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Equivariant quantum differential equation and qKZ equations for a projective space: Stokes bases as exceptional collections, Stokes matrices as Gram matrices, and B-Theorem

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In memory of Boris Dubrovin (1950-2019)

Abstract. In the previous paper by Tarasov and Varchenko the equivariant quantum differential equation (qDE) for a projective space was considered and a compatible system of difference qKZ equations was introduced; the space of solutions to the joint system of the qDE and qKZ equations was identified with the space of the equivariant K-theory algebra of the projective space; Stokes bases in the space of solutions were identified with exceptional bases in the equivariant K-theory algebra. This paper is a continuation of the paper by Tarasov and Varchenko.

We describe the relation between solutions to the joint system of the qDE and qKZ equations and the topological-enumerative solution to the qDE only, definitionned as a generating function of equivariant descendant GromovWitten invariants. The relation is in terms of the equivariant graded Chern character on the equivariant K-theory algebra, the equivariant Gamma class of the projective space, and the equivariant first Chern class of the tangent bundle of the projective space.

We consider a Stokes basis, the associated exceptional basis in the equivariant K-theory algebra, and the associated Stokes matrix. We show that the Stokes matrix equals the Gram matrix of the equivariant GrothendieckEuler-Poincaré pairing wrt to the basis, which is the left dual to the associated exceptional basis.

We identify the Stokes bases in the space of solutions with explicit full exceptional collections in the equivariant derived category of coherent sheaves on the projective space, where the elements of those exceptional collections are just line bundles on the projective space and exterior powers of the tangent bundle of the projective space.

These statements are equivariant analogs of results of G.Cotti, B.Dubrovin, D.Guzzetti, and S.Galkin, V.Golyshev, H.Iritani.

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Contents

1. Introduction

1.1. We consider the equivariant quantum differential equation (*qDE*) of a complex projective space P^{n-1} with the diagonal action of the torus $\mathsf{T} = (\mathbb{C}^*)^n$. This equation is the ordinary differential equation

$$\left(q\frac{d}{dq} - x*_{q,\boldsymbol{z}}\right) I(q,\boldsymbol{z}) = 0 \quad , \tag{1.1}$$

where the unknown function $I(q,\mathsf{z})$ takes values in the equivariant cohomology algebra $H^\bullet_{\mathbb{T}}(\mathbb{P}^{n-1},\mathbb{C})$, and $x*_{q,\boldsymbol{z}}\colon H^\bullet_{\mathbb{T}}(\mathbb{P}^{n-1},\mathbb{C}) \to H^\bullet_{\mathbb{T}}(\mathbb{P}^{n-1},\mathbb{C})$ is the operator of quantum multiplication by the equivariant first Chern class of the tautological line bundle on P^{n-1}. The *qDE* depends on the equivariant parameters $\mathsf{z} = (z_1,\dots,z_n)$ corresponding to the factors of the torus T. The *qDE* has two singularities: a regular singularity at $q = 0$ and an irregular singularity at $q = \infty$.

In [TV19a] a compatible system of difference equations, called the *qKZ* equations, was introduced,

$$I(q,z_1,\dots,z_i-1,\dots,z_n) = K_i(q,z)I(q,z), \qquad i = 1,\dots,n, \tag{1.2}$$

where K_i's are suitable linear operators. In [TV19b] solutions to the joint system of the *qDE* and *qKZ* equations were constructed in the form of q-hypergeometric integrals. In [TV19a] the space of solutions was identified with the space of the equivariant K-theory algebra $K_0^{\mathbb{T}}(\mathbb{P}^{n-1})$. The Stokes bases of the *qDE* at its irregular singular point were described in terms of the exceptional bases of the equivariant K-theory and a suitable braid group action on the set of exceptional bases. In this paper we continue this study.

1.2. We establish relations between the monodromy data of the joint system of the *qDE* and *qKZ* equations for P^{n-1} and characteristic classes of objects of the derived category $\mathcal{D}^b_{\mathbb{T}}(\mathbb{P}^{n-1})$ of equivariant coherent sheaves on P^{n-1}.

Our first result is on the relation between solutions to the joint system of the *qDE* and *qKZ* equations and the topological-enumerative morphism.

The topological-enumerative morphism is the map S^o, which assigns a solution of the *qDE* (only) to every element of $H^\bullet_{\mathbb{T}}(\mathbb{P}^{n-1},\mathbb{C})$ and whose expansion at $q=0$ is the generating function for the equivariant descendant Gromov-Witten invariants of P^{n-1}.

For $E \in K_0^{\mathbb{T}}(\mathbb{P}^{n-1})$ let $\theta(E)$ be the solution to the joint system of the *qDE* and *qKZ* equations, assigned to E in [TV19a]. Our B-Theorem 8.2 says that

$$\theta(E) = \mathcal{S}^o\big(e^{\pi\sqrt{-1}c_1(\mathbb{P}^{n-1})}\widehat{\Gamma}^+_{\mathbb{P}^{n-1}}\mathrm{Ch}_{\mathbb{T}}(E)\big),$$

where $c_1(\mathsf{P}^{n-1})$ is the equivariant first Chern class of the tangent bundle of P^{n-1}, $\widehat{\Gamma}^+_{\mathbb{P}^{n-1}}$ is the equivariant Gamma-class of P^{n-1}, $\mathrm{Ch}_{\mathbb{T}}(E)$ is the equivariant graded Chern character of E. In other words, we have the following commutative diagram:

$$\begin{array}{ccc} & \mathsf{B} & \\ K\text{-Theory} & \longrightarrow & \text{Equiv. Cohomology} \\ \theta \big\downarrow & & \big\downarrow \mathcal{S}^o \\ & \text{Solutions of } qDE & \end{array}$$

where $\mathsf{B}(E) := e^{\pi\sqrt{-1}c_1(\mathbb{P}^{n-1})}\widehat{\Gamma}^+_{\mathbb{P}^{n-1}}\mathrm{Ch}_{\mathbb{T}}(E)$.

Notice that the B-Theorem is an equivariant analog of results of [GGI16, Section 5] and [CDG18, Section 6] for projective spaces. Moreover it is a refinement of the Gamma Theorem in [TV19a,TV19b].

Our second result is the identification of the Stokes bases in the space of solutions to the joint system of the *qDE* and *qKZ* equations with explicit T-full exceptional collections in the derived category $\mathcal{D}^b_{\mathbb{T}}(\mathbb{P}^{n-1})$ of $\mathbb{T}$-equivariant coherent sheaves on P^{n-1}. We show that the elements of these T-full exceptional collections are just line bundles $\mathsf{O}(i)$ on P^{n-1} and exterior powers $\bigwedge^j \mathcal{T}\otimes\mathcal{O}(i)$ of the tangent bundle T of P^{n-1} multiplied by line bundles, see Theorem 7.26, Corollary 7.27, Corollary 7.28 and Theorem 10.15. This result is an equivariant version of [CDG18, Corollary 6.11].

Our third result is on the relation between the Stokes matrices and Gram matrices of the Grothendieck-Euler-Poincaré pairing on $K_0^{\mathbb{T}}(\mathbb{P}^{n-1})$.

Consider a Stokes sector V and the complementary Stokes sector $e^{\pi\sqrt{-1}}\mathcal{V}$. Consider the two exceptional bases in $K_0^{\mathbb{T}}(\mathbb{P}^{n-1})$ assigned V and $e^{\pi\sqrt{-1}}\mathcal{V}$ in [TV19a].

The matrix expressing the second exceptional basis in terms of the first exceptional basis is called the Stokes matrix associated with the Stokes sector V. We show that the second exceptional basis is left dual to the first exceptional basis wrt the Grothendieck-Euler-Poincaré pairing. This fact implies that the Stokes matrix equals the Gram matrix of the Grothendieck-Euler-Poincaré pairing wrt the second exceptional basis, see Theorem 11.7. This is an equivariant analog of [Guz99] (see also [CDG18, Section 6] for some refinements of results in [Guz99]).

1.3. This paper is related to the general theory of D.Maulik and A.Okounkov in [MO19] connecting quantum groups and equivariant quantum cohomology of Nakajima quiver varieties. In that context, it was realized that the *qDE*s of Nakajima quiver varieties admit some compatible difference equations, called the *qKZ* equations.

A special case of Nakajima varieties, namely, the case of the cotangent bundles $T^*\mathrm{F}_\lambda$ of partial flag varieties F_λ was considered in [GRTV13] and [RTV15] [1]. In those papers the *qDE*s and *qKZ* equations for cotangent bundles were identified with the dynamical differential equations and *qKZ* difference equations, associated in representation theory with the evaluation module $\mathsf{C}^N(z_1)\otimes\cdots\otimes\mathsf{C}^N(z_n)$ of the Yangian $Y(\mathrm{gl}_N)$. This identification leads to two constructions of solutions to the joint system of the *qDE* and *qKZ* equations for the cotangent bundles. One construction in [TV14] gave solutions in the form of multidimensional hypergeometric integrals and another construction in [TV19b] gave solutions in the form of multidimensional q-hypergeometric integrals.

Also in [TV19b] a suitable limit of the *qDE*s for cotangent bundles of partial flag varieties was considered. In that limit the *qDE*s for cotangent bundles turn into the *qDE*s for the partial flag varieties themselves. Moreover, in that limit the *qKZ* equations for cotangent bundles survive and turn into new systems of difference equations compatible with the *qDE*s for partial flag varieties. These new systems of difference equations were also called the *qKZ* equations. Furthermore, it was shown in [TV19b] that the q-hypergeometric solutions to the joint systems of the *qDE*s and *qKZ* equations for cotangent bundles have a limit when the *qDE*s and *qKZ* equations turn into the *qDE*s and *qKZ* equations for partial flag varieties.

The special case of that limit was considered in [TV19a] for the partial flag variety P^{n-1}. In [TV19a] the q-hypergeometric solutions to the joint system of the *qDE* and *qKZ* equations for P^{n-1} were applied to study the monodromy properties of the *qDE* for P^{n-1}.

1.4. The paper is organized as follows. The basic notions of the derived category of equivariant coherent sheaves and equivariant Helix theory are collected in Section 2. In Section 3 we describe the equivariant derived category and K-theory of P^{n-1}. In Section 4 we introduce the equivariant cohomology of P^{n-1}. In Section 5 we discuss the equivariant Gromov-Witten theory of P^{n-1}. We introduce the *qDE* and *qKZ* difference equations, and the topological-enumerative morphism S^o. In Section 6 we introduce two fundamental systems of solutions of the *qDE* (only): the Levelt solution and the topological-enumerative solution. We study how they are related, and we describe their monodromy.

[1] Note that the partial flag varieties themselves are not Nakajima varieties

In Section 7 we study solutions to the joint system of the qDE and qKZ equations, their integral representations, their asymptotics. We describe the corresponding objects and exceptional collections in the derived category. In Section 8 we prove our B-Theorem.

In Section 9 we describe the structure of formal solutions to the joint system of the qDE and qKZ difference equations, see Theorem 9.2.

In Section 10 we study the Stokes bases of the space of solutions, their normalizations. We show that the Stokes bases correspond to T-full exceptional collections in $\mathcal{D}^b_{\mathbb{T}}(\mathbb{P}^{n-1})$. In Section 11 we prove that the Stokes matrices coincide with the Gram matrices of the equivariant Grothendieck-Euler-Poincaré pairing.

In Section 12 we study the specialization of the qDE at points z such that $(e^{2\pi\sqrt{-1}z_1}, ..., e^{2\pi\sqrt{-1}z_n})$ are roots of unity. We show that the monodromy group of the qDE is Z_n only for this specialization of the equivariant parameters z.

In Appendix A we prove Theorem A.1 on the formal normal form for a compatible system of a differential equation and a system of difference equations.

In Appendix B we discuss the relation between the equivariant qDE and the isomonodromic system of differential equations attached to the quantum cohomology of P^{n-1}. Such a system plays a central role in Dubrovin's theory of Frobenius manifolds [Dub96,Dub99,Dub98,CDG18].

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2. Equivariant exceptional collections and bases

General references for this Section are [GM03,CG10,GK04].

