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Stocking Under Random Demand and Product Variety: Exact Models and Heuristics

Vashkar Ghosh
Department of Information Systems and Supply Chain Management, University of North Carolina, Greensboro, North Carolina 27402, USA, v_ghosh@uncg.edu

Anand Paul*
Department of Information Systems and Operations Management, Warrington College of Business, University of Florida, Gainesville, Florida 32608, USA, paulaa@ufl.edu

Lingjiong Zhu
Department of Mathematics, Florida State University, Tallahassee, Florida 32306, USA, zhu@math.fsu.edu

Efficient inventory management in the face of product variety is an important part of retail operations management. In this study, we analyze the optimal stocking policy for a retailer, in a setup with a single horizontally differentiated product with an arbitrary number of product variants, stochastic demand, and two-level consumer choice. The demands for individual product variants are negatively correlated conditional on the total demand. We assume that each customer will purchase one unit of a preferred product variant, if it is in stock, and will seek to buy a second choice product, if the former is not in stock. We formulate an exact model, with Poisson customer arrivals. In order to maintain tractability and characterize an optimal policy analytically, we develop a benchmark model which does not explicitly account for the stochastic nature of customer arrival times. In this model, which is a heuristic approximation of the exact model, we find simple conditions under which the objective of maximizing expected profit is jointly concave in the stocking levels of the product variants; under these conditions we prove that the optimal stocking levels are simply scaled versions of the optimal newsvendor quantities. We then analytically establish a connection between the exact and benchmark models. We develop a dynamic Monte Carlo simulation experiment to gain further insights on the impact of different performance measures on the effectiveness of the optimal policy in the benchmark model and its performance in reference to the exact optimal policy.

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*Corresponding author.

1. Introduction

Retailers often have to make critical inventory and assortment related decisions when they offer a wide selection of products, or different varieties of the same product. Customer service in retail operations in terms of product availability (probability of no stock-out) is a key measure of service level, and is an essential part of the retailer's competitive strategy. We study an optimal inventory stocking problem of several product variants within a product category. The distinct product variants are horizontally differentiated from each other. This is a common scenario in retail and grocery outlets, where retailers stock several different variants of the same product within a certain category. As a motivating example consider a retailer stocking T-shirts of a certain brand which has different colors (variants). The retailer manages the inventory of T-shirts and wishes to stock each color (variety) so as to maximize the expected profit. The total demand of the product, in this case T-shirts, in a single period is random. We assume that each customer will purchase one unit of a preferred product variant, if it is in stock, and will seek to buy a second choice product, if the former is not in stock. In the above setting with different variants, the demands for product variant i and j (different colored T-shirts) are negatively correlated, conditional on the sum of the demands of each variety being equal to the total demand D, which is a random variable with a known (but arbitrary) distribution. The retailer has to take into account the substitution effect in the inventory stocking decision, in addition to the price and cost structure of the product varieties.

Past research has underlined that inventory management with dynamic substitution poses a significant challenge as it is dependent on the stochastic nature of customer arrival times. We first develop an exact, dynamic model in which customers arrive stochastically according to a Poisson process and purchase their preferred product variant if it is in stock, or else may substitute and purchase their second choice product variant if available. We formulate the corresponding optimization problem as maximizing the solution of a multi-dimensional integral equation, for a single period,1 thus highlighting the complexity of the exact model. Our model also considers a disutility cost associated with any purchase of a second choice item by a customer.

We then propose a benchmark model, a heuristic approximation of the exact model which allows us to maintain tractability and characterize an optimal policy analytically. This model does not explicitly account for the stochastic nature of customer arrival times at the store to purchase items, and assumes that the first choice demands are satisfied ahead of the second choice demands. We establish a connection between the exact model and our benchmark model to highlight the nature of the approximation.

The benchmark model provides an effective tool for inventory management in the case of substitutable products within a category. We find simple conditions under which the objective of maximizing expected profit in the benchmark model is jointly concave in the stocking levels of the product variants; under this condition we prove that the optimal stocking levels in the benchmark model are simply the optimal newsvendor quantities multiplied by a scale factor. After exploring the theoretical properties of the benchmark model, we develop a dynamic simulation experiment to illustrate the impact of customer heterogeneity, costs and other performance measures on how well the optimal benchmark policy performs in reference to the exact optimal policy. We itemize our main contributions below.

- Our general benchmark model (fully stochastic model) makes no structural assumptions about the consumer choice process by which preferred and second choice products are selected; the first choice and second choice proportions are allowed to be random, without any distributional restrictions.
- We consider a special case of the general benchmark model in which the preferred choices lead to fixed (deterministic) proportions p_i of the total demand to be associated with variant *i*, while the second choice proportions are random; we call this the semi-stochastic model. In this model, we find a simple condition on prices and costs under which the optimal product stocking level of variety *i* in the benchmark model is equal to the optimal newsvendor quantity multiplied by p_i (the proportion of customers whose first choice is product variant *i*).
- The optimal stocking quantities are in general not the optimal newsvendor quantities; our benchmark model provides an efficient heuristic by identifying simple conditions under which the solution to the inventory problem with substitution is to simply stock the classic newsvendor quantities multiplied by a scale factor, which we refer to as the *scaled newsvendor quantities*. This is, we hope, is an interesting contribution; the practical implication of this finding is that purchasing managers do not need recourse to complex optimization software to solve the product stocking problem under product variety in many cases.
- In the fully stochastic model we prove that the optimal stocking policy differs from the scaled newsvendor solution, thus highlighting the impact of stochasticity of first and second choices on the optimal policy in the benchmark model.
- Finally, we develop a dynamic simulation to gain insight into the substitution effect and illustrate the impact of choice fractions, costs, and other performance measures on the effectiveness of the optimal benchmark policy and its performance in reference to the exact optimal policy. The simulation results show that the optimality gap associated with the scaled newsvendor solution is of the order of only 1% to 3% at reasonably high service levels, as desired in a retail setting.

The rest of the paper is organized as follows. In section 2, we review the relevant literature. In section 3, we describe our modeling framework, then present the exact and benchmark models. We analyze the benchmark model in section 4, and also establish a connection between the exact and benchmark models. In section 5 we develop a dynamic simulation experiment to gain further insights and understand the effectiveness of the benchmark policy under different scenarios of total demand, choice preferences, revenue, and cost parameters. Finally, in section 6, we make a few closing remarks summarizing our contributions and making some suggestions for further research in this area.

2. Literature Review

We briefly review the literature on multi-item inventory management for substitutable items. Specifically, we review works that study optimal inventory stocking policies in situations where a product consists of several varieties, consumers may prefer some varieties over others (e.g., each consumer may have a rank ordering of the varieties), and consumers may substitute a first choice option with a less preferred option in case the first choice product variant is not available. Inventory management problems with demand substitution have been extensively studied. Initial research in this area includes Mcgillivray and Silver (1978), Parlar and Goyal (1984), who study dynamic substitution model for two products. Pasternack and Drezner (1991) obtain the optimal stocking policy for two products with full substitution. Ernst and Kouvelis (1999) use a stylized model to study the problem for two products which are not direct substitutes, and can be sold independently or as a package. They use numerical analysis to obtain optimal stocking levels and provide insights into the effect of demand correlation. Other relevant papers under this set of problems with dynamic substitution include Parlar (1988) and Rajaram and Tang (2001) who obtain numerical results using an algorithm for multiple products.

The papers that are most closely connected with our work are Smith and Agrawal (2000), Netessine and Rudi (2003), and Nagarajan and Rajagopalan (2008); we describe their models and results in detail and explain precisely how our work extends their findings.

Smith and Agrawal (2000) study a single-period problem with substitution. The setting is retail, and the model focuses on a given item type and its close substitutes, comprising a fixed set E. The total number of customers demanding an item (of some type) is random, and each customer is assumed to demand exactly one unit of some item. If the demanded item (say of type $i \in E$) is not in stock, the customer will attempt to purchase a unit of item type $j \in E$. If item type j is unavailable, the customer will leave without purchasing an item. A fixed probability distribution governs the probability that a customer has second choice j, given that she has first choice i. The problem is to choose the inventory levels for each item type. It is assumed that the base-stock level for each item is set so as to satisfy a fixed, exogenously given service level. The model is dynamic, in that it tracks the arrival of customers according to a random demand process, and this makes the model formidable to solve analytically. Therefore, the authors focus on developing approximate solutions. The authors also formulate a joint assortment-and-stocking-level model, in which the subset of items from the set E as well as their inventory levels need to be determined jointly; they do this using first choice probabilities and a substitution matrix.

Netessine and Rudi (2003) study a similar model, under both centralized and decentralized optimization regimes. The demand vector for the item set follows a continuous multivariate joint distribution. A deterministic fraction α_{ij} of customers will attempt to purchase their second choice item j if their first choice item i is not in stock; if both item types are out of stock, the customer will not purchase any item. Item costs and prices are in general distinct. In the centralized model, the authors show that the objective function may not be concave. They also show that the newsvendor solution for each item does not in general optimally solve the multivariate optimization problem. The authors obtain a necessary optimality condition for both the centralized and decentralized (competitive) models.

Nagarajan and Rajagopalan (2008) derive the optimal inventory policy in both single-period and multi-period scenarios for substitutable products with negatively correlated demands. In their benchmark model, there are two products with deterministic total demand D. The substitution effect is captured by a parameter γ ($0 < \gamma < 1$); a fixed proportion γ of customers buy their second choice item if their first choice item is not in stock. Assuming symmetric costs and prices, they show that the expected profit function is jointly concave in the stocking levels of the two products, and find explicit formulas whereby the optimal stocking levels—which have a newsvendor like structure—can be found. Under asymmetric costs, the authors show that the objective function is no longer always jointly concave. A generalization of the model to the case of an arbitrary number of items is briefly discussed. In this case, a fraction γ_i will substitute their second choice item for their first choice item in case the latter is out of stock, and the second choice fraction associated with each item is assumed to be equal. Their analysis is mostly limited to two products, and the N-product case is quite restrictive. In the multi-period setting only specific cases are considered.

The second stream of literature related to the multi-item inventory management problem for substitutable items is assortment planning with substitution. Static substitution models have been extensively studied. Static substitution is one in which a consumer choice is only dependent on the assortment (set of alternatives) and is not dependent on product availability or on-hand stock. One of the earliest papers on assortment planning was by Pentico (1974), who considers probabilistic demand with downward substitution and obtains the optimal policy using a dynamic programming formulation. For a comprehensive survey of research on assortment planning, we refer readers to Kök et al. (2015). Another relevant paper on the static substitution model is van Ryzin and Mahajan (1999), who use a multinomial logit model with static substitution and assumes identical cost parameters. Most of the above papers assume that the consumer choice model is known. Cachon et al. (2005) develop consumer choice models and methods for finding assortments that optimally balance revenue expansion with operational costs and incorporates search cost. Gaur and Honhon (2006) consider a single-period problem using location choice models to represent consumer demand and obtain optimal assortment and inventory decisions. They use heuristics to solve the problem under dynamic substitution. Topaloglu (2013) studies a joint stocking and product offer problem via a multinomial logit choice model with static substitution. The problem is formulated as a nonlinear program, which is intractable owing to the large number of decision variables. They propose an alternate formulation based on the structure of the multinomial logit model where the objective function is separable, and solve it through a dynamic program.

Another stream of related literature is dynamic (or stockout based) substitution which is based on availability of products. Mahajan and van Ryzin (2001a, 2001b) study an inventory planning and assortment problem where customers choose dynamically based on current availability. The consumer choice is based on utility maximization. They use a sample path analysis to analyze structural properties of the expected profit function and propose a stochastic gradient algorithm. Kök and Fisher (2007) develop an algorithmic process to study assortment planning problem and present a procedure for estimating substitution parameters. They propose iterative optimization heuristics for solving the assortment planning problem. Honhon et al. (2010) determine the optimal inventory and assortment levels in a single-period setting with stockout-based substitution. The model considers stochastic demand with fixed proportion of customer types which can result from utility maximization. The paper develops an efficient dynamic programming algorithm, and establishes structural properties of the value function of the dynamic program to characterize multiple local maxima. The numerical tests show that the heuristics perform better than previous methods. Akçay et al. (2020) study a single-period inventory planning problem for a category of substitutable products. Their model accounts for the stochastic nature of customer arrivals, and thus considers dynamic substitution. They formulate a stochastic optimization model that minimizes the total stocking cost subject to service level requirements. Considering the challenges involved in solving the problem, they propose a novel optimization model which can accommodate several common stockout-based substitution schemes. There is some related work which considers inventory and pricing with substitution. Xu et al. (2016) consider a two product flexible substitution problem to explore the interaction between price discounts and substitution. The paper develops a stochastic dynamic formulation to show that the optimal policy has a threshold structure. Other relevant work which also considers adjusting price and inventory management in the presence of dynamic substitution includes Dong et al. (2009) and Hopp and Xu (2008).

The above papers provide several insights, and some consider implementing efficient algorithms to find the optimal stocking levels when simple formulas are not available. However our model and analysis which we discuss in the next section yield some novel insights compared with past work.

3. Single-Period Model with Two-Level Consumer Choice

We study the stocking policy of a retailer who has to decide on the stocking quantities of different varieties of a given product within a product category, in the face of random customer demand over a given planning period. The retailer managing the inventory has a single replenishment opportunity at the start of the period. The total demand of the product, summed over all the varieties, is a positive valued random variable D. There are K different varieties of the product, which are indexed as $i = 1, 2, \ldots, K$. Each customer will only buy one unit of a specific product variant, if it is in stock. Customers are choosy, and have a strong preference for the specific product variant (first choice) they prefer to buy. However, each customer may substitute and purchase a second choice item if their preferred first choice item is not in stock. The individual demands for each product variant i and j (different colored T-shirts) are negatively correlated conditional on the total demand, as the sum of the demands of each variety is equal the total demand D, which is a random variable with a known (but arbitrary) distribution. At the end of the period, the retail store calculates its profit using the cost structure stated below. We assume equal unit costs and prices for all product variants in our model. This is a reasonable assumption for horizontally differentiated product varieties and is consistent with past work (Gaur and Honhon 2006, Nagarajan and Rajagopalan 2008).

The unit cost of each product variety stocked is c, and the selling price is s. Any unsold or leftover inventory at the end of the selling period is sold to a discount store at c_0 per unit, where $c_0 < c$. So there is a loss of $c - c_0$ for each item that is leftover. On the other hand the store imputes an opportunity cost of $s - c$ for each item that was demanded but not in stock. Hence stocking too many items as well as stocking too few items is sub-optimal. We make the following standard assumption $s > c > c_0 > 0$. The retailer's objective is to choose stocking quantities for each product variant so as to maximize the expected profit. The retailer stocks x_i units of product variety i. All the customers are assumed to act independently of each other. If the preferred first choice is available (in stock), the customer purchases it otherwise the customer exercises his or her second choice and substitutes to the other variant if it is in stock. Thus lost sales can be either due to stockout of the preferred first choice and customers not finding their second choice option in stock, or customers are not willing to substitute and do not have a second choice option.

Since each customer who purchases his or her second choice variant experiences a disutility as the preferred first choice was not available, the firm attaches a cost $c_p > 0$ for every unit of product that was purchased and was the customer's second choice rather than first. The cost can be seen as a penalty cost or a measure of loss of goodwill as the first choice of the customer was not available. Every unit sold as second choice as a result of substitution generates a profit $s - c - c_p > 0$. The impact of this penalty cost on the overall profit formulation is significant. Past work such as Netessine and Rudi (2003) does not explicitly model this substitution cost. Although Nagarajan and Rajagopalan (2008) do not include this substitution cost in their formulations, they mention that such a cost can be incorporated and that it will correspondingly change their optimal solution. The cost of substitution plays a crucial role in the profit formulation as without this penalty cost, the profit earned from a unit stock is the same irrespective of whether that unit is sold as a first or second choice. The inclusion of the penalty cost requires us to take into account whether an available stock is used to satisfy a first choice or second choice demand. In practice demand for a product as a first or second choice is dependent on the stochastic arrival times of the customers. Thus in the presence of dynamic substitution, an available unit of stock can be used to satisfy a demand originating from second choice ahead of satisfying a first choice demand. Although a retailer cannot infer from the sale of a product whether the sale is first or second choice, this distinction is crucial when considering a substitution cost in the overall expected profit.

The optimal stocking policy under dynamic substitution depends on the stochastic nature of customer arrival times and the stocking level of different product variants at the time of customer arrival. As noted in past work (e.g., Gaur and Honhon 2006), analytically characterizing an optimal policy under dynamic substitution is intractable. In the next section 3.1, we formulate the exact model with Poisson customer arrivals. All proofs are included in the appendix and online electronic companion.

3.1. Exact Model with Poisson Arrivals

The exact model with Poisson customer arrivals can be formulated as follows. Let us first consider a single-period model with the length of the period being $T > 0$.

- Stocking levels $x_1, x_2, \ldots, x_K$ are implemented, where $x_i \in \mathbb{N} \cup \{0\}$ for $i = 1, \ldots, K$.
- Customers arrive according to a Poisson process with constant rate $\lambda > 0$. Customer #1 enters, and picks her preferred option. The firm registers a profit of $s - c$ for the sold item.
- Customers continue to arrive till the end of the period with time $T > 0$. For any customer, the probability that her preferred option is item i is p_i. If her preferred option is in stock, she purchases it, and the firm registers a profit of $s - c$. Otherwise, she substitutes to her second choice with probability p_{ij} with $\sum_{j \neq i} p_{ij} \leq 1$ if her second choice is available or she will not purchase anything. If the second choice is available, the firm registers a profit $s - ct - c_p$.
- Any unsold or leftover inventory at the end of the period is sold to a discount store at c_0 per unit resulting in a loss of $c - c_0$ for each unit that is leftover.

The exact model with the stochastic arrival of customers results in three levels of stochasticity with random demand, first choice and substitution second choice probabilities. Let $V(x_1, \ldots, x_K; t)$ denote the firm's total profit for the single period given that at time t, where $0 \leq t \leq T$, there is an initial stocking of $x_1, \ldots, x_K$ at time t, where $x_1, \ldots, x_k \in \mathbb{N} \cup \{0\}$ such that $V(x_1, \ldots, x_K; t) : (\mathbb{N} \cup \{0\})^K \times [0, T] \to \mathbb{R}$.

Proposition 1. *The optimal stocking levels are given by*

$$\left(x_1^*, \ldots, x_K^*\right) = \arg\max_{x_1, \ldots, x_K \geq 0} V(x_1, \ldots, x_K; 0), \qquad (1)$$

where $V(x_1, \ldots, x_K; t)$ *solves a* $(K+1)$*-dimensional integral equation:*

$$\begin{aligned} V(x_1, \ldots, x_K; t) = & \left(1 - \mathrm{e}^{-\lambda(T-t)}\right) \\ & \times \sum_{i:x_i>0} p_i(s-c) + \left(1 - \mathrm{e}^{-\lambda(T-t)}\right) \\ & \times \sum_{i:x_i=0} p_i \sum_{j:x_j>0} p_{ij}\left(s - c - c_p\right) - \mathrm{e}^{-\lambda(T-t)} \\ & \times \sum_{i=1}^{K} x_i(c - c_0) + \sum_{i:x_i>0} p_i \int_t^T \lambda \mathrm{e}^{-\lambda(u-t)} \\ & \times V(x_1, \ldots, x_{i-1},\ x_i - 1,\ x_{i+1}, \ldots, \\ & x_K; u)du \\ & + \sum_{i:x_i=0} p_i \sum_{j:x_j>0} p_{ij} \int_t^T \lambda \mathrm{e}^{-\lambda(u-t)} \end{aligned}$$

$$\times V(x_1, \ldots, x_{j-1}, x_j - 1, x_{j+1}, \ldots, x_K; u)du, \quad (2)$$

for any $x_1, \ldots, x_K \in \mathbb{N} \cup \{0\}$ *and any* $0 \le t \le T$.

We can see from Proposition 1 that $V(x_1, \ldots, x_K; t)$ solves a $(K+1)$-dimensional integral Equation (2), and we do not expect (2) to yield a simple closed form solution, so we have to rely on Monte Carlo simulations or numerically solving (2) to find the optimal stocking levels. An alternate version of the exact model considering discretized Poisson process which has been used in past literature such as Dong et al. (2009) can also be developed. The online electronic companion includes details of the discretized Poisson process model. The above analysis in the single-period model carries over to the multi-period and infinite horizon setting as well. The details can be found in the online electronic companion.

As evident from the above formulation, gaining further insights or solving the problem analytically with the exact model is challenging. To circumvent this problem, we now develop a benchmark model, to be able to characterize an optimal policy analytically.

3.2. Benchmark Model

Let D_i be the demand for product variant i from first choice, that is, the number of customers whose first preference is variety i such that $\sum_{i=1}^{K} D_i = D$. The total demand D is assumed to have a continuously differentiable density function $f(y) > 0$ and a distribution function $H(y)$, $y \ge 0$. As before the retailer stocks x_i units of product variety i, where $x_i \ge 0$ are real valued.

The consumer choice process: Our model is robust to the process by which customers exercise their choices. For example, every customer can be heterogeneous in choosing a specific product variant based on their idiosyncratic preferences. No matter how and why the customers make the choices they do, it will lead to certain proportion of customers preferring a product variant as their first choice or second choice (substitution); these choice proportions are random in our model. However, the stochastic nature of customer arrival is not considered in the benchmark model, which we discuss next.

