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BERRY-ESSEEN TYPE BOUNDS FOR THE LEFT RANDOM WALK ON $GL_d(\mathbb{R})$ UNDER POLYNOMIAL MOMENT CONDITIONS

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Let $A_n = \varepsilon_n \cdots \varepsilon_1$, where $(\varepsilon_n)_{n\geq 1}$ is a sequence of independent random matrices taking values in $GL_d(\mathbb{R})$, $d \geq 2$, with common distribution μ. In this paper, under standard assumptions on μ (strong irreducibility and proximality), we prove Berry-Esseen type theorems for $\log(\|A_n\|)$ when μ has a polynomial moment. More precisely, we get the rate $\sqrt{\log n}/\sqrt{n}$ when μ has a moment of order 3 and the rate $1/\sqrt{n}$ when μ has a moment of order 4, which significantly improves earlier results in this setting.

1. Introduction.

Let $(\varepsilon_n)_{n\geq 1}$ be independent random matrices taking values in $G = GL_d(\mathbb{R})$, $d \geq 2$ (the group of invertible d-dimensional real matrices) with common distribution μ. Let $\|\cdot\|$ be the euclidean norm on $\mathbb{R}^d$, and for every $A \in GL_d(\mathbb{R})$, let $\|A\| = \sup_{x,\|x\|=1} \|Ax\|$. We shall say that μ has a moment of order $p \geq 1$ if

$$\int_G (\log N(g))^p d\mu(g) < \infty \,,$$

where $N(g) := \max(\|g\|, \|g^{-1}\|)$.

Let $A_n = \varepsilon_n \cdots \varepsilon_1$. It follows from Furstenberg and Kesten [11] that, if μ admits a moment of order 1 then

$$\lim_{n\to\infty} \frac{1}{n} \log \|A_n\| = \lambda_\mu \ \mathbb{P}\text{-a.s.}, \tag{1.1}$$

where $\lambda_\mu := \lim_{n\to\infty} n^{-1}\mathbb{E}\log\|A_n\|$ is the so-called first Lyapunov exponent.

Let now $X := P(\mathbb{R}^d)$ be the projective space of $\mathbb{R}^d$ and write $\bar{x}$ as the projection of $x \in \mathbb{R}^d - \{0\}$ to X. An element A of $G = GL_d(\mathbb{R})$ acts on the projective space X as follows: $A\bar{x} = \overline{Ax}$. Let Γ_μ be the closed semi-group generated by the support of μ. We say that μ is proximal if Γ_μ contains a matrix that admits a unique (with multiplicity 1) eigenvalue of maximal modulus. We say that μ is strongly irreducible if no proper union of subspaces of $\mathbb{R}^d$ is invariant by Γ_μ. Throughout the paper, we assume that μ is strongly irreducible and proximal. In particular, there exists a unique invariant measure ν on $\mathcal{B}(X)$, meaning that for any continuous and bounded function h from X to $\mathbb{R}$,

$$\int_X h(x)d\nu(x) = \int_G \int_X h(g\cdot x)d\mu(g)d\nu(x)\,. \tag{1.2}$$

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Note that, since μ is assumed to be strongly irreducible, the following strong law holds (see for instance [3], Proposition 7.2 page 72): for any $x \in \mathbb{R}^d - \{0\}$,

$$\lim_{n\to\infty} \frac{1}{n} \log \|A_n x\| = \lambda_\mu \ \mathbb{P}\text{-a.s.} \tag{1.3}$$

To specify the rate of convergence in the laws of large numbers (1.1) and (1.3), it is then natural to address the question of the Central Limit Theorem for the two sequences $\log \|A_n\| - n\lambda_\mu$ and $\log \|A_n x\| - n\lambda_\mu$. To specify the limiting variance in these central limit theorems, let introduce some notations: W_0 will denote a random variable with values in the projective space X, independent of $(\varepsilon_n)_{n\geq 1}$ and with distribution ν. By the invariance of ν, we see that the process $(A_n W_0)_{n\geq 1}$ is a strictly stationary process. Denote also by V_0 a random variable such that $\|V_0\| = 1$ and $\bar{V}_0 = W_0$. Setting, $S_n = \log \|A_n V_0\| - n\lambda_\mu$, Benoist and Quint [1] proved that if μ has a moment of order 2, then

$$\lim_{n\to\infty} \frac{1}{n} \mathbb{E}(S_n^2) = s^2 > 0\,, \tag{1.4}$$

$$\lim_{n\to\infty} \sup_{t\in\mathbb{R}} \sup_{x,\|x\|=1} \left| \mathbb{P}\left(\log \|A_n x\| - n\lambda_\mu \leq t\sqrt{n}\right) - \Phi(t/s) \right| = 0\,, \tag{1.5}$$

and

$$\lim_{n\to\infty} \sup_{t\in\mathbb{R}} \left| \mathbb{P}\left(\log \|A_n\| - n\lambda_\mu \leq t\sqrt{n}\right) - \Phi(t/s) \right| = 0\,, \tag{1.6}$$

where Φ is the cumulative distribution function of a standard normal distribution. Let us mention that (1.5) has been firstly established by Le Page [16] under an exponential moment for μ (meaning that $\int_G (N(g))^\alpha d\mu(g) < \infty$ for some $\alpha > 0$, see also [9]) and then by Jan [13] under the condition that μ has a moment of order $p > 2$.

In the present paper, we are interested in Berry-Esseen type bounds in these central limit theorems, under polynomial moments for μ (more precisely we shall focus on the case of moments of order $p = 3$ or $p = 4$). Before giving our main results, let us briefly describe the previous works on this subject.

When μ has an exponential moment, Le Page [16] proved the following inequality: there exists a positive constant C such that

$$\sup_{t\in\mathbb{R}} \sup_{x,\|x\|=1} \left| \mathbb{P}\left(\log \|A_n x\| - n\lambda_\mu \leq t\sqrt{n}\right) - \Phi(t/s) \right| \leq C v_n \ \text{ with } \ v_n = \frac{1}{\sqrt{n}}\,. \tag{1.7}$$

Still in the case of exponential moments, Edgeworth expansions (a strengthening of the Berry-Esseen theorem) have been recently obtained by Fernando and Pène [8] and Xiao et al. [18]. In these three last papers, the assumption that μ has an exponential moment is crucial since it allows to use the strength of the so-called Nagaev-Guivarc'h perturbation method (indeed in case of exponential moments, the associated complex perturbed transfer operator has spectral gap properties).

Now, under the assumption that all the moments of order p of μ are finite, Jan [13] obtained the rate $v_n = n^{-1/2+\varepsilon}$ for any $\varepsilon > 0$ in (1.7). Next, Cuny et al. [4] gave an upper bound of order $v_n = n^{-1/4}\sqrt{\log n}$ in (1.7) provided μ has a moment of order 3 (as a consequence of an upper bound of order $n^{-1/2}\log n$ for the Kantorovich metric). More recently, Jirak [15] proved that, if μ has a moment of order $p > 8$, then there exists a positive constant C such that

$$\sup_{t\in\mathbb{R}} \left| \mathbb{P}\left(\log \|A_n V_0\| - n\lambda_\mu \leq t\sqrt{n}\right) - \Phi(t/s) \right| \leq C v_n \ \text{ with } \ v_n = \frac{1}{\sqrt{n}}\,. \tag{1.8}$$

This result is based on some refinements of the arguments developed in a previous paper of the same author (see [14]) and is based on a completely different method than the perturbation method for the transfer operator. Since our proofs will use a similar scheme let us briefly explain it. First, due to the cocycle property (see the beginning of Section 2), $\log \|A_n V_0\| - n\lambda$ is written as a partial sum associated with functions of a stationary Markov chain, which can be viewed also as a Bernoulli shift (that is a function of iid random elements). Using the conditional expectation, the underlying random variables are then approximated by m-dependent variables, say $X_{k,m}$. Next the partial sum $\sum_{k=1}^n X_{k,m}$ is decomposed in two terms. The first one can be rewritten as the sum of random variables which are defined as blocks, say $Y_j^{(1)}$, of size $2m$ of the $X_{k,m}$'s. These random blocks have the following property: conditionally to $\mathbb{F}_m$ (a particular σ-algebra generated by a part of the ε_k's), they are independent. In addition, for any bounded measurable function h, the random variables $Z_j = E(h(Y_j^{(1)})|\mathbb{F}_m)$ are one-dependent. On another hand, the second term in the decomposition of $\sum_{k=1}^n X_{k,m}$ is $\mathbb{F}_m$-measurable and can be written as a sum of independent blocks of the initial random variables. For both terms in the decomposition, the conditional independence of the blocks comes from the independence of the ε_k's. The next steps of the proof consist first to work conditionally to $\mathbb{F}_m$ and then to give suitable upper bounds for the conditional characteristic function of the blocks $Y_j^{(1)}$.

Concerning matrix norms, we first note that the Berry-Esseen bound of order $n^{-1/4}\sqrt{\log n}$ under a moment of order 3 is still valid for $\log \|A_n\| - n\lambda_\mu$ instead of $\log \|A_n x\| - n\lambda_\mu$ (see the discussion in Section 8 of [4]). Moreover, if μ has an exponential moment, Xiao et al. [19] proved that there exists a positive constant C such that

$$\sup_{t\in\mathbb{R}} \left|\mathbb{P}\left(\log \|A_n\| - n\lambda_\mu \le t\sqrt{n}\right) - \Phi(t/s)\right| \le C w_n \text{ with } w_n = \frac{\log n}{\sqrt{n}} . \tag{1.9}$$

Note that in [19], the authors also proved a similar upper bound for $\log(\rho(A_n))$ where $\rho(A_n)$ is the spectral radius of A_n.

In the present paper, we prove that:

- If μ has a moment of order 3, then the rate in (1.7) (and then in (1.8)) is $v_n = n^{-1/2}(\log n)^{1/2}$ and the rate in (1.9) is $w_n = n^{-1/2}(\log n)^{1/2}$.
- If μ has a moment of order 4, then the rate in (1.7) (and then in (1.8)) is $v_n = n^{-1/2}$ and the rate in (1.9) is $w_n = n^{-1/2}$.

To prove these results, we follow the approach developed in Jirak [14, 15], but with substantial changes. We refer to Comment 3.1 to have a flavor of them. One of the main changes is the use of the dependency coefficients defined in [4] (see also (3.11) below) which are well adapted to the study of the process $(\log \|A_n x\| - n\lambda_\mu)_{n\ge 1}$, instead of the coupling coefficients used in [15],

The paper is organized as follows. In Section 2, we state our main results about Berry-Esseen type bounds in the context of left random walks when μ has either a moment of order 3 or a moment of order 4. All the proofs are postponed to Section 3. Some technical lemmas used in the proofs are stated and proved in Section 4.

In the rest of the paper, we shall use the following notations: for two sequences $(a_n)_{n\ge1}$ and $(b_n)_{n\ge1}$ of positive reals, $a_n \ll b_n$ means that there exists a positive constant C not depending on n such that $a_n \le C b_n$ for any $n \ge 1$. Moreover, given a σ-algebra $\mathcal{F}$, we shall often use the notation $\mathbb{E}_{\mathcal{F}}(\cdot) = \mathbb{E}(\cdot|\mathcal{F})$.

2. Berry-Esseen bounds. Recall the notations of the introduction: let $(\varepsilon_n)_{n\geq 1}$ be independent random matrices taking values in $G = GL_d(\mathbb{R})$, $d \geq 2$, with common distribution μ. Let $A_n = \varepsilon_n \cdots \varepsilon_1$ for $n \geq 1$, and A_0 =Id. We assume that μ is strongly irreducible and proximal, and we denote by ν the unique distribution on $X = P(\mathbb{R}^d)$ satisfying (1.2).

Let now V_0 be a random variable independent of $(\varepsilon_n)_{n\geq 1}$, taking values in $\mathbb{R}^d$, such that $\|V_0\| = 1$ and $\overline{V_0}$ is distributed according to ν.

The behavior of $\log \|A_n V_0\| - n\lambda_\mu$ (where λ_μ is the first Lyapunov exponent defined right after (1.1)) can be handled with the help of an additive cocycle, which can also be viewed as a function of a stationary Markov chain. More precisely, let $W_0 = \overline{V_0}$ (so that W_0 is distributed according to ν), and let $W_n = \varepsilon_n W_{n-1} = A_n W_0$ for any integer $n \geq 1$. By definition of ν, the sequence $(W_n)_{n\geq 0}$ is a strictly stationary Markov chain with values in X. Let now, for any integer $k \geq 1$,

$$X_k := \sigma(\varepsilon_k, W_{k-1}) - \lambda_\mu = \sigma(\varepsilon_k, A_{k-1}W_0) - \lambda_\mu \,, \tag{2.1}$$

where, for any $g \in G$ and any $\bar{x} \in X$,

$$\sigma(g, \bar{x}) = \log\Big(\frac{\|g\cdot x\|}{\|x\|}\Big) .$$

Note that σ is an additive cocycle in the sense that $\sigma(g_1 g_2, \bar{x}) = \sigma(g_1, g_2\bar{x}) + \sigma(g_2, \bar{x})$. Consequently

$$S_n = \sum_{k=1}^n X_k = \log \|A_n V_0\| - n\lambda_\mu \,. \tag{2.2}$$

With the above notations, the following Berry-Esseen bounds hold.

THEOREM 2.1. *Let μ be a proximal and strongly irreducible probability measure on $\mathcal{B}(G)$. Assume that μ has a finite moment of order 3. Then $n^{-1}\mathbb{E}(S_n^2) \to s^2 > 0$ as $n \to \infty$ and, setting $v_n = \sqrt{\log n}/\sqrt{n}$, we have*

$$\sup_{y\in\mathbb{R}} \Big|\mathbb{P}\big(S_n \leq y\sqrt{n}\big) - \Phi(y/s)\Big| \ll v_n \,, \tag{2.3}$$

$$\sup_{y\in\mathbb{R}} \Big|\mathbb{P}\big(\log(\|A_n\|) - n\lambda_\mu \leq y\sqrt{n}\big) - \Phi(y/s)\Big| \ll v_n \,, \tag{2.4}$$

and

$$\sup_{x, \|x\|=1} \sup_{y\in\mathbb{R}} \Big|\mathbb{P}\big(\log \|A_n x\| - n\lambda_\mu \leq y\sqrt{n}\big) - \Phi(y/s)\Big| \ll v_n \,. \tag{2.5}$$

REMARK 2.1. As mentioned in the introduction, the fact that $n^{-1}\mathbb{E}(S_n^2) \to s^2 > 0$ has been proved by Benoist and Quint [1] (see Item (c) of their Theorem 4.11). Let us mention that we also have $s^2 = \mathbb{E}(X_1^2) + 2\sum_{k\geq 2} \mathbb{E}_\nu(X_1 X_k)$, which follows for instance from the proof of item (ii) of Theorem 1 in [4].

Now if μ has a finite moment of order 4 then the following result holds:

THEOREM 2.2. *Let μ be a proximal and strongly irreducible probability measure on $\mathcal{B}(G)$. Assume that μ has a finite moment of order 4. Then $n^{-1}\mathbb{E}(S_n^2) \to s^2 > 0$ as $n \to \infty$ and (2.3), (2.4) and (2.5) hold with $v_n = 1/\sqrt{n}$.*

Recall that the classical Berry-Esseen theorem for independent random variables, which corresponds to the case $d=1$ in our setting, provides the rate $1/\sqrt{n}$ under a finite moment of order 3. Hence, one may wonder whether the conclusion of Theorem 2.2 holds under the assumptions of Theorem 2.1. Note also that we have chosen to focus on the cases where μ has a finite moment of order 3 (since it corresponds to the usual moment assumption for the Berry-Esseen theorem in the iid case) or a finite moment of order 4 (since in this case we reach the rate $1/\sqrt{n}$), but we infer from the proofs that if μ has a finite moment of order $q\in(3,4)$ then the above results hold with $v_n=(\log n)^{(4-q)/2}/\sqrt{n}$.

