theorem, which shows the analysis in Sect. 4.2 is tight.

Theorem 4. *There exists a committee s.t.* $\theta_p = O\left(\frac{1}{\log \min(m,n)}\right)$ *and* $\theta_c = \Theta(1)$.

5 Extension to Arbitrary Utilities and Sizes

We now extend the result in the previous section to the setting where the candidates have general sizes s_j, and voters have arbitrary additive utilities over candidates. We assume voter i has utility $u_{ij} \in \mathbb{Z}^+ \cup \{0\}$ for candidate j. Given a committee W of size at most k, the utility of voter i for the committee is $U_i(W) = \sum_{j\in W} u_{ij}$. We restrict the utilities to be integral, so that if $U_i(T) > U_i(W)$, then $U_i(T) \geq U_i(W) + 1$. Let $A_i = \{j \in C \mid u_{ij} > 0\}$.

LP Formulation. A natural modification to the program in Sect. 2 for θ_c has unbounded integrality gap. We make two modifications to the linear program. First, in the optimal integer solution, we guess the candidate j^* with largest size. This means we set $x_j = 0$ for all j such that $s_j > s_{j^*}$, and delete these items. Since the numerator in the objective is at least s_{j^*}, we can set $x_j = 1$ for all j with $s_j < \frac{s_{j^*}}{m}$, and this only increases the numerator by a constant factor. Let S denote the set of these "small" items; we ignore these items, and set $U_i(W)$ to be $U_i(W) - U_i(S \cap A_i)$. If the latter quantity is smaller than zero, then we can set $z_i = 1$ and delete this voter from further consideration; this only lowers the objective. We let m denote the number of candidates and n denote the number of voters in the residual instance. We now scale the sizes so that the remaining items have sizes in $\left[\frac{1}{m}, 1\right]$. Let $R = \frac{n}{k}$.

Next, we add knapsack cover constraints [13,28]. Let $\hat{U}_i(S) = \max(0, U_i(W) + 1 - U_i(S))$, and let $u_{ijS} = \min(u_{ij}, \hat{U}_i(S))$ The resulting LP is presented below. In this LP, first set of constraints can be interpreted as follows: Even if the x_j for $j \in S$ are all set to 1, so that voter i already has utility $U_i(S)$, if voter i is chosen by the integer program, the remaining $\{y_{ij}\}$ must push the total utility above $U_i(W)$. Further, any utility value u_{ij} on the LHS can be truncated at $\hat{U}_i(S)$ and the constraint should still hold. This constraint is clearly true for any S in the integer program; the LP encodes the fractional version of all of them.

$$\text{Minimize } R \cdot \frac{\sum_{j=1}^{m} s_j x_j}{\sum_{i=1}^{n} z_i}, \text{ s.t.}$$
$$\forall i \in [n], S \subseteq [m], \sum_{j \in A_i \setminus S} u_{ijS} y_{ij} \geq z_i \cdot \hat{U}_i(S);$$
$$\forall i \in [n], \ \forall j \in A_i, \ y_{ij} \leq \min(x_j, z_i);$$
$$\forall i \in [n], j \in [m], \ x_j, z_i, y_{ij} \geq 0.$$

This LP has exponentially many constraints. For any given solution (x, y, z) and fixed voter i, we divide the first set of constraints by z_i and use the polynomial-time dynamic programming procedure exactly as in [13] to find the most violated constraint to a $(1+\epsilon)$ approximation, for constant $\epsilon > 0$. Omitting standard details, this implies the LP can be solved to a $(1 + \epsilon)$ approximation in polynomial time via the Ellipsoid algorithm.

Rounding. The rounding is similar to Sect. 4.2, leading to the following theorem, whose proof is presented in the full paper [31].

Theorem 5. *For the setting with arbitrary additive utilities and sizes, θ_c can be approximated to an $O(\min(\log m, \log n))$ factor in polynomial time.*

6 Auditing Lindahl Priceability

In this section, we study fairness of a committee in terms of closeness to market clearing. The concept is motivated by *Lindahl equilibrium* [23,29], a market clearing concept for public goods. Such market clearing notions have been widely studied as fairness concepts in Economics [7,37]. Our main result is the following novel connection to the core – auditing the approximation of a committee to Lindahl priceability reduces to the LP relaxation for auditing for core stability, hence leading to a polynomial time auditing algorithm.

We consider the Approval Election setting below. The extension to arbitrary utilities and sizes is presented in the full paper [31].

6.1 Lindahl Priceability

As in the definition of core stability, we first scale the entitlements so that the entitlement of each voter is set to 1 instead of k/n. Each candidate now requires $R = n/k$ entitlement to be paid for. A feasible committee of size k corresponds to a total entitlement of n in this scaling.