Fully stochastic model: In our general fully stochastic model, we have

$$D_i = DP_i, \quad 1 \le i \le K,$$

where D_i is the demand of product variety i from first choice, the fractions $P_i \in (0, 1]$ are stochastic and $\sum_{i=1}^{K} P_i = 1$. The demand of product variety i from second choice is given by $\sum_{j \ne i} (D_j - x_j)^+ P_{ji}$. That is if the demand D_j for variety j exceeds the supply x_j of variety j, then a stochastic fraction $P_{ji} \in [0, 1]$ will choose variety i as the second choice. The second choice fractions satisfy $\sum_{j \ne i} P_{ij} \le 1$, as some customers might not have a second choice or not find the second choice option in stock. We assume that D, P_i's and P_{ji}'s are independent of each other. We note that the full stochastic model makes no assumptions about the choice process; every choice process will result in some version of the fully stochastic model. We specialize the general model as follows.

Semi-stochastic model: In this model we assume that

$$D_i = Dp_i, \quad 1 \le i \le K,$$

That is to say that there is a fraction p_i among all the customers who would choose variety i if it was in stock. Therefore, D_i can be interpreted as the demand of product variety i from first choice; the actual number of units of product variety i sold will consist of customers who prefer this variant as well as customers who would rather have bought some other variant. The demand of product variety i from second choice is given by $\sum_{j \ne i} (D_j - x_j)^+ p_{ji}$. That is if the demand D_j for variety j exceeds the supply x_j of variety j, then a fraction p_{ji} will choose variety i as the second choice. In the semi-stochastic model, the fractions p_{ji} may be random while the $p_i > 0$ are deterministic. However, we note that the faction of customers purchasing product variant i (which includes both first and second choice purchases) is random. In the next section we analyze the benchmark model to characterize an optimal policy analytically and then establish a connection between the benchmark and the exact model.

4. Analysis

We proceed to analyze the semi-stochastic model first, followed by the fully stochastic model. We shall highlight the impact of stochasticity of first and second choices on the optimal policy. *We emphasize that when we write 'optimal policy" in this section, we mean "optimal policy in the benchmark model" as opposed to the optimal policy in the exact model.*

4.1. Semi-Stochastic Model

In this sub-section we shall assume that p_{ji} is fixed. However, we shall see in the following section that all the results go through even when the p_{ji}'s are random (please see Theorem 4). Given the supplies, that is, the stocking decision of the retailer $x_1, \ldots, x_K$, the expected profit of the firm is given by:

$$
\begin{aligned}
F(x_1,\ldots,x_K) := & \sum_{i=1}^{K}\mathbb{E}\left[\left((s-c)D_i+(s-c-c_p)\sum_{j\neq i}(D_j-x_j)^+p_{ji}\right)1_{x_i>D_i+\sum_{j\neq i}(D_j-x_j)^+p_{ji}}\right]\\
& -\sum_{i=1}^{K}\mathbb{E}\left[(c-c_0)\left(x_i-D_i-\sum_{j\neq i}(D_j-x_j)^+p_{ji}\right)1_{x_i>D_i+\sum_{j\neq i}(D_j-x_j)^+p_{ji}}\right]\\
& +\sum_{i=1}^{K}\mathbb{E}\left[\big((s-c)D_i+(s-c-c_p)(x_i-D_i)\big)1_{D_i\leq x_i\leq D_i+\sum_{j\neq i}(D_j-x_j)^+p_{ji}}\right]\\
& +\sum_{i=1}^{K}\mathbb{E}[(s-c)x_i 1_{x_i<D_i}]. \qquad (3)
\end{aligned}
$$

We can interpret Equation (3) as follows. When the stock x_i is less than the first choice demand D_i, the entire stock x_i can be sold, and the profit is $(s-c)x_i$, which gives the fourth term in Equation (3). When the stock x_i is greater than the first choice demand D_i but less than the sum of the first choice demand D_i and the second choice demand for variety i, $\sum_{j\neq i}(D_j-x_j)^+p_{ji}$, the first choice demand D_i brings in a profit $(s-c)D_i$, and the second choice demand brings a profit $(s-c-c_p)(x_i-D_i)$, where x_i-D_i is all that is available for the second choice demand, which gives the third term in Equation (3). This is a best-case scenario, since the partitioning of the sold units represented by $D_i+\sum_{j\neq i}(D_j-x_j)^+p_{ji}$ into first choice and second choice sales depends on the stochastic nature of customer arrival times at the store to purchase items. Finally, when the stock x_i is greater than the sum of the first choice and the second choice demands, the first choice demand brings in a profit $(s-c)D_i$ and the second choice demand brings a profit $(s-c-c_p)\sum_{j\neq i}(D_j-x_j)^+p_{ji}$, which gives the first term in Equation (3). There is still left over inventory $x_i-D_i-\sum_{j\neq i}(D_j-x_j)^+p_{ji}$ which is sold at a discount c_0, resulting in a loss $(c-c_0)(x_i-D_i-\sum_{j\neq i}(D_j-x_j)^+p_{ji})$, which gives the second term in Equation (3).

Note: We do not explicitly model the interaction between the order of customer arrivals and the stocking policy in the benchmark model in order to maintain tractability and characterize an optimal policy analytically. For tractability, we assume that the first choice demands are satisfied before the second choice demands. This is a strong assumption, so in section 4.3, we establish a connection with the exact model.

The retailer's objective is to decide how much to stock for each variety so as to maximize the expected profit; thus the optimization problem is:

$$\max_{x_1,\ldots,x_K\geq 0} F(x_1,\ldots,x_K).$$

Theorem 1. *The optimal* $(x_1^*,\ldots,x_K^*)$ *satisfies the equations:*

$$
(s-c_0)\int_0^{G_i(x_1^*,\ldots,x_K^*)} f(y)dy+(s-c_p-c_0)\sum_{\ell\neq i}\int_{\frac{x_i^*}{p_i}}^{G_\ell(x_1^*,\ldots,x_K^*)} p_{i\ell}f(y)dy
+c_p\int_{G_i(x_1^*,\ldots,x_K^*)}^{\frac{x_i^*}{p_i}} f(y)dy=s-c,\quad 1\leq i\leq K, \qquad (4)
$$

where G_i is defined via the equations:

$$x_i=G_ip_i+\sum_{j\neq i}(G_ip_j-x_j)^+p_{ji},\quad 1\leq i\leq K. \qquad (5)$$

Remark 1. For the special case when there is no second choice, that is, $p_{ji}\equiv 0$, then $G_i=x_i/p_i$ and the optimal x_i^* satisfies

$$(s-c_0)\int_0^{\frac{x_i^*}{p_i}} f(y)dy=s-c,$$

which is precisely the newsvendor solution scaled up by p_i. We shall refer to this solution as the **scaled newsvendor solution** and the corresponding order quantities as the **scaled newsvendor quantities**.

Before discussing Theorem 1, we characterize G_i (defined in Equation (5)) in a more explicit form in the following lemma.

Lemma 1. *In Theorem* 1, G_i (*defined in Equation* (5)) *can be characterized as*

$$G_i=\frac{x_i+\sum_{j=1}^{K_i}x_{\pi(j)}p_{\pi(j)i}}{p_i+\sum_{j=1}^{K_i}p_{\pi(j)}p_{\pi(j)i}},\quad 1\leq i\leq K,$$

where π *is a permutation of* $\{1, 2, \ldots, K\}$ *such that* $\frac{x_{\pi(1)}}{p_{\pi(1)}} \leq \frac{x_{\pi(2)}}{p_{\pi(2)}} \leq \cdots \leq \frac{x_{\pi(K)}}{p_{\pi(K)}}$, *and*

$$K_i := \max\left\{k \in \{1, \ldots, K\} : \frac{x_{\pi(k)}}{p_{\pi(k)}} \leq \frac{x_i + \sum_{j=1}^{k} x_{\pi(j)} p_{\pi(j)i}}{p_i + \sum_{j=1}^{k} p_{\pi(j)} p_{\pi(j)i}}\right\}, \quad 1 \leq i \leq K,$$

are functions of $(x_k)_{k=1}^{K}$.

Theorem 1 gives a necessary condition for the optimal stocking policy. We note that the second and third terms on the left-hand side of Equation (4) capture the effect of substitution (second choice). The theorem shows that when substitution is taken into consideration then the scaled newsvendor solution is not necessarily optimal. The optimal stocking quantity is adjusted up or down based on the penalty cost or whether there is high degree of substitution from product i, that is, when $p_{i\ell}$ is large, as a stockout of product i does not necessarily result in a lost sale with substitution.

Remark 1 explains that in the absence of a second choice, the optimal stocking levels for the product options are simply the corresponding scaled newsvendor quantities. The next order of business is to investigate how the optimal stocking levels change when customers exercise a second choice option. Theorem 2 states that under certain conditions, the scaled newsvendor quantities remain optimal.

THEOREM 2.

(a) *Suppose* $F(x_1, \ldots, x_K)$ *is jointly concave in* $(x_1, \ldots, x_K) \in \mathbb{R}_+^K$. *Then the optimal stocking quantity of each product variant is given by the scaled newsvendor solution.*
(b) *Suppose the condition*

$$(s - c_p - c_0)\sum_{\ell \neq i} p_{i\ell} - c_p < 0$$

is satisfied for every $1 \leq i \leq K$. *Then* $F(x_1, \ldots, x_K)$ *is jointly concave in* $(x_1, \ldots, x_K) \in \mathbb{R}_+^K$, *and the optimal stocking quantity of each product variant is given by the scaled newsvendor solution.*

We provide some insight into the condition in Theorem 2 that must be fulfilled in order for the optimal solution to be the scaled newsvendor quantities. Note that c_p represents a penalty cost that the firm imposes for every unit that represents a second choice purchase by a customer. We introduce a parameter θ such that

$$c_p = \theta(s - c).$$

Since $s - c$ presents the profit that the firm makes on the sale of a unit, it is reasonable to assume that $0 < \theta < 1$. The higher the value of θ, greater is the desire of the firm to reduce the frequency of occurrence of the event of having customers pick their second choice product instead of their preferred product. The following corollary can be easily derived from Theorem 2(b).

We define

$$f^* := \frac{(s - c)}{(s - c) + (c - c_0)} = \frac{s - c}{s - c_0}.$$

Note that f^* is equal to the critical fractile in the classic newsvendor formula; therefore, it is the percentile of the demand distribution that corresponds to the scaled newsvendor solution.

COROLLARY 1. *Suppose* $\theta f^* \geq 0.5$. *Then the optimal stocking quantity of each product variant is given by the scaled newsvendor solution.*

The existing literature (e.g., Smith and Agrawal 2000) states that a retail firm will typically target f^* to be around 0.9. In this case, Corollary 1 implies that if $\theta > 0.56$, then the optimal solution is the scaled newsvendor solution. Note that $\theta > 0.56$ means that the firm is very keen on having a satisfied customer base. If a customer buys a second choice variant, the firm levies a stiff penalty on itself on the profit margin from such a sale. This example suggests the following rule of thumb: a firm that is sensitive to the disutility of customers having to avail themselves of their second choice product variant should optimally stock the scaled newsvendor quantities when the target service level is sufficiently high. While target service levels in a retail setting are generally high as suggested by several studies and market research (Gruen et al. 2002), it might not be the case for low margin products. So if $f^* = 0.5$ or lower, which could certainly be the case for low margin products, then Corollary 1 does not apply; we would require $\theta \geq 1$, which is infeasible, by definition.

4.2. Fully Stochastic Model

Given the supplies, that is, the stocking decision of the retailer $x_1, \ldots, x_K$, the expected profit of the firm is given by:

$$
\begin{aligned}
&F(x_1,\ldots,x_K)\\
&:=\sum_{i=1}^{K}\mathbb{E}\left[\left((s-c)DP_i+(s-c-c_p)\sum_{j\neq i}(DP_j-x_j)^+P_{ji}\right)1_{x_i>DP_i+\sum_{j\neq i}(DP_j-x_j)^+P_{ji}}\right]\\
&\quad-\sum_{i=1}^{K}\mathbb{E}\left[(c-c_0)\left(x_i-DP_i-\sum_{j\neq i}(DP_j-x_j)^+P_{ji}\right)1_{x_i>DP_i+\sum_{j\neq i}(DP_j-x_j)^+P_{ji}}\right]\\
&\quad+\sum_{i=1}^{K}\mathbb{E}\left[\left((s-c)DP_i+(s-c-c_p)(x_i-DP_i)\right)1_{DP_i\le x_i\le DP_i+\sum_{j\neq i}(DP_j-x_j)^+P_{ji}}\right]\\
&\quad+\sum_{i=1}^{K}\mathbb{E}[(s-c)x_i1_{x_i<DP_i}],
\end{aligned}
\tag{6}
$$

where the expectations are not only over D, but also over P_i's and P_{ji}'s. The retailer's objective is to decide how much to stock for each variety so as to maximize the expected profit; thus the optimization problem is: $\max_{x_1,\ldots,x_K}F(x_1,\ldots,x_K)$.

By following the same argument as in the semi-stochastic model (section 4.1), we can show the following result.

THEOREM 3. *The optimal* $(x_1^*,\ldots,x_K^*)$ *satisfies the equations:*

$$
\begin{aligned}
&(s-c_0)\mathbb{E}\left[\int_0^{G_i(x_1^*,\ldots,x_K^*;P)}f(y)dy\right]\\
&\quad+(s-c_p-c_0)\mathbb{E}\left[\sum_{\ell\neq i}\int_{\frac{x_i^*}{P_i}}^{G_\ell(x_1^*,\ldots,x_K^*;P)}P_{i\ell}f(y)dy\right]\\
&\quad+c_p\mathbb{E}\left[\int_{G_i(x_1^*,\ldots,x_K^*;P)}^{\frac{x_i^*}{P_i}}f(y)dy\right]=s-c,\qquad 1\le i\le K,
\end{aligned}
\tag{7}
$$

where the expectations are taken over P_i*'s and* $P_{i\ell}$*'s and* $G_i=G_i(\cdot;P)$ *emphasizes the dependence on* P_i*'s and* P_{ji}*'s and is defined via the equations:*

$$x_i=G_iP_i+\sum_{j\neq i}(G_iP_j-x_j)^+P_{ji},\qquad 1\le i\le K. \tag{8}$$

Note that if $P_i\equiv p_i$, $P_{ji}\equiv p_{ji}$, that is, the distribution of P_i is the Dirac delta distribution $\delta_{P_i=p_i}$ and the distribution of P_{ji} is $\delta_{P_{ji}=p_{ji}}$, then it reduces to the semi-stochastic model that we studied earlier in section 4.1, and Theorem 3 reduces to Theorem 1.

If there is no second choice available, the model reduces to the (stochastic) newsvendor problem, that is, $P_{ji}\equiv 0$, then $G_i=x_i/P_i$ and the optimal $(x_1^*,\ldots,x_K^*)$ satisfies the equations:

$$(s-c_0)\mathbb{E}\left[\int_0^{\frac{x_i^*}{P_i}}f(y)dy\right]=s-c,\qquad 1\le i\le K. \tag{9}$$

Next, we assume that $P_i\equiv p_i$ are deterministic whereas P_{ji}'s are possibly stochastic. We have the following result.

THEOREM 4. *Assume that* $P_i\equiv p_i$, $1\le i\le K$.

(a) *Suppose* $F(x_1,\ldots,x_K)$ *is jointly concave in* $(x_1,\ldots,x_K)\in\mathbb{R}_+^K$. *Then the optimal stocking quantity of each product variant is given by the scaled newsvendor solution.*
(b) *Suppose with probability one,* $(s-c_p-c_0)\sum_{\ell\neq i}P_{i\ell}-c_p<0$ *for every* $1\le i\le K$, *then* $F(x_1,\ldots,x_K)$ *is jointly concave in* $(x_1,\ldots,x_K)\in\mathbb{R}_+^K$, *and the optimal stocking quantity of each product variant is given by the scaled newsvendor solution.*

REMARK 2. The above results indicate that if the first choice fractions P_i's are deterministic, then the previous results for the semi-stochastic model hold regardless of the stochasticity of the second choice fractions P_{ji}'s. In particular, Corollary 1 still holds.

Next, we will show that if the first choice fractions P_i's are fully stochastic, then the optimal solution to the second choice problem cannot be the (stochastic) newsvendor solution (9), with a precise statement given in the following theorem.

THEOREM 5. *Assume that the first choice fractions P_i's are fully stochastic in the sense that there exists some $i \neq j$, such that P_i/P_j is stochastic. Then the optimal solution to the second choice problem cannot be the* (*stochastic*) *newsvendor solution* (9).

The above analysis in the single-period model carries over to the infinite horizon setting as well. In the infinite horizon analysis, we rigorously prove that the optimal policy is a base-stock policy, and then show that the optimal base-stock levels are the scaled newsvendor quantities under the same constraint on costs and prices as in the single-period model. We extend the analysis to the fully stochastic model in the infinite horizon setting as well. The details of the infinite horizon setting analysis is included in the online electronic companion. We now establish a connection between the benchmark and the exact model.

4.3. Connection between the Exact and Benchmark Model

We have shown that scaled newsvendor quantities remain optimal with substitution, depending on the stochasticity of the first and second choice fractions. In deriving our results, in order to maintain tractability and characterize an optimal policy, our benchmark model assumes that the first choice demand is always satisfied first and the residual stock (if any) is used to satisfy the second choice demand. However based on the stochastic nature of customer arrival, the second choice demand could certainly eat up a percentage of the demand units that we ascribe to first choice demand, and thereby deflate our profit function (by ignoring penalty cost associated with second choice demand). This scenario transpires when a particular variant is out of stock, and the remaining customers who come in all miss their first choice (since it has already stocked out).

In our benchmark model we obtained explicit solutions for the optimal stocking policy. Next, we discuss the connection between the exact and the benchmark model. We denote the number of units of type i sold to the customers whose first choice is type i till time t as $S_i^1(t)$ and the number of units of type i sold to the customers whose second choice is type i till time t as $S_i^2(t)$, where $0 \le t \le T$ and $1 \le i \le K$. Then $(S_1^1(t), S_1^2(t), S_2^1(t), S_2^2(t), \ldots, S_K^1(t), S_K^2(t))$ is a $2K$-dimensional continuous-time Markov process with the state space $(\mathbb{N} \cup \{0\})^{2K}$ that can be fully characterized by the infinitesimal generator:

$$\begin{aligned}\mathcal{L}f(S_1^1, S_1^2, S_2^1, S_2^2, \ldots, S_K^1, S_K^2) &= \sum_{i=1}^K \lambda p_i \big(f(S_i^1+1) - f(S_i^1)\big) 1_{S_i^1+S_i^2<x_i} \\ &+ \sum_{i=1}^K \sum_{j\neq i} \lambda p_i p_{ij} \Big(f(S_j^2+1) - f(S_j^2)\Big) 1_{S_i^1+S_i^2\ge x_i} 1_{S_j^1+S_j^2<x_j},\end{aligned} \tag{10}$$

where we used $f(S_i^j)$, $1 \le i \le K$, $j = 1, 2$, to denote $f(S_1^1, S_1^2, S_2^1, S_2^2, \ldots, S_K^1, S_K^2)$ with an emphasis on the dependence on S_i^j to ease the notation.

Given the supplies, that is, the stocking decision of the retailer $x_1, \ldots, x_K$, the expected profit of the firm is given by:

$$\begin{aligned}F(x_1, \ldots, x_K) = &\sum_{i=1}^K \mathbb{E}\big[(s-c)S_i^1(T)\big] \\ &+ \sum_{i=1}^K \mathbb{E}\big[(s-c-c_p)S_i^2(T)\big] \\ &- \sum_{i=1}^K (c-c_0)\mathbb{E}\big[x_i - S_i^1(T) - S_i^2(T)\big].\end{aligned} \tag{11}$$

We do not expect (11) can be computed in closed form and one has to rely on numerical methods, such as Monte Carlo simulations, to compute $F(x_1, \ldots, x_K)$. In Proposition 1, we have shown that

$$F(x_1, \ldots, x_K) = V(x_1, \ldots, x_K; 0), \tag{12}$$

where $V(x_1, \ldots, x_K; t)$ satisfies a $(K+1)$-dimensional integral Equation (2) which does not yield a simple closed-form solution. Thus an alternative to Monte Carlo simulation is to numerically solve the integral Equation (2). Since neither (11) or (2) yields closed-form solution, it is natural for us to turn to the simplified benchmark model as an approximation of the exact model to characterize the optimal policy.

Assume D_i's are independent Poisson random variables with mean λp_i. Then, in the exact model, D_i is the total demand for product variant of type i as the first choice from customers within the time period $[0, T]$, and $D = \sum_{i=1}^K D_i$ is the total demand within the time period $[0, T]$. In our benchmark model, we assume D follows a continuous random variable and set $D_i = Dp_i$ in the semi-stochastic model and $D_i = DP_i$ in the fully stochastic model.