3. Proofs.

3.1. *Proof of Theorem 2.1.* As usual, we shall denote by $X_{k,\bar{x}}$ the random variable X_k defined by (2.1) when the Markov chain $(W_n)_{n\geq 0}$ starts from $\bar{x}\in X$. We then define $S_{n,\bar{x}}:=\log\|A_n x\|-n\lambda_\mu=\sum_{k=1}^n X_{k,\bar{x}}$. We shall first prove the upper bound (2.3) and then the upper bounds (2.4) and (2.5) in Sections 3.1.2 and 3.1.3 respectively.

3.1.1. *Proof of the upper bound* (2.3).

As usual, the proof is based on the so-called Berry-Esseen smoothing inequality (see e.g. [10, Ineq. (3.13) p. 538]) stating that, for any positive T,

$$\sup_{x\in\mathbb{R}}\Big|\mathbb{P}_\nu\big(S_n\leq x\sqrt{n}\big)-\Phi(x/s)\Big|\ll\int_{-T}^{T}\frac{\big|\mathbb{E}\big(\mathrm{e}^{\mathrm{i}\xi S_n/\sqrt{n}}\big)-\mathrm{e}^{-\xi^2 s^2/2}\big|}{|\xi|}\mathrm{d}\xi+T^{-1}\,, \tag{3.1}$$

where we recall that S_n has been defined in (2.2).

To take care of the characteristic function of $S_n/\sqrt{n}$ we shall take advantage of the fact that X_k is a function of a stationary Markov chain generated by the iid random elements $(\varepsilon_i)_{i\geq 1}$. As in [14], the first steps of the proof consist in approximating the X_k's by m-dependent random variables $X_{k,m}$, and then in suitably decomposing the partial sum associated with the $X_{k,m}$. This is the subject of the following paragraph.

Step 0. Notations and Preliminaries. We shall adopt most of the time the same notations as in Jirak [14]. Let $\mathcal{E}_i^j=\sigma(\varepsilon_i,\dots,\varepsilon_j)$ for $i\leq j$, and m be a positive integer that will be specified later. For any $k\geq m$, let

$$X_{k,m}=\mathbb{E}(X_k|\mathcal{E}_{k-m+1}^k):=f_m(\varepsilon_{k-m+1},\dots,\varepsilon_k)\,, \tag{3.2}$$

where f_m is a measurable function. More precisely, we have

$$X_{k,m}=\int_X\sigma(\varepsilon_k,A_{k-1}^{k-m+1}\bar{x})d\nu(\bar{x})-\lambda_\mu\,,$$

where we used the notation $A_j^i=\varepsilon_j\cdots\varepsilon_i$ for $i\leq j$. Note that $\mathbb{E}(X_{k,m})=0$.

Next, let N be the positive integer such that $n=2Nm+m'$ with $0\leq m'\leq 2m-1$. The integers N and m are such that $N\sim\kappa_1\log n$ (where κ_1 is a positive constant specified later) and $m\sim(2\kappa_1)^{-1}n(\log n)^{-1}$ (see (3.26) for the selection of κ_1). Define now the following σ-algebra

$$\mathbb{F}_m=\sigma\big((\varepsilon_{(2j-1)m+1},\dots,\varepsilon_{2jm}),j\geq 1\big)\,. \tag{3.3}$$

Let $U_1=\sum_{k=1}^m X_k$ and, for any integer $j\in[2,N]$, define

$$U_j=\sum_{k=(2j-2)m+1}^{(2j-1)m}\big(X_{k,m}-\mathbb{E}(X_{k,m}|\mathbb{F}_m)\big)\,. \tag{3.4}$$

For any integer $j \in [1, N]$, let

$$R_j = \sum_{k=(2j-1)m+1}^{2jm} \left(X_{k,m} - \mathbb{E}(X_{k,m}|\mathbb{F}_m)\right), \tag{3.5}$$

$$Y_j^{(1)} = U_j + R_j \ \text{ and } \ S_{|m}^{(1)} = \sum_{j=1}^{N} Y_j^{(1)}. \tag{3.6}$$

Let also

$$U_{N+1} = \sum_{k=2Nm+1}^{\min(n,(2N+1)m)} \left(X_{k,m} - \mathbb{E}(X_{k,m}|\mathbb{F}_m)\right)$$

and

$$R_{N+1} = \sum_{k=(2N+1)m+1}^{n} \left(X_{k,m} - \mathbb{E}(X_{k,m}|\mathbb{F}_m)\right),$$

where an empty sum has to be interpreted as 0. Note that under $\mathbb{P}_{\mathbb{F}_m}$ (the conditional probability given $\mathbb{F}_m$), the random vectors $(U_j, R_j)_{1\le j\le N+1}$ are independent. Moreover, by stationarity, the r.v.'s $(U_j, R_j)_{2\le j\le N}$ have the same distribution (as well as the r.v.'s $(R_j)_{1\le j\le N}$).

Next, denoting by $S_{|m}^{(2)} = \sum_{k=m+1}^{n} \mathbb{E}(X_{k,m}|\mathbb{F}_m)$, the following decomposition is valid:

$$S_{n,m} := \sum_{k=1}^{m} X_k + \sum_{k=m+1}^{n} X_{k,m} = S_{|m}^{(1)} + S_{|m}^{(2)} + U_{N+1} + R_{N+1}.$$

To simplify the exposition, assume in the rest of the proof that $n = 2Nm$ (so that $m' = 0$). There is no loss of generality by making such an assumption: the only difference would be that since (U_{N+1}, R_{N+1}) does not have the same law as the (U_j, R_j)'s, $2 \le j \le N$, its contribution would have to be treated separately. Therefore, from now we consider $m' = 0$ and then the following decomposition

$$S_{n,m} = S_{|m}^{(1)} + S_{|m}^{(2)}. \tag{3.7}$$

We are now in position to give the main steps of the proof. We start by writing

$$\left|\mathbb{E}\left(\mathrm{e}^{\mathrm{i}\xi S_n/\sqrt{n}}\right) - \mathrm{e}^{-\xi^2 s^2/2}\right| \le \left|\mathbb{E}\left(\mathrm{e}^{\mathrm{i}\xi S_n/\sqrt{n}}\right) - \mathbb{E}\left(\mathrm{e}^{\mathrm{i}\xi S_{n,m}/\sqrt{n}}\right)\right| + \left|\mathbb{E}\left(\mathrm{e}^{\mathrm{i}\xi S_{n,m}/\sqrt{n}}\right) - \mathrm{e}^{-\xi^2 s^2/2}\right|.$$

Next

$$\begin{aligned}
&\left|\mathbb{E}\left(\mathrm{e}^{\mathrm{i}\xi S_{n,m}/\sqrt{n}}\right) - \mathrm{e}^{-\xi^2 s^2/2}\right| \\
&= \left|\mathbb{E}\left(\mathrm{e}^{\mathrm{i}\xi S_{|m}^{(2)}/\sqrt{n}}\left[\mathbb{E}_{\mathbb{F}_m}\left(\mathrm{e}^{\mathrm{i}\xi S_{|m}^{(1)}/\sqrt{n}}\right) - \mathrm{e}^{-\xi^2 s^2/4}\right]\right) + \mathrm{e}^{-\xi^2 s^2/4}\left(\mathbb{E}\left(\mathrm{e}^{\mathrm{i}\xi S_{|m}^{(2)}/\sqrt{n}}\right) - \mathrm{e}^{-\xi^2 s^2/4}\right)\right| \\
&\qquad\qquad \le \left\|\mathbb{E}_{\mathbb{F}_m}\left(\mathrm{e}^{\mathrm{i}\xi S_{|m}^{(1)}/\sqrt{n}}\right) - \mathrm{e}^{-\xi^2 s^2/4}\right\|_1 + \left|\mathbb{E}\left(\mathrm{e}^{\mathrm{i}\xi S_{|m}^{(2)}/\sqrt{n}}\right) - \mathrm{e}^{-\xi^2 s^2/4}\right|.
\end{aligned}$$

Hence, starting from (3.1) and selecting $T = \sqrt{n/\log n}$, Inequality (2.3) of Theorem 2.1 will follow if one can prove that

$$\int_{-T}^{T} \frac{\left|\mathbb{E}\left(\mathrm{e}^{\mathrm{i}\xi S_n/\sqrt{n}}\right) - \mathbb{E}\left(\mathrm{e}^{\mathrm{i}\xi S_{n,m}/\sqrt{n}}\right)\right|}{|\xi|} \mathrm{d}\xi \ll \frac{\sqrt{\log n}}{\sqrt{n}}, \tag{3.8}$$

$$\int_{-T}^{T} \frac{\left\|\mathbb{E}_{\mathbb{F}_m}\left(\mathrm{e}^{\mathrm{i}\xi S^{(1)}_{|m}/\sqrt{n}}\right) - \mathrm{e}^{-\xi^2 s^2/4}\right\|_1}{|\xi|} \mathrm{d}\xi \ll \frac{\sqrt{\log n}}{\sqrt{n}} \tag{3.9}$$

and

$$\int_{-T}^{T} \frac{\left|\mathbb{E}\left(\mathrm{e}^{\mathrm{i}\xi S^{(2)}_{|m}/\sqrt{n}}\right) - \mathrm{e}^{-\xi^2 s^2/4}\right|}{|\xi|} \mathrm{d}\xi \ll \frac{\sqrt{\log n}}{\sqrt{n}} \,. \tag{3.10}$$

The objective is then to prove these three upper bounds, and the main differences compared to [14, 15] lie in the intermediate steps and the technical tools developed for this purpose. They will be based on the following dependence coefficients that are well adapted to our setting. Let $p \geq 1$. For every $k \geq 1$, define

$$\delta^p_{p,\infty}(k) = \sup_{\bar{x},\bar{y}\in X} \mathbb{E}\left|X_{k,\bar{x}} - X_{k,\bar{y}}\right|^p . \tag{3.11}$$

If μ has a finite moment of order $q > 1$, then, by [4, Prop. 3], we know that

$$\sum_{k\geq 1} k^{q-p-1}\, \delta^p_{p,\infty}(k) < \infty \qquad \forall p \in [1, q) \,. \tag{3.12}$$

Hence, since $(\delta_{p,\infty}(k))_{k\geq 1}$ is non increasing, it follows that (if μ has a moment of order $q > 1$)

$$\delta_{p,\infty}(k) = o\big(1/k^{q/p-1}\big) \qquad \forall p \in [1, q) \,. \tag{3.13}$$

COMMENT 3.1. Let us give an idea of the interest of considering these coefficients compared to the coupling coefficients $\vartheta'_k(p)$ and $\vartheta^*_k(p)$ defined in [15, Eq. (7)] (even if $\delta_{p,\infty}(k)$ provides an upper bound for these coefficients). For instance, as we shall see in Lemma 4.3, using a suitable Rosenthal's inequality and the strength of the $\delta_{p,\infty}$ coefficients allowing to control also the infinite norm of conditional expectation (see for instance (4.7)), we have $\|R_1\|_p \ll 1$ for $p \in [2, 3]$ provided that μ has a moment of order $q = p + 1$. As a counterpart, Lemma 5.4 in [15] entails that $\|R_1\|_p \ll \sum_{k=1}^m \delta_{p,\infty}(k)$ and then $\|R_1\|_p \ll 1$ as soon as μ has a moment of order $q > 2p$. Let us mention that requiring $\|R_1\|_p \ll 1$ for some $p \geq 2$ is a key ingredient to take care of the characteristic function of the $Y^{(1)}_j$'s conditionally to $\mathbb{F}_m$ that we will denote by $\varphi_j(t)$ in what follows (see the definition (3.16)). More precisely, if the condition (among others) $\|R_1\|_p \ll 1$ holds for $p = 2$, then we get the upper bound (3.19) and if it holds for $p = 3$ then we get the better upper bound (3.34) (this difference in the upper bounds is the reason why in the statements of Theorem 2.1 we have an extra logarithmic term compared to Theorem 2.2). Note that the upper bounds (3.19) and (3.34) come from Lemmas 4.6, 4.11 and 4.12. Another crucial fact that we would like to point out is the following: Imposing that μ has a moment of order $q = 3$ implies $\|R_1\|_p \ll 1$ only for $p = 2$ and then Lemma 4.5 in [14] cannot be used to prove the upper bound (3.23) which is widely used to prove (3.9). Indeed for [14, Lemma 4.5] to be applied it is necessary that $\|R_1\|_p \ll 1$ for some $p > 2$. The role of our Lemma 4.1 is then to overcome this drawback (see the step 3 below and in particular the control of both $I_{1,N}(\xi)$ and $I_{3,N}(\xi)$).

On another hand, in view of (3.13), it is clear that, as $k \to \infty$, the coefficient $\delta_{r,\infty}(k)$ has a better behavior than $\delta_{p,\infty}(k)$ for any $r \in [1, p[$. Hence, in some cases, it can be preferable to deal with the $\mathbb{L}^r$-norm rather than with the $\mathbb{L}^p$-norm. For instance, in our case, it is much more efficient to control $\|S_n - S_{n,m}\|_1$ (see the forthcoming upper bounds (3.14) and (3.15)) rather than $\|S_n - S_{n,m}\|_3^3$ as done in Jirak [15] (see his upper bound (50)). In both cases these quantities have to be controlled by $1/\sqrt{n}$ and to see the differences between the two upper bounds take m equals to n up to a logarithmic term both in (3.15) and in [15, Ineq. (50)]. This is the reason why we can start directly from Inequality (3.1) and work with the characteristic function rather than using the decomposition given in [15, Lemma 5.11].

Let us now come back to the proof. The next steps will consist in proving the upper bounds (3.8)-(3.10).

Step 1. Proof of (3.8). Note that

$$\int_{-T}^{T} \frac{\big|\mathbb{E}\big(\mathrm{e}^{\mathrm{i}\xi S_n/\sqrt{n}}\big) - \mathbb{E}\big(\mathrm{e}^{\mathrm{i}\xi S_{n,m}/\sqrt{n}}\big)\big|}{|\xi|} \mathrm{d}\xi \le (\log n)^{-1/2} \|S_n - S_{n,m}\|_1 \,.$$

But, by stationarity and [5, Lemma 24],

$$\|S_n - S_{n,m}\|_1 \le n \|X_{m+1} - X_{m+1,m}\|_1 \le n\delta_{1,\infty}(m)\,. \tag{3.14}$$

Hence, by (3.13) and the fact that μ has a moment of order $q > 1$, we derive

$$\|S_n - S_{n,m}\|_1 \ll n m^{-(q-1)}\,. \tag{3.15}$$

So, overall, since $q = 3$, it follows that

$$\int_{-T}^{T} \frac{\big|\mathbb{E}\big(\mathrm{e}^{\mathrm{i}\xi S_n/\sqrt{n}}\big) - \mathbb{E}\big(\mathrm{e}^{\mathrm{i}\xi S_{n,m}/\sqrt{n}}\big)\big|}{|\xi|} \mathrm{d}\xi \ll n(\log n)^{-1/2} m^{-2}\,.$$

The upper bound (3.8) follows from the fact that we will select $m \sim \kappa_2 n (\log n)^{-1}$.