2.1. Basic notions. Let G be a linear algebraic reductive group over C. We denote by

- $Rep(G)$ the category of finite dimensional complex representations of G,
- $R(G) := K_0(Rep(G))$ (resp. $R(G)_{\mathsf{C}} := R(G) \otimes_{\mathsf{Z}} \mathsf{C}$) the ring of finite dimensional complex representations of G with integer (resp. complex) coefficients.

In particular, for a complex torus $\mathsf{T} := (\mathsf{C}^*)^n$ we have $R(\mathbb{T})_{\mathbb{C}} = \mathbb{C}[Z_1^{\pm 1}, \dots, Z_n^{\pm 1}]$. For short, we set $\mathsf{Z} := (Z_1,...,Z_n)$ and $\mathbb{C}[\boldsymbol{Z}^{\pm 1}] := \mathbb{C}[Z_1^{\pm 1}, \dots, Z_n^{\pm 1}]$.

Let X be a smooth complex projective variety equipped with the action of G. We denote by

(1) $\mathrm{D}^b(X)$ its derived category of coherent sheaves, (2) $\mathcal{D}^b_G(X)$ its derived category of G-equivariant coherent sheaves,

(3) $K_0(X)$ (resp. $K_0(X)_{\mathsf{C}}$) its Grothendieck group (resp. complexified),

(4) $K_0^G(X)$ (resp. $K_0^G(X)_{\mathbb{C}}$) its G-equivariant Grothendieck group (resp. complexified).

Any complex of G-equivariant quasi-coherent complexes admit flat and injective resolutions. From this one can deduce that on $\mathrm{D}_G^b(X)$ all standard derived functors are well defined. In particular, we have a well defined left derived tensor product $\otimes: \mathrm{D}_G^b(X) \times \mathrm{D}_G^b(X) \to \mathrm{D}_G^b(X)$, and any $f: X \to Y$, morphism of smooth projective G-varieties, induces left and right derived functors $Lf^*: \mathrm{D}_G^b(Y) \to \mathrm{D}_G^b(X)$ and $Rf_*: \mathcal{D}_G^b(X) \to \mathcal{D}_G^b(Y)$. It is possible to show that all the standard properties of the derived tensor product, the derived pull-back and push-forward functors are valid in the equivariant setting. Moreover, all these equivariant derived functors are compatible with their non-equivariant versions via the forgetful functor [2].

The structural morphism $\pi: X \to \mathrm{Spec}(\mathbb{C})$ endows $K_0(X)$ and $K_0^G(X)$ with a $\mathbb{C}$-algebra and an $R(G)$-algebra structures, respectively. In addition, it induces serveral push-forward morphisms

$$\pi_*: K_0(X) \to K_0(\mathrm{Spec}\,\mathbb{C}) \cong \mathbb{Z}, \quad \pi_*: K_0^G(X) \to R(G),$$

and functors

$$R\pi_*: \mathcal{D}^b(X) \to \mathcal{D}^b(\mathbb{C}), \quad R\pi_*^G: \mathcal{D}_G^b(X) \to \mathcal{D}^b(Rep(G)),$$

which fit into the diagram

$$\begin{array}{ccc} \mathcal{D}_G^b(X) & \xrightarrow{R\pi_*^G} & \mathcal{D}^b(Rep(G)) \\ {\scriptstyle F_X F_{pt}}\downarrow & & \downarrow \\ \mathrm{D}^b(X) & \xrightarrow{R\pi_*} & \mathrm{D}^b(\mathbb{C}) \end{array}$$

where F_X, F_{pt} denote the forgetful functors. If $V \in \mathrm{Ob}(\mathrm{D}^b(X))$ we call a G-equivariant structure on V any object $V' \in \mathrm{Ob}(\mathcal{D}_G^b(X))$ such that $F_X(V') = V$.

2.2. Equivariant Grothendieck-Euler-Poincaré characteristic. The push-forward morphisms

$$\pi_*: K_0(X) \to K_0(\mathrm{Spec}\,\mathbb{C}) \cong \mathbb{Z}, \quad \pi_*^G: K_0^G(X) \to R(G),$$

are respectively given by

$$\pi_*(V) := \sum (-1)^i \mathrm{rk}\ H^i(X, V) \in \mathbb{Z},$$

$$\pi_*^G(V) := \sum (-1)^i [H^i(X, V)] \in R(G),$$

where $[H^i(X,V)]$ denotes the $R(G)$-class of the cohomology space $H^i(X,V)$ seen as a representation of G. These morpshims define the Grothendieck-Euler-Poincaré

[2] For the translation of the theory of derived functors from the non-equivariant setting to the equivariant one, the reader may consult [CG10, Chapter 5], [BL94] for the topological setting, [VV10, Section 1.5], and also [LH09].

characteristic of (the isomorphism class of) an object V, and its equivariant version respectively. They will be denoted by χ, χ^G:

$$\chi(V) := \pi_*(V), \quad \chi^G(V) := \pi^G_*(V).$$

In both cases, using the duality involutions

$$(-)^*\colon K_0(X) \to K_0(X), \quad E \mapsto E^*, \tag{2.1}$$

$$(-)^*\colon K_0^G(X) \to K_0^G(X), \quad E \mapsto E^*, \tag{2.2}$$

we can define a non-symmetric paring, called the Grothendieck-Euler-Poincaré pairing (or also the Mukai pairing):

$$\chi(E,F) := \chi(E^* \otimes F), \qquad \chi^G(E,F) := \chi^G(E^* \otimes F). \tag{2.3}$$

These pairings naturally extend to the complexified algebras $K_0(X)_{\mathbb{C}}$ and $K_0^G(X)_{\mathbb{C}}$.

In the non-equivariant case, the pairing χ is $\mathbb{C}$-bilinear, whereas in the equivariant case the pairing χ^G is $R(G)_{\mathbb{C}}$-sesquilinear wrt the duality involution naturally defined on $R(G)_{\mathbb{C}}$:

$$(-)^*\colon R(G)_{\mathbb{C}} \to R(G)_{\mathbb{C}}, \quad [V] \mapsto [V^*]. \tag{2.4}$$

That is,. $\chi^G(\rho_1 E_1, \rho_2 E_2) = \rho_1^* \rho_2 \, \chi^G(E_1, E_2)$ for $E_1, E_2 \in K_0^G(X)$ and $\rho_1, \rho_2 \in R(G)$

We consider the involutive operation on $n \times n$-matrices

$$(-)^*\colon M_n(R(G)_{\mathbb{C}}) \to M_n(R(G)_{\mathbb{C}}),$$

which consists in applying (2.4) at each entry. For $A \in M_n(R(G)_{\mathbb{C}})$ we define the matrix $A^\dagger \in M_n(R(G)_{\mathbb{C}})$ as follows:

$$(A^\dagger)_{\alpha,\beta} := A^*_{\beta,\alpha}, \quad \alpha, \beta = 1, \dots, n. \tag{2.5}$$

If $G = \mathsf{T}$, then the duality involution acts on $R_G(\mathsf{T})_{\mathbb{C}} = \sim \mathbb{C}[\mathsf{Z}^{\pm 1}]$ by the formula:

$$f(\mathsf{Z})^* = f(\mathsf{Z}^{-1}), \tag{2.6}$$

where $f(\mathsf{Z}) = f(Z_1,\dots,Z_n) \in \mathbb{C}[\mathsf{Z}^{\pm 1}]$ and $f(\mathsf{Z}^{-1}) := f(Z_1^{-1},\dots,Z_n^{-1})$.

2.3. Exceptional collections in $\mathrm{D}_G^b(X)$ and their mutations. Given two objects $E, F \in \mathrm{Ob}(\mathcal{D}_G^b(X))$, we define

$$\mathrm{Hom}_G^\bullet(E,F) := R\pi^G_* (E^* \otimes F) \in \mathrm{Ob}(\mathcal{D}^b(Rep(G))),$$

where $E^* := R\mathcal{H}om(E, \mathcal{O}_X)$ is the ordinary dual sheaf of E.

Definition 2.1. An object $E \in \mathrm{Ob}(\mathrm{D}_G^b(X))$ is called an exceptional object if and only if

$$\mathrm{Hom}_G^\bullet(E,E) \cong \mathbb{C}_G,$$

where $\mathbb{C}_G$ denotes the object of $\mathrm{D}^b(Rep(G))$ given by the trivial complex one dimensional representation of G, concentrated in degree zero.

An ordered collection $(E_1,\dots,E_n)$ is said to be an exceptional collection if and only if

- all objects E_i's are exceptional objects,
- and $\mathrm{Hom}^\bullet_G(E_j,E_i) = 0$ for $j > i$.

The definitions above are the natural equivariant versions of the standard notions of exceptional objects and collections in $\mathrm{D}^b(X)$. The following result, due to A. Elagin, gives an insight on the relationships between ordinary exceptional collections in $\mathrm{D}^b(X)$ and equivariant exceptional collections in $\mathcal{D}^b_G(X)$. Before stating Elagin's result, let us recall that there is a naturally defined operation of tensor product between objects of $\mathcal{D}^b_G(X)$ and $\mathcal{D}^b(Rep(G))$: if $E \in \mathrm{Ob}(\mathcal{D}^b_G(X))$ and $V^\bullet \in \mathrm{Ob}(\mathrm{D}^b(Rep(G)))$, the tensor product $E \otimes V^\bullet$ is defined as the object of $\mathcal{D}^b_G(X)$ given by

$$\bigoplus_i E[-i] \otimes V^i \tag{2.7}$$

.

This extends the obvious operation of tensor product between objects of $Coh_G(X)$ and $Rep(G)$.

If $\mathrm{A}_1,\dots,\mathrm{A}_n$ are subcategories of $\mathrm{D}^b_G(X)$, we denote by $\langle \mathcal{A}_1, \dots, \mathcal{A}_n \rangle$ the smallest full triangulated subcategory of $\mathcal{D}^b_G(X)$ containing $\mathrm{A}_1,\dots,\mathrm{A}_n$.

Definition 2.2. Let $\mathrm{E} := (E_1,\dots,E_n)$ be an exceptional collection in $\mathcal{D}^b_G(X)$. We say that E is G-full if

$$\mathcal{D}^b_G(X) = \langle E_1 \otimes \mathcal{D}^b(Rep(G)), \dots, E_n \otimes \mathcal{D}^b(Rep(G)) \rangle \tag{2.8}$$

.

Remark 2.3. Thus the exceptional collection $(E_1,\dots,E_n)$ is G-full if and only if the collection $\left(E_1 \otimes \mathcal{D}^b(Rep(G)), \dots, E_n \otimes \mathcal{D}^b(Rep(G))\right)$ realizes a so-called semiorthogonal decomposition of $\mathcal{D}^b_G(X)$, see e.g. [Huy06, Chapter 1].

Remark 2.4. Our definition of G-fullness is different from the definition of fullness of exceptional collections in triangulated categories. In the paper [BO18], L.Borisov and D.Orlov studied bounded derived category of T-equivariant coherent sheaves on smooth toric varieties and Deligne-Mumford stacks. In particular, they described and explicitly constructed full exceptional collections in these categories. Notice that their exceptional collections consist of infinite sets of objects, while we collect an infinite set of objects in one symbol $E_i \otimes \mathrm{D}^b(Rep(G))$.

Theorem 2.5 ([Ela09, Theorem 2.6]). Assume that $(E_1,\dots,E_n)$ is a full exceptional collection of $\mathrm{D}^b(X)$, and each object E_i admits a G-equivariant structure E_i. Then, $(\mathrm{E}_1,\dots,\mathrm{E}_n)$ is a G-full exceptional collection in $\mathrm{D}^b_G(X)$.

Being thus important to know under which conditions an exceptional object of $\mathrm{D}^b(X)$ admits a G-equivariant structure, we recall the following result of A. Polishchuk.

Theorem 2.6 ([Pol11, Lemma 2.2]). Let X be a smooth projective complex variety equipped with the action of a linear algebraic connected reductive group G with $\pi_1(G)$ torsion free. If $E \in \mathrm{D}^b(X)$ is an exceptional object, then E admits a G-equivariant structure, which is unique up to tensoring by a character of G.

In the present paper we focus on the case $G = \mathsf{T}$, and the assumption of Theorem 2.6 applies.