Next, we recall that in our semi-stochastic model in section 4.1,

$$
\begin{aligned}
F(x_1,\ldots,x_K) = & \sum_{i=1}^{K}\mathbb{E}[(s-c)\min\{D_i,x_i\}] \\
& + \sum_{i=1}^{K}\mathbb{E}\left[(s-c-c_p)\sum_{j\neq i}(D_j-x_j)^{+}p_{ji}\cdot 1_{x_i>D_i+\sum_{j\neq i}(D_j-x_j)^{+}p_{ji}}\right] \\
& + \sum_{i=1}^{K}\mathbb{E}\left[(s-c-c_p)(x_i-D_i)\cdot 1_{D_i\le x_i\le D_i+\sum_{j\neq i}(D_j-x_j)^{+}p_{ji}}\right] \\
& - \sum_{i=1}^{K}\mathbb{E}\left[(c-c_0)\left(x_i-D_i-\sum_{j\neq i}(D_j-x_j)^{+}p_{ji}\right)1_{x_i>D_i+\sum_{j\neq i}(D_j-x_j)^{+}p_{ji}}\right].
\end{aligned} \tag{13}
$$

In the semi-stochastic model, we approximate $S_i^1(T)$ by $\min\{D_i,x_i\}$ and $S_i^2(T)$ by

$$
\begin{aligned}
& \sum_{j\neq i}(D_j-x_j)^{+}p_{ji}\cdot 1_{x_i>D_i+\sum_{j\neq i}(D_j-x_j)^{+}p_{ji}} + (x_i-D_i)\cdot 1_{D_i\le x_i\le D_i+\sum_{j\neq i}(D_j-x_j)^{+}p_{ji}} \\
& \quad = \min\left\{x_i-D_i,\ \sum_{j\neq i}(D_j-x_j)^{+}p_{ji}\right\}1_{x_i\ge D_i}.
\end{aligned} \tag{14}
$$

We can compute that

$$
\begin{aligned}
x_i - S_i^1(T) - S_i^2(T) &= x_i - \min\{D_i,x_i\} - \min\left\{x_i-D_i,\ \sum_{j\neq i}(D_j-x_j)^{+}p_{ji}\right\}1_{x_i\ge D_i} \\
&= \left(x_i-D_i-\sum_{j\neq i}(D_j-x_j)^{+}p_{ji}\right)1_{x_i>D_i+\sum_{j\neq i}(D_j-x_j)^{+}p_{ji}}.
\end{aligned} \tag{15}
$$

which matches the last term in (13). Hence, we conclude that our semi-stochastic model is an approximation of the exact model by using the approximation:

$$S_i^1(T)\approx\min\{D_i,x_i\}, \tag{16}$$

$$S_i^2(T)\approx\min\left\{x_i-D_i,\ \sum_{j\neq i}(D_j-x_j)^{+}p_{ji}\right\}1_{x_i\ge D_i}. \tag{17}$$

Finally, the same connections between the exact model and the fully stochastic model hold by replacing $D_i = Dp_i$ by DP_i, p_{ji} by P_{ji} in Equations (16) and (17). The online electronic companion also contain details of the connection between the exact model formulation with discretized Poisson arrival process and the exact model in section 3.1.

5. Computational Study with a Dynamic Simulation Model

In the previous sections, we have seen that the newsvendor solution for stocking policy may not be optimal in the presence of substitution. We have demonstrated under various scenarios of random demand and stochastic choice fractions that the scaled newsvendor solution remains optimal when the demands for different product varieties are negatively correlated. Our theoretical analysis in section 3.2 does not model the dynamic nature of customer arrival in real time, and also might seem restrictive as we make certain assumptions in obtaining the optimal policy as the scaled newsvendor solution. In this section, we present a dynamic simulation of precisely this situation. Getting closed-form solution with the exact model in section 3.1 is challenging, thus we resort to extensive numerical analysis to further understand the impact of the assumptions in the benchmark model and explore if our results are reasonably robust.

The goal of this computational study is to analyze the impact of customer heterogeneity, costs, performance measures, and other real world scenarios as would be present in a retail setting. Specifically, we want to examine how well our benchmark approach performs in these scenarios where gaining additional insights with our analytical model is challenging. In this computational study, we simulate a scenario where a retail outlet stocks substitutable products. We consider random demand with stochastic proportions of customers preferring a specific product variant based on individual customer choice. The substitution between the product variants depends on the stochastic nature of customer arrival thus it incorporates stockout-based substitution. The optimization process is that for each and every possible combination of stocking level of the product variants we compute the profits. We then compare these profits to find the maximum profit and note the corresponding optimal stocking level. Due to the computational complexity involved in regard to the possible number of stocking levels based on the customer arrival pattern and number of product variants, we restrict ourselves to two and four products. Next, we describe the simulation setup and problem parameters used in this study.

1. *Demand*: The total random demand for the products D, is drawn from a distribution. The customer demand during a selling period with length $T = 1$ is best simulated using a Poisson distribution with mean λ. Our goal is to see how well the optimal policy derived earlier in the benchmark model matches to those obtained from the simulation. For consistency with our theoretical model, we approximate the total demand for the products as a normal distribution with parameters (mean $\mu = \lambda$, standard deviation $\sigma = \sqrt{\lambda}$). We randomly draw from this distribution to denote the demand for a given period, and our insights are based on this distribution. At the end of this section, we extend our analysis to consider a Poisson demand distribution, and find that the insights are consistent. In the simulation, we only consider integer values; thus the randomly drawn demand value is rounded to the closest integer. Given a random demand we search for the optimal stocking level of the products which maximizes the profit. For any demand realization drawn from this distribution, there can be more than 10^4 possible combinations of stocking levels for two products and 10^8 for four products when the mean demand $\lambda = 100$. We only consider integer values for the stocking levels. A total of 1000, random demand realizations were considered for the simulation to compute the average percentage gap between the simulation results and the policy derived earlier. For each demand realization all possible combinations of stocking levels of the product variants were considered to search for the optimal stocking quantity which maximizes the profit. For example, in case of two products, if the randomly drawn demand is 108, then there can be at most 108^2 possible combinations of stocking levels of the two products. Next we describe the preferred choice for a product variant and the dynamic substitution process.

2. *Preferred choice*: For a random demand realization $D = d$, each customer $m = 1, \ldots, d$ has a preferred first choice among the product variants. Thus the individual demands for the product variants are negatively correlated. Every customer is randomly assigned any one of the product variants as their preferred first choice (drawn from a distribution), thus creating a sample path of random arrivals and preferences. If the preferred first choice is available the customer purchases it and it is noted as a first choice sale. The stocking levels of the product variants are updated accordingly. For example, in case of two products, suppose the stocking level for the two product variants is x_1 and x_2 when the m^{th} customer arrives. The preferred first choice for customer "m" is variety 1, which is in stock. Then the customer purchases it, and the stocking levels are updated to $(x_1 - 1)$ and x_2 for the next—that is the $(m + 1)^{\text{th}}$ customer. Thus the stocking levels are updated dynamically for each customer.

3. *Substitution*: In the event that the preferred product variant is not available when customer "m" arrives, then she randomly chooses to substitute to her second choice item if it is in stock. This process is also stochastic; we randomly assign whether a customer is willing to substitute or not in case of a stockout. If the second choice item is available and the customer substitutes to purchase it, we note it as second choice sale and update the stocking levels accordingly. Thus a lost sale can be either because a customer is not willing to substitute in case of stockout of their preferred product variant or wants to substitute but there is no available stock for the second choice product variant. To illustrate, in case of two products suppose the stocking level for the two product variants is $x_1 = 0$ and x_2 when customer "m" arrives. The preferred first choice for customer "m" is the first variety which is out of stock. The customer may choose to substitute (which is determined randomly for every arriving customer in case of stockout) and purchase the second variant. Then the stocking levels are updated to $x_1 = 0$ and $(x_2 - 1)$ for the $(m + 1)^{\text{th}}$ arriving customer. In case the customer does not substitute, then it results in a lost sale and the stocking levels remain at $x_1 = 0$ and x_2 for the next arriving customer. Thus, the second choice substitution is completely stochastic, and is contingent on whether the preferred first choice product is in stock and the random process of customers willing to substitute based on the availability of second choice item. Essentially the stochastic second choice fractions are dependent on the stocking decision of the retailer and the preferred choices of the customers arriving before a particular customer.

4. *Revenue and cost parameters*: Several previous studies such as Smith and Agrawal (2000) and Gruen et al. (2002) state that in a retail setting, the target service levels are around 90% to 98%. Nagarajan and Rajagopalan (2008) observe similar service levels in their program with the grocery industry. Extensive market research and similar studies in a retail setting by Gruen et al. (2002) suggest that the percentage of stockouts usually ranges between 5% and 10%, with a worldwide average at about 8%. In line with the above empirical evidences, the price and cost parameters were chosen to achieve a newsvendor ratio in the range 0.85 – 0.95, which is generally an acceptable measure of service level for retailers. For instance the price and costs are set as $s = (28, 29, 30, 31, 32)$, $c = 15$, $c_p = 9$ and $c_0 = (12.70, 13.09, 13.33, 13.60, 14.10)$. Table 1 below presents the average percentage gap in profit relative to the simulation for the stocking policy based on our analytical model for different service levels. We also present the average gap in optimal inventory levels of the product variants between scaled newsvendor quantities and those obtained from the simulation.

5. *Solution and insights*: The above setup creates a sample path of random arrivals and preferences for a demand realization D drawn from the distribution. The solution approach is to compute the profit for each and every possible combination of stocking levels of the products for the realized demand and sample path. This is done by an iterative loop for every arriving customer and based on their preferred choices (which is stochastic) the stocking levels are updated dynamically. We can then determine the leftover inventory for a given stocking level and also whether a unit is sold as a first or second choice. We compute the profits for all possible stocking levels and compare them to search for the maximum profit and the corresponding stocking quantity. This process is repeated for 10^3 demand realizations for every service level (newsvendor ratio).

Profit and Inventory gap: The above results show that the optimal policy based on our analytical model (scaled newsvendor quantity) with demand correlation and substitution performs reasonably well. The average percentage gap in profit is between 1% and 3% and the policy performs better at higher service levels. Since we considered every possible stocking levels of the products, we are able to compare the optimal inventory levels with the stocking quantity based on our analytical model for each product variant. We find that the scaled newsvendor policy can carry lower or higher inventory than the optimum level obtained from the simulation. However, on average, the scaled newsvendor policy tends to overstock and carries a higher inventory. As seen in the above table, the average inventory gap for each product variant increases with higher service levels. In terms of the stocking ratio, the scaled newsvendor policy varies between 0.83 and 1.26 times the optimal inventory obtained from the simulation. These results are based on the stochastic nature of customer arrival and their random choices. The benchmark policy in our analytical model is based on service level, which is the newsvendor quantity multiplied by a scale factor and does not explicitly account for the stochastic nature of customer arrival. As a result at higher service levels the scaled newsvendor policy on an average tends to stock more. The overstocking at higher service levels leads to more products being sold as first choice resulting in a higher profit. This explains why the percentage gap in optimal profit is low at higher service levels. Next we analyze the effect of customer heterogeneity and also consider a wider range of service levels.

Table 1 Average Gap in Inventory Levels and Percentage Gap in Profit ($\lambda = 100$)

	$K = 2$		$K = 4$	
Service levels	% Profit difference	Avg. inventory gap	% Profit difference	Avg. inventory gap
0.85	2.65	(5.06, 5.05)	2.63	(2.39, 2.42, 2.48, 2.52)
0.88	2.18	(5.39, 5.42)	2.24	(2.69, 2.72, 2.64, 2.48)
0.90	1.88	(6.55, 6.47)	1.92	(3.02, 2.96, 3.11, 3.16)
0.92	1.43	(6.97, 7.03)	1.59	(3.28, 3.22, 3.34, 3.41)
0.95	1.01	(7.82, 7.86)	1.06	(3.92, 3.82, 3.83, 3.89)

Customer heterogeneity: In the above analysis, the first choice preferences assigned to each customer is randomly drawn from a distribution. The random choices are generated in a manner that for every arriving customer there is an equal likelihood of choosing any one of the products as the preferred choice, that is, Prob.(choosing product i) = Prob.(choosing product j) = $\frac{1}{K}$, where K is the number of product variants. This represents the case when customers are most heterogeneous. However, in practice certain product variants tend to be more popular and thus a larger proportion of customers would prefer that variant as their first choice. So, next we explore the effect of customer preference heterogeneity.

To capture this, the first choice preferences are drawn from a distribution which is skewed such that there is a higher probability of a customer choosing a specific product. Thus on an average a larger proportion of the customers will prefer a specific product. In Table 2, we present the results based on customer preference heterogeneity. The first row is when customers are most heterogeneous and as we move down the table the customers become more

Table 2 Effect of Preference Heterogeneity on Percentage Gap in Profit (λ = 100)

Number of products (*K*)	Choice probabilities	Service levels						
		0.80	0.83	0.85	0.88	0.90	0.92	0.95
2	(0.50, 0.50)	3.38	3.03	2.65	2.18	1.88	1.43	1.01
	(0.70, 0.30)	3.29	2.91	2.69	2.28	1.93	1.56	1.05
	(0.90, 0.10)	3.34	3.08	2.70	2.27	1.89	1.53	1.08
4	(0.25, 0.25, 0.25, 0.25)	3.25	2.96	2.63	2.24	1.92	1.59	1.06
	(0.40, 0.30, 0.20, 0.10)	3.39	3.01	2.67	2.11	1.87	1.55	1.07
	(0.50, 0.30, 0.15. 0.05)	3.36	3.08	2.74	2.17	1.93	1.51	0.94

homogeneous with higher proportions of the customers preferring a specific product variant over the other. To further illustrate, in case of two products the first row is when every arriving customer has Prob.(choosing product 1) = Prob.(choosing product 2) = 0.5 and the last row is when every arriving customer has Prob.(choosing product 1) = 0.9, Prob.(choosing product 2) = 0.1. Thus on an average a larger proportion of customers will choose the first product.

The percentage gap in optimal profit relative to the simulation for the scaled newsvendor policy is fairly consistent for a given service level irrespective of customer preference heterogeneity. This is because the scaled newsvendor policy takes into account these choice probabilities in its formulation as the first choice proportions P_i's are modified accordingly. In the above analysis the stochastic second choice fractions, P_{ij}'s ranged from 0.13 to as high as 0.52, and we observe that the substitution fractions are dependent on customer preference heterogeneity, that is, the first choice proportion of the customers. In general the average substitution fractions vary to a larger extent when the customer preference is more homogeneous. The second choice substitution proportions due to stockout depends on the product category and brand value (Gruen et al. 2002). For example, snacks, perishables, and daily necessities like paper towel have a higher substitution percentage than cosmetics. In a product category consumers may substitute to another variant within the same or a different brand. Typically these values on an average ranges between 15% and 45% and our simulation outcomes are in line with empirical evidence.

Demand characteristics: We now extend our analysis to consider lower service levels and examine the impact of mean demand and variability on the optimal percentage gap. While high service levels is desired, this might not always be the case for certain category of products. The cost and revenue parameters are chosen for a wide range of service levels ensuring all feasibility conditions such as positive margin from substitution sale, that is, $s - c - c_p > 0$, underage and overage costs are satisfied. At low service levels the profit margin or underage costs are relatively small thus the impact of penalty cost becomes more significant, that is, high value of θ. This scenario which can be the case for low margin products, the profit is severely impacted due to substitution resulting from stockout. We consider the case when customers are most heterogeneous, that is, for every arriving customer, there is an equal likelihood of choosing any one of the products.

In Figure 1, we see that the optimal policy based on our theoretical model performs better with larger mean demand and is also consistent with earlier results that the gap improves at higher service levels. The percentage gap decreases at a faster rate with higher demand at low service levels. In all the above analysis we consistently find that optimum policy based on the benchmark model on an average tends to overstock and this varies based on customer heterogeneity and demand variability. As the benchmark model tends to hold higher levels of inventory we also analyzed the impact of the overstocking cost. For this analysis we hold the underage cost to be the same for all product variants but vary the overage cost based on c_0. We find that our policy performs better with lower overage cost as c_0 increases. A low overage cost corresponds to high service levels at which the benchmark model tends to perform better as seen in our earlier analysis. We obtain similar insights as the above reported results.

Next, we analyze how the policy performs under different profit margin and overage costs across the product variants and its interaction with the customer choice probabilities, that is, we investigate the interaction between asymmetric service levels and customer heterogeneity between the product variants. In this analysis we consider two products in order to restrict the number of possible combinations while searching for the optimal solution.

Interaction effects: In this scenario, a less popular product based on customer preference can have a higher service level and vice-versa. From Table 3, it is evident that there is an interaction between the

Figure 1 Effect of Service Level and Demand Variability on Percentage Gap in Profit

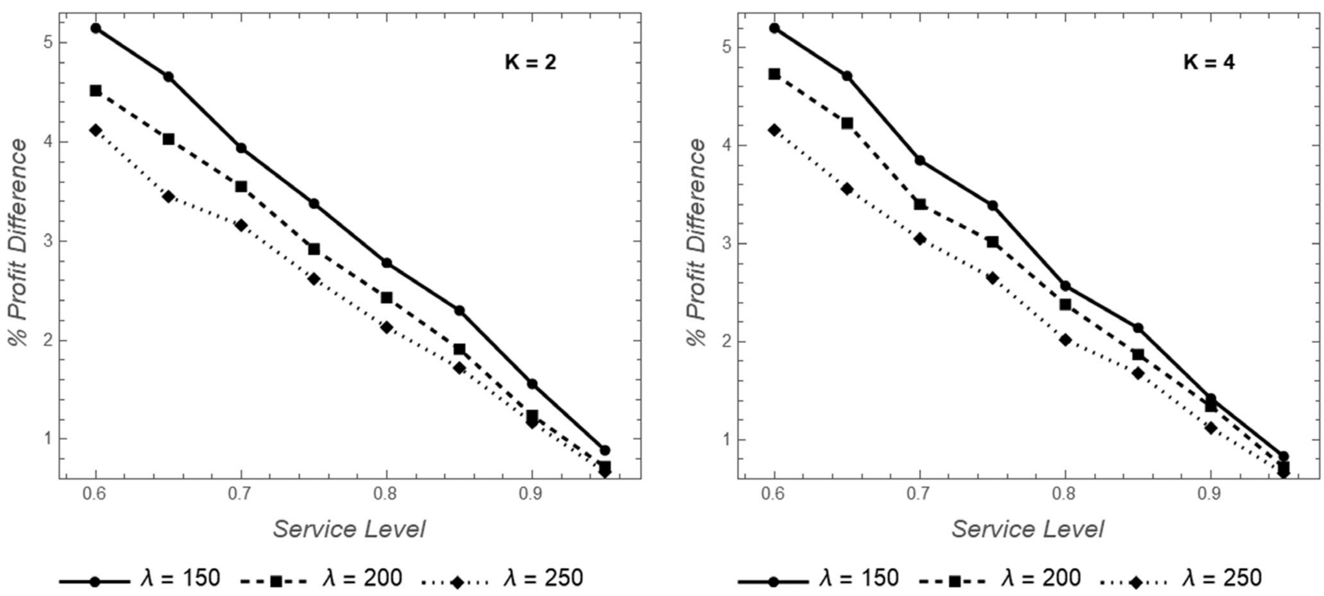


Table 3 Asymmetric Service Level and Preference Heterogeneity ($\lambda = 100$)

Service levels		Demand proportions		
Product 1	Product 2	$P_1 = 0.3$, $P_2 = 0.7$ $P_1 < P_2$	$P_1 = P_2 = 0.5$	$P_1 = 0.7$, $P_2 = 0.3$ $P_1 > P_2$
0.80	0.80	3.39	3.33	3.28
0.80	0.85	2.76	2.93	3.13
0.80	0.90	2.46	2.62	2.84
0.80	0.95	1.54	1.92	2.58
0.85	0.85	2.62	2.66	2.61
0.85	0.90	2.05	2.29	2.41
0.85	0.95	1.42	1.78	2.04
0.90	0.90	1.81	1.87	1.80
0.90	0.95	1.22	1.44	1.56
0.95	0.95	1.07	1.08	1.05

service levels of the two products and customer heterogeneity. The percentage gap in profit is consistent when both products have the same service level for different customer heterogeneity. However, for asymmetric service levels of the two products, when a larger proportion of customers prefer a product variant with higher service level, the percentage gap in profit decreases compared to when there is an equal likelihood of customers choosing any one of the two products. The opposite is seen when a larger proportion of customers prefer a product variant with lower service level. The analysis highlights that the optimal policy based on the analytical model performs better at higher service levels and when customers are more homogeneous in preferring the product with higher service levels.

Table 4 Average Gap in Inventory Levels and Percentage Gap in Profit under Poisson ($\lambda = 100$)

	$K = 2$		$K = 4$	
Service levels	% Profit difference	Avg. inventory gap	% Profit difference	Avg. inventory gap
0.70	4.69	(2.22, 2.19)	4.78	(1.12, 1.04, 1.07, 1.08)
0.75	4.23	(3.69, 3.71)	4.27	(1.77, 1.72, 1.83, 1.85)
0.80	3.54	(4.28, 4.46)	3.46	(2.01, 2.17, 2.07, 2.03)
0.85	2.68	(4.77, 4.69)	2.61	(2.52, 2.54, 2.52, 2.43)
0.90	1.92	(6.61, 6.67)	1.83	(3.07, 3.17, 3.15, 3.35)
0.95	1.04	(8.24, 8.27)	1.10	(4.44, 4.58, 4.55, 4.61)

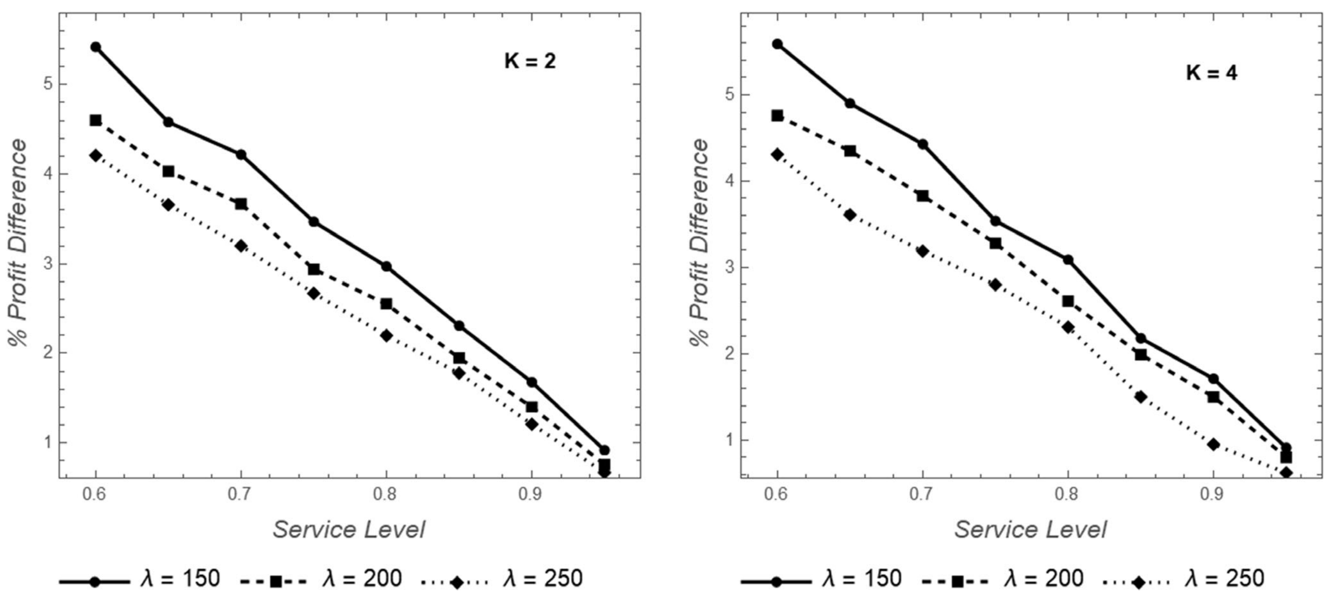
We now extend our analysis by considering a Poisson distribution to examine if there is any significant difference because of the continuous demand distribution assumption in our benchmark model analysis. Table 4 below presents the average gap in optimal inventory levels and percentage profit difference in case of a Poisson distribution. Consistent with our earlier results we find that the benchmark policy performs better at higher service levels and on an average tends to overstock as compared to the optimal inventory levels from the simulation.