Step 2. Proof of (3.9). For any $x \in \mathbb{R}$ and any integer $j \in [1, N]$, let

$$\varphi_j(x) = \mathbb{E}\Big(\mathrm{e}^{\mathrm{i}x Y_j^{(1)}/\sqrt{2m}} | \mathbb{F}_m\Big)\,. \tag{3.16}$$

Since, under $\mathbb{P}_{\mathbb{F}_m}$, the $Y_j^{(1)}$'s are independent we write

$$\big\|\mathbb{E}_{\mathbb{F}_m}\big(\mathrm{e}^{\mathrm{i}\xi S^{(1)}_{|m}/\sqrt{n}}\big) - \mathrm{e}^{-\xi^2 s^2/4}\big\|_1 = \mathbb{E}\Big[\Big|\prod_{j=1}^{N} \varphi_j\Big(\frac{\xi}{\sqrt{N}}\Big) - \prod_{j=1}^{N} \mathrm{e}^{-\xi^2 s^2/(4N)}\Big|\Big] \tag{3.17}$$

As in [14, Section 4.1.1], we use the following basic identity: for any complex numbers $(a_j)_{1\le j\le N}$ and $(b_j)_{1\le j\le N}$, $\prod_{j=1}^N a_j - \prod_{j=1}^N b_j = \sum_{i=1}^n (\prod_{j=1}^{i-1} b_j)(a_i - b_i)(\prod_{j=i+1}^N a_j)$ to handle the right-hand side of (3.17). Taking into account that $(\varphi_j(t))_{1\le j\le N}$ forms a one-dependent sequence and that the r.v.'s $(U_j, R_j)_{2\le j\le N}$ have the same distribution, we then infer that

$$\mathbb{E}\Big[\Big|\prod_{j=1}^{N} \varphi_j\Big(\frac{\xi}{\sqrt{N}}\Big) - \prod_{j=1}^{N} \mathrm{e}^{-\xi^2/(4N)}\Big|\Big] \le I_{1,N}(\xi) + I_{2,N}(\xi) + I_{3,N}(\xi)\,, \tag{3.18}$$

where

$$I_{1,N}(\xi) = (N-1)\|\varphi_2(\xi/\sqrt{N}) - \mathrm{e}^{-\xi^2 s^2/(4N)}\|_1 \Big\| \prod_{j=N/2}^{N-1} \Big|\varphi_j\Big(\frac{\xi}{\sqrt{N}}\Big)\Big|\Big\|_1\,,$$

$$I_{2,N}(\xi) = N \mathrm{e}^{-\xi^2 s^2 (N-6)/(8N)} \|\varphi_2(\xi/\sqrt{N}) - \mathrm{e}^{-\xi^2 s^2/(4N)}\|_1$$

and

$$I_{3,N}(\xi) = \|\varphi_1(\xi/\sqrt{N}) - \mathrm{e}^{-\xi^2 s^2/(4N)}\|_1 \Big\| \prod_{j=N/2}^{N-1} \Big|\varphi_j\Big(\frac{\xi}{\sqrt{N}}\Big)\Big|\Big\|_1\,.$$

To integrate the above quantities, we need to give suitable upper bounds for the two terms $\|\varphi_j(t) - \mathrm{e}^{-s^2t^2/4}\|_1$ and $\|\prod_{j=N/2}^{N-1} |\varphi_j(t)|\|_1$. Applying the first part of Lemma 4.6 and using stationarity, we derive that for any $2 \le j \le N$,

$$\|\varphi_j(t) - \mathrm{e}^{-s^2t^2/4}\|_1 \ll \frac{t^2}{\sqrt{m}} + \frac{|t|}{m^{3/2}}\,. \tag{3.19}$$

Moreover the second part of Lemma 4.6 implies that

$$\|\varphi_1(t) - \mathrm{e}^{-s^2t^2/4}\|_1 \ll \frac{t^2}{\sqrt{m}}\,. \tag{3.20}$$

On another hand, according to [14, Inequality (4.14)], for any integer $\ell \in [1, m]$,

$$\Big\| \prod_{j=N/2}^{N-1} |\varphi_j(t)| \Big\|_1 \le \Big\| \prod_{j \in \mathcal{J}} |\varphi_j^{(\ell)}(t\sqrt{(m-\ell)/(2m)})| \Big\|_1\,,$$

where $\mathcal{J} = [N/2, N-1] \cap 2\mathbb{N}$,

$$\varphi_j^{(\ell)}(x) = \mathbb{E}\Big(\mathrm{e}^{\mathrm{i}xH_{j,m}^{(\ell)}} | \mathcal{H}_{j,m}^{(\ell)}\Big)$$

with $\mathcal{H}_{j,m}^{(\ell)} = \mathbb{F}_m \vee \sigma(\varepsilon_{2(j-1)m+1}, \dots, \varepsilon_{2(j-1)m+\ell})$ and

$$H_{j,m}^{(\ell)} = \frac{1}{\sqrt{m-\ell}}\Big(\sum_{k=2(j-1)m+\ell+1}^{(2j-1)m} (X_{k,m} - \mathbb{E}(X_{k,m}|\mathcal{H}_{j,m}^{(\ell)})) + R_j - \mathbb{E}(R_j|\mathcal{H}_{j,m}^{(\ell)})\Big)\,.$$

We shall apply Lemma 4.1 with

$$A_j = \frac{1}{\sqrt{m-\ell}} \sum_{k=2(j-1)m+\ell+1}^{(2j-1)m} (X_{k,m} - \mathbb{E}(X_{k,m}|\mathcal{H}_{j,m}^{(\ell)})),\ B_j = R_j - \mathbb{E}(R_j|\mathcal{H}_{j,m}^{(\ell)})$$

and $a = (m-\ell)^{-1/2}$. By stationarity, for any $j \in \mathcal{J}$,

$$\begin{aligned}\mathbb{P}\big(\mathbb{E}_{H_{j,m}^{(\ell)}}(A_j^2) \le s^2/4\big) &= \mathbb{P}\big(\mathbb{E}_{H_{2,m}^{(\ell)}}(A_2^2) \le s^2/4\big)\\ &= \mathbb{P}\Big((m-\ell)^{-1}\mathbb{E}_m\Big(\Big(\sum_{k=m+1}^{2m-\ell}(X_{k,m} - \mathbb{E}_m(X_{k,m})\Big)^2\Big) \le s^2/4\Big)\,,\end{aligned}$$

where $\mathbb{E}_m(\cdot)$ means $\mathbb{E}(\cdot|\mathcal{G}_m)$ with $\mathcal{G}_m = \sigma(W_0, \varepsilon_1, \dots, \varepsilon_m)$. Let K be a positive integer and note that

$$\begin{aligned}&\Big\| \sum_{k=m+1}^{m+K} (X_{k,m} - \mathbb{E}_m(X_{k,m}))\Big\|_2 - \Big\| \sum_{k=m+1}^{m+K} X_k \Big\|_2\\ &\quad\le \sum_{k=m+1}^{m+K} \|X_{k,m} - X_k\|_2 + \sum_{k=m+1}^{m+K} \|\mathbb{E}_m(X_{k,m})\|_\infty\\ &\quad\le \sum_{k=m+1}^{m+K} \delta_{2,\infty}(k) + \sum_{k=m+1}^{m+K} \delta_{1,\infty}(k)\,.\end{aligned}$$

Therefore, by taking into account (3.13) and the fact that μ has a moment of order 3, we get that

$$\Big\| \sum_{k=m+1}^{m+K} (X_{k,m} - \mathbb{E}_m(X_{k,m}))\Big\|_2 - \Big\| \sum_{k=m+1}^{m+K} X_k\Big\|_2 = o(K^{1/2})\,.$$

But, using stationarity, we have $K^{-1/2}\Big\| \sum_{k=m+1}^{m+K} X_k\Big\|_2 = K^{-1/2}\Big\| \sum_{k=1}^{K} X_k\Big\|_2 \to s > 0$. Hence provided that $(m-\ell)$ is large enough, we have

$$(m-\ell)^{-1}\mathbb{E}\Big(\Big(\sum_{k=m+1}^{2m-\ell} (X_{k,m} - \mathbb{E}_m(X_{k,m})\Big)^2\Big) > s^2/2\,. \tag{3.21}$$

So, overall, setting $\bar X_{k,m} := X_{k,m} - \mathbb{E}_m(X_{k,m})$, for $(m-\ell)$ large enough, we get

$$\begin{aligned}&\mathbb{P}\big(\mathbb{E}_{H^{(\ell)}_{2,m}}(A_2^2) \le s^2/4\big)\\ &\qquad \le \mathbb{P}\Big((m-\ell)^{-1}\Big|\mathbb{E}_m\Big(\Big(\sum_{k=m+1}^{2m-\ell}\bar X_{k,m}\Big)^2 - \mathbb{E}\Big(\Big(\sum_{k=m+1}^{2m-\ell}\bar X_{k,m}\Big)^2\Big)\Big| \ge \frac{s^2}{4}\Big)\,.\end{aligned}$$

Using Markov's inequality and the same arguments as those used in the proof of Lemma 4.2, we then derive that, for $(m-\ell)$ large enough and any $j\in\mathcal{J}$,

$$\mathbb{P}\big(\mathbb{E}_{H^{(\ell)}_{j,m}}(A_j^2) \le s^2/4\big) \ll (m-\ell)^{-5/7}\,.$$

Hence, provided that $m-\ell$ is large enough, Item (ii) of Lemma 4.1 is satisfied with $u^- = s^2/4$. Note now that by stationarity, for any $j\in\mathcal{J}$,

$$\mathbb{E}(B_j^2) \le 4\mathbb{E}(R_j^2) = 4\mathbb{E}(R_1^2) \ll 1\,,$$

by using Lemma 4.3 with $p=2$. This proves Item (iv) of Lemma 4.1. Next, for $p\ge 2$, using stationarity and [17, Cor. 3.7], we get that for any $j\in\mathcal{J}$,

$$\mathbb{E}(|A_j|^p) \le 2^p(m-\ell)^{-p/2}\Big\|\sum_{k=m+1}^{2m-\ell} X_{k,m}\Big\|_p \ll \Big[\|X_{1+m,m}\|_p + \sum_{k=m+1}^{2m-\ell} k^{-1/2}\|\mathbb{E}_m(X_{k,m})\|_p\Big]^p\,. \tag{3.22}$$

But $\|X_{1+m,m}\|_p \le \|X_1\|_p < \infty$ if $p\le 3$ (indeed recall that it is assumed that μ has a moment of order 3) and $\|\mathbb{E}_m(X_{k+m,m})\|_p \le \|\mathbb{E}_m(X_{k+m,m})\|_\infty \le \delta_{1,\infty}(k)$. Hence, by (3.12) and since μ has a moment of order $q=3$, Item (iii) of Lemma 4.1 is satisfied for any $p\in[2,3]$. So, overall, noticing that $|\mathcal{J}|\ge N/8\ge 16$, we can apply Lemma 4.1 to derive that there exist positive finite constants c_1, c_2 and c_3 depending in particular on s^2 but not on (m,n) such that for $(m-\ell)$ large enough (at least such that $a=(m-\ell)^{-1/2}\le c_1$), we have

$$\Big\|\prod_{j\in\mathcal{J}}|\varphi_j^{(\ell)}(x)|\Big\|_1 \le \mathrm{e}^{-c_3x^2N/8} + \mathrm{e}^{-N/256} \ \text{ for } x^2\le c_2,$$

implying overall that, for $(m-\ell)$ large enough and for $t^2(m-\ell)/(2m)\le c_2$,

$$\Big\|\prod_{j=N/2}^{N-1}|\varphi_j(t)|\Big\|_1 \le \mathrm{e}^{-c_3t^2(m-\ell)N/(16m)} + \mathrm{e}^{-N/256}\,. \tag{3.23}$$

The bounds (3.19), (3.20) and (3.23) allow to give an upper bound for the terms $I_{1,N}(\xi)$, $I_{2,N}(\xi)$ and $I_{3,N}(\xi)$ and next to integrate them over $[-T,T]$ when they are divided by $|\xi|$.

Hence the computations in [14, Sect. 4.1.1., Step 4] are replaced by the following computations. First, as in [14], we select

$$\ell = \ell(\xi) = \mathbf{1}_{\{\xi^2 < Nc_2\}} + (m - [nc_2/(2\xi^2)] + 1)\mathbf{1}_{\{\xi^2 \geq Nc_2\}} . \tag{3.24}$$

Therefore $m-\ell$ is either equal to $m-1$ or to $[nc_2/(2\xi^2)]-1$. Since $|\xi| \leq T = n^{1/2}(\log n)^{-1/2}$, it follows that $nc_2/(2\xi^2) \geq 2^{-1}c_2(\log n)$ (and then for n large enough $m-\ell \geq c_1^{-2}$). So, starting from (3.23) and taking into account the selection of ℓ, we get that for any $|\xi| \leq T$ and n large enough,

$$\Big\| \prod_{j=N/2}^{N-1} |\varphi_j(\xi/\sqrt{N})| \Big\|_1 \ll \mathrm{e}^{-c_3\xi^2/32}\mathbf{1}_{\{\xi^2<Nc_2\}} + \mathrm{e}^{-c_3c_2N/32}\mathbf{1}_{\{\xi^2 \geq Nc_2\}} + \mathrm{e}^{-N/256} . \tag{3.25}$$

Select now

$$N = [\kappa \log n] \text{ with } \kappa > 2\max(256, 32(c_2c_3)^{-1}) \tag{3.26}$$

and then $m \sim (2\kappa)^{-1}n/\log n$. Taking into account (3.19), (3.20) and (3.25), we get

(3.27)
$$\int_{-T}^{T} (I_{1,N}(\xi) + I_{3,N}(\xi))/|\xi| \,\mathrm{d}\xi \ll N \int_0^T \Big(\frac{|\xi|}{N\sqrt{m}} + \frac{1}{\sqrt{N}m^{3/2}} \Big) \Big(\mathrm{e}^{-c_1\xi^2/32} + n^{-2} \Big) \mathrm{d}\xi$$
$$\ll \frac{1}{\sqrt{m}} + \frac{\sqrt{N}}{m\sqrt{m}} + \frac{T^2}{n^2\sqrt{m}} + \frac{T\sqrt{N}}{n^2 m\sqrt{m}} \ll \frac{\sqrt{\log n}}{\sqrt{n}} .$$

Next, using (3.19), we derive

$$I_{2,N}(\xi) \ll \Big(\frac{\xi^2}{\sqrt{m}} + \frac{\sqrt{N}|\xi|}{m^{3/2}} \Big) \times \mathrm{e}^{-s^2\xi^2/16} .$$

Therefore, by the selection of m and N,

$$\int_{-T}^{T} I_{2,N}(\xi)/|\xi| \,\mathrm{d}\xi \ll \frac{\sqrt{\log n}}{\sqrt{n}} . \tag{3.28}$$

Starting from (3.17) and taking into account (3.18), (3.27) and (3.28), the upper bound in (3.9) follows.