Definition 2.7 (Mutations of objects). Let $E \in \mathrm{Ob}(\mathrm{D}^b_G(X))$ be an exceptional object. For any $F \in \mathrm{Ob}(\mathrm{D}^b_G(X))$ we define two new objects

$$\mathsf{L}_E F,\ \mathsf{R}_E F,$$

called the left and right mutations of F with respect to E. These two objects are defined through the distinguished triangles

$$\mathsf{L}_E F[-1] \longrightarrow \mathrm{Hom}^\bullet_G(E,F)^* \otimes E \xrightarrow{j} F \longrightarrow \mathbb{L}_E F, \tag{2.9}$$

$$\mathbb{R}_E F \longrightarrow F \xrightarrow{j_*} \mathrm{Hom}^\bullet_G(F,E)^* \otimes E \longrightarrow \mathbb{R}_E F[1], \tag{2.10}$$

where j^*, j_* denote the natural evaluation and coevaluation morphisms.

Remark 2.8. As in the non-equivariant case, it can be shown that the objects $\mathsf{L}_E F, \mathsf{R}_E F$ are uniquely defined (up to unique isomorphism) by the distinguished triangles above. The key property is the exceptionality of E. We leave the details to the reader, see [CDG18, Section 3.3].

Lemma 2.9. Let $E \in \mathrm{Ob}(\mathrm{D}^b_G(X))$ be an exceptional object. We have

$$\mathrm{Hom}^\bullet_G(E, \mathsf{L}_E F) = 0,\ \mathrm{Hom}^\bullet_G(\mathsf{R}_E F, E) = 0, \text{ for all objects } F \in \mathrm{Ob}(\mathcal{D}^b_G(X)).$$

Proof. Apply the functor $\mathrm{Hom}^\bullet_G(E,-)$ (resp. $\mathrm{Hom}^\bullet_G(-,E)$) to the distinguished triangle (2.9) (resp. (2.10)), and use the exceptionality of E.

Definition 2.10. Let $\mathrm{E} := (E_i)_{i=1}^n$ be an exceptional collection in $\mathcal{D}^b_G(X)$. For any integer i, with $0 < i < n$, we define two new collections

$$\mathsf{L}_i(\mathrm{E}) := (E_1, \dots, \mathsf{L}_{E_i} E_{i+1}, E_i, \dots, E_n),$$

$$\mathsf{R}_i(\mathrm{E}) := (E_1, \dots, E_{i+1}, \mathsf{R}_{E_{i+1}} E_i, \dots, E_n).$$

Proposition 2.11. For any i, with $0 < i < n$, the collections $\mathsf{L}_i(\mathrm{E}), \mathsf{R}_i(\mathrm{E})$ are exceptional. Moreover, the mutation operators $\mathsf{L}_i, \mathsf{R}_i$ satisfy the following identities:

$$\mathsf{L}_i \mathsf{R}_i = \mathsf{R}_i \mathsf{L}_i = \mathrm{Id}, \tag{2.11}$$

$$\mathsf{R}_i \mathsf{R}_j = \mathsf{R}_j \mathsf{R}_i, \quad \text{if } |i-j| > 1, \quad \mathsf{R}_{i+1} \mathsf{R}_i \mathsf{R}_{i+1} = \mathsf{R}_i \mathsf{R}_{i+1} \mathsf{R}_i. \tag{2.12}$$

Proof. The same as in the non-equivariant case, see [GK04], [CDG18, Section 3.3].

Denote by $\tau_1, \dots, \tau_{n-1}$ the generators of the braid group B_n, satisfying the relations

$$\tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1}, \qquad \tau_i \tau_j = \tau_j \tau_i, \qquad \text{if } |i-j| > 1.$$

We define the left action of B_n on the set of exceptional collections of length n by identifying the action of τ_i with R_i, see identities (2.11)-(2.12). For our purposes, we modify this action, by setting

(2.13) $\tau_i(E) := R_{n-i}(E)$, $i = 1,\dots,n-1$, for any exceptional collection

$E = (E_1,\dots,E_n)$.

Remark 2.12. Formula (2.13) is in agreement with the notations of [TV19a], see Remark 3.3.

2.4. Dual exceptional collections and helices.

Definition 2.13 (Dual exceptional collections). Let $E = (E_1,\dots,E_n)$ be an exceptional collection. Define the left and right dual exceptional collections ${}^\vee E$ and $E^\vee$ as the collections

(2.14) $${}^\vee E := \beta(E), \qquad \beta := \tau_1(\tau_2\tau_1)\dots(\tau_{n-2}\dots\tau_1)(\tau_{n-1}\tau_{n-2}\dots\tau_1),$$

(2.15) $$E^\vee := \beta^{-1}(E).$$

Proposition 2.14. Let $E = (E_1,\dots,E_n)$ be an exceptional collection, ${}^\vee E = ({}^\vee E_1,\dots {}^\vee E_n)$ and $E^\vee = (E_1^\vee,\dots,E_n^\vee)$ its left and right dual exceptional collections, respectively. The following orthogonality relations hold true:

$$\bullet_G(E_h, E_k^\vee) = \begin{cases} C_G, & h = n \quad k+1, \\ \mathrm{Hom}\,|\,|\,|\,0, & \text{otherwise,} \end{cases}$$

$$\bullet_G({}^\vee E_k, E_h) = \begin{cases} C_G, & h = n \quad k+1, \\ \mathrm{Hom}\,|\,|\,0, & \text{otherwise.} \end{cases}$$

Moreover, for any $F \in \mathrm{Ob}\left(\mathcal{D}^b_G(X)\right)$ we have the functorial isomorphism (2.16) $\mathrm{Hom}^\bullet_G({}^\vee E_k, F) =\sim \mathrm{Hom}^\bullet_G(F, E_k^\vee)^*$.

Proof. The argument is the same as in the non-equivariant case, see [CDG18, Section 3.6].

Given an exceptional collection E, we introduce the infinite family of exceptional objects called the helix generated by E.

Definition 2.15 (Helix). Let $E = (E_1,\dots,E_n)$ be an exceptional collection. Define the helix generated by E to be the infinite family of objects $(E_i)_{i\in\mathbb{Z}}$ defined by the iterated mutations

$$E_{i+n} := R_{E_{i+n-1}}\dots R_{E_{i+1}}E_i, \qquad E_{i-n} := L_{E_{i-n+1}}\dots L_{E_{i-1}}E_i, \qquad i \in \mathbb{Z}.$$

Such a helix is said to be of period n. Any family of n consecutive objects $(E_i,\dots,E_{i+n})$ is called a foundation of the helix.

2.5. Exceptional bases in equivariant K-theory. In this Section we focus on the K-theoretical counterpart of the notion of exceptional collections introduced in Definition 2.1 and of the action of the braid group on them.

Definition 2.16. An element $e \in K_0^G(X)$ is exceptional if

$$\chi^G(e,e) = C_G.$$

A basis $\varepsilon := (e_i)_{i=1}^n$ of $K_0^G(X)$ as an $R(G)$-module, is exceptional if (2.17)

$$\chi^G(e_i,e_i) = C_G, \qquad \chi^G(e_j,e_i) = 0, \qquad \text{for } j > i.$$

The following result is a K-theoretical analogue of Theorem 2.5.

Theorem 2.17 ([Pol11, Lemma 2.1]). Let $(E_1,\dots,E_n)$ be a full exceptional collection in $D^b(X)$. If each object E_i admits a G-equivariant structure, then the classes $([E_i])_{i=1}^n$ form an exceptional basis of $K_0^G(X)$ as an $R(G)$-module.

Proposition 2.18. Let $E \in \mathrm{Ob}(\mathcal{D}_G^b(X))$ be an exceptional object. For any $F \in \mathrm{Ob}(\mathcal{D}_G^b(X))$ we have

$$[L_E F] = [F] - \chi^G(E,F)\cdot[E], \qquad [R_E F] = [F] - \chi^G(F,E)^*\cdot[E]. \tag{2.18}$$

Proof. From the distinguished triangle (2.9), and equation (2.7), we deduce

$$[L_E F] = [F] - [\mathrm{Hom}_G^\bullet(E,F)\otimes E] = [F] - \left[\bigoplus_i E[-i]\otimes H^i(X, E^*\otimes F)\right]$$
$$= [F] - \left(\sum_i (-1)^i H^i(X, E^*\otimes F)\right)[E].$$

Analogously, from the distinguished triangle (2.10), we deduce

$$[R_E F] = [F] - [\mathrm{Hom}_G^\bullet(F,E)^*\otimes E] = [F] - \left[\bigoplus_i E[-i]\otimes H^i(X, F^*\otimes E)^*\right]$$
$$= [F] - \left(\sum_i (-1)^i H^i(X, F^*\otimes E)\right)^*[E].$$

This completes the proof.

Definition 2.19. Let $e \in K_0^G(X)$ be an exceptional element. Given $f \in K_0^G(X)$, we define its left and right mutations wrt e as the elements

$$L_e f := f - \chi^G(e,f)\cdot e, \quad R_e f := f - \chi^G(f,e)^*\cdot e. \tag{2.19}$$

Lemma 2.20. Let $e \in K_0^G(X)$ be an exceptional element. We have

$$\chi^G(e, L_e f) = 0, \qquad \chi^G(R_e f, e) = 0,$$

for any $f \in K_0{}^G(X)$.

Definition 2.21. Let $K_0^G(X)$ be a free $R(G)$-module of finite rank and $\varepsilon := (e_i)_{i=1}^n$ an exceptional basis of $K_0^G(X)$. For any $0 < i < n$ define the two new exceptional bases

(2.20) $$L_i\varepsilon := (e_1,\dots,e_{i-1},L_{e_i}e_{i+1},e_i,e_{i+2},\dots,e_n),$$

(2.21) $$R_i\varepsilon := (e_1,\dots,e_{i-1},e_{i+1},R_{e_{i+1}}e_i,e_{i+2},\dots,e_n).$$

This construction defines the action of the braid group B_n on the set of exceptional bases of $K_0^G(X)$, in which the action of a generator τ_i, $i = 1,\dots,n-1$, is identified with the action of the mutation R_{n-i}.

2.6. Dual exceptional bases. Let $\varepsilon := (e_i)_{i=1}^n$ be an exceptional basis of $K_0^G(X)$. Define the left and right dual exceptional bases, ${}^\vee\varepsilon$ and $\varepsilon^\vee$, through the mutations

(2.22) $${}^\vee\varepsilon := \beta(\varepsilon), \qquad \beta := \tau_1(\tau_2\tau_1)\dots(\tau_{n-2}\dots\tau_1)(\tau_{n-1}\tau_{n-2}\dots\tau_1),$$

(2.23) $$\varepsilon^\vee := \beta^{-1}\varepsilon.$$

Proposition 2.22. Let $\varepsilon = (e_i)_{i=1}^n$ be an exceptional basis of $K_0^G(X)$, ${}^\vee\varepsilon = ({}^\vee e_i)_{i=1}^n$ and $\varepsilon^\vee = (e_i^\vee)_{i=1}^n$ its left and right dual exceptional basis, respectively. The following orthogonality relations hold true

(2.24) $$\chi^G(e_h, e_k^\vee) = \delta_{h+k,n+1}, \quad \chi^G({}^\vee e_k, e_h) = \delta_{h+k,n+1},$$

for $k = 1,\dots,n$. In particular, for any $v \in K_0^G(X)$ we have

(2.25) $$v = \sum_{h=1}^n \chi^G(v, e_h^\vee)^*\ e_{n+1-h}, \quad v = \sum_{h=1}^n \chi^G({}^\vee e_h, v)\ e_{n+1-h}.$$

Proof. We prove the first identity in (2.24), the proof of the second is analogous. We have

$$\chi^G(e_h, e_k^\vee) = 0, \qquad \text{for } h = 1,\dots,n-k,$$

by Lemma 2.20.

If $e, f \in K_0^G(X)$, e is exceptional and $\chi^G(f,e) = 0$, then $\chi^G(f,v) = \chi^G(f,L_e v)$ for any $v \in K_0^G(X)$. By iteration of this identity, we deduce

$$\chi^G(e_{h'}, {}^\vee e_k) = 0, \qquad \text{for } h = n-k+2,\dots,n,$$

and

$$\chi^G(e_{n-k+1}, {}^\vee e_k) = \chi^G(e_{n-k+1}, e_{n-k+1}) = 1.$$

Identities (2.25) follow from the sesquilinearity of χ^G.