A committee W of size at most k is Lindahl priceable if there exists a price system $\{p_{ij}\}$ from voters to candidates, such that the following hold:

1. $\forall j \in [m],\ \sum_i p_{ij} \leq R$, and
2. $\forall i \in [n],\ T \subseteq C$, if $|T \cap A_i| \geq |W \cap A_i| + 1$, then $\sum_{j \in T} p_{ij} > 1$.

The first condition above means that for each candidate, the prices from all voters sum up to at most $R = n/k$, so that each candidate is not "over-paid". Note that the first set of constraints can be made equalities by raising the prices $\{p_{ij}\}$, so the candidates are exactly paid for. The second condition means a voter cannot afford any committee that she strictly prefers to W.

Lindahl priceability can be viewed as an integral version of the *gradient optimality conditions* in the fractional Lindahl equilibrium [23]. As mentioned before, this makes our definition subtly different from a related concept in [33]. Analogous to the fractional Lindahl equilibrium, the following proposition holds, and we present a proof later in this section.

Proposition 1. *If a committee is Lindahl priceable, it lies in the core.*

6.2 Auditing via Duality

As with core stability, we now define the best approximation to Lindahl priceability achievable by a committee W. Formally, we only allow a voter to use $\theta_p < 1$ endowment if they want to deviate to a committee with larger utility.

Definition 2 (θ-Approximate Lindahl Priceability). *A committee W of size at most k is* θ-approximate Lindahl priceable *if there exists a price system $\{p_{ij}\}$ from voters to candidates, such that the following conditions hold:*

1. $\forall j \in [m],\ \sum_i p_{ij} \leq R$, and
2. $\forall i \in [n],\ T \subseteq C$, if $|T \cap A_i| \geq |W \cap A_i| + 1$, then $\sum_{j \in T} p_{ij} > \theta$.

The Lindahl priceability ratio of a committee W is the smallest θ for which the committee is not θ-approximate Lindahl priceable. Our main result is the following theorem that ties Lindahl priceability ratio to the fractional relaxation of θ_c. As a corollary, this shows that determining if a committee W is Lindahl priceable is polynomial time solvable.

Theorem 6. *For a committee W, its Lindahl priceability ratio is θ_p from Eq. (1).*

Proof. For simplicity, let $u_i = U_i(W)$. Let the Lindahl priceability ratio of the instance be θ_ℓ. Fix the prices $\{p_{ij}\}$ achieving this. Then the minimum entitlement needed for a voter i to deviate to a committee of utility larger than u_i is captured by the following linear program:

$$\text{Minimize} \sum_{j \in A_i} p_{ij}\gamma_{ij}, \text{ s.t.}$$
$$\sum_{j \in A_i} \gamma_{ij} \geq u_i + 1;$$
$$\forall j \in A_i,\ \gamma_{ij} \leq 1;$$
$$\forall j \in A_i,\ \gamma_{ij} \geq 0.$$

Here, the variable γ_{ij} corresponds to the fraction to which this voter chooses candidate j. In the optimal solution, these variables will be integers. Since the Lindahl priceability ratio is θ_ℓ, Condition (2) of Definition 2 implies objective of the above LP is at least θ_ℓ for any $i \in [n]$.

Now take the dual of the above, where the dual variable for the first constraint is λ_i and the dual variable for the second constraint is α_{ij}. We obtain:

$$\text{Maximize } \theta_i, \text{ s.t.}$$
$$\forall j \in A_i, \lambda_i - \alpha_{ij} \leq p_{ij};$$
$$(u_i + 1)\lambda_i - \sum_{j \in A_i} \alpha_{ij} \geq \theta_i;$$
$$\forall j \in [m],\ \lambda_i, \alpha_{ij} \geq 0.$$

Since the optimal $\theta_i \geq \theta_\ell$, this solution satisfies $(u_i + 1)\lambda_i - \sum_{j \in A_i} \alpha_{ij} \geq \theta_\ell$. Since $\{p_{ij}\}$ satisfy Condition (1) in Definition 2, $\{p_{ij}\}, \{\alpha_{ij}\}, \{\lambda_i\}$ and $\theta = \theta_\ell$ are feasible for the following program:

$$\text{Maximize } \theta, \text{ s.t.}$$
$$\forall i \in [n], j \in A_i, \lambda_i - \alpha_{ij} \leq p_{ij};$$
$$\forall i \in [n],\ (u_i + 1)\lambda_i - \sum_{j \in A_i} \alpha_{ij} \geq \theta;$$
$$\forall j \in [m],\ \sum_{i \in T_j} p_{ij} \leq R;$$
$$\forall i \in [n], j \in [m],\ \lambda_i, \alpha_{ij}, p_{ij} \geq 0.$$

We now claim that the optimal solution to the above program must be exactly θ_ℓ. If it is larger, this larger value θ' must be feasible for the per-voter duals, which means the per-voter primals have value at least θ'. Then the Lindahl priceability is at least θ', contradicting the definition of θ_ℓ.