Step 3. Proof of (3.10). Recall that $S^{(2)}_{|m} = \sum_{k=m+1}^n \mathbb{E}(X_{k,m}|\mathbb{F}_m)$, and recall that we assume that $2Nm = n$. Denoting

$$Y_j^{(2)} = U_j^{(2)} + R_j^{(2)} \text{ for } j = 1, \ldots, N ,$$

where $U_N^{(2)} = \sum_{k=(2N-1)m+1}^n \mathbb{E}(X_{k,m}|\mathbb{F}_m)$, $R_N^{(2)} = 0$,

$$U_j^{(2)} = \sum_{k=(2j-1)m+1}^{2jm} \mathbb{E}(X_{k,m}|\mathbb{F}_m) \text{ and } R_j^{(2)} = \sum_{k=2jm+1}^{(2j+1)m} \mathbb{E}(X_{k,m}|\mathbb{F}_m) \text{ for } j = 1, \ldots, N-1 ,$$

we have $S^{(2)}_{|m} = \sum_{j=1}^N Y_j^{(2)}$. Note that the random vectors $(U_j^{(2)}, R_j^{(2)})_{1 \leq j \leq N}$ are independent. The proof of (3.10) can be done by using similar (but even simpler) arguments to those developed in the step 2. In this part, one of the important fact is to notice that the $R_j^{(2)}$'s also have a negligible contribution. Indeed, for any $2m+1 \le k \le 3m$,

$$\|\mathbb{E}(X_{k,m}|\mathbb{F}_m)\|_\infty = \Big\| \iint \Big(f_m(\varepsilon_{k-m+1}, \dots, \varepsilon_{2m}, a_{2m+1}, \dots, a_k) \\ - f_m(b_{k-m+1}, \dots, b_{2m}, b_{2m+1}, \dots, b_k) \Big) \prod_{i=2m+1}^{k} d\mu(a_i) \prod_{i=k-m+1}^{k} d\mu(b_i) \Big\|_\infty \\ \le \sup_{\bar{x}} \Big| \mathbb{E}(X_{k-2m}|W_0 = \bar{x}) - \int \mathbb{E}(X_{k-2m}|W_0 = \bar{y}) d\nu(\bar{y}) \Big| \le \delta_{1,\infty}(k-2m) \,.$$

Hence by stationarity and (3.12) we derive that $\|R_j^{(2)}\|_\infty \ll 1$ for any $j = 1, \dots, N$.

To complete the proof of the upper bound (2.3), we just have to put together the results in the steps 1, 2 and 3. □

3.1.2. *Proof of the upper bound* (2.4). Recall the notation $S_{n,\bar{u}} := \sum_{k=1}^n X_{k,\bar{u}}$ where $X_{k,\bar{u}}$ denotes the random variable X_k defined by (2.1) when the Markov chain $(W_n)_{n\ge 0}$ starts from $\bar{u} \in X$. Our starting point is the following upper bound:

$$\sup_{n\ge 1} \Big\| \log(\|A_n\|) - n\lambda_\mu - \int_X S_{n,\bar{u}} d\nu(\bar{u}) \Big\|_\infty < \infty \,. \tag{3.29}$$

The proof of (3.29) is outlined in Section 8.1 in [4] but, since it is a key ingredient in the proof of (2.4), we shall provide more details here. Let $g \in G$ and $\bar{u} \in X$. By item (i) of Lemma 4.7 of [1], there exists $\bar{v}(g)$ such that

$$\log \|g\| - \sigma(g, \bar{u}) \le -\log \delta\big(\bar{u}, \bar{v}(g)\big) \,,$$

where $\delta(\bar{u}, \bar{v}) := \frac{|\langle u,v\rangle|}{\|u\|\,\|v\|}$. Integrating with respect to ν, it follows that

$$0 \le \log \|g\| - \int_X \sigma(g, \bar{u})\, d\nu(\bar{u}) \le \sup_{\bar{v}\in X} \int_X |\log \delta(\bar{u}, \bar{v})|\, d\nu(\bar{u}) \,. \tag{3.30}$$

But, according to Proposition 4.5 in [1], since μ has a polynomial moment of order 2, $\sup_{\bar{v}\in X} \int_X \big|\log \delta(\bar{u}, \bar{v})\big|\, d\nu(\bar{u}) < \infty$. Therefore, (3.29) comes from an application of (3.30) with $g = A_n$.

Now, using (3.29) and Lemma 1 in [2], the upper bound (2.4) will follow if one can prove that

$$\sup_{y\in\mathbb{R}} \Big| \mathbb{P}\Big(\int_X S_{n,\bar{u}} d\nu(\bar{u}) \le y\sqrt{n} \Big) - \Phi(y/s) \Big| \ll (\log n)/\sqrt{n} \,. \tag{3.31}$$

We proceed as for the proof of the upper bound (2.3) with the following differences. First we consider

$$S_{n,m} = \sum_{k=1}^{m} \int_X X_{k,\bar{u}} d\nu(\bar{u}) + \sum_{k=m+1}^{n} X_{k,m} \,,$$

where $X_{k,m}$ is defined by (3.2). Hence

$$\Big\| \int_X S_{n,\bar{u}} d\nu(\bar{u}) - S_{n,m} \Big\|_1 \le \int_X \sum_{k=m+1}^{n} \|X_{k,\bar{u}} - X_{k,m}\|_1 d\nu(\bar{u}) \le n\delta_{1,\infty}(m) \,.$$

It follows that the step 1 of the previous subsection is unchanged. Next, we use the same notation as in subsection 3.1.1 with the following change: U_1 is now defined by

$$U_1 = \sum_{k=1}^{m} \int_X X_{k,\bar u} d\nu(\bar u)\,, \tag{3.32}$$

and then, when $n = 2mN$, the decomposition (3.7) is still valid for $S_{n,m}$. The step 3 is also unchanged. Concerning the step 2, the only difference concerns the upper bound of the quantity $\|\varphi_1(t) - \mathrm{e}^{-s^2t^2/4}\|_1$ since the definition of U_1 is now given by (3.32). To handle this term, we note that for $f(x) \in \{\cos x, \sin x\}$, we have

$$\Big\|\mathbb{E}_{\mathbb{F}_m}\Big[f\Big(t\frac{\sum_{k=1}^m \int_X X_{k,\bar u} d\nu(\bar u) + R_1}{\sqrt{2m}}\Big)\Big] - \mathbb{E}_{\mathbb{F}_m}\Big[f\Big(t\frac{\sum_{k=1}^m X_k + R_1}{\sqrt{2m}}\Big)\Big]\Big\|_1 \\ \le \frac{|t|}{\sqrt{2m}} \int_X \sum_{k=m+1}^{n} \|X_{k,\bar u} - X_k\|_1 d\nu(\bar u) \le \frac{|t|}{\sqrt{2m}} \sum_{k=1}^{m} \delta_{1,\infty}(k) \ll \frac{|t|}{\sqrt{m}}\,.$$

The last upper bound comes from (3.12) together with the fact that μ is assumed to have a moment of order $q = 3$. Next, by taking into account (3.25), note that

$$\int_{-T}^{T} \frac{|\xi|}{\sqrt{N}\sqrt{m}} \Big\| \prod_{j=N/2}^{N-1} |\varphi_j(\xi/\sqrt{N})| \Big\|_1 d\xi \ll 1/\sqrt{n}\,.$$

This implies in particular that (3.27) still holds. Compared to subsection 3.1.1 the rest of the proof is unchanged. □

3.1.3. *Proof of the upper bound* (2.5). Once again we highlight the differences with respect to the proof given in Subsection 3.1.1. For $x \in S^{d-1}$, we consider

$$S_{n,m,\bar x} = \sum_{k=1}^{m} X_{k,\bar x} + \sum_{k=m+1}^{n} X_{k,m}\,,$$

and we note that

$$\sup_{\bar x \in X} \|S_{n,\bar x} - S_{n,m,\bar x}\|_1 \le \sum_{k=m+1}^{n} \sup_{\bar x \in X} \|X_{k,\bar x} - X_{k,m}\|_1 \le n\delta_{1,\infty}(m)\,.$$

Once again Step 1 of Subsection 3.1.1 is unchanged. Next, U_1 is now defined by

$$U_{1,\bar x} = U_1 = \sum_{k=1}^{m} X_{k,\bar x}\,, \tag{3.33}$$

and the step 3 is also unchanged. Concerning the step 2, due to the new definition (3.33) of U_1, the only difference concerns the upper bound of the quantity $\|\varphi_1(t) - \mathrm{e}^{-s^2t^2/4}\|_1$. To handle this term, we note that for $f(y) \in \{\cos y, \sin y\}$, we have, by using (3.12) together with the fact that μ is assumed to have a moment of order $q = 3$.

$$\sup_{\bar x \in X} \Big\|\mathbb{E}_{\mathbb{F}_m}\Big[f\Big(t\frac{\sum_{k=1}^m X_{k,\bar x} + R_1}{\sqrt{2m}}\Big)\Big] - \mathbb{E}_{\mathbb{F}_m}\Big[f\Big(t\frac{\sum_{k=1}^m X_k + R_1}{\sqrt{2m}}\Big)\Big]\Big\|_1 \\ \le \frac{|t|}{\sqrt{2m}} \sum_{k=m+1}^{n} \sup_{\bar x \in X} \|X_{k,\bar x} - X_k\|_1 \ll \frac{|t|}{\sqrt{m}}\,.$$

We then end the proof as in subsection 3.1.1. □

3.2. *Proof of Theorem 2.2.* Let us point out the differences compared to the proof of Theorem 2.1 (the selections of N and m being identical). To get the upper bound (3.25), we still establish an upper bound similar to (3.23) valid for any $\ell \in [1, m]$ and any t such that $t^2(m-\ell)/(2m) \le C$. Since μ has a finite moment of order $q = 4$, according to Lemma 4.3, $\|R_1\|_3 \ll 1$. Hence, using Lemma 4.1 with $a = 0$ (here Lemma 4.5 in [14] can also be used), the desired upper bound follows and the constant C appearing above in the restriction for t can be taken equal to c_2 (which is the constant appearing in Lemma 4.1). The fact that $a = 0$ implies that we do not need to verify, as in the proof of Theorem 2.1 that $(m-\ell)^{-1/2} \le c_1$. Next, we select ℓ as in (3.24). This selection makes sense if $\xi^2 \le nc_2/2$. Therefore, we use (3.1) by selecting $T = \eta\sqrt{n}$ with η small enough (more precisely such that $c_2/(2\eta^2)$ is large enough for (3.21) to be satisfied when $m - \ell$ is of order $c_2/(2\eta^2)$). Therefore, for any $|\xi| \le T$, the upper bound (3.25) is still valid. The second difference, in addition to the choice of T, is that instead of using Lemma 4.6 we use Lemmas 4.11 and 4.12 with $r = 3$ which then entail that for any $j \ge 1$,

$$\|\varphi_j(\xi/\sqrt{N}) - \mathrm{e}^{-s^2\xi^2/(4N)}\|_1 \ll N^{-1}|\xi|^3 n^{-1/2} + |\xi| n^{-1/2} m^{-3/14} . \qquad (3.34) \quad \square$$

Note that the upper bound (42) in Jirak [15] with $p = 3$ has the same order as (3.34) and is obtained provided $\sum_{k\ge1} k^a \delta_{3,\infty}(k) < \infty$ for some $a > 0$ (indeed [15, Lemma 5.8 (iii)] is a key ingredient to get (42)). Now using (3.12) we see that $\sum_{k\ge1} k^a \delta_{3,\infty}(k) < \infty$ for some $a > 0$ as soon as μ has a moment of order $q > 6$. As we shall see [15, Lemma 5.8] is not needed in its full generality to get an upper bound as (3.34). Indeed our Lemmas 4.11 and 4.12 are rather based on an estimate as (4.22) which involves the $\mathbb{L}^1$-norm rather than the $\mathbb{L}^{3/2}$-norm .

4. Technical lemmas. Suppose that we have a sequence of random vectors $\{(A_j, B_j)\}_{1\le j\le J}$ and a filtration $\{\mathcal{H}_j\}_{1\le j\le J}$ such that

$$\Big(\mathbb{E}_{\mathcal{H}_j}(A_j^2), \mathbb{E}_{\mathcal{H}_j}(|A_j|^p), \mathbb{E}_{\mathcal{H}_j}(B_j^2)\Big)_{j\in J}$$

is a sequence of independent random vectors (with values in $\mathbb{R}^3$). For any real a, let $H_j(a) = A_j + aB_j$ and

$$\varphi^{\mathcal{H}}_{j,a}(x) = \mathbb{E}\big(\exp(\mathrm{i}xH_j(a))|\mathcal{H}_j\big) .$$

With the notations above, the following modification of [14, Lemma 4.5] holds:

LEMMA 4.1. *Let $p > 2$. Let $J \ge 16$ be an integer. Assume the following*
(i) $\mathbb{E}_{\mathcal{H}_j}(A_j) = \mathbb{E}_{\mathcal{H}_j}(B_j) = 0$, for any $1 \le j \le J$
(ii) there exists $u^- > 0$ such that $\mathbb{P}(\mathbb{E}_{\mathcal{H}_j}(A_j^2) \le u^-) < 1/2$, for any $1 \le j \le J$,
(iii) $\sup_{j\ge1} \mathbb{E}(|A_j|^p) < \infty$,
(iv) $\sup_{j\ge1} \mathbb{E}(B_j^2) < \infty$.
Then there exist positive finite constants c_1, c_2 and c_3 depending only on p, u^-, $\sup_{j\ge1} \mathbb{E}(|A_j|^p)$ and $\sup_{j\ge1} \mathbb{E}(B_j^2)$ such that for any $a \in [0, c_1]$ and any $x^2 \le c_2$,

$$\mathbb{E}\Big(\prod_{j=1}^{J} |\varphi^{\mathcal{H}}_{j,a}(x)|\Big) \le \mathrm{e}^{-c_3x^2J} + \mathrm{e}^{-J/32} .$$

Proof of Lemma 4.1. The beginning of the proof proceeds as the proof of [14, Lemma 4.5].

Let $1 \le j \le J$ be fixed for the moment. Using a Taylor expansion we have

$$\mathbb{E}\big(\exp(\mathrm{i}xH_j(a))|\mathcal{H}_j\big) = 1 - \mathbb{E}_{\mathcal{H}_j}(H_j^2(a))x^2/2 + x^2/2\int_0^1 (1-s)I(s,x)ds ,$$

where, for any $h>0$ and any $s\in[0,1]$,

$$\begin{aligned}|I(s,x)| &\le 4a^2\mathbb{E}_{\mathcal{H}_j}(B_j^2)+2\mathbb{E}_{\mathcal{H}_j}\big(A_j^2\big|(\cos(sxH_j(a))-\cos(0))+\mathrm{i}(\sin(sxH_j(a))-\sin(0))\big|\big)\\ &\le 4a^2\mathbb{E}_{\mathcal{H}_j}(B_j^2)+8\mathbb{E}_{\mathcal{H}_j}(A_j^2)|xh|+4\mathbb{E}_{\mathcal{H}_j}(A_j^2\mathbf{1}_{|H_j(a)|\ge 2h})\,.\end{aligned}$$

Using the fact that for any reals u and v, $u^2\mathbf{1}_{|u+v|\ge 2h}\le u^2\mathbf{1}_{|u|\ge h}+v^2$, we get

$$\begin{aligned}|I(s,x)| &\le 8a^2\mathbb{E}_{\mathcal{H}_j}(B_j^2)+8\mathbb{E}_{\mathcal{H}_j}(A_j^2)|xh|+4\mathbb{E}_{\mathcal{H}_j}(A_j^2\mathbf{1}_{|A_j|\ge h})\\ &\le 8a^2\mathbb{E}_{\mathcal{H}_j}(B_j^2)+8\mathbb{E}_{\mathcal{H}_j}(A_j^2)|xh|+4h^{2-p}\mathbb{E}_{\mathcal{H}_j}(|A_j|^p)\,.\end{aligned}$$

Now for any $\alpha>0$,

$$|\mathbb{E}_{\mathcal{H}_j}(H_j^2(a))-\big(\mathbb{E}_{\mathcal{H}_j}(A_j^2)+a^2\mathbb{E}_{\mathcal{H}_j}(B_j^2)\big)|\le\alpha^{-1}\mathbb{E}_{\mathcal{H}_j}(A_j^2)+\alpha a^2\mathbb{E}_{\mathcal{H}_j}(B_j^2)\,.$$

So, overall, for any $h>0$ and any $\alpha>0$,

$$\begin{aligned}\Big|\mathbb{E}\big(\exp(\mathrm{i}xH_j(a))|\mathcal{H}_j\big)-1+\mathbb{E}_{\mathcal{H}_j}(A_j^2))x^2/2\Big| &\le x^2(4a^2+\alpha a^2)\mathbb{E}_{\mathcal{H}_j}(B_j^2)\\ &\quad+\mathbb{E}_{\mathcal{H}_j}(A_j^2)(x^2\alpha^{-1}/2+4h|x|^3)+x^2h^{2-p}\mathbb{E}_{\mathcal{H}_j}(|A_j|^p)\,.\end{aligned}$$

Let us take $h=|x|^{-1/(p-1)}$ and $\alpha=a^{-1}$. Set $\delta(p):=(p-2)/(p-1)$.