Corollary 2.23. Let $\varepsilon = (e_i)_{i=1}^n$ be an exceptional basis of $K_0^G(X)$, and G the Gram matrix of χ^G wrt ε. Then the Gram matrix of χ^G wrt ${}^\vee\varepsilon$ equals the Gram matrix of χ^G wrt $\varepsilon^\vee$ and equals

$$J \cdot (G^\dagger)^{-1} \cdot J, \qquad \text{where } J_{\alpha,\beta} = \delta_{\alpha+\beta,n+1}.$$

Proof. Let $X = (X_k^j)_{j,k=1}^n$ be the matrix defined by $e_k^\vee := \sum_{j=1}^n X_k^j e_j$. Then X satisfies the equation $GX = J$ by formula (2.24). Hence the Gram matrix of χ^G wrt $\varepsilon^\vee$ equals

$$X^\dagger \cdot G \cdot X = J \cdot (G^\dagger)^{-1} \cdot J.$$

The case of ${}^{\vee}\varepsilon$ is analogous.

2.7. Serre functor and canonical operator. A Serre functor $\mathrm{K}\colon \mathrm{D}^b(X) \to \mathrm{D}^b(X)$ is a functor defined (uniquely up to canonical isomorphism) by the condition (2.26) $\mathrm{Hom}^\bullet(E,F)^* = \sim \mathrm{Hom}^\bullet(F,\mathrm{K}(E))$, $E,F \in \mathrm{Ob}(\mathrm{D}^b(X))$.

We can take

$$\mathrm{K} = (\omega_X \otimes -)[\dim_{\mathbb{C}} X]\colon \mathrm{D}^b(X) \to \mathrm{D}^b(X), \tag{2.27}$$

where ω_X denotes the canonical sheaf of X. Analogously, in the equivariant case a Serre functor $\mathcal{K}\colon \mathcal{D}^b_G(X) \to \mathcal{D}^b_G(X)$ is defined by the condition

$$\mathrm{Hom}^\bullet_G(E,F)^* \cong \mathrm{Hom}^\bullet_G(F,\mathcal{K}(E)), \quad E,F \in \mathrm{Ob}(\mathcal{D}^b_G(X)). \tag{2.28}$$

We can take

$$\mathrm{K} = (\omega_X{}^G \otimes -)[\dim_{\mathbb{C}} X]\colon \mathrm{D}^b_G(X) \to \mathrm{D}^b_G(X), \tag{2.29}$$

where ω_X^G is the G-equivariant canonical sheaf of X. By abuse of language, we will call (2.27) (and its equivariant version (2.29)) the Serre functor in $\mathrm{D}^b(X)$ (and $\mathcal{D}^b_G(X)$, respectively).

The Serre duality (2.28) implies the Serre periodicity,

$$\mathrm{Hom}^\bullet_G(E,F) \cong \mathrm{Hom}^\bullet_G(\mathcal{K}(E),\mathcal{K}(F)), \quad E,F \in \mathrm{Ob}(\mathcal{D}^b_G(X)). \tag{2.30}$$

Proposition 2.24. Let $\mathrm{E} = (E_i)_{i=1}^n$ be an exceptional collection of length n of $\mathrm{D}^b_G(X)$. The following operations are equivalent, i.e. produce the same exceptional collection when applied to E:

(1) to act on E with the braid $(\tau_1 \dots \tau_{n-1})^{-n}$,

(2) to take the double right-dual exceptional collection $(\mathrm{E}^\vee)^\vee$,

(3) to apply the Serre functor to each object of E.

Proof. The equivalence of points (1) and (2) follows from the well-known identity of braids in B_n

$$(\tau_1 \dots \tau_{n-1})^n = \beta^2, \tag{2.31}$$

where $\beta := \tau_1(\tau_2\tau_1)\dots(\tau_{n-2}\dots\tau_1)(\tau_{n-1}\tau_{n-2}\dots\tau_1)$ is the braid that appears in (2.14) and (2.22), see [KT08, Theorem 1.24]. The equivalence of (2) and (3) follows from the functorial isomorphism (2.16).

Remark 2.25. Note that the element (2.31) of B_n is the generator of its center $Z(\mathrm{B}_n)$, see [KT08, Theorem 1.24].

The K-theoretical version of the Serre functor is the so-called canonical operator $k\colon K_0^G(X) \to K_0^G(X)$, defined through the identity

$$\chi^G(e,f)^* = \chi^G(f,k(e)), \quad e,f \in K_0^G(X). \tag{2.32}$$

The K-theoretical analog of (2.30), i.e.

(2.33) $\chi^G(e,f) = \chi^G(k(e),k(f)), \quad e,f \in K_0^G(X)$, shows that the canonical operator k is a χ^G-isometry.

Proposition 2.26. Let $\varepsilon = (e_i)_{i=1}^n$ be a basis of $K_0^G(X)$, and G the Gram matrix of χ^G wrt ε. Then the matrix of the canonical operator $k\colon K_0^G(X) \to K_0^G(X)$ wrt the basis ε is equal to

(2.34) $$G^{-1}G^{\dagger}.$$

Proof. It follows from identity (2.32), written in matrix notation.

3. Equivariant derived category, exceptional collections and K-theory of $\mathbb{P}^{n-1}$

3.1. Symmetric functions. Consider the algebra $\mathbb{C}[\boldsymbol{Z}^{\pm 1}] = \mathbb{C}[Z_1^{\pm 1},\dots,Z_n^{\pm 1}]$ of Laurent polynomials in n indeterminates. The elementary and complete symmetric functions are defined as the elements

(3.1) $$s_k(\boldsymbol{Z}) := \sum_{1\le i_1<\dots<i_k\le n} \prod_{j=1}^{k} Z_{i_j}, \quad k = 1,\dots,n,$$

3.2. $$m_k(\boldsymbol{Z}) := \sum_{\substack{i_1\ge 0,\dots i_n\ge 0\\ i_1+\dots+i_n=k}} Z_1^{i_1}\dots Z_n^{i_n}, \quad k\in\mathbb{Z}_{>0}.$$

Put $s_0 = 1$, $m_0 = 1$. We have

3.3. $$\sum_{i=0}^{k}(-1)^i m_i(\boldsymbol{Z}) s_{k-i}(\boldsymbol{Z}) = 0, \quad k\in\mathbb{Z}_{>0}.$$

3.2. Torus action. Let $n \ge 2$. Consider the diagonal action of $\mathsf{T} = (\mathbb{C}^*)^n$ on the space $\mathbb{C}^n$. Such an action induces an action of T on $\mathbb{P}^{n-1}$, the projective space parametrizing the one dimensional subspaces $F \subset \mathbb{C}^n$. If $(u_1,\dots,u_n)$ denote the standard basis of $\mathbb{C}^n$, denote by $pt_I \in \mathbb{P}^{n-1}$, with $I = 1,\dots,n$, the point corresponding to the coordinate line spanned by u_I. The points pt_I, $I = 1,\dots,n$ are the fixed points of the T-action.

3.3. Derived category. The action of T on $\mathbb{C}^n$ induces naturally a T-structure on the structural sheaf $\mathcal{O}_{\mathbb{P}^{n-1}}$ and the tautological line bundle $\mathcal{O}(-1)$ on $\mathbb{P}^{n-1}$. Any vector bundle obtained from $\mathcal{O}_{\mathbb{P}^{n-1}}$ and $\mathcal{O}(-1)$ through tensorial operations inherits a "natural" T-structure.

The derived category $D^b(\mathbb{P}^{n-1})$ admits a well-known full exceptional collection, the Beilinson exceptional collection

$$\mathrm{B} := (\mathsf{O}, \mathsf{O}(1), \ldots, \mathsf{O}(n-1)).$$

Such an exceptional collection, with its natural T-structure, is an exceptional collection in $\mathrm{D}^b_{\mathsf{T}}(\mathsf{P}^{n-1})$. Its K-theoretical counterpart $([\mathcal{O}(i-1)])_{i=1}^n$ defines an exceptional basis of $K_0^{\mathbb{T}}(\mathbb{P}^{n-1})$ (in accordance with Theorem 2.5, Theorem 2.6 and Theorem 2.17).

3.4. Equivariant K-theory. The equivariant K-theory algebra $K_0^{\mathbb{T}}(\mathbb{P}^{n-1})_{\mathbb{C}}$ admits the following presentation

$$K_0^{\mathbb{T}}(\mathbb{P}^{n-1})_{\mathbb{C}} \cong \mathbb{C}[X^{\pm 1}, \boldsymbol{Z}^{\pm 1}] \Big/ \langle \prod_{j=1}^n (X - Z_j) \rangle, \tag{3.4}$$

where the variable X corresponds to the tautological line bundle $\mathsf{O}(-1)$ over P^{n-1}, and the variables $Z_1, \ldots, Z_n$ are the equivariant parameters corresponding to the factors of the torus $\mathsf{T} = (\mathsf{C}^*)^n$.

Under the presentation (3.4), the duality involution (2.2) is given by

$$f(X, \boldsymbol{Z})^* := f(X^{-1}, \boldsymbol{Z}^{-1}), \quad f \in K_0^{\mathbb{T}}(\mathbb{P}^{n-1})_{\mathbb{C}}.$$

The equivariant Grothendieck-Euler-Poincaré pairing $\chi^{\mathbb{T}}$ on $K_0^{\mathbb{T}}(\mathbb{P}^{n-1})_{\mathbb{C}}$ is given by the formula

$$\chi^{\mathbb{T}}(f,g) = \sum_{a=1}^n \frac{f(Z_a^{-1}, \boldsymbol{Z}^{-1}) g(Z_a, \boldsymbol{Z})}{\prod_{j \neq a}(1 - Z_a/Z_j)} = -\sum_{a=1}^n \mathrm{Res}_{X=Z_a} \frac{f(X^{-1}, \boldsymbol{Z}^{-1}) g(X, \boldsymbol{Z})}{X \prod_{j=1}^n (1 - X/Z_j)}, \tag{3.5}$$

by the Atiyah-Bott equivariant localization theorem [AB84].

Remark 3.1. By putting $Z_i = 1$, for $i = 1, \ldots, n$, in (3.4) and (3.5), we obtain the presentation of the non-equivariant K-theory of P^{n-1} and its non-equivariant Grothendieck-Euler-Poincaré pairing.

The class of the T-equivariant canonical sheaf $[\omega^{\mathbb{T}}_{\mathbb{P}^{n-1}}]$ is obtained by twisting the class $X^n = [\mathsf{O}(-n)]$ with a character of $\mathbb{T}^n$:

$$[\omega^{\mathbb{T}}_{\mathbb{P}^{n-1}}] = \frac{X}{\prod_{j=1}^n Z_j} \quad \text{in } K_0^{\mathbb{T}}(\mathbb{P}^{n-1}). \tag{3.6}$$

Lemma 3.2. For $i,j \in \mathsf{Z}$ we have

$$\chi^{\mathbb{T}}([\mathcal{O}(i)], [\mathcal{O}(j)]) = \begin{cases} m_{j-i}(\boldsymbol{Z}^{-1}), & i \leq j, \\ 0, & j < i < j+n, \\ (-1)^{n-1} m_{i-j-n}(\boldsymbol{Z}) \prod_{j=1}^n Z_j, & i \geq j+n. \end{cases}$$

Remark 3.3. In [TV19a], instead of the pairing $\chi^{\mathbb{T}}$ on $K_0^{\mathbb{T}}(\mathbb{P}^{n-1})_{\mathbb{C}}$, it is studied another non-symmetric pairing A defined by the formula

$$A(f,g) := \pi_* \left(f^* \cdot g \cdot (-1)^{n-1} \frac{X^n}{\prod_{j=1}^n Z_j} \right), \tag{3.7}$$

where $f, g \in K_0^{\mathbb{T}}(\mathbb{P}^{n-1})_{\mathbb{C}}$ and $\pi \colon \mathbb{P}^{n-1} \to \mathrm{Spec}(\mathsf{C})$. In [TV19a, Section 6], a notion of exceptional bases of $K_0^{\mathbb{T}}(\mathbb{P}^{n-1})_{\mathbb{C}}$ wrt the pairing A, analogous to Definition 2.16, is given. From (2.29), (2.32) and (3.6) we deduce the following relationships between A and χ^G:

(3.8) $$A(f,g) = \chi^{\mathsf{T}}(g,f)^{*}.$$

This implies, in particular, that A-exceptional bases of $K_0^{\mathbb{T}}(\mathbb{P}^{n-1})_{\mathbb{C}}$ are exactly χ^{T}exceptional bases, although ordered in the opposite order. Moreover, if we denote by $\mathbb{L}^A, \mathbb{R}^A$(resp.$\mathbb{L}^{\chi^{\mathbb{T}}}, \mathbb{R}^{\chi^{\mathbb{T}}}$) the morphisms of left/right mutations wrt A (resp. χ^{T}), then

(3.9) $$\mathbb{L}^A = \mathbb{R}^{\chi^{\mathbb{T}}}, \quad \mathbb{R}^A = \mathbb{L}^{\chi^{\mathbb{T}}}.$$

3.5. Diophantine constraints on Gram matrices. In this section, we show that the Gram matrices of χ^{T} wrt exceptional bases of$K_0^{\mathbb{T}}(\mathbb{P}^{n-1})$ satisfy certain Diophantine constraints.