Finally, take the dual for the LP above, let y_{ij}, z_i, x_j respectively be the dual variable of the three constraints. The dual is the following:

$$\text{Minimize} R \cdot \sum_{j=1}^{m} x_j, \text{ s.t.}$$

$$\forall i \in [n], \sum_{j \in A_i} y_{ij} \geq z_i \cdot (u_i + 1);$$

$$\forall i \in [n], \forall j \in A_i, y_{ij} \leq x_j;$$

$$\forall i \in [n], \forall j \in A_i, y_{ij} \leq z_i;$$

$$\sum_i z_i \geq 1;$$

$$\forall i \in [n], j \in [m], z_i, x_j, y_{ij} \geq 0.$$

This optimal value (which is θ_ℓ) is also the definition of θ_p, completing the proof.

Note that if $\theta_\ell > 1$, then since $\theta_c \geq \theta_p = \theta_\ell > 1$, we have $\theta_c > 1$. Therefore, if a committee is Lindahl priceable, it lies in the core, showing Proposition 1.

7 Sub-core for Approval Elections

Given our approximation results for auditing the core, we can ask if such results can also be derived for weaker fairness notions. Such an auditing notion would be interesting to a practitioner in addition to auditing the core, since it is quite likely an implemented rule and resulting committee would be closer to satisfying a weaker but still reasonable notion of fairness compared to the core. We present a new weakening of the core for Approval elections, that we term the sub-core, that we show also admits approximate auditing. Note that this result is not implied by our results for the core; indeed, there are weakenings of the core, such as EJR, that we do not know how to efficiently audit.

7.1 Weak Priceability

In the multiwinner election setting, suppose the final condition in Lindahl priceability is relaxed so that each voter is only allowed to add candidates to its deviating committee, we get the following relaxed version of priceability. Recall that $R = n/k$, where n is the total number of voters.

Definition 3 (Weak Priceability). *A committee W of size at most k is* weakly priceable *if there exists a set of prices $\{p_{ij}\}$ from each voter v_i to each candidate c_j, such that the following two conditions hold:*

1. $\forall j \in [m], \sum_i p_{ij} \leq R.$
2. $\forall i \in [n], d \in A_i \setminus W, p_{id} + \sum_{j \in A_i \cap W} p_{ij} > 1.$

This notion is equivalent to "priceability" as proposed in [34]. Unlike Lindahl priceability, there are many natural and greedy voting rules, such as the Phragmén rule [10], that satisfy weak priceability, making it a desirable property to study in practice.

7.2 Sub-core

If we proceed as in the proof of Theorem 6 and take the dual of the weak priceability ratio, we obtain a new concept of fairness that we call the *sub-core.*

Definition 4 (Sub-core). *A committee W lies in the* sub-core *if there is no $S \subseteq V$ and committee T with $|T| \le \frac{|S|}{n} \cdot k$, s.t. $A_i \cap W \subsetneq A_i \cap T$ for all $i \in S$.*

The sub-core prevents any group of voters from deviating to a new committee in which each voter's approved candidates forms a proper superset of the approved candidates in the original committee.

Clearly, any committee that lies in the core also lies in the sub-core. The following proposition shows the sub-core is a weakening of weak priceability.

Proposition 2. *If a committee is weakly priceable, then it lies in the sub-core.*

Since weakly priceable committees can be easily found by greedy procedures [34], this shows that the sub-core is always non-empty.

Hardness of Auditing Sub-core. Though the sub-core seems like a weak and satisfiable fairness condition (it insists voters have no greedy deviation to a better committee), we show that deciding if a given committee W lies in the sub-core is actually NP-Hard. Towards this end, we observe that the core and sub-core coincide when each voter approves at most 2 candidates (i.e., for all voters i, we have $|A_i| \le 2$). To see this, suppose the original committee was W and a subset of voters deviate to T. If a deviating voter had original utility zero, then $A_i \cap W = \varnothing$, so that $A_i \cap T \supsetneq A_i \cap W$. Similarly, if $|A_i \cap W| = 1$ and $|A_i \cap T| = 2$, then $A_i \cap T = A_i \supsetneq A_i \cap W$. This shows any deviation satisfies the sub-core property, so that the core coincides with the sub-core.