Let $\tilde u, u^+$ be positive numbers to be chosen later.

Recall that by the conditional Jensen inequality, $\mathbb{E}_{\mathcal{H}_j}(A_j^2)\le\big(\mathbb{E}_{\mathcal{H}_j}(|A_j|^p)\big)^{2/p}$ $\mathbb{P}$-almost surely. For the sake of simplicity, we shall assume that this inequality takes place everywhere.

From the above computations, we infer that, on the set $\{\mathbb{E}_{\mathcal{H}_j}(B_j^2)\le\tilde u\}\cap\{\mathbb{E}_{\mathcal{H}_j}(|A_j|^p)\le u^+\}$, one has

$$\begin{aligned}&\Big|\mathbb{E}\big(\exp(\mathrm{i}xH_j(a))|\mathcal{H}_j\big)-1+\mathbb{E}_{\mathcal{H}_j}(A_j^2))x^2/2\Big|\\ &\qquad\le x^2(4a^2+a)\tilde u+x^2(u^+)^{2/p}a/2+|x|^{2+\delta(p)}(4(u^+)^{2/p}+u^+)\,.\end{aligned}$$

Set

$$u(x):=x(4a^2+a)\tilde u+x^2(u^+)^{2/p}a/2+|x|^{\delta(p)}(4(u^+)^{2/p}+u^+)\,.$$

Let u^- be a positive number (u^- will be given by (ii) but it is unimportant at this stage). We infer that, for every $x^2\le 2/u^-$, on the set

$$\Gamma_j:=\{\mathbb{E}_{\mathcal{H}_j}(B_j^2)\le\tilde u\}\cap\{\mathbb{E}_{\mathcal{H}_j}(A_j^2)>u^-\}\cap\{\mathbb{E}_{\mathcal{H}_j}(|A_j|^p)\le u^+\}$$

one has

$$\Big|\mathbb{E}\big(\exp(\mathrm{i}xH_j(a))|\mathcal{H}_j\big)\Big|\le 1-u^-x^2/2+x^2u(x)\,.$$

Since $0<u^-,u^+,\tilde u<\infty$, there exist positive constants $c_1,c_2<\infty$ (depending only on $(u^-,u^+,\tilde u)$) such that

$$\begin{aligned}a\le c_1 &\Rightarrow (4a^2+a)\tilde u+(u^+)^{2/p}a/2\le u^-/8\,,\\ x^2\le c_2 &\Rightarrow |x|^{\delta(p)}(4(u^+)^{2/p}+u^+)\le u^-/8\,.\end{aligned}$$

Therefore, there exist constants $0 < c_1, c_2 < \infty$ (depending only on $(\tilde u, u^-, u^+)$) such that for any $a \le c_1$, any $x^2 \le c_2$, on the set Γ_j,

$$\left|\mathbb{E}\big(\exp(\mathrm{i}xH_j(a))|\mathcal{H}_j\big)\right| \le 1 - u^- x^2/4 \le e^{-u^- x^2/4}\,.$$

Set also $\Sigma_J := \sum_{j=1}^J \mathbf{1}_{\Gamma_j}$ and $\Lambda_J := \{\Sigma_J \ge J/8\}$.

From the previous computations and the trivial bound $\left|\mathbb{E}\big(\exp(\mathrm{i}xH_j(a))|\mathcal{H}_j\big)\right| \le 1$, we see that, for any $0 < \tilde u, u^-, u^+ < \infty$, there exists positive contants c_1, c_2, c_3 such that for every $x^2 \le c_2$ and every $a \le c_1$, on the set Γ_J, one has (recall that $J \ge 16$),

$$\mathbb{E}\Big(\prod_{j=1}^J |\varphi^{\mathcal{H}}_{j,a}(x)|\Big) \le \mathrm{e}^{-u^- x^2 [J/8]/2} \le \mathrm{e}^{-u^- x^2 J/32}\,.$$

Using the the above trivial bound again, the lemma will be proved if, with u^- given by (ii), one can chose $\tilde u, u^+ > 0$ such that $\mathbb{P}(\Lambda_J) \le \mathrm{e}^{-J/32}$.

By Markov's inequality,

$$\mathbb{P}(\mathbb{E}_{\mathcal{H}_j}(B_j^2) > \tilde u) \le \frac{\sup_{j\in J}\mathbb{E}(B_j^2)}{\tilde u} \underset{\tilde u \to +\infty}{\longrightarrow} 0\,.$$

Hence there exists $\tilde u > 0$ such that, for any $1 \le j \le J$, $\mathbb{P}(\mathbb{E}_{\mathcal{H}_j}(B_j^2) > \tilde u) \le 1/8$.

Similarly, there exists $u^+ > 0$ such that , for any $1 \le j \le J$, $\mathbb{P}(\mathbb{E}_{\mathcal{H}_j}(|A_j|^p) > u^+) \le 1/8$.

By assumption (ii) and by definition of $\tilde u$ and u^+, we have

$$\mathbb{E}(\Sigma_J) \ge \sum_{j=1}^J (1 - (1/2 + 1/8 + 1/8)) = J/4\,.$$

Hence,

$$\begin{aligned}\mathbb{P}(\Lambda_J^c) = \mathbb{P}(\Sigma_J < J/8) &= \mathbb{P}(\Sigma_J - \mathbb{E}(\Sigma_J) < J/8 - \mathbb{E}(\Sigma_J)) \\ &\le \mathbb{P}(\Sigma_J - \mathbb{E}(\Sigma_J) < -J/8) = \mathbb{P}(-\Sigma_J + \mathbb{E}(\Sigma_J) > J/8)\,.\end{aligned}$$

Hence, using Hoeffding's inequality (see [12, Theorem 2]),

$$\mathbb{P}(\Lambda_J^c) \le \mathrm{e}^{\frac{-2(J/8)^2}{J}} = \mathrm{e}^{-J/32}\,.$$

□

LEMMA 4.2. *Assume that μ has a moment of order $q = 3$. Let $X_{k,m}$ be defined by (3.2). Then, setting $\bar X_{k,m} = X_{k,m} - \mathbb{E}_m(X_{k,m})$, we have*

$$\Big\|\mathbb{E}_m\Big(\sum_{k=m+1}^{2m} \bar X_{k,m}\Big)^2 - \mathbb{E}\Big(\sum_{k=m+1}^{2m} \bar X_{k,m}\Big)^2\Big\|_1 \ll m^{2/7}\,,$$

where $\mathbb{E}_m(\cdot)$ means $\mathbb{E}(\cdot|\mathcal{G}_m)$ with $\mathcal{G}_m = \sigma(W_0, \varepsilon_1, \ldots, \varepsilon_m)$. In addition if $q = 4$, then

$$\Big\|\mathbb{E}_m\Big(\sum_{k=m+1}^{2m} \bar X_{k,m}\Big)^2 - \mathbb{E}\Big(\sum_{k=m+1}^{2m} \bar X_{k,m}\Big)^2\Big\|_1 \ll 1\,.$$

Proof of Lemma 4.2. Note first that

$$\Big\|\mathbb{E}_m\Big(\sum_{k=m+1}^{2m}\bar{X}_{k,m}\Big)^2-\mathbb{E}\Big(\sum_{k=m+1}^{2m}\bar{X}_{k,m}\Big)^2\Big\|_1$$
$$\le \Big\|\mathbb{E}_m\Big(\sum_{k=m+1}^{2m}X_{k,m}\Big)^2-\mathbb{E}\Big(\sum_{k=m+1}^{2m}X_{k,m}\Big)^2\Big\|_1+2\Big\|\mathbb{E}_m\Big(\sum_{k=m+1}^{2m}X_{k,m}\Big)\Big\|_2^2$$
$$:= I_m+I\!I_m\,.$$

But

$$\Big\|\mathbb{E}_m\Big(\sum_{k=m+1}^{2m}X_{k,m}\Big)\Big\|_2\le\sum_{k=m+1}^{2m}\|\mathbb{E}_m(X_{k,m})\|_\infty\ll\sum_{k=1}^{m}\delta_{1,\infty}(k)\ll 1\,,\tag{4.1}$$

by taking into account (3.12) and the fact that $q>2$. It remains to handle I_m. With this aim, we first write the following decomposition: for any $\gamma\in(0,1]$

$$\begin{aligned}I_m\le{}&\sum_{k=1}^{m}\|\mathbb{E}_m(X^2_{k+m,m})-\mathbb{E}(X^2_{k+m,m})\|_1\\&+2\sum_{\ell=1}^{m}\ell^\gamma\sup_{\ell\le j<i\le 2\ell}\|\mathbb{E}_m(X_{i+m,m}X_{j+m,m})-\mathbb{E}(X_{i+m,m}X_{j+m,m})\|_1\\&\quad+2\sum_{\ell=1}^{m}\sum_{k=\ell^\gamma+1}^{m-\ell}\|\mathbb{E}_m(X_{\ell+m,m}X_{\ell+k+m,m})-\mathbb{E}(X_{\ell+m,m}X_{\ell+k+m,m})\|_1\,.\end{aligned}\tag{4.2}$$

Note that for $1\le i,j\le m$,

$$\|\mathbb{E}_m(X_{i+m,m}X_{j+m,m})-\mathbb{E}(X_{i+m,m}X_{j+m,m})\|_1\le\sup_{\substack{\bar{x}_1,\bar{x}_2\in X\\ \bar{y}_1,\bar{y}_2\in X}}\mathbb{E}\big|X_{i,\bar{x}_1}X_{j,\bar{x}_2}-X_{i,\bar{y}_1}X_{j,\bar{y}_2}\big|$$

With the same arguments as those developed in the proof of [4, Prop. 4], we infer that, if μ has a moment of order $q=3$,

$$\sum_{k\ge m+1}\|\mathbb{E}_m(X^2_{k,m})-\mathbb{E}(X^2_{k,m})\|_1\ll 1\,,\tag{4.3}$$

and, for every $\beta<1/3$,

$$\sum_{\ell\ge m+1}\ell^\beta\sup_{\ell\le j<i\le 2\ell}\|\mathbb{E}_m(X_{i+m,m}X_{j+m,m})-\mathbb{E}(X_{i+m,m}X_{j+m,m})\|_1\ll 1\,.\tag{4.4}$$

On another hand, with the same arguments as those used to prove [4, Relation (34)], we have

$$\begin{aligned}&\sum_{\ell=1}^{m}\sum_{k=\ell^\gamma+1}^{m-\ell}\|\mathbb{E}_m(X_{\ell+m,m}X_{\ell+k+m,m})-\mathbb{E}(X_{\ell+m,m}X_{\ell+k+m,m})\|_1\\&\ll\Big(\sum_{\ell=m+1}^{2m}\|\mathbb{E}_m(X_{\ell,m})\|_2\Big)^2+\sum_{i=m}^{2m-1}\|\mathbb{E}_m(X_{i,m})\|_2\times\sum_{k=1}^{m}k^{1/\gamma-1/2}\|\mathbb{E}_m(X_{k+m,m})\|_2\,.\end{aligned}\tag{4.5}$$

But, by taking into account (3.12) and the fact that μ has a moment of order $q=3$, we have, for any $1\le k\le m$,

$$\|\mathbb{E}_m(X_{k+m,m})\|_2\le\|\mathbb{E}_m(X_{k+m,m})\|_\infty\le\delta_{1,\infty}(k)\le k^{-2}\,.$$

Hence, using this upper bound in (4.5) and considering the estimates (4.3) and (4.4), we get, for any $\gamma \in (0,1]$ and any $\beta < 1/3$,

$$I_m \le 1 + m^{\gamma-\beta}\mathbf{1}_{\gamma\ge\beta} + \sum_{k=1}^{m} k^{1/\gamma-5/2}\,.$$

Hence, selecting $\gamma = 2\beta$ and $\beta = 2/7$, we derive that $I_m \le m^{2/7}$ which gives the desired inequality, when μ has a moment of order $q = 3$.

Assume now that μ has a moment of order $q = 4$. In this case, with the same arguments as those developed in the proof of [4, Prop. 4], we infer that (4.4) holds with $\beta < 1 + 1/4$. Then, selecting $\gamma = 1$ in the decomposition (4.2) and using similar computations as above, the desired upper bound follows. □

LEMMA 4.3. *Let $p \in [2,3]$. Assume that μ has a moment of order $q = p+1$. Then $\|R_1\|_p \ll 1$, where R_1 is defined by* (3.5)*.*

Proof of Lemma 4.3. Setting $\tilde X_{k,m} = X_{k,m} - \mathbb{E}_{\mathbb{F}_m}(X_{k,m})$ and applying [6, Proposition 3.1] with $N = 1$, we have

$$\|R_1\|_p \le \Big(2(p-1)\sum_{i=m+1}^{2m}\gamma_{i,m}\Big)^{1/2} + \Big(\sum_{k=m+1}^{2m}\|\tilde X_{k,m}\|_{p,\nu}^p + p(p-1)\sum_{i=m+2}^{2m}\alpha_{i,m}\Big)^{1/p},$$

where

$$\gamma_{i,m} = \frac{1}{2}\|\tilde X_{i,m}\|_2^2 + \sum_{j=m+1}^{i-1}\|\tilde X_{j,m}\mathbb{E}(\tilde X_{i,m}|\mathcal{E}_2^j)\|_{p/2}$$

and

$$\alpha_{i,m} = \frac{1}{2}\sum_{j=m+1}^{i-1}\||\tilde X_{j,m}|^{p-2}\mathbb{E}(\tilde X_{i,m}^2 - \mathbb{E}(\tilde X_{i,m}^2)|\mathcal{E}_2^j)\|_1\,.$$

But, for any integer $k \in [m+1, 2m]$ and any $p \ge 1$,

$$\begin{aligned}
&\mathbb{E}\big|\tilde X_{k,m}\big|^p \\
&= \mathbb{E}\Big|f_m(\varepsilon_{k-m+1},\dots,\varepsilon_m,\varepsilon_{m+1},\dots,\varepsilon_k) - \int f_m(v_{k-m+1},\dots,v_m,\varepsilon_{m+1},\dots,\varepsilon_k)\prod_{i=k-m+1}^{m} d\mu(v_i)\Big|^p \\
&\le \int \mathbb{E}\Big|f_m(\varepsilon_{k-m+1},\dots,\varepsilon_m,\varepsilon_{m+1},\dots,\varepsilon_k) - f_m(v_{k-m+1},\dots,v_m,\varepsilon_{m+1},\dots,\varepsilon_k)\Big|^p\prod_{i=k-m+1}^{m} d\mu(v_i)\,.
\end{aligned}$$