Given $G \in GL(n,\mathbb{Z}[\mathbb{Z}^{\pm 1}])$, denote

(3.10) $$p_{\mathcal{G}}(\lambda) := \det\left(\lambda \cdot \mathbb{1} - \mathcal{G}^{-1}\mathcal{G}^{\dagger}\right) \in \mathbb{Z}[\boldsymbol{Z}^{\pm 1}, \lambda].$$

Lemma 3.4. We have

(3.11) $$p_{G^{-1}}(\lambda) = p_G(\lambda)^{*},$$

where for any $f(\mathbb{Z},\lambda) \in \mathbb{Z}[\mathbb{Z}^{\pm 1},\lambda]$ we define $f(\boldsymbol{Z},\lambda)^{*} := f\left(\boldsymbol{Z}^{-1},\lambda\right)$.

Proof. Notice that$p_{\mathcal{G}^{-1}}(\lambda) = \det\left(\left(\lambda \cdot \mathbb{1} - \mathcal{G}^{-1}\mathcal{G}^{\dagger}\right)^{\dagger}\right) = p_{\mathcal{G}}(\lambda)^{*}$.

Theorem 3.5. *Let* $\varepsilon := (e_i)_{i=1}^n$ be a basis of$K_0^{\mathbb{T}}(\mathbb{P}^{n-1})$, and let G be the Gram matrix of χ_{T} wrt ε. The following identity holds true:

(3.12) $$p_{\mathcal{G}}(\lambda) = \sum_{j=0}^{n} (-1)^j \lambda^{n-j} s_j\left((-1)^{n-1}\frac{Z_1^n}{s_n(\boldsymbol{Z})}, \ldots, (-1)^{n-1}\frac{Z_n^n}{s_n(\boldsymbol{Z})}\right).$$

Proof. From presentation (3.4) and equation (3.6), it is readily seen that the eigenvalues of the canonical operator k are

$$(-1)^{n-1}\frac{Z_1^n}{s_n(\boldsymbol{Z})}, \ldots, (-1)^{n-1}\frac{Z_n^n}{s_n(\boldsymbol{Z})}.$$

Then, identity (3.12) follows from Proposition 2.26.

If we expand (3.10) in powers of λ, i.e.

(3.13) $$p_{\mathcal{G}}(\lambda) = \sum_{j=0}^{n} (-1)^j \lambda^{n-j} p_j(\mathcal{G}),$$

for suitable polynomial functions $p_j(G)$ of the entries of G and $G^{\dagger}$, from the identity (3.12) we deduce the validity of n constraints:

(3.14) $$p_j(\mathcal{G}) = s_j\left((-1)^{n+1}\frac{Z_1^n}{s_n(\boldsymbol{Z})}, \ldots, (-1)^{n+1}\frac{Z_n^n}{s_n(\boldsymbol{Z})}\right), \quad j = 1, \ldots, n.$$

If G is a Gram matrix of χ^{T}, then detG = 1, and we have

(3.15) $$p_{\mathcal{G}}\left(\frac{1}{\lambda}\right) = \frac{(-1)^n}{\lambda^n} p_{\mathcal{G}}(\lambda)^{*},$$

so that

(3.16) $$p_{n-j}(G) = p_j(G)^*, \qquad j = 0,\dots,n.$$

Thus, we are left with $\lceil \frac{n}{2} \rceil$ constraints for the entries of G. Let us write the constraints for $n = 2,3,4$.

Proposition 3.6. Let $\varepsilon := (e_1,e_2)$ be an exceptional basis of $K_0^{\mathbb{T}}(\mathbb{P}^1)$, and let

$$\mathcal{G} = \begin{pmatrix} 1 & g \\ 0 & 1 \end{pmatrix}, \quad g \in \mathbb{Z}[Z_1^{\pm 1}, Z_2^{\pm 1}],$$

be the Gram matrix of χ^T wrt ε. Then, the Laurent polynomial g is a solution of the equation

(3.17) $$gg^* = \frac{(Z_1 + Z_2)^2}{Z_1 Z_2}.$$

All the solutions of (3.17) are of the form

(3.18) $g(Z_1,Z_2) = Z_1^{\alpha} Z_2^{\beta}(Z_1 + Z_2) \in \mathbb{Z}[Z_1^{\pm 1}, Z_2^{\pm 1}]$, where $\alpha,\beta \in \mathbb{Z}$.

Proof. We have $p_G(\lambda) = \lambda^2 + (gg^* - 2)\lambda + 1$, and the only non-trivial constraint (3.14) is

(3.19) $$gg^* - 2 = \frac{Z_1^2 + Z_2^2}{Z_1 Z_2},$$

which coincides with (3.17). Notice that g is a solution of (3.17) if and only if $\gamma := g \cdot s_1(Z_1,Z_2)^{-1}$ is a solution of $\gamma\gamma^* = 1$, whose solutions are $\gamma(Z_1, Z_2) = Z_1^{\alpha} Z_2^{\beta}$, with $\alpha,\beta \in \mathbb{Z}$.

Remark 3.7. By Lemma 3.2, the matrix G corresponding to the solution (3.18) coincide with the Gram matrix wrt the exceptional basis $([p \otimes O], [q \otimes O(1)])$, where $p, q \in R(\mathbb{T}) \cong \mathbb{Z}[Z_1^{\pm 1}, Z_2^{\pm 1}]$ are characters of T such that $p^* q = Z_1^{\alpha+1} Z_2^{\beta+1}$.

Proposition 3.8. Let $\varepsilon = (e_1,e_2,e_3)$ be an exceptional basis of $K_0^{\mathbb{T}}(\mathbb{P}^2)$, and let

$$\mathcal{G} = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}, \quad a, b, c \in \mathbb{Z}[Z_1^{\pm 1}, Z_2^{\pm 1}, Z_3^{\pm 1}],$$

be the Gram matrix of χ^T wrt ε. Then, the triple (a,b,c) is a solution of the Markov-type equations

(3.20) $$aa^* + bb^* + cc^* - ab^* c = 3 - \frac{Z_1^3 + Z_2^3 + Z_3^3}{Z_1 Z_2 Z_3},$$

(3.21) $$aa^* + bb^* + cc^* - a^* bc^* = 3 - \frac{Z_2^3 Z_3^3 + Z_1^3 Z_3^3 + Z_2^3 Z_3^3}{Z_1^2 Z_2^2 Z_3^2}.$$

Notice that the triple $(a,b,c) = (s_1(Z),s_2(Z),s_1(Z))$ gives a solutions of (3.20) and (3.21).

The properties of the Markov-type equations (3.20)-(3.21) and its solutions are discussed in [CV20].

Remark 3.9. In the non-equivariant case, the Gram matrices wrt exceptional bases are upper triangular matrices with ones on the diagonal and integer entries (a,b,c) satisfying the equation

$$a^2 + b^2 + c^2 - abc = 0, \tag{3.22}$$

see [GK04,Bon04]. This Diophantine equation is equivalent to the famous Markov equation,

$$a^2 + b^2 + c^2 - 3abc = 0, \tag{3.23}$$

see [Aig13]. A triple of integers (a,b,c) is a solution of (3.23) if and only if the triple of integers $(3a,3b,3c)$ is a solution of (3.22).

If $a, b, c \in \mathbb{Z}[Z_1^{\pm 1}, Z_2^{\pm 1}, Z_3^{\pm 1}]$ is a solution of equations (3.20) and (3.21), then putting $Z_1 = Z_2 = Z_3 = 1$ in the Laurent polynomials a,b,c we obtain a triple of integers satisfying the Markov equation (3.23). For example, the solution

$$(s_1(Z),s_2(Z),s_1(Z))$$

gives the minimal Markov triple $(3,3,3)$. Thus equations (3.20) and (3.21) may be considered as a Laurent polynomial deformation of the classical Markov equation.

Proposition 3.10. Let $\varepsilon = (e_1,e_2,e_3,e_3)$ be an exceptional basis of $K_0^{\mathrm{T}}(\mathbb{P}^3)$, and let

$$\mathcal{G} = \begin{pmatrix} 1 & a & b & c \\ 0 & 1 & d & e \\ 0 & 0 & 1 & f \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad a, b, c, d, e, f \in \mathbb{Z}[Z_1^{\pm 1}, Z_2^{\pm 1}, Z_3^{\pm 1}, Z_4^{\pm 1}],$$

be the Gram matrix of χ^{T} wrt ε. Then, (a,b,c,d,e,f) is a solution of the equations

$$\begin{aligned} & aa^* + bb^* + cc^* + dd^* + ee^* + ff^* \\ & \quad - a^*bd^* - a^*ce^* - b^*cf^* - d^*ef^* + a^*cd^*f^* \\ & \quad = 4 + \frac{Z_2Z_4Z_3}{Z_1^3} + \frac{Z_1Z_4Z_3}{Z_2^3} + \frac{Z_1Z_2Z_4}{Z_3^3} + \frac{Z_1Z_2Z_3}{Z_4^3} \end{aligned}, \tag{3.24}$$

$$\begin{aligned} & -2aa^* - 2bb^* - 2cc^* - 2dd^* - 2ee^* - 2ff^* \\ & \quad + ab^*d + a^*bd^* + ac^*e + a^*ce^* + b^*cf^* + bc^*f + de^*f + d^*ef^* \\ & \quad - ab^*ef^* - a^*be^*f - bc^*d^*e - b^*cde^* \\ & \quad + aa^*ff^* + bb^*ee^* + cc^*dd^* \\ & \quad = -6 + \frac{Z_4^2Z_2^2}{Z_1^2Z_3^2} + \frac{Z_3^2Z_2^2}{Z_1^2Z_4^2} + \frac{Z_1^2Z_2^2}{Z_3^2Z_4^2} + \frac{Z_3^2Z_4^2}{Z_1^2Z_2^2} + \frac{Z_1^2Z_4^2}{Z_2^2Z_3^2} + \frac{Z_1^2Z_3^2}{Z_2^2Z_4^2}, \end{aligned} \tag{3.25}$$

$$\begin{aligned} & aa^* + bb^* + cc^* + dd^* + ee^* + ff^* \\ & \quad - ab^*d - ac^*e - bc^*f - de^*f + ac^*df \\ & \quad = 4 + \frac{Z_1^3}{Z_2Z_3Z_4} + \frac{Z_2^3}{Z_1Z_3Z_4} + \frac{Z_3^3}{Z_1Z_2Z_4} + \frac{Z_4^3}{Z_1Z_2Z_3} \end{aligned}. \tag{3.26}$$

Remark 3.11. In the corresponding non-equivariant case, these Diophantine constraints on the Gram matrices wrt exceptional collection reduce to the equations

$$a^2 + b^2 + c^2 + d^2 + e^2 + f^2 - abd - ace - bcf - def + acdf = 8, \tag{3.27}$$

(3.28) $$(af - be + cd)^2 = 16,$$

on the integers (a,b,c,d,e,f), see e.g. [Bon04]. These constraints may be reobtained by putting $Z_1 = Z_2 = Z_3 = Z_4 = 1$ in equations (3.24)-(3.26).