Theorem 7. *If each voter only approves at most two candidates, deciding whether a committee W does not lie in the sub-core (or core) is* NP-Complete.

Approximately Auditing Sub-core Property. Similar to θ_c, we can now define a parameter θ_{sc} showing how close a committee is to the sub-core.

Definition 5. *For $\theta \le 1$, a committee W of size k lies in the θ-*approximate sub-core *if for all subsets of voters $T \subseteq [n]$, there is no* deviating committee T' *with size at most $\theta \cdot |T| \cdot \frac{k}{n}$, such that for all $i \in T$, we have $A_i \cap W \subsetneq A_i \cap T'$.*

Given a committee W, θ_{sc} is defined as the smallest θ such that W is not in the θ-approximate sub-core. The following theorem shows the sub-core can be approximately audited. This positive result on auditing makes the sub-core a desirable weakening of the core property.

Theorem 8. *Given any committee W, θ_{sc} has an $O(\min(\log m, \log n))$ approximation in polynomial time, where m, n are the total number of candidates and voters respectively.*

8 Conclusion

Note that our theoretical approximation results for auditing are worst case guarantees. In practice, the linear program value θ_p will provide a lower bound on θ_c, and if this can be rounded so that the integer solution has value $\alpha\theta_p$ for some small $\alpha \geq 1$, then this sandwiches $\theta_c \in [\theta_p, \alpha\theta_p]$. Further, the rounding outputs a deviating set of voters and their chosen committee, which will be of interest as a demographic that is not well-represented by the current committee.

The main open question arising from this work is closing the gap between the positive and hardness results for auditing the core. As mentioned before, showing such results for density objectives is challenging [27]. A related question is existence results: A major open question in social choice is whether there is a committee in the core for Approval Elections. Though there are voting rules that find committees in the approximate core [16,25,34], these results do not translate to the exact core. Even more specifically, it is an open question whether there is always a committee that is Lindahl priceable.

Finally, it would be interesting to use the techniques in this paper to approximately audit other notions of fairness or efficiency in social choice. For instance, consider the notion of *extended justified representation* (or EJR, [5]), where a group of $t \cdot n/k$ voters can only deviate if they all approve at least t candidates in common. Since this notion is weaker than the core, it is easier to show existence – indeed the PAV rule [36] satisfies EJR but fails the core. However, imposing restrictions on the deviation does not necessarily make it easier to audit such notions [33], and we do not know how to audit EJR approximately. We showed a particular weakening of the core, the sub-core, that can be approximately audited, and it would be interesting to study the landscape of efficient auditing more systematically.

References

1. The Stanford Participatory Budgeting Platform. https://pbstanford.org
2. Aziz, H., Brandt, F., Elkind, E., Skowron, P.: Computational social choice: the first ten years and beyond. In: Steffen, B., Woeginger, G. (eds.) Computing and Software Science. LNCS, vol. 10000, pp. 48–65. Springer, Cham (2019). https://doi.org/10.1007/978-3-319-91908-9_4
3. Aziz, H., Brill, M., Conitzer, V., Elkind, E., Freeman, R., Walsh, T.: Justified representation in approval-based committee voting. Soc. Choice Welfare **48**(2), 461–485 (2017). https://doi.org/10.1007/s00355-016-1019-3
4. Aziz, H., Elkind, E., Huang, S., Lackner, M., Fernández, L.S., Skowron, P.: On the complexity of extended and proportional justified representation. In: AAAI, pp. 902–909 (2018)
5. Aziz, H., Lang, J., Monnot, J.: Computing Pareto optimal committees. In: Kambhampati, S. (ed.) IJCAI, pp. 60–66 (2016)
6. Aziz, H., Shah, N.: Participatory budgeting: models and approaches. In: Rudas, T., Péli, G. (eds.) Pathways Between Social Science and Computational Social Science. CSS, pp. 215–236. Springer, Cham (2021). https://doi.org/10.1007/978-3-030-54936-7_10