Hence, for any integer $k \in [m+1, 2m]$ and any $p \ge 1$,

$$\begin{aligned}
\mathbb{E}\big|\tilde X_{k,m}\big|^p &\le \iint \mathbb{E}\Big|f_m(u_{k-m+1},\dots,u_m,\varepsilon_{m+1},\dots,\varepsilon_k) \\
&\qquad - f_m(v_{k-m+1},\dots,v_m,\varepsilon_{m+1},\dots,\varepsilon_k)\Big|^p\prod_{i=k-m+1}^{m} d\mu(v_i)\prod_{i=k-m+1}^{m} d\mu(u_i) \\
&\le \sup_{\bar x,\bar y\in X}\mathbb{E}|X_{k-m,\bar x} - X_{k-m,\bar y}|^p = \delta_{p,\infty}^p\,(k-m)\,. \qquad (4.6)
\end{aligned}$$

By taking into account (3.12) and the fact that μ has a moment of order $q = p+1$, it follows that

$$\sum_{k=m+1}^{2m} \|\tilde{X}_{k,m}\|_{p,\nu}^p \le \sum_{k\ge 1} \delta_{p,\infty}^p(k) < \infty \,.$$

On another hand, for $m+1 \le j < i \le 2m$,

$$\|\mathbb{E}(\tilde{X}_{i,m}|\mathcal{E}_2^j)\|_\infty \le 2\delta_{1,\infty}(i-j)\,. \tag{4.7}$$

By taking into account (3.12) and the fact that μ has a finite moment of order $q = p+1$ (and then $q \ge 3$ and $q > p$), it follows that

$$\sum_{i=m+1}^{2m} \gamma_{i,m} \le 2^{-1} \sum_{i=1}^{m} \delta_{2,\infty}^2(i) + 2 \sum_{i=m+1}^{2m} \sum_{j=m+1}^{i-1} \big(\delta_{p/2,\infty}(j-m)\big)\delta_{1,\infty}(i-j)$$
$$\ll \sum_{i=1}^{m} \delta_{2,\infty}^2(i) + \sum_{j=1}^{m} \delta_{p/2,\infty}(j) \sum_{i=1}^{m} \delta_{1,\infty}(i) \ll 1\,.$$

On another hand, for any $m+1 \le i \le 2m$, by Lemma 4.4,

$$\|\mathbb{E}(\tilde{X}_{i,m}^2 - \mathbb{E}(\tilde{X}_{i,m}^2)|\mathcal{E}_2^j)\|_\infty \le 4 \sup_{\substack{\bar{x}_1,\bar{x}_2 \in X \\ \bar{y}_1,\bar{y}_2 \in X}} \mathbb{E}\big|X_{i-j,\bar{x}_1} X_{i-j,\bar{x}_2} - X_{i-j,\bar{y}_1} X_{i-j,\bar{y}_2}\big|\,.$$

Again, by (3.12) and the fact that μ has a moment of order $q = p+1$ (and then $q \ge 3$ and $q \ge p-1$) and by using (4.8), it follows that

$$\sum_{i=m+2}^{2m} \alpha_{i,m} \ll \sum_{j=m+1}^{2m} \delta_{p-2,\infty}^{p-2}(j-m) \sum_{k=1}^{m} \sup_{\substack{\bar{x}_1,\bar{x}_2 \in X \\ \bar{y}_1,\bar{y}_2 \in X}} \mathbb{E}\big|X_{k,\bar{x}_1} X_{k,\bar{x}_2} - X_{k,\bar{y}_1} X_{k,\bar{y}_2}\big| \ll 1\,.$$

Putting together all the computations above we get the lemma. □

LEMMA 4.4. *Let $\tilde{X}_{k,m} = X_{k,m} - \mathbb{E}_{\mathbb{F}_m}(X_{k,m})$. For any $m+1 \le j < i \le 2m$,*

$$\|\mathbb{E}(\tilde{X}_{i,m}^2 - \mathbb{E}(\tilde{X}_{i,m}^2)|\mathcal{E}_2^j)\|_\infty \le 4 \sup_{\substack{\bar{x}_1,\bar{x}_2 \in X \\ \bar{y}_1,\bar{y}_2 \in X}} \mathbb{E}\big|X_{i-j,\bar{x}_1} X_{i-j,\bar{x}_2} - X_{i-j,\bar{y}_1} X_{i-j,\bar{y}_2}\big|\,.$$

In addition, if μ has a finite moment of order $q > 2$,

$$\sum_{k\ge 1} k^{q-3} \sup_{\substack{\bar{x}_1,\bar{x}_2 \in X \\ \bar{y}_1,\bar{y}_2 \in X}} \mathbb{E}\big|X_{k,\bar{x}_1} X_{k,\bar{x}_2} - X_{k,\bar{y}_1} X_{k,\bar{y}_2}\big| < \infty\,. \tag{4.8}$$

Proof of Lemma 4.4. The upper bound (4.8) can be proved by using the same arguments as those used to show Equation (8) in [4]. Let us prove the first part of the lemma. Let $A_j^i = \varepsilon_i \cdots \varepsilon_j$. For any integer i in $[m+1, 2m]$, write that

$$\tilde{X}_{i,m} = \int_X \sigma(\varepsilon_i, A_{i-m+1}^{i-1}\bar{x})d\nu(\bar{x}) - \int_X \int_G \sigma(\varepsilon_i, A_{m+1}^{i-1} g_m \cdots g_{i-m+1}\bar{x})d\nu(\bar{x}) \prod_{k=i-m+1}^{m} d\mu(g_k)$$
$$:= Y_{i,m} - Z_{i,m}\,.$$

Now, for any $m+1 \le j < i \le 2m$,

$$\mathbb{E}(Y_{i,m}^2|\mathcal{E}_2^j) = \int \sigma(g_i, g_{i-1}\cdots g_{j+1}A_{i-m+1}^j\bar{x})\sigma(g_i, g_{i-1}\cdots g_{j+1}A_{i-m+1}^j\bar{y})d\nu(\bar{x})d\nu(\bar{y}) \prod_{k=j+1}^{i} d\mu(g_k)\,,$$

and

$$\mathbb{E}(Y_{i,m}^2) = \int \sigma(g_i, g_{i-1} \cdots g_{j+1} g_j \cdots g_{i-m+1} \bar{x}) \times \sigma(g_i, g_{i-1} \cdots g_{j+1} g_j \cdots g_{i-m+1} \bar{y}) d\nu(\bar{x}) d\nu(\bar{y}) \prod_{k=i-m+1}^{i} d\mu(g_k) .$$

Hence, using stationarity, we get

$$\|\mathbb{E}(Y_{i,m}^2 | \mathcal{E}_2^j) - \mathbb{E}(Y_{i,m}^2)\|_\infty \le \sup_{\substack{\bar{x}_1, \bar{x}_2 \in X \\ \bar{y}_1, \bar{y}_2 \in X}} \mathbb{E}\big| X_{i-j,\bar{x}_1} X_{i-j,\bar{x}_2} - X_{i-j,\bar{y}_1} X_{i-j,\bar{y}_2} \big| .$$

Next, for any $m+1 \le j < i \le 2m$,

$$\mathbb{E}(Y_{i,m} Z_{i,m} | \mathcal{E}_2^j) = \int \sigma(u_i, u_{i-1} \cdots u_{j+1} A_{i-m+1}^j \bar{x}) \times \sigma(u_i, u_{i-1} \cdots u_{j+1} A_{m+1}^j g_m \cdots g_{i-m-1} \bar{y}) d\nu(\bar{x}) d\nu(\bar{y}) \prod_{k=j+1}^{i} d\mu(u_k) \prod_{k=i-m-1}^{m} d\mu(g_k) ,$$

By stationarity, we derive

$$\|\mathbb{E}(Y_{i,m} Z_{i,m} | \mathcal{E}_2^j) - \mathbb{E}(Y_{i,m} Z_{i,m})\|_\infty \le \sup_{\substack{\bar{x}_1, \bar{x}_2 \in X \\ \bar{y}_1, \bar{y}_2 \in X}} \mathbb{E}\big| X_{i-j,\bar{x}_1} X_{i-j,\bar{x}_2} - X_{i-j,\bar{y}_1} X_{i-j,\bar{y}_2} \big| .$$

We get a similar upper bound for $\|\mathbb{E}(Z_{i,m}^2 | \mathcal{E}_2^j) - \mathbb{E}(Z_{i,m}^2)\|_\infty$. The first part of the lemma follows by taking into account all the above computations. □

LEMMA 4.5. *Assume that μ has a finite moment of order $q \ge 2$. Then $\big\| \sum_{k=m+1}^{2m} X_k \big\|_q \ll \sqrt{m}$ and $\big\| \sum_{k=m+1}^{2m} X_{k,m} \big\|_q \ll \sqrt{m}$.*

Proof of Lemma 4.5. The two upper bounds are proved similarly. Let us prove the second one. As to get (3.22), we use [17, Cor. 3.7], to derive that

$$\Big\| \sum_{k=m+1}^{2m} X_{k,m} \Big\|_q \ll \sqrt{m} \Big[\|X_{1+m,m}\|_q + \sum_{k=m+1}^{2m} k^{-1/2} \|\mathbb{E}_m(X_{k,m})\|_q \Big] ,$$

where $\mathbb{E}_m(\cdot)$ means $\mathbb{E}(\cdot | \mathcal{G}_m)$ with $\mathcal{G}_m = \sigma(W_0, \varepsilon_1, \ldots, \varepsilon_m)$. But $\|X_{1+m,m}\|_q \le \|X_1\|_q < \infty$ and $\|\mathbb{E}_m(X_{k+m,m})\|_q \le \|\mathbb{E}_m(X_{k+m,m})\|_\infty \le \delta_{1,\infty}(k)$. Hence, the lemma follows by considering (3.12). □

For the next lemma, we recall the notations (3.3) and (3.6) for $\mathbb{F}_m$ and $Y_j^{(1)}$.

LEMMA 4.6. *Assume that μ has a finite moment of order $q = 3$. Then for $f(x) \in \{\cos x, \sin x\}$, we have*

$$\Big\| \mathbb{E}_{\mathbb{F}_m} \Big[f\Big(t \frac{Y_2^{(1)}}{\sqrt{2m}} \Big) \Big] - \mathbb{E}\big[f(tsN/\sqrt{2}) \big] \Big\|_1 \ll \frac{t^2}{\sqrt{m}} + \frac{|t|}{m^{3/2}} .$$

In addition

$$\Big\| \mathbb{E}_{\mathbb{F}_m} \Big[f\Big(t \frac{Y_1^{(1)}}{\sqrt{2m}} \Big) \Big] - \mathbb{E}\big[f(tsN/\sqrt{2}) \big] \Big\|_1 \ll \frac{t^2}{\sqrt{m}} .$$

Proof of Lemma 4.6. Since the derivative of $x \mapsto f(tx)$ is t^2-Lipschitz, making use of a Taylor expansion as done in the proof of Item (2) of [7, Lemma 5.2], we have

$$\Big\| \mathbb{E}_{\mathbb{F}_m}\Big[f\Big(t\frac{Y_2^{(1)}}{\sqrt{2m}}\Big)\Big] - \mathbb{E}\big[f(tsN/\sqrt{2})\big]\Big\|_1 \leq \Big\| \mathbb{E}_{\mathbb{F}_m}\Big[f\Big(t\frac{U_2}{\sqrt{2m}}\Big)\Big] - \mathbb{E}\big[f(tsN/\sqrt{2})\big]\Big\|_1 + \frac{t^2}{2m}\big(\|R_2\|_2\|U_2\|_2 + \|R_2\|_2^2\big)\,. \tag{4.9}$$

Now note that $U_2 = \sum_{k=2m+1}^{3m} \tilde{X}_{k,m}$ where

$$\tilde{X}_{k,m} = X_{k,m} - \mathbb{E}_{\mathbb{F}_m}(X_{k,m})\,.$$

In the above formula, recall that $X_{k,m} = \mathbb{E}(X_k|\mathcal{E}_{k-m+1}^k) := f_m(\varepsilon_{k-m+1}, \ldots, \varepsilon_k)$. Let $(\varepsilon_k^*)_k$ be an independent copy of $(\varepsilon_k)_k$. Define

$$\tilde{X}_{k,m}^* = f_m(\varepsilon_{k-m+1}^*, \ldots, \varepsilon_{2m}^*, \varepsilon_{2m+1}, \ldots, \varepsilon_k) \text{ and } U_2^* = \sum_{k=2m+1}^{3m} \tilde{X}_{k,m}^*\,. \tag{4.10}$$

Clearly U_2^* is independent of $\mathbb{F}_m$. Using again the fact that the derivative of $x \mapsto f(tx)$ is t^2-Lipschitz, we get

$$\Big\| \mathbb{E}_{\mathbb{F}_m}\Big[f\Big(t\frac{U_2}{\sqrt{2m}}\Big)\Big] - \mathbb{E}\big[f(tsN/\sqrt{2})\big]\Big\|_1 \ll \Big| \mathbb{E}\Big[f\Big(t\frac{U_2^*}{\sqrt{2m}}\Big)\Big] - \mathbb{E}\big[f(tsN/\sqrt{2})\big]\Big| + \frac{t^2}{2m}\big(\|U_2 - U_2^*\|_2\|U_2^*\|_2 + \|U_2 - U_2^*\|_2^2\big)\,. \tag{4.11}$$

But, by stationarity, $\|R_2\|_2 = \|R_1\|_2$, and by Lemma 4.3, since μ has a moment of order $q = 3$, we have $\|R_1\|_2 \ll 1$. Moreover, by using Lemma 4.5 and the fact that $\tilde{X}_{k,m}^*$ is distributed as $X_{k,m}$, we get that $\|U_2\|_2 + \|U_2^*\|_2 \ll \sqrt{m}$. On another hand, setting $\mathcal{G}_{k,m} = \sigma(\varepsilon_{k-m+1}^*, \ldots, \varepsilon_{2m}^*, \varepsilon_{k-m+1}, \ldots, \varepsilon_{2m}, \varepsilon_{2m+1}, \ldots, \varepsilon_k)$, we have

$$\|U_2 - U_2^*\|_2^2 \leq \sum_{k=2m+1}^{3m} \|\tilde{X}_{k,m} - \tilde{X}_{k,m}^*\|_2^2 + 2\sum_{k=2m+1}^{3m}\sum_{\ell=k+1}^{3m} \|(\tilde{X}_{k,m} - \tilde{X}_{k,m}^*)\mathbb{E}(\tilde{X}_{\ell,m} - \tilde{X}_{\ell,m}^*|\mathcal{G}_{k,m})\|_1\,.$$

Now, for $p \geq 1$, $\|\tilde{X}_{k,m} - \tilde{X}_{k,m}^*\|_p^p \leq \delta_{p,\infty}^p(k-2m)$ and, for $\ell > k$,

$$\|\mathbb{E}(\tilde{X}_{k,m} - \tilde{X}_{k,m}^*|\mathcal{G}_{k,m})\|_\infty \leq \delta_{1,\infty}(\ell - k)\,.$$

Since μ has finite moment of order $q = 3$, by (3.12), we obtain

$$\|U_2 - U_2^*\|_2^2 \ll \sum_{k=1}^{m} \delta_{2,\infty}^2(k) + \Big(\sum_{k=1}^{m} \delta_{1,\infty}(k)\Big)^2 \ll 1\,.$$

So, the inequalities (4.9) and (4.11) together with the above considerations, lead to

$$\Big\| \mathbb{E}_{\mathbb{F}_m}\Big[f\Big(t\frac{Y_2^{(1)}}{\sqrt{2m}}\Big)\Big] - \mathbb{E}\big[f(tsN/\sqrt{2})\big]\Big\|_1 \ll \Big| \mathbb{E}\Big[f\Big(t\frac{U_2^*}{\sqrt{2m}}\Big)\Big] - \mathbb{E}\big[f(tsN/\sqrt{2})\big]\Big| + \frac{t^2}{\sqrt{m}}\,. \tag{4.12}$$