4. Equivariant cohomology of P^{n-1}

4.1. Equivariant cohomology. Consider the T-equivariant cohomology algebra $H^\bullet_{\mathbb{T}}(\mathbb{P}^{n-1}, \mathbb{C})$. Denote

- by x the first equivariant Chern class of the tautological line bundle $\mathsf{O}(-1)$ on P^{n-1} with its standard T-structure,
- by $F \subset \mathbb{C}^n = (y_1, \dots, y_{n-1})$ the equivariant Chern roots of the quotient bundle Q (if $\subset$ is the line represented by $p \in \mathsf{P}^{n-1}$, then the fiber Q_p is the quotient C^n/F),
- by $\mathsf{z} = (z_1,\dots,z_n)$ the equivariant parameters corresponding to the factors of the torus T,
- by Ω the complement in C^n to the union of the hyperplanes

$$z_i - z_j = m, \quad i,j = 1,\dots,n, \quad i \neq j, \quad m \in \mathbb{Z}.$$

It is well known that

$$H^\bullet_{\mathbb{T}}(\mathbb{P}^{n-1}, \mathbb{C}) \cong \mathbb{C}[x, \boldsymbol{z}] \Big/ \left\langle \prod_{i=1}^{n} (x - z_i) \right\rangle \tag{4.1}$$

$$\cong \mathbb{C}[x, \boldsymbol{y}, \boldsymbol{z}]^{\mathfrak{S}_{n-1}} \Big/ \left\langle (u-x) \prod_{j=1}^{n-1} (u - y_j) - \prod_{a=1}^{n} (u - z_a) \right\rangle, \tag{4.2}$$

where $\mathsf{C}[x,y,z]^{S_{n-1}}$ is the algebra of polynomials in x,y,z symmetric in the variables $y_1,\dots,y_{n-1}$. The equivariant cohomology $H^\bullet_{\mathbb{T}}(\mathbb{P}^{n-1}, \mathbb{C})$ is a module over the ring $H^\bullet_{\mathbb{T}}(\mathrm{pt}, \mathbb{C}) \cong \mathbb{C}[\boldsymbol{z}]$. By setting all the equivariant parameter z_i's to zero in (4.1), we obtain the presentation of the classical cohomology algebra

$$H^\bullet(\mathbb{P}^{n-1}, \mathbb{C}) \cong \mathbb{C}[x]/\langle x^n \rangle.$$

4.2. Extension of scalars. Denote by O_Ω the ring of holomorphic functions on the domain Ω. This ring is a module over the ring $H_\mathsf{T}^\bullet(\mathrm{pt},\mathsf{C}) =\sim \mathsf{C}[z]$. Set (4.3) $H_{\mathsf{T}\Omega}(\mathsf{P}^{n-1}) := H_\mathsf{T}^\bullet(\mathsf{P}^{n-1},\mathsf{C}) \otimes_{H_\mathsf{T}^\bullet(\mathrm{pt},\mathsf{C})} \mathsf{O}_\Omega$.

A class $\alpha \in H_\mathsf{T}^\Omega(\mathsf{P}^{n-1})$ is uniquely determined by the restrictions $\alpha|_{pt_i} \in \mathsf{O}_\Omega$ at fixed points.

Following the notations of [TV19a], we will use three different bases of $H_{\mathsf{T}\Omega}(\mathsf{P}^{n-1})$:

(1) the standard basis $(1,x,\dots,x_{n-1})$, where $x_\alpha := x^\alpha$;

(2) the basis $(g_1,\dots,g_n)$ defined by

$$g_i := \prod_{a=i+1}^{n} (x - z_a), \quad i = 1, \dots, n-1 \quad \text{and } g_n := 1; \tag{4.4}$$

(3) the idempotent basis $(\Delta_1,\dots,\Delta_n)$ defined by the Lagrange inteprolating polynomials

$$\Delta_i := \prod_{j \neq i} \frac{x - z_j}{z_i - z_j}, \quad i = 1, \dots, n, \tag{4.5}$$

We have

$$\Delta_i \cdot \Delta_j = \delta_{ij} \Delta_i. \tag{4.6}$$

4.3. Poincaré pairing and D-matrix. Denote by

$$\eta \colon H^\bullet_{\mathbb{T}}(\mathbb{P}^{n-1}, \mathbb{C}) \times H^\bullet_{\mathbb{T}}(\mathbb{P}^{n-1}, \mathbb{C}) \to H^\bullet_{\mathbb{T}}(\mathrm{pt}, \mathbb{C}) \cong \mathbb{C}[\boldsymbol{z}] \tag{4.7}$$

the equivariant Poincaré metric given by equivariant integration

$$\eta(v, w) := \int_{\mathbb{P}^{n-1}}^{eq} v \cdot w = \sum_{a=1}^{n} \frac{v(z_a, \boldsymbol{z}) w(z_a, \boldsymbol{z})}{\prod_{j \neq a} (z_a - z_j)}, \quad v, w \in H^\bullet_{\mathbb{T}}(\mathbb{P}^{n-1}, \mathbb{C}) \tag{4.8}$$

The equivariant cohomology $H_T^\bullet(\mathbb{P}^{n-1},\mathbb{C})$ with the equivariant Poincaré metric η is a Frobenius algebra over the ring $H_T^\bullet(\mathrm{pt},\mathbb{C}) = \mathbb{C}[z]$:

$$\eta(a \cdot b, c) = \eta(a, b \cdot c), \quad a, b, c \in H^\bullet_{\mathbb{T}}(\mathbb{P}^{n-1}, \mathbb{C}). \tag{4.9}$$

By bilinearity, we extend the Poincaré pairing to $H_T^\Omega(\mathbb{P}^{n-1})$. The idempotent vectors are pairwise orthogonal:

$$\eta(\Delta_i, \Delta_j) = \eta(\Delta_i \cdot \Delta_j, 1) = \int_{\mathbb{P}^{n-1}} \Delta_i \delta_{ij} = \delta_{ij} \chi_i, \quad \chi_i := \frac{1}{\prod_{j \neq i} (z_i - z_j)}. \tag{4.10}$$

Define the matrix $D = (D_{j\alpha})$ as the matrix of the base change

$$x_\alpha = \sum_{j=1}^{n} \mathcal{D}_{j\alpha} \Delta_j, \quad \alpha = 0, \dots, n-1.$$

Lemma 4.1. We have

$$D_{j\alpha} = z_j^\alpha, \qquad \alpha = 0,\dots,n-1, \qquad j = 1,\dots,n.$$

Thus D is the Vandermonde matrix

$$\mathcal{D} = \begin{pmatrix} 1 & z_1 & z_1^2 & \dots & z_1^{n-1} \\ 1 & z_2 & z_2^2 & \dots & z_2^{n-1} \\ \vdots & & & \vdots & \\ 1 & z_n & z_n^2 & \dots & z_n^{n-1} \end{pmatrix}.$$

Its inverse D^{-1} is

$$(\mathcal{D}^{-1})_{\alpha j} = \begin{cases} (-1)^{\alpha} \dfrac{s^j_{\alpha+1}(z)}{\prod_{m\neq j}(z_j - z_m)}, & 0 \leq \alpha < n-1 \\ \\ \dfrac{1}{\prod_{m\neq j}(z_j - z_m)}, & \alpha = n-1, \end{cases},$$

where

$$s^j_k(z) := \sum_{\substack{1\leq m_1<\cdots<m_{n-k}\leq n \\ m_1,\ldots,m_{n-k}\neq j}} z_{m_1}\ldots z_{m_{n-k}}.$$

Proof. The identity $x = z_1\Delta_1 + \cdots + z_n\Delta_n$ implies the identity $x_\alpha = z_1^\alpha \Delta_1 + \ldots z_n^\alpha \Delta_n$. □

Lemma 4.2. Let $\eta = (\eta_{\alpha\beta})_{\alpha,\beta}$, with

$$\eta_{\alpha\beta} := \eta(x_\alpha, x_\beta),$$

be the Gram matrix of the equivariant Poincaré metric. We have

$$\eta_{\alpha\beta} = \begin{cases} 0 & \text{if } \alpha+\beta < n-1 \\ 1 & \text{if } \alpha+\beta = n-1, \\ m_{\alpha+\beta-n+1}(\boldsymbol{z}) & \text{if } \alpha+\beta > n-1. \end{cases},$$

Proof. It readily follows from the identity $\mathrm{D}^T \cdot \mathrm{diag}(\chi_1,\ldots,\chi_n) \cdot \mathrm{D} = \eta$.

4.4. Equivariant characteristic classes. Consider a T-equivariant vector bundle V of rank r on P^{n-1}, with equivariant Chern roots $\xi_1,\ldots,\xi_r$.

Definition 4.3. Define the graded equivariant Chern character of V as the characteristic class

Ch. $$_{\mathbb{T}}(V) := \sum_{j=1}^{r} \exp(2\pi\sqrt{-1}\xi_j) \in H^\Omega_{\mathbb{T}}(\mathbb{P}^{n-1})$$

Example 4.4. For $V = \mathsf{O}(k)$, $k \in \mathsf{Z}$, the graded Chern character $\mathrm{Ch}_{\mathsf{T}}(V)$ is the class

$$\mathrm{Ch}_{\mathbb{T}}(\mathcal{O}(k)) = \exp(-2\pi\sqrt{-1}kx).$$

This is the element of $H_{\mathsf{T}}^{\Omega}(\mathsf{P}^{n-1})$ whose restriction at the fixed point pt_I is $\exp(-2\pi \sqrt{-1} \; _I)$, for $= 1,\ldots,n\sqrt{\quad} kz \quad I \quad .$

Lemma 4.5. Let $V \in K_0^{\mathbb{T}}(\mathbb{P}^{n-1})_{\mathbb{C}}$ and $Q(\mathsf{Z}) \in \mathsf{C}[\mathsf{Z}^{\pm 1}]$. We have

$$\mathrm{Ch}_{\mathsf{T}}(Q(\mathsf{Z})V) = Q(\mathsf{Z}')\mathrm{Ch}_{\mathsf{T}}(V),\ \mathsf{Z}' := (e^{2\pi\sqrt{-1}z_1},\ldots,e^{2\pi\sqrt{-1}z_n}).$$

Proof. By additivity it is sufficient to prove the lemma for a monomial $Q(\mathsf{Z}) = Z_1^{\alpha_1}\ldots Z_n^{\alpha_n}$. If $(\xi_i)_i$ are the equivariant Chern roots of V, then $(\xi_i + \sum_{j=1}^n \alpha_j z_j)_i$ are the equivariant Chern roots of $Q(\mathsf{Z})V$.

Definition 4.6. Given any meromorphic function F on C, holomorphic at 0, with Taylor expansion of the form

$$F(t) = 1 + \sum_{k=1}^{\infty} F_k t^k,$$

we define the *F*-class of *V* as the characteristic class

$$\widehat{F}(V) := \prod_{j=1}^{r} F(\xi_j).$$

Remark 4.7. We consider only the vector bundles *V* and functions $F(t)$ such that $\widehat{F}(V)$ are elements of $H_T^{\Omega}(\mathsf{P}^{n-1})$.

Definition 4.8 (Gamma classes). The $\widehat{\Gamma}^{\pm}$-classes of *V* are defined as the characteristic classes of *V* obtained from the Taylor expansions

$$\Gamma(1 \pm t) = \exp\left(\mp\gamma t + \sum_{k=2}^{\infty} (\mp 1)^k \frac{\zeta(k)}{k} t^k \right).$$

Definitions 4.3, 4.6, 4.8 naturally extend to objects of the equivariant derived category $\mathcal{D}^b_{\mathbb{T}}(\mathbb{P}^{n-1})$.

We will denote it by $F_{\mathsf{P}^{n-1}}$. Since the Chern roots of $T\mathsf{P}^{n-1}$ are $(y_a - x)_{a=1}^{n-1}$, such Remark 4.9. If $V = T\mathsf{P}^{n-1}$, the *F*-class of *V* is called the *F*-class of P^{n-1}. a class is given by

$$\widehat{F}_{\mathbb{P}^{n-1}} = \prod_{a=1}^{n-1} F(y_a - x).$$

This is the class whose restriction at the fixed point pt_I is $\prod_{a \neq I} F(z_a - z_I)$. It is an element of $H_T^{\Omega}(\mathsf{P}^{n-1})$ if $F(t)$ has poles only at points of Z. This is the case for $\widehat{\Gamma}^{\pm}_{\mathbb{P}^{n-1}}$.