Next we consider customer preference heterogeneity under Poisson. Similar to our earlier results the percentage gap in optimal profit under the benchmark policy is consistent for a given service level regardless of the choice probabilities. Table 5 below presents the percentage profit difference in case of a Poisson distribution based on customer preference heterogeneity.

Table 5 Effect of Preference Heterogeneity on Percentage Gap in Profit under Poisson (λ = 100)

Number of products (K)	Choice probabilities	Service levels 0.70	0.75	0.80	0.85	0.90	0.95
2	(0.50, 0.50)	4.69	4.23	3.54	2.68	1.92	1.04
	(0.70, 0.30)	4.76	4.26	3.51	2.63	1.87	1.09
	(0.90, 0.10)	4.66	4.18	3.44	2.71	1.89	1.08
4	(0.25, 0.25, 0.25, 0.25)	4.78	4.27	3.46	2.61	1.83	1.10
	(0.40, 0.30, 0.20, 0.10)	4.71	4.34	3.32	2.72	1.88	1.07
	(0.50, 0.30, 0.15. 0.05)	4.74	4.30	3.38	2.67	1.93	1.06

Figure 2 Effect of Service Level and Demand Variability on Percentage Gap in Profit under Poisson



Finally we examine the effect of demand variability and lower service levels under Poisson when customers are most heterogeneous. As seen earlier, with larger mean demand the benchmark policy performs better and the percentage gap decreases at a faster rate with higher mean demand at lower service levels (Figure 2).

6. Concluding Remarks

In this study, we have extended past work on the problem of selecting the optimal stocking levels of horizontally differentiated product variants. The paper considers the demand for different product variants to be negatively correlated, and studies the impact of substitution between the products. Customers substitute one product variant for another if their first choice option is not in stock. We develop an exact model in which customers arrive stochastically according to a Poisson process. In order to characterize an optimal policy analytically, we propose a general benchmark model in which demand, first choice and second choice proportions, are stochastic. We identify conditions under which the optimal stocking policy in the benchmark model is the scaled newsvendor solution and show that this depends on the stochasticity of the first and second choices. We then establish a connection between the exact and benchmark models.

The dynamic simulation accounts for the stochastic nature of the customer arrival and choice process. In the computational study, we examine the impact of choice fractions, costs, and other performance measures on the optimal policy as compared to the benchmark model to better understand the impact of our modeling assumptions on the optimality gap. Our results have practical implications in a retail setting, where a retailer has to decide on optimal stocking levels of different product variants within a category. Correlated demand and substitution across different product variants in a retail setting is quite common, and we have been able to find simple conditions on price and cost when the optimum stocking policy is given by the scaled newsvendor solution. Future extensions of this work include adding the effect of demand correlation and customer arrival process in the assortment decision.

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Appendix

PROOF OF PROPOSITION 1. If $x_1, \ldots, x_K > 0$, then we have

$$V(x_1,\ldots,x_K;t) = \sum_{i=1}^{K} p_i \int_t^T \lambda e^{-\lambda(u-t)}(s - c + V(x_1,\ldots, x_{i-1}, x_i - 1, x_{i+1},\ldots,x_K;u))du - \sum_{i=1}^{K} x_i(c-c_0)\int_T^{\infty} \lambda e^{-\lambda(u-t)}du. \quad \text{(A1)}$$

First we elaborate on how we obtained Equation (A1). Recall that customers arrive according to a Poisson process with arrival rate $\lambda > 0$. The arrival time of the first customer has probability density function $\lambda e^{-\lambda(u-t)}$ for arrival at time $u \ge t$. If $t \le u \le T$, then at time u, with probability p_i, the customer purchases a product of type i, which is available given that $x_i > 0$. The seller collects a profit $s - c$, and the available number of units for product i reduces to $x_i - 1$ from x_i, and by the strong Markov property, the expected profit for the seller at time u after selling a product of type i is given by $V(x_1,\ldots,x_{i-1},x_i-1,x_{i+1},\ldots,x_K;u)$, which explains the first term on the right-hand side of Equation (A1). If $u > T$, then no customer arrives before the end of the period, and thus the firm has to sell the remaining stocks $\sum_{i=1}^{K} x_i$ at a loss $(c - c_0)$ for each unit, which explains the second term on the right-hand side of Equation (A1).

Next, we can simplify Equation (A1) such that for any $x_1,\ldots,x_K > 0$

$$V(x_1,\ldots,x_K;t) = \left(1 - e^{-\lambda(T-t)}\right)\sum_{i=1}^{K} p_i(s-c) - e^{-\lambda(T-t)} \times \sum_{i=1}^{K} x_i(c-c_0) + \sum_{i=1}^{K} p_i \int_t^T \lambda e^{-\lambda(u-t)} \times V(x_1,\ldots,x_{i-1},x_i-1, x_{i+1},\ldots,x_K;u)du. \quad \text{(A2)}$$

More generally, for any $x_1,\ldots,x_K \in \mathbb{N}\cup\{0\}$ and any $0 \le t \le T$, we have

$$\begin{aligned} V(x_1,\ldots,x_K;t) &= \sum_{i:x_i>0} p_i \int_t^T \lambda e^{-\lambda(u-t)}(s - c + V(x_1,\ldots,x_{i-1},x_i-1, x_{i+1},\ldots,x_K;u))du \\ &+ \sum_{i:x_i=0} p_i \sum_{j:x_j>0} p_{ij}\int_t^T \lambda e^{-\lambda(u-t)}\left(s - c - c_p + V(x_1,\ldots, x_{j-1},\ x_j - 1,\ x_{j+1},\ldots,x_K;u)\right)du \\ &- \sum_{i=1}^{K} x_i(c-c_0)\int_T^{\infty}\lambda e^{-\lambda(u-t)}du, \end{aligned} \quad \text{(A3)}$$

To obtain Equation (A3), recall that customers arrive according to a Poisson process with arrival rate $\lambda > 0$ and with probability p_i, a customer's first preferred product is of type i. We consider the arrival time of the first customer whose probability density function is $\lambda e^{-\lambda(u-t)}$ for $u \ge t$. If $t \le u \le T$, and the customer's first choice of type i is available, that is, $x_i > 0$, then we obtain the first part in the right-hand side of Equation (A3) following the argument for Equation (A1). If type i products are not available, that is, $x_i = 0$, then with probability p_{ij}, the customer will substitute to the second choice product type j if they are available, that is, $x_j > 0$, we can use similar argument as Equation (A1) to obtain the second part in the right-hand side of Equation (A3). Finally, if $u > T$, then the third part in the right-hand side of Equation (A3) can be obtained in the same way as the second part in the right-hand side of Equation (A1).

Moreover, by simplifying Equation (A3), we have for any $x_1,\ldots,x_K \in \mathbb{N}\cup\{0\}$ and any $0 \le t \le T$,

$$\begin{aligned} V(x_1,\ldots,x_K;t) &= \left(1 - e^{-\lambda(T-t)}\right)\sum_{i:x_i>0} p_i(s-c) \\ &+ \left(1 - e^{-\lambda(T-t)}\right)\sum_{i:x_i=0} p_i \sum_{j:x_j>0} p_{ij}(s-c-c_p) \\ &- e^{-\lambda(T-t)}\sum_{i=1}^{K} x_i(c-c_0) \\ &+ \sum_{i:x_i>0} p_i \int_t^T \lambda e^{-\lambda(u-t)} V(x_1,\ldots,x_{i-1},x_i-1, x_{i+1},\ldots,x_K;u)du + \sum_{i:x_i=0} p_i \sum_{j:x_j>0} p_{ij}\int_t^T \lambda e^{-\lambda(u-t)} \end{aligned}$$

$$\times V(x_1, \ldots, x_{j-1}, x_j - 1, x_{j+1}, \ldots, x_K; u)du. \tag{A4}$$

Hence, we conclude that the optimal stocking levels are given by

$$(x_1^*, \ldots, x_K^*) = \arg\max_{x_1,\ldots,x_K \geq 0} V(x_1, \ldots, x_K; 0), \tag{A5}$$

where $V(x_1, \ldots, x_K; t)$ solves a $(K+1)$-dimensional integral Equation (A4). □

PROOF OF THEOREM 1. The function $F(x_1, \ldots, x_K)$ defined in Equation (3), can be simplified and re-written as:

$$\begin{aligned}
F(x_1, \ldots, x_K) &= \sum_{i=1}^{K} \mathbb{E}\left[\left((s-c)D_i + (s-c-c_p)\sum_{j\neq i}(D_j - x_j)^+ p_{ji}\right) 1_{x_i > D_i + \sum_{j\neq i}(D_j - x_j)^+ p_{ji}}\right] \\
&\quad - \sum_{i=1}^{K} \mathbb{E}\left[(c-c_0)\left(x_i - D_i - \sum_{j\neq i}(D_j - x_j)^+ p_{ji}\right) 1_{x_i > D_i + \sum_{j\neq i}(D_j - x_j)^+ p_{ji}}\right] \\
&\quad + \sum_{i=1}^{K} \mathbb{E}\left[\left((s-c)D_i + (s-c-c_p)(x_i - D_i)\right) 1_{D_i \leq x_i \leq D_i + \sum_{j\neq i}(D_j - x_j)^+ p_{ji}}\right] \\
&\quad + \sum_{i=1}^{K} \mathbb{E}[(s-c)x_i] - \sum_{i=1}^{K} \mathbb{E}[(s-c)x_i 1_{x_i} \geq D_i] \\
&= \sum_{i=1}^{K} \mathbb{E}\left[\left((s-c)(D_i - x_i) + (s-c-c_p)\sum_{j\neq i}(D_j - x_j)^+ p_{ji}\right) 1_{x_i > D_i + \sum_{j\neq i}(D_j - x_j)^+ p_{ji}}\right] \\
&\quad - \sum_{i=1}^{K} \mathbb{E}\left[(c-c_0)\left(x_i - D_i - \sum_{j\neq i}(D_j - x_j)^+ p_{ji}\right) 1_{x_i > D_i + \sum_{j\neq i}(D_j - x_j)^+ p_{ji}}\right] \\
&\quad + \sum_{i=1}^{K} \mathbb{E}\left[\left((s-c)(D_i - x_i) + (s-c-c_p)(x_i - D_i)\right) 1_{D_i \leq x_i \leq D_i + \sum_{j\neq i}(D_j - x_j)^+ p_{ji}}\right] \\
&\quad + \sum_{i=1}^{K} \mathbb{E}[(s-c)x_i] \\
&= \sum_{i=1}^{K} \mathbb{E}\left[\left((s-c_0)(D_i - x_i) + (s-c-c_p)\sum_{j\neq i}(D_j - x_j)^+ p_{ji}\right) 1_{x_i > D_i + \sum_{j\neq i}(D_j - x_j)^+ p_{ji}}\right] \\
&\quad + (c-c_0)\sum_{i=1}^{K} \mathbb{E}\left[\sum_{j\neq i}(D_j - x_j)^+ p_{ji} \cdot 1_{x_i > D_i + \sum_{j\neq i}(D_j - x_j)^+ p_{ji}}\right] \\
&\quad + \sum_{i=1}^{K} \mathbb{E}\left[c_p(D_i - x_i) 1_{D_i \leq x_i \leq D_i + \sum_{j\neq i}(D_j - x_j)^+ p_{ji}}\right] + \sum_{i=1}^{K} \mathbb{E}[(s-c)x_i],
\end{aligned} \tag{A6}$$

which yields that we can further simplify $F(x_1, \ldots, x_K)$ as

$$\begin{aligned}
F(x_1,\ldots,x_K) &= (s-c_0)\sum_{i=1}^{K} \mathbb{E}\left[(D_i - x_i)\cdot 1_{x_i > D_i + \sum_{j\neq i}(D_j - x_j)^+ p_{ji}}\right] \\
&\quad + (s-c_p-c_0)\sum_{i=1}^{K} \mathbb{E}\left[\sum_{j\neq i}(D_j - x_j)^+ p_{ji}\cdot 1_{x_i > D_i + \sum_{j\neq i}(D_j - x_j)^+ p_{ji}}\right] \\
&\quad + c_p \sum_{i=1}^{K} \mathbb{E}\left[(D_i - x_i)\cdot 1_{D_i \leq x_i \leq D_i + \sum_{j\neq i}(D_j - x_j)^+ p_{ji}}\right] + (s-c)\sum_{i=1}^{K} x_i.
\end{aligned} \tag{A7}$$

As stated earlier $D_i = Dp_i$ and recall that $f(y)$ is the probability density function of demand D, which is assumed to be continuously differentiable, thus the objective function $F(x_1, \ldots, x_K)$ can be stated as:

$$
\begin{aligned}
F(x_1,\ldots,x_K) = {} & (s-c_0)\sum_{i=1}^{K}\mathbb{E}\left[(Dp_i - x_i)\cdot 1_{x_i > Dp_i + \sum_{j\neq i}(Dp_j - x_j)^+ p_{ji}}\right] \\
& + (s-c_p-c_0)\sum_{i=1}^{K}\mathbb{E}\left[\sum_{j\neq i}(Dp_j - x_j)^+ p_{ji}\cdot 1_{x_i > Dp_i + \sum_{j\neq i}(Dp_j - x_j)^+ p_{ji}}\right] \\
& + c_p\sum_{i=1}^{K}\mathbb{E}\left[(Dp_i - x_i)\cdot 1_{Dp_i \le x_i \le Dp_i + \sum_{j\neq i}(Dp_j - x_j)^+ p_{ji}}\right] + (s-c)\sum_{i=1}^{K} x_i, \qquad \text{(A8)}
\end{aligned}
$$

which is equivalent to

$$
\begin{aligned}
F(x_1,\ldots,x_K) = {} & (s-c_0)\sum_{i=1}^{K}\int\limits_{x_i > yp_i + \sum_{j\neq i}(yp_j - x_j)^+ p_{ji}} (yp_i - x_i) f(y)dy \\
& + (s-c_p-c_0)\sum_{i=1}^{K}\int\limits_{x_i > yp_i + \sum_{j\neq i}(yp_j - x_j)^+ p_{ji}} \sum_{j\neq i}(yp_j - x_j)^+ p_{ji} f(y)dy \\
& + c_p\sum_{i=1}^{K}\int\limits_{yp_i \le x_i \le yp_i + \sum_{j\neq i}(yp_j - x_j)^+ p_{ji}} (yp_i - x_i) f(y)dy + (s-c)\sum_{i=1}^{K} x_i. \qquad \text{(A9)}
\end{aligned}
$$

Recall from Equation (5) that $G_i(x_1,\ldots,x_K)$ is defined as the value of y such that $x_i = yp_i + \sum_{j\neq i}(yp_j - x_j)^+ p_{ji}$. Then

$$
\begin{aligned}
F(x_1,\ldots,x_K) = {} & (s-c_0)\sum_{i=1}^{K}\int_0^{G_i(x_1,\ldots,x_K)} (yp_i - x_i) f(y)dy \\
& + (s-c_p-c_0)\sum_{i=1}^{K}\int_0^{G_i(x_1,\ldots,x_K)} \sum_{j\neq i}(yp_j - x_j)^+ p_{ji} f(y)dy \\
& + c_p\sum_{i=1}^{K}\int_{G_i(x_1,\ldots,x_K)}^{\frac{x_i}{p_i}} (yp_i - x_i) f(y)dy + (s-c)\sum_{i=1}^{K} x_i, \qquad \text{(A10)}
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\frac{\partial F}{\partial x_i} = {} & -(s-c_0)\int_0^{G_i(x_1,\ldots,x_K)} f(y)dy \\
& + (s-c_0)\sum_{\ell=1}^{K}\frac{\partial G_\ell}{\partial x_i}\left(G_\ell(x_1,\ldots,x_K)p_\ell - x_\ell\right) f(G_\ell(x_1,\ldots,x_K)) \\
& - (s-c_p-c_0)\sum_{\ell=1}^{K}\frac{\partial G_\ell}{\partial x_i}\left(G_\ell(x_1,\ldots,x_K)p_\ell - x_\ell\right) f(G_\ell(x_1,\ldots,x_K)) \\
& - (s-c_p-c_0)\sum_{\ell\neq i}\int_{\frac{x_i}{p_i}}^{G_\ell(x_1,\ldots,x_K)} p_{i\ell} f(y)dy \\
& - c_p\int_{G_i(x_1,\ldots,x_K)}^{\frac{x_i}{p_i}} f(y)dy - c_p\sum_{\ell=1}^{K}\frac{\partial G_\ell}{\partial x_i}\left(G_\ell(x_1,\ldots,x_K)p_\ell - x_\ell\right) f(G_\ell(x_1,\ldots,x_K)) + (s-c) \\
= {} & -(s-c_0)\int_0^{G_i(x_1,\ldots,x_K)} f(y)dy - (s-c_p-c_0)\sum_{\ell\neq i}\int_{\frac{x_i}{p_i}}^{G_\ell(x_1,\ldots,x_K)} p_{i\ell} f(y)dy \\
& - c_p\int_{G_i(x_1,\ldots,x_K)}^{\frac{x_i}{p_i}} f(y)dy + (s-c). \qquad \text{(A11)}
\end{aligned}
$$

Finally, let us show that the optimal $(x_1^*, \ldots, x_K^*)$ satisfies the first-order condition. Given any x_j, $j \neq i$, we can compute that

$$\left.\frac{\partial F}{\partial x_i}\right|_{x_i=0} = s - c > 0,$$

since $G_i = 0$ when $x_i = 0$. Moreover, as $x_i \to \infty$, $G_i \to \infty$ and thus

$$\left.\frac{\partial F}{\partial x_i}\right|_{x_i=\infty} = -(s - c_0) + s - c = c_0 - c < 0,$$

and hence the optimal $x_i^* \in (0, \infty)$, and the optimal $(x_1^*, \ldots, x_K^*)$ satisfies the first-order condition $\frac{\partial F}{\partial x_i}|_{(x_1,\ldots,x_K)=(x_1^*,\ldots,x_K^*)} = 0$, $1 \leq i \leq K$, which gives Equation (4). The proof is complete. □

Proof of Lemma 1. First, we notice that the right-hand side of Equation (5) is monotonically increasing in G_i, it is zero when $G_i = 0$ and infinity when $G_i = \infty$, and thus there exists a unique positive value G_i that solves this equation. Note that $(G_i p_j - x_j)^+ = G_i p_j - x_j$ for any $\frac{x_j}{p_j} \leq G_i$ so that we obtain

$$G_i = \frac{x_i + \sum_{j:x_j/p_j \leq G_i} x_j p_{ji}}{p_i + \sum_{j:x_j/pj \leq G_i} p_j p_{ji}}.$$

Since there exists a unique positive G_i, there exists a unique set $S_i \subseteq \{1, 2, \ldots, K\} \backslash \{i\}$ such that $\frac{x_j}{p_j} \leq \frac{x_i + \sum_{j \in S_i} x_j p_{ji}}{p_i + \sum_{j \in S_i} p_j p_{ji}}$ if and only if $j \in S_i$. Then, we get

$$G_i = \frac{x_i + \sum_{j \in S_i} x_j p_{ji}}{p_i + \sum_{j \in S_i} p_j p_{ji}}, \quad 1 \leq i \leq K,$$

where the set S_i is a function of $(x_k)_{k=1}^K$.

There exists a permutation π of $\{1, 2, \ldots, K\}$ such that $\frac{x_{\pi(1)}}{p_{\pi(1)}} \leq \frac{x_{\pi(2)}}{p_{\pi(2)}} \leq \cdots \leq \frac{x_{\pi(K)}}{p_{\pi(K)}}$. Then $S_i = \{\pi(1), \pi(2), \ldots, \pi(K_i)\}$ for some unique $1 \leq K_i \leq K$, so that

$$G_i = \frac{x_i + \sum_{j=1}^{K_i} x_{\pi(j)} p_{\pi(j)i}}{p_i + \sum_{j=1}^{K_i} p_{\pi(j)} p_{\pi(j)i}}, \quad 1 \leq i \leq K,$$

where

$$K_i := \max\left\{k \in \{1, \ldots, K\} : \frac{x_{\pi(k)}}{p_{\pi(k)}} \leq \frac{x_i + \sum_{j=1}^{k} x_{\pi(j)} p_{\pi(j)i}}{p_i + \sum_{j=1}^{k} p_{\pi(j)} p_{\pi(j)i}}\right\}$$

is a function of $(x_k)_{k=1}^K$. The proof is complete. □

Before we proceed to the proof of Theorem 2, let us first recall a standard lemma characterizing concave functions.