Next, note that $U_2^* =^{\mathcal{D}} \sum_{k=1}^m X_{k+m,m}$ and $S_m =^{\mathcal{D}} S_{2m} - S_m$. So, taking into account that $x \mapsto f(tx)$ is t-Lipschitz, it follows that

$$\Big|\mathbb{E}\Big[f\Big(t\frac{U_2^*}{\sqrt{2m}}\Big)\Big] - \mathbb{E}\Big[f\Big(t\frac{S_m}{\sqrt{2m}}\Big)\Big]\Big| \le \frac{|t|}{\sqrt{2m}}\Big\|\sum_{k=1}^m (X_{k+m,m} - X_{k+m})\Big\|_1 .$$

But, by stationarity, [5, Lemma 24] and (3.12), we have

$$\Big\|\sum_{k=1}^m (X_{k+m,m} - X_{k+m})\Big\|_1 \le m\delta_{1,\infty}(m) \ll 1/m ,$$

implying that

$$\Big|\mathbb{E}\Big[f\Big(t\frac{U_2^*}{\sqrt{2m}}\Big)\Big] - \mathbb{E}\Big[f\Big(t\frac{S_m}{\sqrt{2m}}\Big)\Big]\Big| \ll \frac{|t|}{m^{3/2}} . \tag{4.13}$$

Hence starting from (4.12) and taking into account (4.13), we derive that

$$\begin{aligned}(4.14)\quad &\Big\|\mathbb{E}_{\mathbb{F}_m}\Big[f\Big(t\frac{Y_2^{(1)}}{\sqrt{2m}}\Big)\Big] - \mathbb{E}\big[f(tsN/\sqrt{2})\big]\Big\|_1 \\ &\qquad \ll \Big|\mathbb{E}\Big[f\Big(t\frac{S_m}{\sqrt{2m}}\Big)\Big] - \mathbb{E}\big[f(tsN/\sqrt{2})\big]\Big| + \frac{t^2}{\sqrt{m}} + \frac{|t|}{m^{3/2}} .\end{aligned}$$

Next note that $x \mapsto f(tx)$ is such that its first derivative is t^2-Lipshitz. Hence, by the definition of the Zolotarev distance of order 2 (see for instance the introduction of [7] for the definition of those distances),

$$\Big|\mathbb{E}\Big[f\Big(t\frac{S_m}{\sqrt{2m}}\Big)\Big] - \mathbb{E}\big[f(tsN/\sqrt{2})\big]\Big| \le t^2\zeta_2\big(P_{S_m/\sqrt{2m}}, G_{s^2/2}\big) .$$

Next we apply [7, Theorem 3.2] and derive that (since μ has a finite moment of order $q=3$),

$$\zeta_2\big(P_{S_m/\sqrt{2m}}, G_{s^2/2}\big) \ll m^{-1/2} .$$

Note that the fact that the conditions (3.1), (3.4) and (3.5) required in [7, Theorem 3.2] hold when μ has a finite moment of order $q=3$ has been established in the proof of [4, Theorem 2]. Hence

$$\Big|\mathbb{E}\Big[f\Big(t\frac{S_m}{\sqrt{2m}}\Big)\Big] - \mathbb{E}\big[f(tsN/\sqrt{2})\big]\Big| \ll \frac{t^2}{\sqrt{m}} . \tag{4.15}$$

Starting from (4.14) and considering (4.15), the first part of Lemma 4.6 follows. Now to prove the second part, we note that

$$\begin{aligned}&\Big\|\mathbb{E}_{\mathbb{F}_m}\Big[f\Big(t\frac{Y_1^{(1)}}{\sqrt{2m}}\Big)\Big] - \mathbb{E}\big[f(tsN/\sqrt{2})\big]\Big\|_1 \\ &\qquad \le \Big\|\mathbb{E}\Big[f\Big(t\frac{S_m}{\sqrt{2m}}\Big)\Big] - \mathbb{E}\big[f(tsN/\sqrt{2})\big]\Big\|_1 + \frac{t^2}{2m}\big(\|R_1\|_2\|S_m\|_2 + \|R_1\|_2^2\big) ,\end{aligned}$$

where we used the fact that S_m is independent of $\mathbb{F}_m$. Hence the second part of Lemma 4.6 follows by using (4.15) and by taking into account Lemma 4.3 and the fact that, by Lemma 4.5, $\|S_m\|_2 \ll \sqrt{m}$. □

LEMMA 4.7. *Let $p \in [2,3]$. Assume that μ has a finite moment of order $q = p+1$. Then $\|U_2 - U_2^*\|_p \ll 1$, where U_2 is defined by (3.4) and U_2^* is defined by (4.10).*

Proof of Lemma 4.7. Let $Z_{k,m} := X_{k,m} - \tilde{X}^*_{k,m}$ where $\tilde{X}^*_{k,m}$ is defined by (4.10). Using once again [6, Proposition 3.1] with $N = 1$, we get

$$\|U_2 - U_2^*\|_p \le \sum_{k=2m+1}^{3m} \|\mathbb{E}(X_{k,m}|\mathbb{F}_m)\|_p + \Big(2(p-1)\sum_{i=2m+1}^{3m} \gamma^*_{i,m}\Big)^{1/2} + \Big(\sum_{k=2m+1}^{3m} \|Z_{k,m}\|_p^p + p(p-1)\sum_{i=2m+2}^{3m} \alpha^*_{i,m}\Big)^{1/p},$$

where, setting $\mathcal{F}^Z_j = \sigma(\varepsilon_{m+2}, \dots, \varepsilon_j, \varepsilon^*_{m+2}, \dots, \varepsilon^*_{2m})$, we have

$$\gamma^*_{i,m} = \frac{1}{2}\|Z_{i,m}\|_2^2 + \sum_{j=2m+1}^{i-1} \|Z_{j,m}\mathbb{E}(Z_{i,m}|\mathcal{F}^Z_j)\|_{p/2,\nu}$$

and

$$\alpha^*_{i,m} = \frac{1}{2}\sum_{j=2m+1}^{i-1} \||Z_{j,m}|^{p-2}\mathbb{E}(Z^2_{i,m} - \mathbb{E}(Z^2_{i,m})|\mathcal{F}^Z_j)\|_1 .$$

But, for any integer k in $[2m+1, 3m]$,

$$\|\mathbb{E}(X_{k,m}|\mathbb{F}_m)\|_p = \|\mathbb{E}(X_{k,m}|\mathcal{F}_{2m})\|_p \le \|\mathbb{E}(X_{k,m}|\mathcal{F}_{2m})\|_\infty \le \delta_{1,\infty}(k-2m)\,,$$

and, for $2m+1 \le j \le i-1$,

$$\|\mathbb{E}(Z_{i,m}|\mathcal{F}^Z_j)\|_\infty \le 2\|\mathbb{E}(X_{i,m}|\mathcal{F}_j)\|_\infty \le \delta_{1,\infty}(i-j)\,.$$

In addition, we infer that, for $2m+1 \le j \le i-1$,

$$\|\mathbb{E}(Z^2_{i,m} - \mathbb{E}(Z^2_{i,m})|\mathcal{F}^Z_j)\|_\infty \le 4 \sup_{\substack{\bar{x}_1,\bar{x}_2 \in X \\ \bar{y}_1,\bar{y}_2 \in X}} \mathbb{E}\big|X_{i-j,\bar{x}_1}X_{i-j,\bar{x}_2} - X_{i-j,\bar{y}_1}X_{i-j,\bar{y}_2}\big| := 4\eta(i-j)\,.$$

On another hand, for any $r \ge 1$ and any integer $k \in [2m+1, 3m]$,

$$\begin{aligned}\mathbb{E}\big|Z_{k,m}\big|^r &= \mathbb{E}\Big|f_m(\varepsilon_{k-m+1}, \dots, \varepsilon_m, \varepsilon_{2m+1}, \dots, \varepsilon_k) - f_m(\varepsilon^*_{k-m+1}, \dots, \varepsilon^*_{2m}, \varepsilon_{2m+1}, \dots, \varepsilon_k)\Big|^r \\ &\le \int\!\!\int \mathbb{E}\Big|f_m(u_{k-m+1}, \dots, u_{2m}, \varepsilon_{2m+1}, \dots, \varepsilon_k) \\ &\qquad - f_m(v_{k-m+1}, \dots, v_{2m}, \varepsilon_{2m+1}, \dots, \varepsilon_k)\Big|^r \prod_{i=k-m+1}^{2m} d\mu(v_i) \prod_{i=k-m+1}^{2m} d\mu(u_i) \\ &\le \sup_{\bar{x},\bar{y}\in X} \mathbb{E}|X_{k-2m,\bar{x}} - X_{k-2m,\bar{y}}|^r = \delta^r_{r,\infty}\,(k-2m)\,. \end{aligned} \tag{4.16}$$

So, taking into account the above computations, we infer that

$$\|U_2 - U_2^*\|_p \ll \sum_{k=1}^m \delta_{1,\infty}(k) + \Big(\sum_{i=1}^m \delta^2_{2,\infty}(i) + \sum_{j=1}^m \delta_{p/2,\infty}(j)\sum_{i=1}^m \delta_{1,\infty}(i)\Big)^{1/2} + \Big(\sum_{k=1}^m \delta^p_{p,\infty}(k) + \sum_{j=1}^m \delta^{p-2}_{p-2,\infty}(j)\sum_{i=1}^m \eta(i)\Big)^{1/p},$$

The lemma follows by taking into account (3.12), (4.8) and the fact that μ has a moment of order $q = p+1$. □

For the lemmas below, we recall the definitions (3.4), (3.5), (3.6) and (4.10) for U_2, R_2, $Y_2^{(1)}$ and U_2^*.

LEMMA 4.8. *Let $r \in]2,3]$. Assume that μ has a finite moment of order $r+1$. Let $\alpha_m = \sqrt{\frac{\mathbb{E}_{\mathbb{F}_m}((U_2+R_2)^2)}{\mathbb{E}_{\mathbb{F}_m}((U_2^*)^2)}}$. Then for $f(x) \in \{\cos x, \sin x\}$, we have*

$$\Big\| \mathbb{E}_{\mathbb{F}_m}\Big[f\Big(t\frac{Y_2^{(1)}}{\sqrt{2m}}\Big)\Big] - \mathbb{E}_{\mathbb{F}_m}\Big[f\Big(t\alpha_m \frac{U_2^*}{\sqrt{2m}}\Big)\Big]\Big\|_1 \ll |t|^r m^{-1/2} .$$

Proof of Lemma 4.8. Note that $h = f/2^{3-r}$ is such that $|h''(x) - h''(y)| \le |x-y|^{r-2}$. Using the arguments developed in the proof of [7, Lemma 5.2, Item 3] and setting $V = U_2 + R_2 - U_2^*$ and $\tilde V = V + (1-\alpha_m)U_2^*$, we get

$$\begin{aligned}(4.17)\quad 2^{r-3}(r-1)\times(2m)^{r/2}\Big|\mathbb{E}_{\mathbb{F}_m}\Big[f\Big(t\frac{Y_2^{(1)}}{\sqrt{2m}}\Big)\Big] - \mathbb{E}_{\mathbb{F}_m}\Big[f\Big(t\alpha_m \frac{U_2^*}{\sqrt{2m}}\Big)\Big]\Big| \\ \le |t|^r\Big\{ \alpha_m^{r-1}\big(\mathbb{E}_{\mathbb{F}_m}(|\tilde V|^r)\big)^{1/r}\big(\mathbb{E}(|U_2^*|^r)\big)^{(r-1)/r} \\ + \alpha_m^{r-2}\big(\mathbb{E}_{\mathbb{F}_m}(|\tilde V|^r)\big)^{2/r}\big(\mathbb{E}(|U_2^*|^r)\big)^{(r-2)/r} + \mathbb{E}_{\mathbb{F}_m}(|\tilde V|^r)\Big\} .\end{aligned}$$

Next, note that, by Hölder's inequality,

$$\begin{aligned}\mathbb{E}\big(\alpha_m^{r-1}\big(\mathbb{E}_{\mathbb{F}_m}(|\tilde V|^r)\big)^{1/r}\big) &\le \mathbb{E}\big(\alpha_m^{r-1}\big(\mathbb{E}_{\mathbb{F}_m}(|V|^r)\big)^{1/r}\big) + \mathbb{E}\big(\alpha_m^{r-1}\times|1-\alpha_m|\big)\|U_2^*\|_r \\ &\le \|\alpha_m\|_r^{r-1}\|V\|_r + \|\alpha_m\|_r^{r-1}\|1-\alpha_m\|_r\|U_2^*\|_r .\end{aligned}$$

Proceeding similarly for the two last terms in (4.17) and taking the expectation, we derive

$$\begin{aligned}2^{r-3}(r-1)\times(2m)^{r/2}\Big\|\mathbb{E}_{\mathbb{F}_m}\Big[f\Big(t\frac{Y_2^{(1)}}{\sqrt{2m}}\Big)\Big] - \mathbb{E}_{\mathbb{F}_m}\Big[f\Big(t\alpha_m \frac{U_2^*}{\sqrt{2m}}\Big)\Big]\Big\|_1 \\ \le |t|^r\|\alpha_m\|_r^{r-1}\|V\|_r\|U_2^*\|_r^{r-1} + |t|^r\|\alpha_m\|_r^{r-1}\|1-\alpha_m\|_r\|U_2^*\|_r^r \\ + 2|t|^r\|\alpha_m\|_r^{r-2}\|V\|_r^2\|U_2^*\|_r^{r-2} + 2|t|^r\|\alpha_m\|_r^{r-2}\|1-\alpha_m\|_r^2\|U_2^*\|_r^r \\ + 2^{r-1}|t|^r\|V\|_r^r + 2^{r-1}|t|^r\|1-\alpha_m\|_r^r\|U_2^*\|_r^r .\end{aligned}$$

According to Lemmas 4.3 and 4.7, since μ has a moment of order $r+1$, $\|V\|_r \ll 1$. Moreover $\|U_2^*\|_r = \|U_2\|_r \le \sqrt{m}$. On another hand,

$$\begin{aligned}\|U_2^*\|_2 \times \|1-\alpha_m\|_r &= \Big\| \sqrt{\mathbb{E}_{\mathbb{F}_m}((U_2+R_2)^2)} - \sqrt{\mathbb{E}_{\mathbb{F}_m}((U_2^*)^2)}\Big\|_r \\ &\le \Big\| \sqrt{\mathbb{E}_{\mathbb{F}_m}((U_2+R_2-U_2^*)^2)}\Big\|_r \le \|V\|_r \ll 1 .\end{aligned}$$

Since $\lim_{m\to\infty} m^{-1}\|U_2^*\|_2^2 = s^2 > 0$, it follows that for m large enough $\|1-\alpha_m\|_r \ll m^{-1/2}$. The lemma follows from all the above considerations. □

LEMMA 4.9. *Let $r \in]2,3]$. Assume that μ has a finite moment of order $r+1$. Recall the notation $\alpha_m = \sqrt{\frac{\mathbb{E}_{\mathbb{F}_m}((U_2+R_2)^2)}{\mathbb{E}_{\mathbb{F}_m}((U_2^*)^2)}}$. Then for $f(x) \in \{\cos x, \sin x\}$, we have*

$$\Big\| \mathbb{E}_{\mathbb{F}_m}\Big[f\Big(t\alpha_m \frac{U_2^*}{\sqrt{2m}}\Big)\Big] - \mathbb{E}_{\mathbb{F}_m}\Big[f\Big(t\alpha_m \frac{s_m N}{\sqrt{2}}\Big)\Big]\Big\|_1 \ll |t|^r m^{-1/2} + |t| m^{-5/2} ,$$

where $s_m^2 = \mathbb{E}(S_m^2)/m$ and N is a standard Gaussian random variable independent of $\mathbb{F}_m$.