5. Equivariant quantum cohomology of P^{n-1}

5.1. Equivariant Gromov-Witten invariants. For a given $d \in H_2(\mathsf{P}^{n-1},\mathsf{Z})$ and given integers $g,m \geq 0$, denote by $\overline{\mathcal{M}}_{g,m}(\mathbb{P}^{n-1}, d)$ the moduli stack of genus g stable maps to P^{n-1} with degree d and m marked points. We assume that either $d > 0$ or $2g + m > 2$ so that $\overline{\mathcal{M}}_{g,m}(\mathbb{P}^{n-1}, d)$ is non-empty. The

T-action on P^{n-1} induces a T-action on $\mathsf{M}_{g,m}(\mathsf{P}^{n-1},d)$. Given m cohomological classes

$$\gamma_1,\ldots,\gamma_m \in H_T^{\bullet}(\mathsf{P}^n,\mathsf{C}),$$

scendant Gromov-Witten invariantsand integers $d_1,\ldots,d_n \in \mathsf{Z}_{\geq 0}$, we define theof P_{n-1} to be the polynomialsgenus g, degree d, T-equivariant de-

(5.1)
$$\langle \tau_{d_1}(\gamma_1), \dots, \tau_{d_m}(\gamma_m) \rangle_{g,n,d}^{\mathbb{P}^{n-1},\mathbb{T}} := \left(\int_{[\overline{\mathcal{M}}_{g,m}(\mathbb{P}^{n-1},d)]^{\mathrm{vir}}_{\mathbb{T}}}^{eq} \prod_{j=1}^{m} \psi_j^{d_j} \mathrm{ev}_j^*(\gamma_j) \right) \in H^\bullet_{\mathbb{T}}(\mathrm{pt},\mathbb{C}),$$

where

- $[\overline{\mathcal{M}}_{g,m}(\mathbb{P}^{n-1},d)]^{\mathrm{vir}}_{\mathbb{T}} \in A^{\mathbb{T}}_{D_{\mathrm{vir}}}(\overline{\mathcal{M}}_{g,m}(\mathbb{P}^{n-1},d))$, with $D_{\mathrm{vir}} := nd+(n-4)(1-g)+m$, is the equivariant virtual fundamental class[3],
- the map $\mathrm{ev}_j : \overline{\mathcal{M}}_{g,m}(\mathbb{P}^{n-1},d) \to \mathbb{P}^{n-1}$ is the evaluation at the j-th marked point, which is $\mathbb{T}$-equivariant,
- the classes $\psi_j \in A^{\mathbb{T}}_1(\overline{\mathcal{M}}_{g,m}(\mathbb{P}^{n-1},d))$ denote any equivariant lift of the first Chern classes of the universal cotangent line bundles $\mathbb{L}_j$ on $\mathbb{M}_{g,m}(\mathbb{P}^{n-1},d)$.

We refer the interested reader to the expository article [LS17], and references therein, for details. If all d_i's are zero, then the polynomials above are called primary equivariant Gromov-Witten invariants.

5.2. Equivariant Gromov-Witten potential. Consider the standard basis $(x_\alpha)_{\alpha=0}^{n-1}$ of $H^\bullet_{\mathbb{T}}(\mathbb{P}^{n-1},\mathbb{C})$, seen as a $H^\bullet_{\mathbb{T}}(\mathrm{pt},\mathbb{C})$-module. Denote by $\mathbf{t} := (t^0,\dots,t^{n-1})$ the corresponding dual coordinates on $H^\bullet_{\mathbb{T}}(\mathbb{P}^{n-1},\mathbb{C})$, so that the generic element of $H^\bullet_{\mathbb{T}}(\mathbb{P}^{n-1},\mathbb{C})$ is
$$\gamma = \sum_{\alpha=0}^{n-1} t^\alpha x_\alpha.$$

Consider the generating function $F_0^{\mathbb{P}^{n-1},\mathbb{T}} \in H^\bullet_{\mathbb{T}}(\mathrm{pt},\mathbb{C})[\![t^0,\dots,t^n]\!] \cong \mathbb{C}[\boldsymbol{z}][\![\boldsymbol{t}]\!]$, called equivariant Gromov-Witten potential of $\mathbb{P}^{n-1}$, defined by

$$F_0^{\mathbb{P}^{n-1},\mathbb{T}}(\boldsymbol{t}) := \sum_{m=0}^{\infty}\sum_{d=0}^{\infty} \frac{1}{m!} \langle \underbrace{\gamma,\dots,\gamma}_{m \text{ times}} \rangle_{0,m,d}^{\mathbb{P}^{n-1},\mathbb{T}}$$

(5.2)
$$= \sum_{m=0}^{\infty}\sum_{d=0}^{\infty}\sum_{\alpha_1,\dots\alpha_m=0}^{n-1} \frac{t^{\alpha_1}\dots t^{\alpha_m}}{m!} \langle x_{\alpha_1},\dots,x_{\alpha_m} \rangle_{0,m,d}^{\mathbb{P}^{n-1},\mathbb{T}}.$$

Theorem 5.1 ([Giv96, Theorem 3.1]). The function $F_0^{\mathbb{P}^{n-1},\mathbb{T}}(t)$ satisfies the *WDVV*-equations
$$\frac{\partial^3 F_0^{\mathbb{P}^{n-1},\mathbb{T}}}{\partial t^\alpha \partial t^\beta \partial t^\lambda} \eta^{\lambda\mu} \frac{\partial^3 F_0^{\mathbb{P}^{n-1},\mathbb{T}}}{\partial t^\mu \partial t^\delta \partial t^\nu} = \frac{\partial^3 F_0^{\mathbb{P}^{n-1},\mathbb{T}}}{\partial t^\nu \partial t^\beta \partial t^\lambda} \eta^{\lambda\mu} \frac{\partial^3 F_0^{\mathbb{P}^{n-1},\mathbb{T}}}{\partial t^\mu \partial t^\delta \partial t^\alpha}.$$

5.3. Equivariant quantum cohomology. The big equivariant quantum product $*$ defined by

[3] Its existence is ensured by the properness of $\mathbb{M}_{g,m}(\mathbb{P}^{n-1},d)$. From this property, it also follows that equivariant Gromov-Witten invariants are polynomials in $\mathbf{z}$, see [LS17, Section 3] and references therein.

$$x_\alpha * x_\beta = \sum_{\lambda,\mu} \frac{\partial^3 F_0^{\mathbb{P}^{n-1},\mathbb{T}}}{\partial t^\alpha \partial t^\beta \partial t^\lambda} \eta^{\lambda\mu} x_\mu \tag{5.3}$$

defines on $H^\bullet_{\mathbb{T}}(\mathbb{P}^{n-1},\mathbb{C})[\![\boldsymbol{t}]\!]$ a Frobenius algebra structure, namely a commutative, associative algebra with unit (the element 1) whose product is compatible with the equivariant Poincaré metric (4.8), that is

$$\eta(a*b,c) = \eta(a,b*c), \quad a,b,c \in H^\bullet_{\mathbb{T}}(\mathbb{P}^{n-1},\mathbb{C}). \tag{5.4}$$

This algebra structure on $H^\bullet_{\mathbb{T}}(\mathbb{P}^{n-1},\mathbb{C})[\![\boldsymbol{t}]\!]$ is called the big equivariant quantum cohomology of $\mathbb{P}^{n-1}$. It gives an example of a formal Frobenius manifold [Man99, Chapter III]. The quantum product (5.3) is a deformation of the product in classical cohomology. It is customary to denote the big quantum product also by $*_t$ to emphasize its dependence on parameters t^i's.

5.4. Quantum connection. The quantum connection of the equivariant quantum cohomology of $\mathbb{P}^{n-1}$ is defined by the formula

$$\nabla^{\mathrm{quant}}_{\alpha,\kappa} : H^\bullet_{\mathbb{T}}(\mathbb{P}^{n-1},\mathbb{C})[\![\boldsymbol{t}]\!] \to H^\bullet_{\mathbb{T}}(\mathbb{P}^{n-1},\mathbb{C})[\![\boldsymbol{t}]\!], \quad \alpha = 0,\dots,n-1, \tag{5.5}$$

$$\nabla^{\mathrm{quant}}_{\alpha,\kappa} := \kappa \frac{\partial}{\partial t^\alpha} - x_\alpha *_{\boldsymbol{t}}, \tag{5.6}$$

where $\kappa \in \mathbb{C}^*$ is the spectral parameter. The associativity of the quantum multiplication $*_t$, i.e. Theorem 5.1, is equivalent to the flatness condition of the quantum connection $\nabla^{\mathrm{quant}}{}_{\alpha,\kappa}$, for all $\kappa \in \mathbb{C}^*$:

$$'\nabla^{\mathrm{quant}}{}_{\alpha,\kappa}, \nabla^{\mathrm{quant}}{}_{\beta,\kappa}(= 0, \qquad \alpha,\beta = 0,\dots,n-1, \qquad \kappa \in \mathbb{C}_*. \tag{5.7}$$

The system of equations for flat sections of the quantum connection is called the system of equivariant quantum differential equations.

Definition 5.2. The T-equivariant topological-enumerative morphism is the element

$$S(t,\kappa) \in \mathrm{End}(H^\bullet_{\mathbb{T}}(\mathbb{P}^{n-1},\mathbb{C}))[\![\boldsymbol{t}]\!][\![\kappa^{-1}]\!]$$

defined by the formula

$$\eta(\mathcal{S}(\boldsymbol{t},\kappa)u,v) = \eta(u,v) + \sum_{d=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\alpha_1,\dots,\alpha_m=0}^{n-1} \frac{t^{\alpha_1}\dots t^{\alpha_m}}{m!} \left\langle u, x_{\alpha_1},\dots,x_{\alpha_m}, \frac{v}{\kappa-\psi} \right\rangle^{\mathbb{P}^{n-1},\mathbb{T}}_{0,m+2,d},$$

where $u,v \in H^\bullet_{\mathbb{T}}(\mathbb{P}^{n-1},\mathbb{C})$, $\psi := c_1(\mathcal{L}_{m+2})$ and the term $\frac{1}{\kappa-\psi}$ has to be expanded in power series $\sum_{j=0}^{\infty} \psi^j \kappa^{-j-1}$.

Definition 5.3. The (big) T-equivariant *J*-function of $\mathbb{P}^{n-1}$ is the cohomologyvalued function defined by the identity

$$\eta(J(\boldsymbol{t},\kappa),a) = \eta(1,\mathcal{S}(\boldsymbol{t},\kappa)a), \quad a \in H^\bullet_{\mathbb{T}}(\mathbb{P}^{n-1},\mathbb{C}). \tag{5.8}$$

Theorem 5.4 ([Giv96,Giv98]). For any $\kappa \in \mathbb{C}^*$, and any $\alpha \in H^\bullet_{\mathbb{T}}(\mathbb{P}^{n-1}, \mathbb{C})$, the cohomology class

$$S(t,\kappa)\alpha$$

is a flat section of the quantum connection $\nabla^{\text{quant}}_{\alpha,\kappa}$, namely it satisfies the following system of differential equations

$$\kappa \frac{\partial}{\partial t^\beta} \mathcal{S}(\boldsymbol{t}, \kappa)\alpha = x_\beta *_{\boldsymbol{t}} \mathcal{S}(\boldsymbol{t}, \kappa)\alpha, \quad \beta = 0, \dots, n-1. \tag{5.9}$$

Proof. The validity of equations (5.9) is equivalent to the topological recursion relations in genus 0 for Gromov-Witten invariants with descendants [Wit90]. For the proof in the non-equivariant case, see [Dub92], [Dub96, Lecture 6], [Dub99, Lecture 2], [CK99, Chapter 10], [CDG20, Section 7]. For the adaptation to the equivariant case see [Giv96, Section 6] and [Giv98, Sections 1 and 2].

5.5. Small equivariant quantum product for $\mathbb{P}^{n-1}$.

Definition 5.5. The small quantum product of $\mathbb{P}^{n-1}$ is obtained by specializing the parameters t^i's of the big quantum product (5.3) as follows: $t^i = 0$ for $i \neq 1$.

It is customary to put $q := \exp(t^1)$ and to denote by $*_q$ the small quantum product. Following the notations of [TV19a], we denote by $*_{q,z}$ the small quantum product, underlining its dependence on the equivariant parameters z.