Lemma 2. *The function $q : \mathbb{R}^n \mapsto \mathbb{R}$ is concave if and only if the function $r : \mathbb{R} \mapsto \mathbb{R}$, defined as $r(t) = q(x + tv)$, is a concave function of the single real variable t, where the domain of $r(\cdot)$ is the set of points t such that $x + tv$ belongs to the domain of $q(\cdot)$.*

Now, we are ready to prove Theorem 2.

Proof of Theorem 2. We present a consolidated proof of both parts of the theorem, in two steps. First, in Step 1, we shall show that under the stated condition, the objective function is jointly concave in the decision variables. Then, in Step 2, we shall show that when the objective function is concave, the newsvendor solution is optimal.

Step 1: Recall that from the proof of Theorem 1, we have

$$\begin{aligned} F(x_1, \ldots, x_K) &= (s - c_0) \sum_{i=1}^{K} \int_0^{G_i(x_1,\ldots,x_K)} (yp_i - x_i) f(y) dy \\ &+ (s - c_p - c_0) \sum_{i=1}^{K} \int_0^{G_i(x_1,\ldots,x_K)} \sum_{j \neq i} (yp_j - x_j)^+ p_{ji} f(y) dy \\ &+ c_p \sum_{i=1}^{K} \int_{G_i(x_1,\ldots,x_K)}^{\frac{x_i}{p_i}} (yp_i - x_i) f(y) dy + (s - c) \sum_{i=1}^{K} x_i. \end{aligned}$$

Recall from Equation (5) that G_i is defined via the equations

$$x_i = G_i p_i + \sum_{j \neq i} (G_i p_j - x_j)^+ p_{ji}, \quad 1 \leq i \leq K.$$

Let us define for any t such that $(x_1 + tz_1, \ldots, x_K + tz_K) \in \mathbb{R}_+^K$:

$$
\begin{aligned}
g(t) &:= F((x_1+tz_1),\ldots,(x_K+tz_K)) \\
&= (s-c_0)\sum_{i=1}^{K}\int_0^{G_i((x_1+tz_1),\ldots,(x_K+tz_K))}(yp_i-(x_i+tz_i))f(y)dy \\
&\quad + (s-c_p-c_0)\sum_{i=1}^{K}\int_0^{G_i((x_1+tz_1),\ldots,(x_K+tz_K))}\sum_{j\neq i}(yp_j-(x_j+tz_j))^+p_{ji}f(y)dy \\
&\quad + c_p\sum_{i=1}^{K}\int_{G_i((x_1+tz_1),\ldots,(x_K+tz_K))}^{\frac{x_i+tz_i}{p_i}}(yp_i-(x_i+tz_i))f(y)dy+(s-c)\sum_{i=1}^{K}(x_i+tz_i). \qquad \text{(A12)}
\end{aligned}
$$

We can compute that

$$
\begin{aligned}
\frac{\partial g}{\partial t} &= -(s-c_0)\sum_{i=1}^{K}\int_0^{G_i((x_1+tz_1),\ldots,(x_K+tz_K))}z_if(y)dy \\
&\quad + (s-c_0)\sum_{\ell=1}^{K}\frac{\partial G_\ell}{\partial t}(G_\ell((x_1+tz_1),\ldots,(x_K+tz_K))p_\ell-(x_\ell+tz_\ell)) \\
&\quad \cdot f(G_\ell((x_1+tz_1),\ldots,(x_K+tz_K))) \\
&\quad - (s-c_p-c_0)\sum_{\ell=1}^{K}\frac{\partial G_\ell}{\partial t}(G_\ell((x_1+tz_1),\ldots,(x_K+tz_K))p_\ell-(x_\ell+tz_\ell)) \\
&\quad \cdot f(G_\ell((x_1+tz_1),\ldots,(x_K+tz_K))) \\
&\quad - (s-c_p-c_0)\sum_{i=1}^{K}\sum_{\ell\neq i}\int_{\frac{x_\ell+tz_\ell}{p_\ell}}^{G_i((x_1+tz_1),\ldots,(x_K+tz_K))}z_\ell p_{\ell i}f(y)dy \\
&\quad - c_p\sum_{i=1}^{K}\int_{G_i((x_1+tz_1),\ldots,(x_K+tz_K))}^{\frac{x_i+tz_i}{p_i}}z_if(y)dy \\
&\quad - c_p\sum_{\ell=1}^{K}\frac{\partial G_\ell}{\partial t}(G_\ell((x_1+tz_1),\ldots,(x_K+tz_K))p_\ell-(x_\ell+tz_\ell)) \\
&\quad \cdot f(G_\ell((x_1+tz_1),\ldots,(x_K+tz_K)))+(s-c)\sum_{i=1}^{K}z_i,
\end{aligned}
$$

which yields that

$$
\begin{aligned}
\frac{\partial g}{\partial t} &= -(s-c_0)\sum_{i=1}^{K}\int_0^{G_i((x_1+tz_1),\ldots,(x_K+tz_K))}z_if(y)dy \\
&\quad - (s-c_p-c_0)\sum_{i=1}^{K}\sum_{\ell\neq i}\int_{\frac{x_\ell+tz_\ell}{p_\ell}}^{G_i((x_1+tz_1),\ldots,(x_K+tz_K))}z_\ell p_{\ell i}f(y)dy \\
&\quad - c_p\sum_{i=1}^{K}\int_{G_i((x_1+tz_1),\ldots,(x_K+tz_K))}^{\frac{x_i+tz_i}{p_i}}z_if(y)dy+(s-c)\sum_{i=1}^{K}z_i.
\end{aligned}
$$

Furthermore, we can compute that

$$
\begin{aligned}
\frac{\partial^2 g}{\partial t^2} = & -(s-c_0)\sum_{i=1}^{K}\frac{\partial G_i}{\partial t} z_i f(G_i((x_1+tz_1),\ldots,(x_K+tz_K))) \\
& -(s-c_p-c_0)\sum_{i=1}^{K}\sum_{\ell\neq i}\left[\frac{\partial G_i}{\partial t} f(G_i((x_1+tz_1),\ldots,(x_K+tz_K))) - \frac{z_\ell}{p_\ell} f\left(\frac{x_\ell+tz_\ell}{p_\ell}\right)\right] z_\ell p_{\ell i} \\
& - c_p \sum_{i=1}^{K}\left[\frac{z_i}{p_i} f\left(\frac{x_i+tz_i}{p_i}\right) - \frac{\partial G_i}{\partial t} f(G_i((x_1+tz_1),\ldots,(x_K+tz_K)))\right] z_i \\
= & -(s-c_p-c_0)\sum_{i=1}^{K}\frac{\partial G_i}{\partial t} z_i f(G_i((x_1+tz_1),\ldots,(x_K+tz_K))) \\
& -(s-c_p-c_0)\sum_{i=1}^{K}\sum_{\ell\neq i}\left[\frac{\partial G_i}{\partial t} f(G_i((x_1+tz_1),\ldots,(x_K+tz_K))) - \frac{z_\ell}{p_\ell} f\left(\frac{x_\ell+tz_\ell}{p_\ell}\right)\right] z_\ell p_{\ell i} \\
& - c_p \sum_{i=1}^{K}\left[\frac{z_i}{p_i} f\left(\frac{x_i+tz_i}{p_i}\right)\right] z_i \\
= & -(s-c_p-c_0)\sum_{i=1}^{K}\frac{\partial G_i}{\partial t} z_i f(G_i((x_1+tz_1),\ldots,(x_K+tz_K))) \\
& -(s-c_p-c_0)\sum_{i=1}^{K}\sum_{\ell\neq i}\left[\frac{\partial G_i}{\partial t} f(G_i((x_1+tz_1),\ldots,(x_K+tz_K)))\right] z_\ell p_{\ell i} \\
& +(s-c_p-c_0)\sum_{i=1}^{K}\sum_{\ell\neq i}\left[\frac{z_\ell}{p_\ell} f\left(\frac{x_\ell+tz_\ell}{p_\ell}\right)\right] z_\ell p_{\ell i} - c_p \sum_{i=1}^{K}\left[\frac{z_i}{p_i} f\left(\frac{x_i+tz_i}{p_i}\right)\right] z_i \\
= & -(s-c_p-c_0)\sum_{i=1}^{K}\frac{\partial G_i}{\partial t} z_i f(G_i((x_1+tz_1),\ldots,(x_K+tz_K))) \\
& -(s-c_p-c_0)\sum_{i=1}^{K}\sum_{\ell\neq i}\left[\frac{\partial G_i}{\partial t} f(G_i((x_1+tz_1),\ldots,(x_K+tz_K)))\right] z_\ell p_{\ell i} \\
& + \sum_{i=1}^{K}\left[\frac{z_i}{p_i} f\left(\frac{x_i+tz_i}{p_i}\right)\right] z_i \left((s-c_p-c_0)\sum_{\ell\neq i} p_{i\ell} - c_p\right).
\end{aligned}
$$

Recall from Equation (5) that G_i is defined via the equations

$$x_i = G_i p_i + \sum_{j\neq i}(G_i p_j - x_j)^+ p_{ji}, \quad 1 \le i \le K.$$

This is equivalent to

$$\frac{x_i}{p_i} = G_i + \sum_{j\neq i}\left(G_i - \frac{x_j}{p_j}\right)^+ \frac{p_j}{p_i} p_{ji}, \quad 1 \le i \le K,$$

where

$$\sum_{j\neq i}\left(G_i - \frac{x_j}{p_j}\right)^+ \frac{p_j}{p_i} p_{ji} = \sum_{j\neq i}\max\left(G_i - \frac{x_j}{p_j},\ 0\right)\frac{p_j}{p_i} p_{ji}.$$

Define $\tau_j := G_i - \frac{x_j}{p_j}$. We consider the following three cases. Note that Case 1, Case 2, and Case 3 are mutually exclusive and collectively exhaustive.

Case 1: Suppose $\tau_j \le 0$, for every $j \neq i$. Then it follows from the above equation that:

$$G_i(x_1,\ldots,x_K) = \frac{x_i}{p_i}, \quad 1 \le i \le K.$$

For any t such that $(x_1+tz_1),\ldots,(x_K+tz_K) \in \mathbb{R}_+^K$, let

$$h(t) = G_i((x_1+tz_1),\ldots,(x_K+tz_K)) = \frac{x_i+tz_i}{p_i}, \quad 1 \le i \le K.$$

Then we have

$$\frac{\partial h}{\partial t} = \frac{\partial G_i}{\partial t} > 0.$$

Case 2: Suppose $\tau_j > 0$, for every $j \neq i$. This implies $G_i > \frac{x_j}{p_j}$, for every $j \neq i$, and hence

$$\frac{x_i}{p_i} = G_i + \sum_{j \neq i} \left(G_i - \frac{x_j}{p_j} \right) \frac{p_j}{p_i} p_{ji}, \quad 1 \leq i \leq K,$$

which yields that

$$G_i = \frac{\frac{x_i}{p_i} + \sum_{j \neq i} \left(\frac{x_j}{p_j} \right) \frac{p_j}{p_i} p_{ji}}{1 + \sum_{j \neq i} \frac{p_j}{p_i} p_{ji}}, \quad 1 \leq i \leq K.$$

For any t such that $(x_1 + tz_1), \ldots, (x_K + tz_K) \in \mathbb{R}_+^K$, let

$$h(t) = G_i((x_1 + tz_1), \ldots, (x_K + tz_K)) = \frac{\frac{x + tz_i}{p_i} + \sum_{j \neq i} \left(\frac{x_j + tz_j}{p_j} \right) \frac{p_j}{p_i} p_{ji}}{1 + \sum_{j \neq i} \frac{p_j}{p_i} p_{ji}}, \quad 1 \leq i \leq K.$$

Then we have

$$\frac{\partial h}{\partial t} = \frac{\partial G_i}{\partial t} > 0.$$

Case 3: Suppose $\tau_j \leq 0$, $j = i_1, i_2, \ldots, i_m$ and $\tau_j > 0$, $j \neq i_1, i_2, \ldots, i_m$. Recall that Case 1, Case 2, and Case 3 are mutually exclusive and collectively exhaustive. This implies that $G_i \leq \frac{x_j}{p_j}$, $j = i_1, i_2, \ldots, i_m$ if and only if $G_i = \frac{x_i}{p_i}$, $1 \leq i \leq K$, and $G_i > \frac{x_j}{p_j}$, $j \neq i_1, i_2, \ldots, i_m$. The proof that $\frac{\partial G_i}{\partial t} > 0$ is similar for Case 3 and is thus omitted. Moreover, the following term is negative, that is,

$$\sum_{i=1}^{K} \left[\frac{z_i}{p_i} f\left(\frac{x_i + tz_i}{p_i} \right) \right] z_i \left((s - c_p - c_0) \sum_{\ell \neq i} p_{i\ell} - c_p \right) < 0,$$

which ensures that $\frac{\partial^2 g}{\partial t^2} < 0$. That is $(s - c_p - c_0) \sum_{\ell \neq i} p_{i\ell} - c_p < 0$ or $c_p > \frac{(s - c_0) \sum_{\ell \neq i} p_{i\ell}}{1 + \sum_{\ell \neq i} p_{i\ell}}$, for every $1 \leq i \leq K$, ensures the objective function is jointly concave in the decision variables by Lemma 2.

Step 2: The optimum stocking policy with substitution, that is, the second choice model is characterized by the equations in Theorem 1. Recall from Equation (5) that G_i is defined via the equations

$$x_i = G_i p_i + \sum_{j \neq i} (G_i p_j - x_j)^+ p_{ji}, \quad 1 \leq i \leq K.$$

This is equivalent to

$$\frac{x_i}{p_i} = G_i + \sum_{j \neq i} \left(G_i - \frac{x_j}{p_j} \right)^+ \frac{p_j}{p_i} p_{ji}, \quad 1 \leq i \leq K,$$

where

$$\sum_{j \neq i} \left(G_i - \frac{x_j}{p_j} \right)^+ \frac{p_j}{p_i} p_{ji} = \sum_{j \neq i} \max\left(G_i - \frac{x_j}{p_j}, 0 \right) \frac{p_j}{p_i} p_{ji}, \quad 1 \leq i \leq K.$$

We separately treat the same three cases that we did earlier, which are mutually exclusive and collectively exhaustive. Recall that $\tau_j = G_i - \frac{x_j}{p_j}$.

Case 1: Suppose $\tau_j \leq 0$, for every $j \neq i$. This implies $G_i \leq \frac{x_j}{p_j}$, for every $j \neq i$ so that $G_i = \frac{x_i}{p_i}$, $1 \leq i \leq K$.

Case 2: Suppose $\tau_j > 0$, for every $j \neq i$. This implies $G_i > \frac{x_j}{p_j}$, for every $j \neq i$, and hence

$$\frac{x_i}{p_i} = G_i + \sum_{j \neq i} \left(G_i - \frac{x_j}{p_j} \right) \frac{p_j}{p_i} p_{ji}, \quad 1 \leq i \leq K,$$

which yields

$$G_i = \frac{\frac{x_i}{p_i} + \sum_{j \neq i} \left(\frac{x_j}{p_j} \right) \frac{p_j}{p_i} p_{ji}}{1 + \sum_{j \neq i} \frac{p_j}{p_i} p_{ji}}, \quad 1 \leq i \leq K.$$

Case 3: Suppose $\tau_j \leq 0$, $j = i_1, i_2, \ldots, i_m$ and $\tau_j > 0$, $j \neq i_1, i_2, \ldots, i_m$.

Recall that Case 1, Case 2, and Case 3 are mutually exclusive and collectively exhaustive.

This implies that if $G_i \leq \frac{x_j}{p_j}$, $j = i_1, i_2, \ldots, i_m$, then $G_i = \frac{x_i}{p_i}$, $1 \leq i \leq K$, and if $G_i > \frac{x_j}{p_j}$, for $j \neq i_1, i_2, \ldots, i_m$ then

$$\frac{x_i}{p_i} = G_i + \sum_{j \neq i_1, i_2, \ldots, i_m} \left(G_i - \frac{x_j}{p_j} \right) \frac{p_j}{p_i} p_{ji},$$

which yields that

$$G_i = \frac{\frac{x_i}{p_i} + \sum_{j \neq i_1, i_2, \ldots, i_m} \left(\frac{x_j}{p_j} \right) \frac{p_j}{p_i} p_{ji}}{1 + \sum_{j \neq i_1, i_2, \ldots, i_m} \frac{p_j}{p_i} p_{ji}}.$$

The logic of the proof is as follows: we consider the vector of newsvendor order quantities of the K product variants and show that it also satisfies the necessary conditions for an optimal solution to the substitution model. We remark that if the objective function is jointly concave in the stocking levels of all the product variants, then the necessary conditions are also sufficient.

Suppose the optimal stocking policy when there is no second choice or substitution satisfies the equations defined in Theorem 1 which characterizes the optimal stocking policy with substitution. The optimal stocking policy without substitution is to stock the newsvendor quantity for each product variant (Remark 1). Hence, we have

$$H\left(\frac{x_i^*}{p_i}\right) = \frac{s-c}{s-c_0}, \quad 1 \le i \le K,$$

where $H(\cdot)$ is the distribution function of the demand. This implies

$$\frac{x_1}{p_1} = \frac{x_2}{p_2} = \cdots = \frac{x_K}{p_K},$$

where we used the assumption that $f(\cdot) > 0$ so that $H(\cdot)$ is invertible.

We again separate our analysis into three cases, exactly as before. Recall that $\tau_j = G_i - \frac{x_j}{p_j}$.

Case 1: When $\tau_j \le 0$, for every $j \neq i$, we have $G_i = \frac{x_i}{p_i}$, $1 \le i \le K$, so that

$$G_1 = G_2 = \cdots = G_K = \frac{x_1}{p_1} = \frac{x_2}{p_2} = \cdots = \frac{x_K}{p_K}.$$

Case 2: When $\tau_j > 0$, for every $j \neq i$, we have

$$G_i = \frac{\frac{x_i}{p_i} + \sum_{j\neq i}\left(\frac{x_j}{p_j}\right)\frac{p_j}{p_i}p_{ji}}{1 + \sum_{j\neq i}\frac{p_j}{p_i}p_{ji}} = \frac{\frac{x_i}{p_i} + \left(\frac{x_i}{p_i}\right)\sum_{j\neq i}\frac{p_j}{p_i}p_{ji}}{1 + \sum_{j\neq i}\frac{p_j}{p_i}p_{ji}}$$
$$= \frac{x_i}{p_i}, \quad 1 \le i \le K,$$

which implies that

$$G_1 = G_2 = \cdots = G_K = \frac{x_1}{p_1} = \frac{x_2}{p_2} = \cdots = \frac{x_K}{p_K}.$$

Case 3: When $\tau_j \le 0, \;\; j = i_1, i_2, \ldots, i_m$, we have

$$G_1 = G_2 = \cdots = G_m = \frac{x_1}{p_1} = \frac{x_2}{p_2} = \cdots = \frac{x_m}{p_m},$$

and $\tau_j > 0, \;\; j \neq i_1, i_2, \ldots, i_m$, we have

$$G_i = \frac{\frac{x_i}{p_i} + \sum_{j\neq i_1,i_2,\ldots,i_m}\left(\frac{x_j}{p_j}\right)\frac{p_j}{p_i}p_{ji}}{1 + \sum_{j\neq i_1,i_2,\ldots,i_m}\frac{p_j}{p_i}p_{ji}} = \frac{\frac{x_i}{p_i} + \left(\frac{x_i}{p_i}\right)\sum_{j\neq i_1,i_2,\ldots,i_m}\frac{p_j}{p_i}p_{ji}}{1 + \sum_{j\neq i_1,i_2,\ldots,i_m}\frac{p_j}{p_i}p_{ji}}$$
$$= \frac{x_i}{p_i}.$$

Thus

$$G_1 = G_2 = \cdots = G_K = \frac{x_1}{p_1} = \frac{x_2}{p_2} = \cdots = \frac{x_K}{p_K}.$$

We recall the left-hand side of the Equation (4) from Theorem 1 is given by:

$$(s-c_0)\int_0^{G_i(x_1^*,\ldots,x_K^*)} f(y)dy + (s-c_p-c_0)$$
$$\times \sum_{\ell\neq i}\int_{\frac{x_i^*}{p_i}}^{G_\ell(x_1^*,\ldots,x_K^*)} p_{i\ell}f(y)dy$$
$$+\, c_p \int_{G_i(x_1^*,\ldots,x_K^*)}^{\frac{x_i^*}{p_i}} f(y)dy,$$

for every $1 \le i \le K$. As $G_\ell = \frac{x_\ell}{p_\ell} = \frac{x_i}{p_i} = G_i, \;\; \ell \neq i$, the left-hand side of the Equation (4) reduces to

$$(s-c_0)\int_0^{\frac{x_i^*}{p_i}} f(y)dy, \quad 1 \le i \le K.$$

Now

$$\int_0^{\frac{x_i^*}{p_i}} f(y)dy = H\left(\frac{x_i^*}{p_i}\right) = \frac{s-c}{s-c_0},$$

where $H(\cdot)$ is the distribution function of the demand. Thus the left-hand side reduces to $(s-c)$, which is equal to the right-hand side of the Equation (4) in Theorem 1. The proof is complete. □

Proof of Corollary 1. Since $(s - c_p - c_0)\sum_{\ell\neq i}p_{i\ell} - c_p < 0$, for every $1 \le i \le K$, is a sufficient condition for the optimal solution to be the scaled newsvendor solution, the corollary will follow if

$$\frac{c_p}{(s-c_p-c_0)} \ge 1. \tag{A13}$$

We recall the definition

$$f^* = \frac{(s-c)}{(s-c)+(c-c_0)} = \frac{s-c}{s-c_0}.$$

Thus f^* is equal to the critical fractile in the classic newsvendor formula. We rewrite Equation (A13) as

$$\frac{\theta f^*}{(1-\theta f^*)} \ge 1, \tag{A14}$$

which simplifies to

$$\theta \geq \frac{1}{2f^*}.$$

The corollary follows. □

PROOF OF THEOREM 3. The proof is similar to Theorem 1, hence omitted. □

PROOF OF THEOREM 4. Recall the assumption that $P_i \equiv p_i$ and P_{ji}'s are possibly stochastic. In this case, the fractions for the first choice are deterministic and the fractions for the second choice may be stochastic. Then, we can easily check that the deterministic newsvendor solution

$$(s - c_0) \int_0^{\frac{x_i^*}{p_i}} f(y)dy = s - c, \quad 1 \leq i \leq K, \tag{A15}$$

such that $\frac{x_1^*}{p_1} = \cdots = \frac{x_K^*}{p_K}$ satisfies the Equation (7) with $G_i \equiv x_i^*/p_i$. The conclusions then follows from the same arguments as in the proof of Theorem 2 for the semi-stochastic model. □

PROOF OF THEOREM 5. First, we recall from Theorem 3 that the optimal solution $(x_1^*, \ldots, x_K^*)$ to the second choice problem satisfies the first-order condition:

$$\begin{aligned}(s - c_0)\mathbb{E}\left[\int_0^{G_i(x_1^*,\ldots,x_K^*;P)} f(y)dy\right] &+ (s - c_p - c_0)\mathbb{E}\left[\sum_{\ell \neq i} \int_{\frac{x_i^*}{P_i}}^{G_\ell(x_1^*,\ldots,x_K^*;P)} P_{i\ell} f(y)dy\right] \\ &+ c_p \mathbb{E}\left[\int_{G_i(x_1^*,\ldots,x_K^*;P)}^{\frac{x_i^*}{P_i}} f(y)dy\right] = s - c, \quad 1 \leq i \leq K,\end{aligned} \tag{A16}$$

where the expectations are taken over P_i's and $P_{i\ell}$'s and $G_i = G_i(\cdot; P)$ emphasizes the dependence on P_i's and P_{ji}'s and is defined via the equations: $x_i = G_i P_i + \sum_{j \neq i}(G_i P_j - x_j)^+ P_{ji}$, $1 \leq i \leq K$.