7. Brainard, W.C., Scarf, H.E.: How to compute equilibrium prices in 1891. Am. J. Econ. Sociol. **64**(1), 57–83 (2005)
8. Brams, S.J., Kilgour, D.M., Sanver, M.R.: A minimax procedure for electing committees. Public Choice **132**(3), 401–420 (2007). https://doi.org/10.1007/s11127-007-9165-x
9. Brandt, F., Conitzer, V., Endriss, U., Lang, J., Procaccia, A.D.: Handbook of Computational Social Choice, 1st edn. Cambridge University Press, Cambridge (2016)
10. Brill, M., Freeman, R., Janson, S., Lackner, M.: Phragmén's voting methods and justified representation. In: AAAI, pp. 406–413 (2017)
11. Brill, M., Gölz, P., Peters, D., Schmidt-Kraepelin, U., Wilker, K.: Approval-based apportionment. In: AAAI, pp. 1854–1861 (2020)
12. Cabannes, Y.: Participatory budgeting: a significant contribution to participatory democracy. Environ. Urban. **16**(1), 27–46 (2004)
13. Carr, R.D., Fleischer, L.K., Leung, V.J., Phillips, C.A.: Strengthening integrality gaps for capacitated network design and covering problems. In: SODA, pp. 106–115 (2000)
14. Chamberlin, J.R., Courant, P.N.: Representative deliberations and representative decisions: proportional representation and the Borda rule. Am. Polit. Sci. Rev. **77**(3), 718–733 (1983)
15. Charikar, M.: Greedy approximation algorithms for finding dense components in a graph. In: Jansen, K., Khuller, S. (eds.) APPROX 2000. LNCS, vol. 1913, pp. 84–95. Springer, Heidelberg (2000). https://doi.org/10.1007/3-540-44436-X_10
16. Chen, X., Fain, B., Lyu, L., Munagala, K.: Proportionally fair clustering. In: ICML, pp. 1032–1041 (2019)
17. Droop, H.R.: On methods of electing representatives. J. Stat. Soc. Lond. **44**(2), 141–202 (1881)
18. Endriss, U.: Trends in Computational Social Choice. Lulu.com (2017)
19. Fain, B., Goel, A., Munagala, K.: The core of the participatory budgeting problem. In: Cai, Y., Vetta, A. (eds.) WINE 2016. LNCS, vol. 10123, pp. 384–399. Springer, Heidelberg (2016). https://doi.org/10.1007/978-3-662-54110-4_27
20. Fain, B., Munagala, K., Shah, N.: Fair allocation of indivisible public goods. In: EC (2018)
21. Feige, U.: A threshold of ln n for approximating set cover. J. ACM (JACM) **45**(4), 634–652 (1998)
22. Fernández, L.S., et al.: Proportional justified representation. In: AAAI, pp. 670–676 (2017)
23. Foley, D.K.: Lindahl's solution and the core of an economy with public goods. Econometrica J. Econometric Soc. **38**(1), 66–72 (1970)
24. Goel, A., Krishnaswamy, A.K., Sakshuwong, S., Aitamurto, T.: Knapsack voting for participatory budgeting. ACM Trans. Econ. Comput. **7**(2), 1–27 (2019)
25. Jiang, Z., Munagala, K., Wang, K.: Approximately stable committee selection. In: STOC, pp. 463–472 (2020)
26. Kearns, M., Neel, S., Roth, A., Wu, Z.S.: Preventing fairness gerrymandering: auditing and learning for subgroup fairness. In: ICML, pp. 2564–2572 (2018)
27. Khot, S.: Ruling out PTAS for graph min-bisection, dense k-subgraph, and bipartite clique. SIAM J. Comput. **36**(4), 1025–1071 (2006)
28. Kolliopoulos, S.G., Young, N.E.: Approximation algorithms for covering/packing integer programs. J. Comput. Syst. Sci. **71**(4), 495–505 (2005)
29. Lindahl, E.: Just taxation-a positive solution. In: Classics in the Theory of Public Finance, pp. 168–176 (1958)

30. Monroe, B.L.: Fully proportional representation. Am. Polit. Sci. Rev. **89**(4), 925–940 (1995)
31. Munagala, K., Shen, Y., Wang, K.: Auditing for core stability in participatory budgeting (2022). https://doi.org/10.48550/ARXIV.2209.14468. https://arxiv.org/abs/2209.14468
32. Munagala, K., Shen, Y., Wang, K., Wang, Z.: Approximate core for committee selection via multilinear extension and market clearing. In: SODA, pp. 2229–2252 (2022)
33. Peters, D., Pierczyński, G., Shah, N., Skowron, P.: Market-based explanations of collective decisions. In: AAAI, vol. 35, no. 6, pp. 5656–5663 (2021)
34. Peters, D., Skowron, P.: Proportionality and the limits of welfarism. In: EC, pp. 793–794 (2020)
35. Scarf, H.E.: The core of an N person game. Econometrica **35**(1), 50–69 (1967)
36. Thiele, T.N.: Om flerfoldsvalg. Oversigt over det Kongelige Danske Videnskabernes Selskabs Forhandlinger **1895**, 415–441 (1895)
37. Varian, H.R.: Two problems in the theory of fairness. J. Public Econ. **5**(3), 249–260 (1976)