Proof of Lemma 4.9. Let W_0^* be distributed as W_0 and independent of W_0. Let $(\varepsilon_k^*)_{k\geq 1}$ be an independent copy of $(\varepsilon_k)_{k\geq 1}$, independent of (W_0^*, W_0). Define $S_m^* = \sum_{k=m+1}^{2m} X_k^*$ where $X_k^* = \sigma(\varepsilon_k^*, W_{k-1}^*) - \lambda_\mu$. Note that S_m^* is independent of $\mathbb{F}_m$ and has the same law as S_m. In addition

$$\Big\|\mathbb{E}_{\mathbb{F}_m}\Big[f\Big(t\alpha_m \frac{S_m^*}{\sqrt{2m}}\Big)\Big] - \mathbb{E}_{\mathbb{F}_m}\Big[f\Big(t\alpha_m \frac{U_2^*}{\sqrt{2m}}\Big)\Big]\Big\|_1 \\ \ll \frac{|t|}{\sqrt{2m}}\mathbb{E}|\alpha_m| \times \sum_{k=m+1}^{2m} \|X_{k,m} - X_k\|_1 \ll \frac{|t|}{\sqrt{m}} \times m\delta_{1,\infty}(m) \ll |t| m^{-5/2}\,. \tag{4.18}$$

On another hand, let $h = f/2^{3-r}$ and note that $|h''(x) - h''(y)| \leq |x-y|^{r-2}$. Hence, by the definition of the Zolotarev distance of order r,

$$\Big\|\mathbb{E}_{\mathbb{F}_m}\Big[f\Big(t\alpha_m \frac{S_m^*}{\sqrt{2m}}\Big)\Big] - \mathbb{E}_{\mathbb{F}_m}\Big[f\Big(t\alpha_m \frac{s_m N}{\sqrt{2}}\Big)\Big]\Big\|_1 \leq 2^{3-r}|t|^r \times \|\alpha_m\|_r^r \zeta_r\big(P_{S_m/\sqrt{2m}}, G_{s_m^2/2}\big)\,.$$

Next we apply [7, Theorem 3.2, Item 3.] and derive that since μ has a moment of order at least 3,

$$\zeta_r\big(P_{S_m/\sqrt{2m}}, G_{s_m^2/2}\big) \ll m^{-1/2}\,.$$

As we mentioned before, the fact that the conditions (3.1), (3.4) and (3.5) required in [7, Theorem 3.2] hold when μ has a moment of order at least 3 has been proved in the proof of [4, Theorem 2]. Hence, since we have previously proved that $\|\alpha_m\|_r \ll 1$,

$$\Big\|\mathbb{E}_{\mathbb{F}_m}\Big[f\Big(t\alpha_m \frac{S_m^*}{\sqrt{2m}}\Big)\Big] - \mathbb{E}_{\mathbb{F}_m}\Big[f\Big(t\alpha_m \frac{s_m N}{\sqrt{2}}\Big)\Big]\Big\|_1 \ll \frac{|t|^r}{\sqrt{m}}\,. \tag{4.19}$$

Considering the upper bounds (4.18) and (4.19), the lemma follows. □

LEMMA 4.10. *Let $r \in]2,3]$. Assume that μ has a finite moment of order $q = r+1$. Recall the notations $\alpha_m = \sqrt{\frac{\mathbb{E}_{\mathbb{F}_m}((U_2+R_2)^2)}{\mathbb{E}_{\mathbb{F}_m}((U_2^*)^2)}}$ and $s_m^2 = \mathbb{E}(S_m^2)/m$. . Then, for $f(x) \in \{\cos x, \sin x\}$,*

$$\Big\|\mathbb{E}_{\mathbb{F}_m}\Big[f\Big(t\alpha_m \frac{s_m N}{\sqrt{2}}\Big)\Big] - \mathbb{E}_{\mathbb{F}_m}\Big[f\Big(t \frac{sN}{\sqrt{2}}\Big)\Big]\Big\|_1 \ll \frac{|t|}{m^{1/2+\eta}}\,.$$

where $\eta = \min(\frac{3}{14}, \frac{r-2}{2})$ and N is a standard Gaussian random variable independent of $\mathbb{F}_m$.

Proof of Lemma 4.10. We have

$$\Big\|\mathbb{E}_{\mathbb{F}_m}\Big[f\Big(t\alpha_m \frac{s_m N}{\sqrt{2}}\Big)\Big] - \mathbb{E}_{\mathbb{F}_m}\Big[f\Big(t \frac{sN}{\sqrt{2}}\Big)\Big]\Big\|_1 \\ \leq |t|\mathbb{E}|N|\big(\|\alpha_m\|_1 |s - s_m| + s \times \|1-\alpha_m\|_1\big)\,. \tag{4.20}$$

But, since $\lim_{m\to\infty} m^{-1}\|U_2^*\|_2^2 = s^2 > 0$,

$$\|1-\alpha_m^2\|_1 \sim \frac{1}{s^2 m}\big\|\mathbb{E}_{\mathbb{F}_m}((U_2+R_2)^2) - \mathbb{E}_{\mathbb{F}_m}((U_2^*)^2)\big\|_1\,.$$

On another hand

$$\big\|\mathbb{E}_{\mathbb{F}_m}((U_2+R_2)^2) - \mathbb{E}_{\mathbb{F}_m}((U_2^*)^2)\big\|_1 \leq \big\|\mathbb{E}_{\mathbb{F}_m}(U_2^2) - \mathbb{E}(U_2^2)\big\|_1 + \|R_2\|_2^2 + 2\|\mathbb{E}_{\mathbb{F}_m}(U_2 R_2)\|_1\,.$$

But, by stationarity,

$$\big\|\mathbb{E}_{\mathbb{F}_m}(U_2^2) - \mathbb{E}(U_2^2)\big\|_1 = \Big\|\mathbb{E}_m\Big(\sum_{k=m+1}^{2m} \tilde{X}_{k,m}\Big)^2 - \mathbb{E}\Big(\sum_{k=m+1}^{2m} \tilde{X}_{k,m}\Big)^2\Big\|_1\,.$$

Hence, by Lemma 4.2, since $q=r+1$,

$$\big\|\mathbb{E}_{\mathbb{F}_m}(U_2^2)-\mathbb{E}(U_2^2)\big\|_1 \ll m^{2/7}\mathbf{1}_{r\neq 3}+\mathbf{1}_{r=3}\,.$$

By stationarity and Lemma 4.3, we also have $\|R_2\|_2=\|R_1\|_2\ll 1$. Therefore

$$\big\|\mathbb{E}_{\mathbb{F}_m}((U_2+R_2)^2)-\mathbb{E}_{\mathbb{F}_m}((U_2^*)^2)\big\|_1 \ll m^{2/7}\mathbf{1}_{r\neq 3}+\mathbf{1}_{r=3}+\|\mathbb{E}_{\mathbb{F}_m}(U_2R_2)\|_1\,. \tag{4.21}$$

Next, note that

$$\|\mathbb{E}_{\mathbb{F}_m}(U_2R_2)\|_1=\Big\|\mathbb{E}_{\mathbb{F}_m}\Big(R_2\sum_{k=2m+1}^{3m}X_{k,m}\Big)\Big\|_1\,.$$

Let $h(m)$ be a positive integer less than m. Using stationarity, Lemma 4.3 and similar arguments as those developed in the proof of Lemma 4.5, we first notice that

$$\Big\|\mathbb{E}_{\mathbb{F}_m}\Big(R_2\sum_{k=3m-h(m)+1}^{3m}X_{k,m}\Big)\Big\|_1\le\|R_2\|_2\Big\|\sum_{k=3m-h(m)+1}^{3m}X_{k,m}\Big\|_2\ll\sqrt{h(m)}\,.$$

We handle now the term $\|\mathbb{E}_{\mathbb{F}_m}\big(R_2\sum_{k=2m+1}^{3m-h(m)}X_{k,m}\big)\|_1$. Let $(\varepsilon_k^*)_{k\geq 1}$ be an independent copy of $(\varepsilon_k)_{k\geq 1}$. For $2m+1\le k\le 3m$, define

$$X_{k,m}^*=f_m(\varepsilon_{k-m+1}^*,\dots,\varepsilon_{2m}^*,\varepsilon_{2m+1},\dots\varepsilon_k)\,,$$

where we recall that f_m is defined as follows: $X_{k,m}=\mathbb{E}(X_k|\mathcal{E}_{k-m+1}^k):=f_m(\varepsilon_{k-m+1},\dots,\varepsilon_k)$. Using (3.13), note that

$$\sum_{k=2m+1}^{3m-h(m)}\|X_{k,m}-X_{k,m}^*\|_2\le\sum_{k=2m+1}^{3m}\delta_{r',\infty}(k-2m)\ll\sum_{k=1}^{m}k^{-(q/2-1)}\,.$$

Hence

$$\Big\|\sum_{k=2m+1}^{3m-h(m)}(X_{k,m}-X_{k,m}^*)\Big\|_2\ll m^{(3-r)/2}\mathbf{1}_{r<3}+\mathbf{1}_{r=3}\log(m)\,.$$

This estimate combined with $\|R_2\|_2\ll 1$ entails

$$\Big\|\mathbb{E}_{\mathbb{F}_m}\Big(R_2\sum_{k=2m+1}^{3m-h(m)}X_{k,m}\Big)\Big\|_1\ll m^{(3-r)/2}\mathbf{1}_{r<3}+\mathbf{1}_{r=3}\log(m)+\Big\|\mathbb{E}_{\mathbb{F}_m}\Big(R_2\sum_{k=2m+1}^{3m-h(m)}X_{k,m}^*\Big)\Big\|_1.$$

Since $(X_{k,m}^*)_{2m+1\le k\le 3m}$ is independent of $\mathbb{F}_m$, we have $\mathbb{E}(X_{k,m}^*|\mathbb{F}_m)=0$ for any $2m+1\le k\le 3m$. Hence

$$\Big\|\mathbb{E}_{\mathbb{F}_m}\Big(R_2\sum_{k=2m+1}^{3m-h(m)}X_{k,m}^*\Big)\Big\|_1=\Big\|\mathbb{E}_{\mathbb{F}_m}\Big(\sum_{k=2m+1}^{3m-h(m)}X_{k,m}^*\sum_{\ell=3m+1}^{4m}X_{\ell,m}\Big)\Big\|_1.$$

Next, note that if $\ell-m+1\geq k+1$, conditionally to $\mathbb{F}_m$, $X_{k,m}^*$ is independent of $X_{\ell,m}$, which implies that $\mathbb{E}_{\mathbb{F}_m}(X_{k,m}^*X_{\ell,m})=0$. Hence

$$\Big\|\mathbb{E}_{\mathbb{F}_m}\Big(\sum_{k=2m+1}^{3m-h(m)}X_{k,m}^*\sum_{\ell=3m+1}^{4m}X_{\ell,m}\Big)\Big\|_1=\Big\|\mathbb{E}_{\mathbb{F}_m}\Big(\sum_{k=2m+1}^{3m-h(m)}X_{k,m}^*\sum_{\ell=3m+1}^{4m-h(m)-1}X_{\ell,m}\Big)\Big\|_1.$$

Now, for any $3m+1\le\ell\le 4m-h(m)-1$, let

$$X_{\ell,m}^{(h(m),*)}=f_m(\varepsilon_{\ell-m+1}^*,\dots,\varepsilon_{3m-h(m)}^*,\varepsilon_{3m-h(m)+1},\dots\varepsilon_\ell)\,,$$

and note that $\mathbb{E}_{\mathbb{F}_m}(X^*_{k,m}X^{(h(m),*)}_{\ell,m}) = 0$ for any $k \leq 3m - h(m)$ and any $\ell \geq 3m+1$. So, overall, setting $q' = q/(q-1)$,

$$\Big\|\mathbb{E}_{\mathbb{F}_m}\Big(R_2\sum_{k=2m+1}^{3m-h(m)} X^*_{k,m}\Big)\Big\|_1 = \Big\|\mathbb{E}_{\mathbb{F}_m}\Big(\sum_{k=2m+1}^{3m-h(m)} X^*_{k,m}\sum_{\ell=3m+1}^{4m-h(m)-1}(X_{\ell,m} - X^{(h(m),*)}_{\ell,m})\Big)\Big\|_1$$
$$\leq \Big\|\sum_{k=2m+1}^{3m-h(m)} X^*_{k,m}\Big\|_q \sum_{\ell=3m+1}^{4m-h(m)-1}\|X_{\ell,m} - X^{(k,*)}_{\ell,m}\|_{q'}\,.$$

But $\|X_{\ell,m} - X^{(k,*)}_{\ell,m}\|_{q'} \leq \delta_{q',\infty}(\ell - 3m + h(m))$. Hence, taking into account (3.13) and Lemma 4.5, we get

$$\Big\|\mathbb{E}_{\mathbb{F}_m}\Big(R_2\sum_{k=2m+1}^{3m-h(m)} X^*_{k,m}\Big)\Big\|_1 \ll \sqrt{m}\sum_{\ell\geq h(m)}\frac{1}{\ell^{q-2}} \ll \sqrt{m}(h(m))^{2-r}\,.$$

Taking into account all the above considerations and selecting $h(m) = m^{1/(2r-3)}$, we derive

$$m\|1-\alpha_m\|_1 \ll (m^{(3-r)/2} + m^{2/7})\mathbf{1}_{r<3} + m^{1/(4r-6)} \ll m^{(3-r)/2} + m^{2/7}\,. \tag{4.22}$$

On another hand, since $s^2 > 0$, $|s - s_m| \leq s^{-1}|s^2 - s_m^2|$. Hence by using Remark 2.1 and the definition of s_m^2, we derive that

$$|s - s_m| \leq \frac{2}{sm}\sum_{k\geq 1}k|\mathrm{Cov}(X_0, X_k)|\,.$$

By the definition of $\delta_{1,\infty}$, $|\mathrm{Cov}(X_0,X_k)| \leq \|X_0\|_1\delta_{1,\infty}(k)$. So, using (3.12), it follows that

$$|s - s_m| \ll m^{-1}\,. \tag{4.23}$$

Starting from (4.20) and taking into account (4.22) and (4.23), the lemma follows. □

Combining Lemmas 4.8, 4.9 and 4.10, we derive

LEMMA 4.11. *Let $r \in]2,3]$. Assume that μ has a finite moment of order $q = r+1$. Then, for $f(x) \in \{\cos x, \sin x\}$,*

$$\Big\|\mathbb{E}_{\mathbb{F}_m}\Big[f\Big(t\frac{Y_2^{(1)}}{\sqrt{2m}}\Big)\Big] - \mathbb{E}\big[f(tsN/\sqrt{2})\big]\Big\|_1 \ll |t|^r m^{-1/2} + |t|m^{-(1/2+\eta)}\,,$$

where $\eta = \min(\frac{3}{14}, \frac{r-2}{2})$.

Let R_1 be defined by (3.5). Proceeding similarly as to derive the previous lemma, we get

LEMMA 4.12. *Let $r \in]2,3]$. Assume that μ has a finite moment of order $q = r+1$. Then for $f(x) \in \{\cos x, \sin x\}$,*

$$\Big\|\mathbb{E}_{\mathbb{F}_m}\Big[f\Big(t\frac{\sum_{k=1}^m X_k + R_1}{\sqrt{2m}}\Big)\Big] - \mathbb{E}\big[f(tsN/\sqrt{2})\big]\Big\|_1 \ll |t|^r m^{-1/2} + |t|m^{-(1/2+\eta)}\,,$$

where $\eta = \min(\frac{3}{14}, \frac{r-2}{2})$.

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