Let us prove by contradiction. Assume the optimal solution was indeed the (stochastic) newsvendor solution:

$$(s - c_0)\mathbb{E}\left[\int_0^{\frac{x_i^*}{P_i}} f(y)dy\right] = s - c, \quad 1 \leq i \leq K. \tag{A17}$$

Then, by plugging Equation (A17) into Equation (A16), we get

$$\begin{aligned}(s - c_0)\mathbb{E}\left[\int_0^{G_i(x_1^*,\ldots,x_K^*;P)} f(y)dy\right] &+ (s - c_p - c_0)\mathbb{E}\left[\sum_{\ell \neq i} \int_{\frac{x_i^*}{P_i}}^{G_\ell(x_1^*,\ldots,x_K^*;P)} P_{i\ell} f(y)dy\right] \\ &+ c_p \mathbb{E}\left[\int_{G_i(x_1^*,\ldots,x_K^*;P)}^{\frac{x_i^*}{P_i}} f(y)dy\right] \\ &= (s - c_0)\mathbb{E}\left[\int_0^{\frac{x_i^*}{P_i}} f(y)dy\right], \quad 1 \leq i \leq K,\end{aligned} \tag{A18}$$

which implies that

$$(s - c_p - c_0)\mathbb{E}\left[\int_{\frac{x_i^*}{P_i}}^{G_i(x_1^*,\ldots,x_K^*;P)} f(y)dy\right] + (s - c_p - c_0)\mathbb{E}\left[\sum_{\ell \neq i} \int_{\frac{x_i^*}{P_i}}^{G_\ell(x_1^*,\ldots,x_K^*;P)} P_{i\ell} f(y)dy\right] = 0, \tag{A19}$$

for any $1 \leq i \leq K$. Since the unit profit selling second choice product $s - c - c_p > 0$ and $c > c_0$, we have $s - c_0 - c_p > 0$ and Equation (A19) becomes:

$$\mathbb{E}\left[\int_{\frac{x_i^*}{P_i}}^{G_i(x_1^*,\ldots,x_K^*;P)} f(y)dy + \sum_{\ell \neq i} \int_{\frac{x_i^*}{P_i}}^{G_\ell(x_1^*,\ldots,x_K^*;P)} P_{i\ell} f(y)dy\right] = 0, \quad 1 \leq i \leq K. \tag{A20}$$

Since $x_i^* = G_iP_i + \sum_{j\neq i}(G_iP_j - x_j^*)^+P_{ji}$, $1 \le i \le K$, we have $G_i \le x_i^*/P_i$ and therefore

$$0 = \mathbb{E}\left[\int_{\frac{x_i^*}{P_i}}^{G_i(x_1^*,\ldots,x_K^*;P)} f(y)dy + \sum_{\ell\neq i}\int_{\frac{x_i^*}{P_i}}^{G_\ell(x_1^*,\ldots,x_K^*;P)} P_{i\ell}f(y)dy\right] \tag{A21}$$

$$\le \mathbb{E}\left[\int_{\frac{x_i^*}{P_i}}^{\frac{x_i^*}{P_i}} f(y)dy + \sum_{\ell\neq i}\int_{\frac{x_i^*}{P_i}}^{\frac{x_\ell^*}{P_\ell}} P_{i\ell}f(y)dy\right] \tag{A22}$$

$$= \mathbb{E}\left[\sum_{\ell\neq i}\int_{\frac{x_i^*}{P_i}}^{\frac{x_\ell^*}{P_\ell}} P_{i\ell}f(y)dy\right] \tag{A23}$$

$$= \sum_{\ell\neq i}\mathbb{E}[P_{i\ell}]\mathbb{E}\left[\int_{\frac{x_i^*}{P_i}}^{\frac{x_\ell^*}{P_\ell}} f(y)dy\right] = 0, \quad 1 \le i \le K, \tag{A24}$$

where we used independence between P_i's and $P_{i\ell}$'s and the equality

$$\mathbb{E}\left[\int_{\frac{x_i^*}{P_i}}^{\frac{x_\ell^*}{P_\ell}} f(y)dy\right] = \mathbb{E}\left[\int_0^{\frac{x_\ell^*}{P_\ell}} f(y)dy\right] - \mathbb{E}\left[\int_0^{\frac{x_i^*}{P_i}} f(y)dy\right] = 0, \tag{A25}$$

which holds since it follows from Equation (A17) that

$$\mathbb{E}\left[\int_0^{\frac{x_1^*}{P_1}} f(y)dy\right] = \mathbb{E}\left[\int_0^{\frac{x_2^*}{P_2}} f(y)dy\right] = \cdots = \mathbb{E}\left[\int_0^{\frac{x_K^*}{P_K}} f(y)dy\right] = \frac{s-c}{s-c_0}. \tag{A26}$$

It follows from Equations (A21)–(A24) that the inequality in Equation (A22) must be the equality, which implies that with probability 1, we have $G_i = x_i^*/P_i$ and $G_\ell = x_\ell^*/P_\ell$, for any $\ell \neq i$. Since $x_i^* = G_iP_i + \sum_{j\neq i}(G_iP_j - x_j^*)^+P_{ji}$, $1 \le i \le K$, we have that with probability 1,

$$\sum_{j\neq i}\left(\frac{x_i^*}{P_i}P_j - x_j^*\right)^+ P_{ji} = 0, \quad 1 \le i \le K, \tag{A27}$$

which implies that with probability 1, $\frac{x_i^*}{P_i} \le \frac{x_j^*}{P_j}$, $1 \le i$, $j \le K$, which further implies that with probability 1,

$$\frac{x_1^*}{P_1} = \frac{x_2^*}{P_2} = \cdots = \frac{x_K^*}{P_K},$$

which leads to the contradiction since x_i^* are deterministic and P_i, $1 \le i \le K$ are fully stochastic in the sense that there exists some $i \neq j$, such that P_i/P_j is stochastic. The proof is complete.

Note

[1]The multi-period, and an infinite horizon setting formulation is included in the online electronic companion. We also develop an exact model in which customers arrive stochastically according to a discretized Poisson (Bernoulli) process in the online electronic companion, which can be viewed as a discrete-time approximation of the exact model with Poisson customer arrivals.

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Supporting Information

Additional supporting information may be found online in the Supporting Information section at the end of the article.

Appendix A: A Exact Model with Discretized Poisson Arrivals.

Appendix B: Multi-Period Exact Model with Poisson Arrivals.

Appendix C: Infinite Horizon.

Appendix D: Simulation Data Table for Figures 1 and 2.

Appendix E: Technical Proofs of Results in Sections A, B and C.

Electronic Companion - Stocking Under Random Demand and Product Variety: Exact Models and Heuristics

Vashkar Ghosh [1], Anand Paul [2], Lingjiong Zhu [3]

A Exact Model with Discretized Poisson Arrivals

We recall that in the exact model with Poisson customer arrivals (section 3.1), the objective function involves solving a high-dimensional integral equation which is not analytically tractable (Proposition 1). In this section, we introduce the exact model with Bernoulli customer arrivals which serves as a discrete-time approximation of the exact model with Poisson arrivals. Solving the problem to be able to find simple analytical solutions still remains challenging. The objective function involves an equation of linear recursion, which is in principle easier to solve numerically than a high-dimensional integral equation. All proofs are in Section E

A.1 Single-period model

In this section, we consider a single period model with the length of the period being $T > 0$. Within this time period $[0, T]$, we divide it into $T/\Delta t$ sub-periods each with length $\Delta t > 0$. Without loss of generality, we assume that $T/\Delta t$ is a positive integer. The customers arrive according to a Bernoulli process in the sense that in each sub-period, with probability $p = \lambda\Delta t$, one customer arrives, and with probability $1 - p = 1 - \lambda\Delta t$, no customer arrives. This discrete-time Bernoulli process on $[0, T]$ approximates the continuous-time Poisson process on $[0, T]$ as $\Delta t \to 0$.

The exact model with Bernoulli (discretized Poisson) arrivals can be formulated as follows.

- Stocking levels $x_1, x_2, \ldots, x_K$ are implemented, where $x_i \in \mathbb{N} \cup \{0\}$ for $i = 1, \ldots, K$.
- Customers arrive according to a Bernoulli (discretized Poisson) process. In each sub-period $[m\Delta t, (m+1)\Delta]$, with $m = 0, 1, 2, \ldots, (T/\Delta t) - 1$, with probability $p = \lambda\Delta t$, exactly one customer arrives, and with probability $1 - p = 1 - \lambda\Delta t$, no customer arrives. When customer #1 enters, and picks her preferred option. The firm registers a profit of $s - c$ for the sold item.

[1] Department of Information Systems and Supply Chain Management, University of North Carolina, Greensboro; v_ghosh@uncg.edu

[2] Department of Information Systems and Operations Management, Warrington College of Business, University of Florida; paulaa@ufl.edu

[3] Department of Mathematics, Florida State University; zhu@math.fsu.edu.

– Customers continue to arrive till the end of the period with time $T > 0$. For any customer, the probability that her preferred option is item i is p_i. If her preferred option is in stock, she purchases it, and the firm registers a profit of $s - c$. Otherwise, she substitutes to her second choice with probability p_{ij} with $\sum_{j \neq i} p_{ij} \leq 1$ if her second choice is available or she will not purchase anything. If the second choice is available, the firm registers a profit $s - c - c_p$.

– Any unsold or leftover inventory at the end of the period is sold to a discount store at c_0 per unit resulting in a loss of $c - c_0$ for each unit that is leftover.

Let $V_m(x_1, \dots, x_K)$ denote the firm's total profit for the single period given that at the start of time $m\Delta t$, there is an initial stocking of $x_1, \dots, x_K$ at time $m\Delta$, so that there are $(T/\Delta t) - m$ sub-periods left till the end of the period T, with $m = 0, 1, 2, \dots, T/\Delta t$, where $x_1, \dots, x_k \in \mathbb{N} \cup \{0\}$ such that $V_m(x_1, \dots, x_K) : (\mathbb{N} \cup \{0\})^K \to \mathbb{R}$, for every $m = 0, 1, 2, \dots, T/\Delta t$.

Proposition A.1. *The optimal stocking levels are given by*

$$(x_1^*, \dots, x_K^*) = \arg \max_{x_1, \dots, x_K \geq 0} V_0(x_1, \dots, x_K), \tag{A.1}$$

where $V_m(x_1, \dots, x_K)$, $m = 0, 1, 2, \dots, T/\Delta t$, solves a $(K+1)$-dimensional recursive equation:

$$\begin{aligned} V_m(x_1, \dots, x_K) = {} & (1 - \lambda\Delta t) V_{m+1}(x_1, \dots, x_K) \\ & + (\lambda\Delta t) \sum_{i: x_i > 0} p_i \left(V_{m+1}(x_1, \dots, x_{i-1}, x_i - 1, x_{i+1}, \dots, x_K) + s - c \right) \\ & + (\lambda\Delta t) \sum_{i: x_i = 0} p_i \sum_{j: x_j > 0} p_{ij} \left(V_{m+1}(x_1, \dots, x_{j-1}, x_j - 1, x_{j+1}, \dots, x_K) + s - c - c_p \right), \end{aligned} \tag{A.2}$$

with the terminal condition $V_{T/\Delta t}(x_1, \dots, x_K) = -\sum_{i=1}^K x_i (c - c_0)$.

We can see from Proposition A.1 that $V(x_1, \dots, x_K; t)$ solves a $(K+1)$-dimensional recursive equation (A.2), and we do not expect (A.2) to yield a simple closed form solution, so we have to rely on Monte Carlo simulations or numerically solving (A.2) to find optimal stocking levels.

A.2 Connection to the exact model with Poisson arrivals

In this section, we discuss the connection between the exact model with Bernoulli (discretized Poisson) arrivals to the exact model with Poisson arrivals (section 3.1). Let us recall that in the exact model with Poisson arrivals, the optimal stocking levels are given by

$$(x_1^*, \ldots, x_K^*) = \arg \max_{x_1, \ldots, x_K \geq 0} V(x_1, \ldots, x_K; 0), \tag{A.3}$$

where $V(x_1, \ldots, x_K; t)$ solves a $(K+1)$-dimensional integral equation:

$$\begin{aligned} V(x_1, \ldots, x_K; t) = &\left(1 - e^{-\lambda(T-t)}\right) \sum_{i:x_i>0} p_i(s-c) + \left(1 - e^{-\lambda(T-t)}\right) \sum_{i:x_i=0} p_i \sum_{j:x_j>0} p_{ij}(s - c - c_p) \\ &- e^{-\lambda(T-t)} \sum_{i=1}^{K} x_i(c - c_0) \\ &+ \sum_{i:x_i>0} p_i \int_t^T \lambda e^{-\lambda(u-t)} V(x_1, \ldots, x_{i-1}, x_i - 1, x_{i+1}, \ldots, x_K; u) du \\ &+ \sum_{i:x_i=0} p_i \sum_{j:x_j>0} p_{ij} \int_t^T \lambda e^{-\lambda(u-t)} V(x_1, \ldots, x_{j-1}, x_j - 1, x_{j+1}, \ldots, x_K; u) du, \end{aligned} \tag{A.4}$$

for any $x_1, \ldots, x_K \in \mathbb{N} \cup \{0\}$ and any $0 \leq t \leq T$. By letting $t = T$ in (A.4), we can see the terminal value is given by

$$V(x_1, \ldots, x_K; T) = -\sum_{i=1}^{K} x_i(c - c_0). \tag{A.5}$$

If we differentiate both hand sides of (A.4) w.r.t. t, we obtain

$$\begin{aligned} \frac{\partial}{\partial t} V(x_1, \ldots, x_K; t) = &-\lambda e^{-\lambda(T-t)} \sum_{i:x_i>0} p_i(s-c) - \lambda e^{-\lambda(T-t)} \sum_{i:x_i=0} p_i \sum_{j:x_j>0} p_{ij}(s - c - c_p) \\ &- \lambda e^{-\lambda(T-t)} \sum_{i=1}^{K} x_i(c - c_0) \\ &+ \lambda \sum_{i:x_i>0} p_i \int_t^T \lambda e^{-\lambda(u-t)} V(x_1, \ldots, x_{i-1}, x_i - 1, x_{i+1}, \ldots, x_K; u) du \\ &- \sum_{i:x_i>0} p_i \lambda V(x_1, \ldots, x_{i-1}, x_i - 1, x_{i+1}, \ldots, x_K; t) \\ &+ \lambda \sum_{i:x_i=0} p_i \sum_{j:x_j>0} p_{ij} \int_t^T \lambda e^{-\lambda(u-t)} V(x_1, \ldots, x_{j-1}, x_j - 1, x_{j+1}, \ldots, x_K; u) du \\ &- \sum_{i:x_i=0} p_i \sum_{j:x_j>0} p_{ij} \lambda V(x_1, \ldots, x_{j-1}, x_j - 1, x_{j+1}, \ldots, x_K; t). \end{aligned} \tag{A.6}$$

By plugging (A.4) into (A.6), we get

$$\begin{aligned}\frac{\partial}{\partial t}V(x_1,\ldots,x_K;t) =& -\sum_{i:x_i>0} p_i\lambda V(x_1,\ldots,x_{i-1},x_i-1,x_{i+1},\ldots,x_K;t)\\ &-\sum_{i:x_i=0} p_i \sum_{j:x_j>0} p_{ij}\lambda V(x_1,\ldots,x_{j-1},x_j-1,x_{j+1},\ldots,x_K;t)\\ &+\lambda\left(V(x_1,\ldots,x_K;t)-\sum_{i:x_i>0}p_i(s-c)-\sum_{i:x_i=0}p_i\sum_{j:x_j>0}p_{ij}(s-c-c_p)\right),\end{aligned}$$

which yields that

$$\begin{aligned}\frac{\partial}{\partial t}V(x_1,\ldots,x_K;t) =& -\lambda\sum_{i:x_i>0} p_i\left(V(x_1,\ldots,x_{i-1},x_i-1,x_{i+1},\ldots,x_K;t)+s-c\right)\\ &-\lambda\sum_{i:x_i=0} p_i \sum_{j:x_j>0} p_{ij}\left(V(x_1,\ldots,x_{j-1},x_j-1,x_{j+1},\ldots,x_K;t)+s-c-c_p\right)\\ &+\lambda V(x_1,\ldots,x_K;t).\end{aligned} \tag{A.7}$$

We notice that in the exact model with Bernoulli (discretized Poisson) arrivals, (A.2) can be re-written as:

$$\begin{aligned}&\frac{V_{m+1}(x_1,\ldots,x_K)-V_m(x_1,\ldots,x_K)}{\Delta t}\\ &=\lambda V_{m+1}(x_1,\ldots,x_K)\\ &\quad-\lambda\sum_{i:x_i>0}p_i\left(V_{m+1}(x_1,\ldots,x_{i-1},x_i-1,x_{i+1},\ldots,x_K)+s-c\right)\\ &\quad-\lambda\sum_{i:x_i=0}p_i\sum_{j:x_j>0}p_{ij}\left(V_{m+1}(x_1,\ldots,x_{j-1},x_j-1,x_{j+1},\ldots,x_K)+s-c-c_p\right).\end{aligned} \tag{A.8}$$

Therefore, (A.8) is the Euler discretization of (A.7). Hence as $\Delta t \to 0$, the objective function in the exact model with Bernoulli (discretized Poisson) arrivals converges to the corresponding objective function in the exact model with Poisson arrivals.

B Multi-Period Exact Model with Poisson Arrivals

In this section, we consider a multi-period model in the setting of the exact model with Poisson customer arrivals (section 3.1). We assume that there are N periods in total and each period is of length $T > 0$. We introduce a discount factor $\rho \in (0, 1)$ and a holding cost $h > 0$ per unit for each product variant. At the end of the final period, any unsold unit will be sold at a loss $c - c_0$.

Proposition B.1. *For any $n = 1, 2, \ldots, N-1$, given any initial stocking $x_1, \ldots, x_K$ at the beginning of the n-th period, the optimal stockings to be added to the inventory are given by*

$$(y^*_{n1}, \ldots, y^*_{nK}) = \arg \max_{y_1,\ldots,y_K \geq 0} \tilde{V}_n(x_1 + y_1, \ldots, x_K + y_K; 0), \tag{B.1}$$

where $\tilde{V}_n$, $n = 1, 2, \ldots, N-1$ are defined backward recursively that satisfy the integral equations:

$$\begin{aligned}
\tilde{V}_n(x_1, \ldots, x_K; t) = {} & \left(1 - e^{-\lambda(T-t)}\right) \sum_{i:x_i>0} p_i(s-c) + \left(1 - e^{-\lambda(T-t)}\right) \sum_{i:x_i=0} p_i \sum_{j:x_j>0} p_{ij}(s-c-c_p) \\
& + \rho e^{-\lambda(T-t)} V^*_{n+1}(x_1, \ldots, x_K) - e^{-\lambda(T-t)} \sum_{i=1}^{K} x_i h \\
& + \sum_{i:x_i>0} p_i \int_t^T \lambda e^{-\lambda(u-t)} \tilde{V}_n(x_1, \ldots, x_{i-1}, x_i - 1, x_{i+1}, \ldots, x_K; u) du \\
& + \sum_{i:x_i=0} p_i \sum_{j:x_j>0} p_{ij} \int_t^T \lambda e^{-\lambda(u-t)} \tilde{V}_n(x_1, \ldots, x_{j-1}, x_j - 1, x_{j+1}, \ldots, x_K; u) du,
\end{aligned} \tag{B.2}$$

for any $x_1, \ldots, x_K \in \mathbb{N} \cup \{0\}$ and any $0 \leq t \leq T$, where

$$V^*_{n+1}(x_1, \ldots, x_K) = \max_{y_1,\ldots,y_K \geq 0} \tilde{V}_{n+1}(x_1 + y_1, \ldots, x_K + y_K; 0), \tag{B.3}$$

for any $x_1, \ldots, x_K \in \mathbb{N} \cup \{0\}$ and $\tilde{V}_N$ satisfies the integral equation (2).

If we assume in Proposition B.1 that the initial stocking level at the beginning of the first period is zero prior to making any decision, then the optimal expected profit of the firm at time zero is given by $V^*_1(0, 0, \ldots, 0)$. We do not expect (B.2)-(B.3) to yield a simple closed form solution, so we have to rely on Monte Carlo simulations or numerically solving (B.2)-(B.3) to find optimal stocking levels.

Finally, we remark that the exact model with Bernoulli (discretized Poisson) arrivals (section A) can be extended to the multi-period case just as in the exact model with Poisson arrivals. We omit the details here and do not include discussions on the multi-period case for the benchmark model (section 3.2) since closed-form solutions do not seem to be available for the multi-period benchmark model.

C Infinite Horizon

C.1 Exact model with Poisson arrivals

In this section, we consider an infinite-horizon exact model with Poisson arrivals which is equivalent to a multi-period model with $N = \infty$ and each period has length $T > 0$. We recall that $0 < \rho < 1$ is a discount factor so that the present value of 1 USD from the n-th period is ρ^{n-1} USD and $h > 0$ is a holding cost per unit for each product variant.

Proposition C.1. *Given any initial stocking $x_1, \dots, x_K$ at time zero, the optimal stocking level is given by*

$$(y_1^*, \dots, y_K^*) = \arg \max_{y_1,\dots,y_K \geq 0} \tilde{V}_\infty(x_1 + y_1, \dots, x_K + y_K; 0), \tag{C.1}$$

where $\tilde{V}_\infty$ satisfies the integral equation:

$$\begin{aligned}\tilde{V}_\infty(x_1, \dots, x_K; t) = \left(1 - e^{-\lambda(T-t)}\right) \sum_{i:x_i>0} p_i(s-c) + \left(1 - e^{-\lambda(T-t)}\right) \sum_{i:x_i=0} p_i \sum_{j:x_j>0} p_{ij}(s - c - c_p) \\ + \rho e^{-\lambda(T-t)} V_\infty^*(x_1, \dots, x_K) - e^{-\lambda(T-t)} \sum_{i=1}^{K} x_i h \\ + \sum_{i:x_i>0} p_i \int_t^T \lambda e^{-\lambda(u-t)} \tilde{V}_\infty(x_1, \dots, x_{i-1}, x_i - 1, x_{i+1}, \dots, x_K; u) du \\ + \sum_{i:x_i=0} p_i \sum_{j:x_j>0} p_{ij} \int_t^T \lambda e^{-\lambda(u-t)} \tilde{V}_\infty(x_1, \dots, x_{j-1}, x_j - 1, x_{j+1}, \dots, x_K; u) du,\end{aligned} \tag{C.2}$$

for any $x_1, \dots, x_K \in \mathbb{N} \cup \{0\}$ and any $0 \leq t \leq T$, where V_∞^* satisfies

$$V_\infty^*(x_1, \dots, x_K) = \max_{y_1,\dots,y_K \geq 0} \tilde{V}_\infty(x_1 + y_1, \dots, x_K + y_K; 0), \tag{C.3}$$

for any $x_1, \dots, x_K \in \mathbb{N} \cup \{0\}$.

It follows from Proposition C.1 that given the initial stocking levels $x_1, \dots, x_K$, the optimal expected profit for the firm at time zero is given by $V_\infty^*(x_1, \dots, x_K)$ defined in (C.3). In particular, if the stocking level is zero prior to making any decision about the initial stocking level at the beginning of the first period, then the value function is given by $V_\infty^*(0, 0, \dots, 0)$. We do not expect (C.2)-(C.3) to yield a simple closed form solution, so we have to rely on Monte Carlo simulations or numerically solving (C.2)-(C.3) to find optimal stocking levels.

Finally, we remark that we can extend the exact model with Bernoulli (discretized Poisson) arrivals to the infinite-horizon case just as in the exact model with Poisson arrivals. We omit the details here.

C.2 Benchmark model

We consider the same problem as in the previous section, but for the benchmark model setting (section 3.2). At time zero, assume that there is a given inventory $(a_1, \ldots, a_K)$ where $a_1, \ldots, a_K \geq 0$ prior to any management decision, and let $V(a_1, \ldots, a_K)$ be its associated value function, that is the expected present value of the profit of the firm over an infinite horizon given the initial inventory $(a_1, \ldots, a_K)$[4]. As in the single-period model, let us recall that the unit cost of each product variety stocked is c, the selling price is s. We first consider the semi-stochastic model, whose first choice demand for product variety i is D_i, where $D_i = Dp_i$, and D is the total demand with a continuously differentiable probability density function $f(\cdot) > 0$, and the demand of product variety i from second choice is given by $\sum_{j \neq i}(D_j - x_j)^+ p_{ji}$. Let us define:

$$\begin{aligned}
(x_1^*, \ldots, x_K^*) := \arg \max_{x_1, \ldots, x_K \geq 0} \Bigg\{ & -c \sum_{i=1}^K x_i \\
& + \sum_{i=1}^K \mathbb{E}\left[\left(sD_i + (s - c_p)\sum_{j \neq i}(D_j - x_j)^+ p_{ji}\right) 1_{x_i > D_i + \sum_{j \neq i}(D_j - x_j)^+ p_{ji}}\right] \\
& - \sum_{i=1}^K \mathbb{E}\left[h\left(x_i - D_i - \sum_{j \neq i}(D_j - x_j)^+ p_{ji}\right) 1_{x_i > D_i + \sum_{j \neq i}(D_j - x_j)^+ p_{ji}}\right] \\
& + \sum_{i=1}^K \mathbb{E}\left[(sD_i + (s - c_p)(x_i - D_i))1_{D_i < x_i < D_i + \sum_{j \neq i}(D_j - x_j)^+ p_{ji}}\right] \\
& + \sum_{i=1}^K \mathbb{E}[s x_i 1_{x_i < D_i}] + \rho c \sum_{i=1}^K \mathbb{E}\left[\left(x_i - D_i - \sum_{j \neq i}(D_j - x_j)^+ p_{ji}\right)^+\right] \Bigg\}. \qquad \text{(C.4)}
\end{aligned}$$

We will show later that $(x_1^*, \ldots, x_K^*)$ is indeed an optimal base-stock policy.

Theorem C.1. *For any $a_i \leq x_i^*$, $1 \leq i \leq K$, we have*

$$V(a_1, \ldots, a_K) = V(0, \ldots, 0) + c \sum_{i=1}^K a_i,$$

[4]In our model, there is no inventory prior to any management decision, and hence $V(0, \ldots, 0)$ is the value function we need. We introduce a more general $V(a_1, \ldots, a_K)$ here only because we want to use dynamic programming.

where

$$
\begin{aligned}
V(0,\ldots,0) = \frac{1}{1-\rho} \max_{x_1,\ldots,x_K \geq 0} \Bigg\{ & -c\sum_{i=1}^{K} x_i \\
& + \sum_{i=1}^{K} \mathbb{E}\left[\left(sD_i + (s-c_p)\sum_{j\neq i}(D_j - x_j)^+ p_{ji} \right) 1_{x_i > D_i + \sum_{j\neq i}(D_j - x_j)^+ p_{ji}} \right] \\
& - \sum_{i=1}^{K} \mathbb{E}\left[h\left(x_i - D_i - \sum_{j\neq i}(D_j - x_j)^+ p_{ji} \right) 1_{x_i > D_i + \sum_{j\neq i}(D_j - x_j)^+ p_{ji}} \right] \\
& + \sum_{i=1}^{K} \mathbb{E}\left[(sD_i + (s-c_p)(x_i - D_i)) 1_{D_i < x_i < D_i + \sum_{j\neq i}(D_j - x_j)^+ p_{ji}} \right] \\
& + \sum_{i=1}^{K} \mathbb{E}[s x_i 1_{x_i < D_i}] + \rho c \sum_{i=1}^{K} \mathbb{E}\left[\left(x_i - D_i - \sum_{j\neq i}(D_j - x_j)^+ p_{ji} \right)^+ \right] \Bigg\},
\end{aligned} \tag{C.5}
$$

and the optimal strategy is a base-stock policy $(x_1^*, \ldots, x_K^*)$.

Given a base-stock policy with base-stock levels $(x_1, \ldots, x_K)$, let $R(x_1, \ldots, x_K)$ be the expected value (i.e. the expected present value of the profit of the firm over an infinite horizon) given that there is zero inventory prior to any management decision at time zero and a base-stock policy $(x_1, \ldots, x_K)$, then, we have the following result which computes $R(x_1, \ldots, x_K)$ and as a corollary the value function equals to maximizing the expected value over all the base-stock policy and as a result, the optimal policy is indeed a base-stock policy.

Theorem C.2.

$$
\begin{aligned}
R(x_1,\ldots,x_K) = & \frac{1}{1-\rho}\sum_{i=1}^{K} \mathbb{E}\left[\left(sD_i + (s-c_p)\sum_{j\neq i}(D_j - x_j)^+ p_{ji} \right) 1_{x_i > D_i + \sum_{j\neq i}(D_j - x_j)^+ p_{ji}} \right] \\
& - \frac{1}{1-\rho}\sum_{i=1}^{K} \mathbb{E}\left[h\left(x_i - D_i - \sum_{j\neq i}(D_j - x_j)^+ p_{ji} \right) 1_{x_i > D_i + \sum_{j\neq i}(D_j - x_j)^+ p_{ji}} \right] \\
& + \frac{1}{1-\rho}\sum_{i=1}^{K} \mathbb{E}\left[(sD_i + (s-c_p)(x_i - D_i)) 1_{D_i < x_i < D_i + \sum_{j\neq i}(D_j - x_j)^+ p_{ji}} \right] \\
& + \frac{1}{1-\rho}\sum_{i=1}^{K} \mathbb{E}[s x_i 1_{x_i < D_i}] \\
& + \frac{\rho}{1-\rho} c \sum_{i=1}^{K} \mathbb{E}\left[\left(x_i - D_i - \sum_{j\neq i}(D_j - x_j)^+ p_{ji} \right)^+ \right] - \frac{c}{1-\rho}\sum_{i=1}^{K} x_i,
\end{aligned}
$$

so that $V(0,\dots,0)=\max_{x_1,\dots,x_K\ge 0} R(x_1,\dots,x_K)$ *and*

$$(x_1^*,\dots,x_K^*)=\arg\max_{x_1,\dots,x_K\ge 0} R(x_1,\dots,x_K).$$

In the next result, we solve for the optimal $(x_1^*,\dots,x_K^*)$.

Theorem C.3. *The optimal* $(x_1^*,\dots,x_K^*)$ *satisfies the equations:*

$$\begin{aligned}(s-c\rho+h)\int_0^{G_i(x_1^*,\dots,x_K^*)} f(y)dy+(s-c_p-c\rho+h)\sum_{\ell\neq i}\int_{\frac{x_i^*}{p_i}}^{G_\ell(x_1^*,\dots,x_K^*)} p_{i\ell}f(y)dy \\ +c_p\int_{G_i(x_1^*,\dots,x_K^*)}^{\frac{x_i^*}{p_i}} f(y)dy=s-c,\qquad 1\le i\le K,\end{aligned}\tag{C.6}$$

where G_i *is defined via the equations:*

$$x_i=G_ip_i+\sum_{j\neq i}(G_ip_j-x_j)^+p_{ji},\qquad 1\le i\le K.\tag{C.7}$$

Finally, we can extend the result in Theorem C.3 to the setting of the fully stochastic model.

Theorem C.4. *The optimal* $(x_1^*,\dots,x_K^*)$ *satisfies the equations:*

$$\begin{aligned}(s-c\rho+h)\mathbb{E}\left[\int_0^{G_i(x_1^*,\dots,x_K^*;P)} f(y)dy\right]+(s-c_p-c\rho+h)\sum_{\ell\neq i}\mathbb{E}\left[\int_{\frac{x_i^*}{P_i}}^{G_\ell(x_1^*,\dots,x_K^*;P)} P_{i\ell}f(y)dy\right] \\ +c_p\mathbb{E}\left[\int_{G_i(x_1^*,\dots,x_K^*;P)}^{\frac{x_i^*}{P_i}} f(y)dy\right]=s-c,\qquad 1\le i\le K,\end{aligned}$$

where the expectations are taken over P_i*'s and* $P_{i\ell}$*'s and* $G_i=G_i(\cdot;P)$ *emphasizes the dependence on* P_i*'s and* P_{ji}*'s and is defined via the equations:*

$$x_i=G_iP_i+\sum_{j\neq i}(G_iP_j-x_j)^+P_{ji},\qquad 1\le i\le K.$$

Remark C.1. *It follows from Theorem C.3 and the analysis in the single-period model that the analogous results of Theorem 2 hold in the infinite horizon semi-stochastic model as well and it follows from Theorem C.4 and the analysis in the single-period model that the analogous results of Theorem 4 and Theorem 5 hold in the infinite horizon fully stochastic model as well.*

D Simulation Data Table for Figure 1 and 2

Number of Products (K)	Demand λ	Service Levels 0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
	150	5.15	4.66	3.94	3.38	2.78	2.30	1.56	0.89
2	200	4.52	4.03	3.55	2.92	2.43	1.91	1.24	0.72
	250	4.12	3.45	3.16	2.62	2.13	1.72	1.17	0.67
	150	5.20	4.71	3.85	3.39	2.57	2.14	1.42	0.83
4	200	4.73	4.23	3.40	3.02	2.38	1.87	1.34	0.72
	250	4.16	3.56	3.05	2.65	2.02	1.68	1.12	0.66

Table 1: Data Table Figure 1.

Number of Products (K)	Demand λ	Service Levels 0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
	150	5.42	4.58	4.22	3.47	2.97	2.31	1.68	0.92
2	200	4.60	4.03	3.67	2.94	2.55	1.95	1.40	0.76
	250	4.21	3.66	3.20	2.67	2.20	1.78	1.21	0.67
	150	5.59	4.90	4.43	3.54	3.09	2.18	1.71	0.91
4	200	4.76	4.35	3.83	3.28	2.61	1.99	1.50	0.80
	250	4.31	3.61	3.19	2.80	2.31	1.50	0.95	0.62

Table 2: Data Table Figure 2.

E Technical Proofs of Results in Sections A, B and C

Proof of Proposition A.1 In the sub-period $[m\Delta t, (m+1)\Delta t]$, with probability $p = \lambda\Delta t$, one customer arrives, and with probability $1 - p = 1 - \lambda\Delta t$, no customer arrives. In no customer arrives, nothing happens during this sub-period, and the stock carries onto the next sub-period. If one customer arrives, the probability that this particular customer's preferred option is item i is p_i. If item i in stock, she purchases it, and the firm registers a profit of $s-c$. Otherwise, she substitutes to her second choice with probability p_{ij} with $\sum_{j\neq i} p_{ij} \leq 1$ if her second choice is available or she will not purchase anything. If the second choice is available, the firm registers a profit $s - c - c_p$.

Therefore, we obtain the following recursion:

$$\begin{aligned}V_m(x_1,\ldots,x_K) &= (1-\lambda\Delta t)V_{m+1}(x_1,\ldots,x_K)\\ &+ (\lambda\Delta t)\sum_{i:x_i>0} p_i\left(V_{m+1}(x_1,\ldots,x_{i-1},x_i-1,x_{i+1},\ldots,x_K)+s-c\right)\\ &+ (\lambda\Delta t)\sum_{i:x_i=0} p_i \sum_{j:x_j>0} p_{ij}\left(V_{m+1}(x_1,\ldots,x_{j-1},x_j-1,x_{j+1},\ldots,x_K)+s-c-c_p\right).\end{aligned}$$

Finally, notice that if we are already at the end of last sub-period, all the remaining items will be sold to a discount store at a loss of $c-c_0$ for each unit and that gives us

$$V_{T/\Delta t}(x_1,\ldots,x_K) = -\sum_{i=1}^{K} x_i(c-c_0).$$

The proof is complete. □

Proof of Proposition B.1. Let $V_n(x_1,\ldots,x_K;t)$ denote the firm's total profit for the n-th period given that time t has passed since the start of the n-th period, where $0 \le t \le T$, there is an initial stocking level $x_1,\ldots,x_K$ at time t. By following the result from the single period model (Proposition 1), we have

$$\begin{aligned}&V_N(x_1,\ldots,x_K;t)\\ &= \left(1-e^{-\lambda(T-t)}\right)\sum_{i:x_i>0} p_i(s-c) + \left(1-e^{-\lambda(T-t)}\right)\sum_{i:x_i=0} p_i \sum_{j:x_j>0} p_{ij}(s-c-c_p)\\ &\qquad - e^{-\lambda(T-t)}\sum_{i=1}^{K} x_i(c-c_0)\\ &\quad + \sum_{i:x_i>0} p_i \int_t^T \lambda e^{-\lambda(u-t)} V_N(x_1,\ldots,x_{i-1},x_i-1,x_{i+1},\ldots,x_K;u)du\\ &\quad + \sum_{i:x_i=0} p_i \sum_{j:x_j>0} p_{ij} \int_t^T \lambda e^{-\lambda(u-t)} V_N(x_1,\ldots,x_{j-1},x_j-1,x_{j+1},\ldots,x_K;u)du, \qquad \text{(E.1)}\end{aligned}$$

with the terminal condition $V_N(x_1,\ldots,x_K;T) = -\sum_{i=1}^{K} x_i(c-c_0)$.

Next, consider $n = 1,2,\ldots,N-1$. Suppose at the beginning of the n-th period, stocking levels are boosted by $y_{n1}, \dots, y_{nK}$, where $n = 1, 2, \dots, N$. Then, for any $n = 1, 2, \dots, N-1$, we have

$$\begin{aligned} &V_n(x_1, \dots, x_K; t) \\ &= \left(1 - e^{-\lambda(T-t)}\right) \sum_{i:x_i>0} p_i(s-c) + \left(1 - e^{-\lambda(T-t)}\right) \sum_{i:x_i=0} p_i \sum_{j:x_j>0} p_{ij}(s-c-c_p) \\ &\qquad\qquad + \rho e^{-\lambda(T-t)} V_{n+1}(x_1 + y_{(n+1)1}, \dots, x_K + y_{(n+1)K}; 0) - e^{-\lambda(T-t)} \sum_{i=1}^{K} x_i h \\ &\quad + \sum_{i:x_i>0} p_i \int_t^T \lambda e^{-\lambda(u-t)} V_n(x_1, \dots, x_{i-1}, x_i - 1, x_{i+1}, \dots, x_K; u) du \\ &\quad + \sum_{i:x_i=0} p_i \sum_{j:x_j>0} p_{ij} \int_t^T \lambda e^{-\lambda(u-t)} V_n(x_1, \dots, x_{j-1}, x_j - 1, x_{j+1}, \dots, x_K; u) du. \end{aligned} \tag{E.2}$$

The result (E.2) essentially follows from the proof of Proposition 1 and the only part unique here is the term $\rho e^{-\lambda(T-t)} V_{n+1}(x_1 + y_{(n+1)1}, \dots, x_K + y_{(n+1)K}; 0) - e^{-\lambda(T-t)} \sum_{i=1}^{K} x_i h$, which is obtained by considering the event that no customer arrives in the n-th period which occurs with the probability $e^{-\lambda(T-t)}$, and then $V_{n+1}(x_1 + y_{(n+1)1}, \dots, x_K + y_{(n+1)K}; 0)$ gives the expected profit at the beginning of the $(n+1)$-th period with the stocking levels increased by $y_{(n+1)1}, \dots, y_{(n+1)K}$, and $-e^{-\lambda(T-t)} \sum_{i=1}^{K} x_i h$ computes the holding cost at the end of the n-th period if nothing is sold during the n-th period, and finally the factor ρ is used the discount the value from the $(n+1)$-th period to the n-th period.

Next, we consider the optimal stocking levels $y^*_{n1}, \dots, y^*_{nK}$ for every $n = 1, 2, \dots, N$. Given any initial stocking $x_1, \dots, x_K$ at the beginning of the N-th period, we have

$$(y^*_{N1}, \dots, y^*_{NK}) = \arg \max_{y_1, \dots, y_K} V_N(x_1 + y_1, \dots, x_K + y_K; 0), \tag{E.3}$$

and we define

$$V^*_N(x_1, \dots, x_K) = \max_{y_1, \dots, y_K} V_N(x_1 + y_1, \dots, x_K + y_K; 0). \tag{E.4}$$

Moreover, for any $n = 1, 2, \dots, N-1$, given any initial stocking $x_1, \dots, x_K$ at the beginning of the n-th period we have

$$(y^*_{n1}, \dots, y^*_{nK}) = \arg \max_{y_1, \dots, y_K} \tilde{V}_n(x_1 + y_1, \dots, x_K + y_K; 0), \tag{E.5}$$

and

$$V_n^*(x_1, \ldots, x_K) = \max_{y_1,\ldots,y_K} \tilde{V}_n(x_1 + y_1, \ldots, x_K + y_K; 0), \tag{E.6}$$

where $\tilde{V}_n$ satisfies the integral equation:

$$\begin{aligned}\tilde{V}_n(x_1, \ldots, x_K; t) = \left(1 - e^{-\lambda(T-t)}\right) \sum_{i:x_i>0} p_i(s-c) + \left(1 - e^{-\lambda(T-t)}\right) \sum_{i:x_i=0} p_i \sum_{j:x_j>0} p_{ij}(s - c - c_p) \\ + \rho e^{-\lambda(T-t)} V_{n+1}^*(x_1, \ldots, x_K) - e^{-\lambda(T-t)} \sum_{i=1}^{K} x_i h \\ + \sum_{i:x_i>0} p_i \int_t^T \lambda e^{-\lambda(u-t)} \tilde{V}_n(x_1, \ldots, x_{i-1}, x_i - 1, x_{i+1}, \ldots, x_K; u) du \\ + \sum_{i:x_i=0} p_i \sum_{j:x_j>0} p_{ij} \int_t^T \lambda e^{-\lambda(u-t)} \tilde{V}_n(x_1, \ldots, x_{j-1}, x_j - 1, x_{j+1}, \ldots, x_K; u) du,\end{aligned} \tag{E.7}$$

and $\tilde{V}_N := V_N$ which is defined in (E.1). □

Proof of Proposition C.1. The infinite-horizon model can be viewed as a multi-period model with $N = \infty$ and each period has length $T > 0$. Therefore, the result can be obtained following similar argument as in the proof of Proposition B.1 and hence the proof is omitted here. □

Proof of Theorem C.1. We shall first prove that there exists an optimal base-stock policy; then, we shall restrict ourselves to the class of base-stock policies $(x_1, \ldots, x_K)$ and optimize over $(x_1, \ldots, x_K)$ to find the optimal base-stock levels $(x_1^*, \ldots, x_K^*)$. At time zero, if there is a given inventory $(a_1, \ldots, a_K)$ prior to any management decision, where $a_1, \ldots, a_K \geq 0$, then recall that $V(a_1, \ldots, a_K)$ is the associated value function. Using the Bellman recursion from dynamic programming, we have

$$
\begin{aligned}
&V(a_1, \ldots, a_K) \qquad \text{(E.8)}\\
&= \max_{y_1,\ldots,y_K \geq 0} \Bigg\{ -c \sum_{i=1}^{K} y_i \\
&\quad + \sum_{i=1}^{K} \mathbb{E}\left[\left(sD_i + (s-c_p) \sum_{j \neq i} (D_j - a_j - y_j)^+ p_{ji} \right) 1_{a_i + y_i > D_i + \sum_{j \neq i}(D_j - a_j - y_j)^+ p_{ji}} \right] \\
&- \sum_{i=1}^{K} \mathbb{E}\left[h \left(a_i + y_i - D_i - \sum_{j \neq i} (D_j - a_j - y_j)^+ p_{ji} \right) 1_{a_i + y_i > D_i + \sum_{j \neq i}(D_j - a_j - y_j)^+ p_{ji}} \right] \\
&\quad + \sum_{i=1}^{K} \mathbb{E}\left[(sD_i + (s-c_p)(a_i + y_i - D_i)) 1_{D_i < a_i + y_i < D_i + \sum_{j \neq i}(D_j - a_j - y_j)^+ p_{ji}} \right] \\
&\quad + \sum_{i=1}^{K} \mathbb{E}[s(a_i + y_i) 1_{a_i + y_i < D_i}] \\
&\qquad + \rho \mathbb{E} V \Bigg(\left(a_1 + y_1 - D_1 - \sum_{j \neq 1} (D_j - a_j - y_j)^+ p_{j1} \right)^+, \\
&\qquad\qquad \ldots, \left(a_K + y_K - D_K - \sum_{j \neq K} (D_j - a_j - y_j)^+ p_{jK} \right)^+ \Bigg) \Bigg\}.
\end{aligned}
$$

We claim that

$$
V(a_1, \ldots, a_K) = V(0, \ldots, 0) + c \sum_{i=1}^{K} a_i, \qquad \text{(E.9)}
$$

where $V(0, \ldots, 0)$ has an expression given in equation (C.5).

Plugging equation (E.9) into equation (E.8), we obtain

$$
\begin{aligned}
&V(0,\dots,0) + c\sum_{i=1}^{K} a_i \\
&= \max_{y_1,\dots,y_K \geq 0} \Bigg\{ - c\sum_{i=1}^{K} y_i \\
&\quad + \sum_{i=1}^{K} \mathbb{E}\left[\left(sD_i + (s - c_p)\sum_{j\neq i}(D_j - a_j - y_j)^+ p_{ji} \right) 1_{a_i + y_i > D_i + \sum_{j\neq i}(D_j - a_j - y_j)^+ p_{ji}} \right] \\
&- \sum_{i=1}^{K} \mathbb{E}\left[h\left(y_i - D_i - \sum_{j\neq i}(D_j - y_j)^+ p_{ji} \right) 1_{y_i > D_i + \sum_{j\neq i}(D_j - y_j)^+ p_{ji}} \right] \\
&\quad + \sum_{i=1}^{K} \mathbb{E}\left[(sD_i + (s - c_p)(a_i + y_i - D_i)) 1_{D_i < a_i + y_i < D_i + \sum_{j\neq i}(D_j - a_j - y_j)^+ p_{ji}} \right] \\
&\quad + \sum_{i=1}^{K} \mathbb{E}[s(a_i + y_i) 1_{a_i + y_i < D_i}] \\
&\qquad + \rho c \sum_{i=1}^{K} \mathbb{E}\left[\left(a_i + y_i - D_i - \sum_{j\neq i}(D_j - a_j - y_j)^+ p_{ji} \right)^+ \right] + \rho V(0,\dots,0) \Bigg\},
\end{aligned}
$$

which is equivalent to

$$
\begin{aligned}
&V(0,\dots,0) \\
&= \frac{1}{1-\rho} \max_{y_1,\dots,y_K \geq 0} \Bigg\{ - c\sum_{i=1}^{K} (a_i + y_i) \\
&\quad + \sum_{i=1}^{K} \mathbb{E}\left[\left(sD_i + (s - c_p)\sum_{j\neq i}(D_j - a_j - y_j)^+ p_{ji} \right) 1_{a_i + y_i > D_i + \sum_{j\neq i}(D_j - a_j - y_j)^+ p_{ji}} \right] \\
&\qquad - \sum_{i=1}^{K} \mathbb{E}\left[h\left(y_i - D_i - \sum_{j\neq i}(D_j - y_j)^+ p_{ji} \right) 1_{y_i > D_i + \sum_{j\neq i}(D_j - y_j)^+ p_{ji}} \right] \\
&\quad + \sum_{i=1}^{K} \mathbb{E}\left[(sD_i + (s - c_p)(a_i + y_i - D_i)) 1_{D_i < a_i + y_i < D_i + \sum_{j\neq i}(D_j - a_j - y_j)^+ p_{ji}} \right] \\
&\quad + \sum_{i=1}^{K} \mathbb{E}[s(a_i + y_i) 1_{a_i + y_i < D_i}] + \rho c \sum_{i=1}^{K} \mathbb{E}\left[\left(a_i + y_i - D_i - \sum_{j\neq i}(D_j - a_j - y_j)^+ p_{ji} \right)^+ \right] \Bigg\}.
\end{aligned}
$$

Let

$$
\begin{aligned}
(x_1^*, \ldots, x_K^*) = \arg \max_{x_1,\ldots,x_K \geq 0} \Bigg\{ & - c \sum_{i=1}^{K} x_i \\
& + \sum_{i=1}^{K} \mathbb{E} \left[\left(sD_i + (s - c_p) \sum_{j \neq i} (D_j - x_j)^+ p_{ji} \right) 1_{x_i > D_i + \sum_{j \neq i} (D_j - x_j)^+ p_{ji}} \right] \\
& - \sum_{i=1}^{K} \mathbb{E} \left[h \left(x_i - D_i - \sum_{j \neq i} (D_j - x_j)^+ p_{ji} \right) 1_{x_i > D_i + \sum_{j \neq i} (D_j - x_j)^+ p_{ji}} \right] \\
& + \sum_{i=1}^{K} \mathbb{E} \left[(sD_i + (s - c_p)(x_i - D_i)) 1_{D_i < x_i < D_i + \sum_{j \neq i} (D_j - x_j)^+ p_{ji}} \right] \\
& + \sum_{i=1}^{K} \mathbb{E}[s x_i 1_{x_i < D_i}] + \rho c \sum_{i=1}^{K} \mathbb{E} \left[\left(x_i - D_i - \sum_{j \neq i} (D_j - x_j)^+ p_{ji} \right)^+ \right] \Bigg\}.
\end{aligned}
$$

Then, for any $a_i \leq x_i^*$, $1 \leq i \leq K$, we have

$$
\begin{aligned}
V(0, \ldots, 0) = \frac{1}{1 - \rho} \max_{x_1,\ldots,x_K \geq 0} \Bigg\{ & - c \sum_{i=1}^{K} x_i \\
& + \sum_{i=1}^{K} \mathbb{E} \left[\left(sD_i + (s - c_p) \sum_{j \neq i} (D_j - x_j)^+ p_{ji} \right) 1_{x_i > D_i + \sum_{j \neq i} (D_j - x_j)^+ p_{ji}} \right] \\
& - \sum_{i=1}^{K} \mathbb{E} \left[h \left(x_i - D_i - \sum_{j \neq i} (D_j - x_j)^+ p_{ji} \right) 1_{x_i > D_i + \sum_{j \neq i} (D_j - x_j)^+ p_{ji}} \right] \\
& + \sum_{i=1}^{K} \mathbb{E} \left[(sD_i + (s - c_p)(x_i - D_i)) 1_{D_i < x_i < D_i + \sum_{j \neq i} (D_j - x_j)^+ p_{ji}} \right] \\
& + \sum_{i=1}^{K} \mathbb{E}[s x_i 1_{x_i < D_i}] + \rho c \sum_{i=1}^{K} \mathbb{E} \left[\left(x_i - D_i - \sum_{j \neq i} (D_j - x_j)^+ p_{ji} \right)^+ \right] \Bigg\},
\end{aligned}
$$

which is exactly the expression in equation (C.5). The proof is complete. □

Proof of Theorem C.2. Using a base-stock policy $(x_1, \ldots, x_K)$, we have

$$\begin{aligned}
R(x_1,\ldots,x_K) = &-c\sum_{i=1}^K x_i + \sum_{i=1}^K \mathbb{E}\left[\left(sD_i + (s-c_p)\sum_{j\neq i}(D_j-x_j)^+ p_{ji}\right) 1_{x_i > D_i + \sum_{j\neq i}(D_j-x_j)^+ p_{ji}}\right] \\
&- \sum_{i=1}^K \mathbb{E}\left[h\left(x_i - D_i - \sum_{j\neq i}(D_j-x_j)^+ p_{ji}\right) 1_{x_i > D_i + \sum_{j\neq i}(D_j-x_j)^+ p_{ji}}\right] \\
&+ \sum_{i=1}^K \mathbb{E}\left[(sD_i + (s-c_p)(x_i - D_i))1_{D_i < x_i < D_i + \sum_{j\neq i}(D_j-x_j)^+ p_{ji}}\right] \\
&+ \sum_{i=1}^K \mathbb{E}[s x_i 1_{x_i < D_i}] - \rho c \sum_{i=1}^K \mathbb{E}\left[\min\left\{x_i, D_i + \sum_{j\neq i}(D_j-x_j)^+ p_{ji}\right\}\right] \\
&+ \rho\left(R(x_1,\ldots,x_K) + c\sum_{i=1}^K x_i\right),
\end{aligned}$$

which implies that

$$\begin{aligned}
R(x_1,\ldots,x_K) = &\frac{1}{1-\rho}\sum_{i=1}^K \mathbb{E}\left[\left(sD_i + (s-c_p)\sum_{j\neq i}(D_j-x_j)^+ p_{ji}\right) 1_{x_i > D_i + \sum_{j\neq i}(D_j-x_j)^+ p_{ji}}\right] \\
&- \sum_{i=1}^K \mathbb{E}\left[h\left(x_i - D_i - \sum_{j\neq i}(D_j-x_j)^+ p_{ji}\right) 1_{x_i > D_i + \sum_{j\neq i}(D_j-x_j)^+ p_{ji}}\right] \\
&+ \frac{1}{1-\rho}\sum_{i=1}^K \mathbb{E}\left[(sD_i + (s-c_p)(x_i - D_i))1_{D_i < x_i < D_i + \sum_{j\neq i}(D_j-x_j)^+ p_{ji}}\right] \\
&+ \frac{1}{1-\rho}\sum_{i=1}^K \mathbb{E}[s x_i 1_{x_i < D_i}] \\
&+ \frac{\rho}{1-\rho} c \sum_{i=1}^K \mathbb{E}\left[\left(x_i - D_i - \sum_{j\neq i}(D_j-x_j)^+ p_{ji}\right)^+\right] - \frac{c}{1-\rho}\sum_{i=1}^K x_i.
\end{aligned}$$

Therefore, by comparing with equation (C.5), we get

$$V(0,0,\ldots,0) = \max_{x_1,\ldots,x_K \geq 0} R(x_1,\ldots,x_K), \tag{E.10}$$

and by comparing with equation (C.4), we get

$$(x_1^*,\ldots,x_K^*) = \arg\max_{x_1,\ldots,x_K \geq 0} R(x_1,\ldots,x_K). \tag{E.11}$$

The proof is complete. □

Proof of Theorem C.3. We can rewrite $R(x_1, \ldots, x_K)$ as

$$
\begin{aligned}
&R(x_1, \ldots, x_K) \\
&= \frac{1}{1-\rho} \sum_{i=1}^{K} \mathbb{E}\left[\left(s(D_i - x_i) + (s - c_p) \sum_{j \neq i} (D_j - x_j)^+ p_{ji}\right) 1_{x_i > D_i + \sum_{j \neq i} (D_j - x_j)^+ p_{ji}}\right] \\
&\quad - \frac{1}{1-\rho} \sum_{i=1}^{K} \mathbb{E}\left[h\left(x_i - D_i - \sum_{j \neq i} (D_j - x_j)^+ p_{ji}\right) 1_{x_i > D_i + \sum_{j \neq i} (D_j - x_j)^+ p_{ji}}\right] \\
&\quad + \frac{1}{1-\rho} \sum_{i=1}^{K} \mathbb{E}\left[(s(D_i - x_i) + (s - c_p)(x_i - D_i)) 1_{D_i < x_i < D_i + \sum_{j \neq i} (D_j - x_j)^+ p_{ji}}\right] \\
&\quad + \frac{s}{1-\rho} \sum_{i=1}^{K} x_i - \frac{c}{1-\rho} \sum_{i=1}^{K} x_i \\
&\quad + \frac{\rho}{1-\rho} c \sum_{i=1}^{K} \mathbb{E}\left[\left(x_i - D_i - \sum_{j \neq i} (D_j - x_j)^+ p_{ji}\right) 1_{x_i > D_i + \sum_{j \neq i} (D_j - x_j)^+ p_{ji}}\right],
\end{aligned}
$$

which is equivalent to

$$
\begin{aligned}
R(x_1, \ldots, x_K) = &\frac{s - c\rho + h}{1-\rho} \sum_{i=1}^{K} \mathbb{E}\left[(D_i - x_i) 1_{x_i > D_i + \sum_{j \neq i} (D_j - x_j)^+ p_{ji}}\right] \\
&+ \frac{s - c_p - c\rho + h}{1-\rho} \sum_{i=1}^{K} \mathbb{E}\left[\sum_{j \neq i} (D_j - x_j)^+ p_{ji} 1_{x_i > D_i + \sum_{j \neq i} (D_j - x_j)^+ p_{ji}}\right] \\
&+ \frac{c_p}{1-\rho} \sum_{i=1}^{K} \mathbb{E}\left[(D_i - x_i) 1_{D_i < x_i < D_i + \sum_{j \neq i} (D_j - x_j)^+ p_{ji}}\right] + \frac{s - c}{1-\rho} \sum_{i=1}^{K} x_i.
\end{aligned}
$$

Let us recall that under our assumptions, $D_i = Dp_i$, $1 \leq i \leq K$, and $f(\cdot)$ is the probability density function of demand D, which is assumed to be continuously differentiable. Then the objection function $R(x_1, \ldots, x_K)$ becomes:

$$
\begin{aligned}
R(x_1, \ldots, x_K) = & \frac{s - c\rho + h}{1-\rho} \sum_{i=1}^{K} \mathbb{E}\left[(Dp_i - x_i) \cdot 1_{x_i > Dp_i + \sum_{j \neq i} (Dp_j - x_j)^+ p_{ji}} \right] \\
& + \frac{s - c_p - c\rho + h}{1-\rho} \sum_{i=1}^{K} \mathbb{E}\left[\sum_{j \neq i} (Dp_j - x_j)^+ p_{ji} \cdot 1_{x_i > Dp_i + \sum_{j \neq i} (Dp_j - x_j)^+ p_{ji}} \right] \\
& + \frac{c_p}{1-\rho} \sum_{i=1}^{K} \mathbb{E}\left[(Dp_i - x_i) \cdot 1_{Dp_i \leq x_i \leq Dp_i + \sum_{j \neq i} (Dp_j - x_j)^+ p_{ji}} \right] + \frac{s-c}{1-\rho} \sum_{i=1}^{K} x_i, \\
= & \frac{s - c\rho + h}{1-\rho} \sum_{i=1}^{K} \int_{x_i > yp_i + \sum_{j \neq i} (yp_j - x_j)^+ p_{ji}} (yp_i - x_i) f(y) dy \\
& + \frac{s - c_p - c\rho + h}{1-\rho} \sum_{i=1}^{K} \int_{x_i > yp_i + \sum_{j \neq i} (yp_j - x_j)^+ p_{ji}} \sum_{j \neq i} (yp_j - x_j)^+ p_{ji} f(y) dy \\
& + \frac{c_p}{1-\rho} \sum_{i=1}^{K} \int_{yp_i \leq x_i \leq yp_i + \sum_{j \neq i} (yp_j - x_j)^+ p_{ji}} (yp_i - x_i) f(y) dy + \frac{s-c}{1-\rho} \sum_{i=1}^{K} x_i.
\end{aligned}
\tag{E.12}
$$

Define $G_i(x_1, \ldots, x_K)$ as the value of y such that

$$
x_i = yp_i + \sum_{j \neq i} (yp_j - x_j)^+ p_{ji}. \tag{E.13}
$$

Then, we have

$$
\begin{aligned}
R(x_1, \ldots, x_K) = & \frac{s - c\rho + h}{1-\rho} \sum_{i=1}^{K} \int_0^{G_i(x_1, \ldots, x_K)} (yp_i - x_i) f(y) dy \\
& + \frac{s - c_p - c\rho + h}{1-\rho} \sum_{i=1}^{K} \int_0^{G_i(x_1, \ldots, x_K)} \sum_{j \neq i} (yp_j - x_j)^+ p_{ji} f(y) dy \\
& + \frac{c_p}{1-\rho} \sum_{i=1}^{K} \int_{G_i(x_1, \ldots, x_K)}^{\frac{x_i}{p_i}} (yp_i - x_i) f(y) dy + \frac{s-c}{1-\rho} \sum_{i=1}^{K} x_i,
\end{aligned}
\tag{E.14}
$$

which implies that

$$
\begin{aligned}
\frac{\partial R}{\partial x_i} =& -\frac{s-c\rho+h}{1-\rho}\int_0^{G_i(x_1,\ldots,x_K)} f(y)dy \\
&+ \frac{s-c\rho+h}{1-\rho}\sum_{\ell=1}^{K}\frac{\partial G_\ell}{\partial x_i}(G_\ell(x_1,\ldots,x_K)p_\ell - x_\ell)f(G_\ell(x_1,\ldots,x_K)) \\
&- \frac{s-c_p-c\rho+h}{1-\rho}\sum_{\ell=1}^{K}\frac{\partial G_\ell}{\partial x_i}(G_\ell(x_1,\ldots,x_K)p_\ell - x_\ell)f(G_\ell(x_1,\ldots,x_K)) \\
&- \frac{s-c_p-c\rho+h}{1-\rho}\sum_{\ell\neq i}\int_{\frac{x_i}{p_i}}^{G_\ell(x_1,\ldots,x_K)} p_{i\ell}f(y)dy - \frac{c_p}{1-\rho}\int_{G_i(x_1,\ldots,x_K)}^{\frac{x_i}{p_i}} f(y)dy \\
&- \frac{c_p}{1-\rho}\sum_{\ell=1}^{K}\frac{\partial G_\ell}{\partial x_i}(G_\ell(x_1,\ldots,x_K)p_\ell - x_\ell)f(G_\ell(x_1,\ldots,x_K)) + \frac{s-c}{1-\rho} \\
=& -\frac{s-c\rho+h}{1-\rho}\int_0^{G_i(x_1,\ldots,x_K)} f(y)dy - \frac{s-c_p-c\rho+h}{1-\rho}\sum_{\ell\neq i}\int_{\frac{x_i}{p_i}}^{G_\ell(x_1,\ldots,x_K)} p_{i\ell}f(y)dy \\
&- \frac{c_p}{1-\rho}\int_{G_i(x_1,\ldots,x_K)}^{\frac{x_i}{p_i}} f(y)dy + \frac{s-c}{1-\rho}.
\end{aligned}
$$

Finally, let us show that the optimal $(x_1^*,\ldots,x_K^*)$ satisfies the first-order condition. Given any x_j, $j\neq i$, we can compute that

$$\left.\frac{\partial R}{\partial x_i}\right|_{x_i=0} = \frac{s-c}{1-\rho} > 0, \tag{E.15}$$

since $G_i = 0$ when $x_i = 0$. Moreover, as $x_i \to \infty$, $G_i \to \infty$ and thus

$$\left.\frac{\partial R}{\partial x_i}\right|_{x_i=\infty} = -\frac{s-c\rho+h}{1-\rho} + \frac{s-c}{1-\rho} = -c - \frac{h}{1-\rho} < 0, \tag{E.16}$$

and hence the optimal $x_i^* \in (0,\infty)$, and the optimal $(x_1^*,\ldots,x_K^*)$ satisfies the first-order condition $\frac{\partial R}{\partial x_i}|_{(x_1,\ldots,x_K)=(x_1^*,\ldots,x_K^*)} = 0$, $1\leq i\leq K$, which gives equation (C.6). The proof is complete. □

Proof of Theorem C.4. The proof is similar to Theorem C.3, hence omitted. □