A detailed study of the equivariant Gromov-Witten invariants of $\mathbb{P}^{n-1}$ (and more general flag varieties) and its small quantum cohomology can be found for example in [GK95, Kim96, Mih06]. For a fixed $q \in \mathbb{C}^*$, the small quantum product operator

$$x*_{q,\boldsymbol{z}} \colon H^\bullet_{\mathbb{T}}(\mathbb{P}^{n-1}, \mathbb{C}) \to H^\bullet_{\mathbb{T}}(\mathbb{P}^{n-1}, \mathbb{C}), \tag{5.10}$$

is the $\mathbb{C}[z]$-linear morphism defined by the identities

$$x *_{q,\boldsymbol{z}} x_j = x_{j+1}, \quad j = 0, \dots, n-2, \tag{5.11}$$

$$x *_{q,\boldsymbol{z}} x_{n-1} = q + \sum_{i=1}^{n} (-1)^{i-1} s_i(\boldsymbol{z}) x_{n-i}, \tag{5.12}$$

where $s_i(z)$ are the elementary symmetric polynomials in z.

Remark 5.6. In the basis $(g_1,\dots,g_n)$, the operator $x*_{q,z}$ is given by

$$x *_{q,z} g_i = z_i g_i + g_{i-1}, \qquad i = 2,\dots,n, \tag{5.13}$$

$$x *_{q,z} g_1 = z_1 g_1 + q g_n. \tag{5.14}$$

5.6. R-matrices and qKZ operators. For $a,b \in \{1,\dots,n\}$, with $a \neq b$, we define a family of $\mathbb{C}[z]$-linear operators, called the R-matrices,

$$R_{ab}(u)\colon H^\bullet_{\mathbb{T}}(\mathbb{P}^{n-1},\mathbb{C}) \to H^\bullet_{\mathbb{T}}(\mathbb{P}^{n-1},\mathbb{C}),$$

depending on a parameter $u \in \mathbb{C}$, and defined by the formulae

$$R_{ab}(u)g_i := g_i, \quad i \neq a,b,$$
$$R_{ab}(u)g_b := g_a, \quad R_{ab}(u)g_a := g_b + ug_a.$$

The R-matrices satisfy the Yang-Baxter equation

$$R_{ab}(u-v)R_{ac}(u)R_{bc}(v) = R_{bc}(v)R_{ac}(u)R_{ab}(u-v),$$

for a,b,c all distinct, and the inversion relation

$$R_{ab}(u)R_{ba}(-u) = 1.$$

Define the operators $E_1,\dots,E_n$ such that

$$E_i g_j := \delta_{ij} g_j.$$

Define the *qKZ* operators $K_1,\dots,K_n$ by the formula

$$K_i\colon H^\bullet_{\mathbb{T}}(\mathbb{P}^{n-1},\mathbb{C}) \to H^\bullet_{\mathbb{T}}(\mathbb{P}^{n-1},\mathbb{C}),$$

(5.15)

(5.16)

$$K_i := R_{i,i-1}(z_i - z_{i-1} - 1)\dots R_{i,1}(z_i - z_1 - 1)q^{-E_i}R_{i,n}(z_i - z_n)\dots R_{i,i+1}(z_i - z_{i+1}).$$

5.7. Equivariant *qDE* and *qKZ* difference equations. Consider the vector bundle H over the base space $\mathbb{C}^n$, with fiber over z_0 given by the equivariant cohomology algebra (4.1) specialized at $z = z_0$, i.e.

$$H^\bullet_{\mathbb{T}}(\mathbb{P}^{n-1},\mathbb{C})|_{z=z_0}$$

(5.17) × →).

Denote by pr: $\mathbb{C}^* \mathbb{C}^n \mathbb{C}^n$ the natural projection. Consider the pull-back vector bundle pr$^*H$.

The quantum connection described in Section 5.4 defines a differential operator

$$\nabla_{q\frac{d}{dq},\kappa} := \kappa q\frac{d}{dq} - x*_{q,z}$$

(5.18) ,

acting on sections $I(q,z)$ of the vector bundle pr$^*H$. Following [TV19a], we fix $\kappa = 1$.

The (small) equivariant quantum differential equation (*qDE* for short) of $\mathbb{P}^{n-1}$ is the differential equation

$$\nabla_{q\frac{d}{dq},\kappa=1} I(q,z) = \left(q\frac{d}{dq} - x*_{q,z}\right) I$$

(5.19) ,

where I is a section of the vector bundle pr$^*H$. The *qDE* is thus the equation for flat sections of pr$^*H$.

Definition 5.7. Fix $q \in \mathbb{C}^*$, $z, z' \in \mathbb{C}^n$. Define the isomoprhism of vector spaces

$$\Pi_{\boldsymbol{z},\boldsymbol{z}'}\colon \mathrm{pr}^* H_{q,\boldsymbol{z}} \to \mathrm{pr}^* H_{q,\boldsymbol{z}'}, \quad x_\alpha|_{q,\boldsymbol{z}} \mapsto x_\alpha|_{q,\boldsymbol{z}'} \quad (5.20) \quad ,$$

$\alpha = 0, \dots, n-1.$

for

Definition 5.8. Fix $q \in \mathbb{C}^*$, $\boldsymbol{z}, \boldsymbol{z}' \in \mathbb{C}^n$. For $i = 1,\dots,n$, define the isomorphisms of vector spaces

(5.21) $\Theta_{\boldsymbol{z},\boldsymbol{z}'}\colon \mathrm{pr}^* H_{q,\boldsymbol{z}} \to \mathrm{pr}^* H_{q,\boldsymbol{z}'}, \quad g_j|_{q,\boldsymbol{z}}$

,

for $j = 1,\dots,n$, where g_j's are the elements of bases (4.4).

For $z \in C^n$, $i = 1,\dots,n$, we use the following notations:

(5.22) $z_i^{\pm} := (z_1,\dots,z_i \pm 1,\dots,z_n) \in C^n,$

(5.23) $\Theta_{\pm z,i} := \Theta_{z,z_i^{\pm}}\colon \mathrm{pr}_* H_{q,z}$

$\to \mathrm{pr}_* H_{q,z_i^{\pm}},$ (5.24)

$\pm\Theta_{z,i} := \Theta_{z_i^{\pm},z}\colon$

$\mathrm{pr}_* H_{q,z_i^{\pm}} \to \mathrm{pr}_* H_{q,z}.$

For every fiber $\mathrm{pr}^* H_{q,z}$ we have the *qKZ*-operators $K_1,\dots,K_n$ defined by equation (5.16).

Definition 5.9. Fix $q \in C^*$, $z \in C^n$. The *qKZ*-discrete connection on the bundle H is given by the datum of the isomorphisms of vector spaces

(5.25) $\Theta_{-z,i} \circ K_i(q,z)\colon \mathrm{pr}_* H_{q,z} \to \mathrm{pr}_* H_{q,z_i-}.$

The system of difference equations

(5.26) $I(q,z_1,\dots,z_i-1,\dots,z_n) = {}^*\Theta^{-}{}_{z,i} \circ K_i(q,z) + I(q,z), \quad i = 1,\dots,n,$

is called the system of the *qKZ* difference equations. These are equations for flat sections for the *qKZ* discrete connection.

Theorem 5.10 ([TV19a, Theorem 3.1]). The joint system of equation (5.19) and (5.26) is compatible.

Remark 5.11. The *qKZ* difference equations (5.26), can be written in the equivalent form

(5.27) $$I(q, z_1, \dots, z_i + 1, \dots, z_n) = \left[\Theta^+_{\boldsymbol{z},i} \circ K'_i(q,\boldsymbol{z})\right] I(q,\boldsymbol{z}), \quad i = 1, \dots, n,$$

where the operators

(5.28) $$K'_i(q,\boldsymbol{z}) := {}^+\Theta_{\boldsymbol{z},i} \circ K_i(q, \boldsymbol{z}_i^+)^{-1} \circ \Theta^+_{\boldsymbol{z},i}$$

act on the fiber $\mathrm{pr}^* H_{q,z}$. In terms of the R-matrices we have

$$K_i(q,z_{i+})^{-1} = R_{i+1,i}(z_{i+1}-z_i-1)\dots R_{n,i}(z_n-z_i-1)q^{E_i}\cdot$$

$$\cdot R_{1,i}(z_1-z_i)\dots R_{i-1,i}(z_{i-1}-z_i). \tag{5.29}$$

Remark 5.12. The *qKZ* operators are defined in the g-basis (4.4). That basis is the limit of the stable envelope basis of the equivariant cohomology of the cotangent bundle $T^*\mathsf{P}^{n-1}$, in the limit, in which the equivariant cohomology of the cotangent bundle $T^*\mathsf{P}^{n-1}$ turns into the equivariant cohomology of the base space P^{n-1}. See [RTV15] on the stable envelopes for the cotangent bundle $T^*\mathsf{P}^{n-1}$, see [GRTV13, Section 7] and [TV19b, Section 11.4] on that limit.

6. Equivariant *qDE* of P^{n-1} and its topological-enumerative solution

6.1. Equivariant quantum differential equation. We consider the equivariant quantum differential equation (5.19) written wrt the standard basis $(x_\alpha)_{\alpha=0}^{n-1}$, namely,

$$\frac{dY}{dq} = \mathcal{A}(q,z)Y, \quad \mathcal{A}(q,z) := \frac{1}{q}\begin{pmatrix} 0 & & \dots & 0 & q+(-1)^{n-1}s_n(z) \\ 1 & 0 & \dots & 0 & (-1)^{n-2}s_{n-1}(z) \\ 0 & 1 & \dots & 0 & (-1)^{n-3}s_{n-2}(z) \\ & & \ddots & & \vdots \\ & & & 1 & s_1(z) \end{pmatrix}. \tag{6.1}$$

We have

$$\mathcal{A}(q,z) = \mathcal{A}_0 + \frac{1}{q}\mathcal{A}_1(z), \tag{6.2}$$

where

$$\mathcal{A}_0 := \begin{pmatrix} 0 & & \dots & 0 & 1 \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ & & \ddots & & \vdots \\ & & & 0 & 0 \end{pmatrix}, \quad \mathcal{A}_1(z) := \begin{pmatrix} 0 & & \dots & 0 & (-1)^{n-1}s_n(z) \\ 1 & 0 & \dots & 0 & (-1)^{n-2}s_{n-1}(z) \\ 0 & 1 & \dots & 0 & (-1)^{n-3}s_{n-2}(z) \\ & & \ddots & & \vdots \\ & & & 1 & s_1(z) \end{pmatrix}.$$

The eigenvalues of the matrix $A_1(z)$ are exactly $z_1,\dots,z_n$, as it easily follows from Viète formulae. Notice that $A_1(z)$ denotes the matrix of equivariant multipilcation

$$H^\bullet_{\mathbb{T}}(\mathbb{P}^{n-1},\mathbb{C}) \to H^\bullet_{\mathbb{T}}(\mathbb{P}^{n-1},\mathbb{C}), \quad f \mapsto x\cdot f,$$

whereas A_0 represents the quantum correction terms of the product. Moreover, we have

$$D\cdot A_1(z)\cdot D^{-1} = Z := \mathrm{diag}(z_1,\dots,z_n), \tag{6.3}$$

since the classes Δ_i's are the idempotents of the equivariant cohomology algebra.

The differential system (6.1) has a regular singularity at $q = 0$ and an irregular singularity (of Poincaré rank 1) at $q = \infty$.

6.2. Levelt Solution.

Theorem 6.1. There exist unique $n \times n$-matrix valued functions $(G_k(z))_{k=1}^\infty$, meromorphic on $\mathbb{C}^n$ and regular on Ω, such that the gauge transformation

$$Y(q,z) = G(q,z)\tilde{Y}(q,z), \quad G(q,z) = \mathcal{D}^{-1}\left(\mathbb{1} + \sum_{k=1}^{\infty} G_k(z)q^k\right), \tag{6.4}$$

transforms the differential system (6.1) into the differential equation

$$\frac{d}{dq}\tilde{Y} = \frac{1}{q}\mathcal{Z}\tilde{Y}. \tag{6.5}$$

Moreover, the formal power series $G(q,z)$ converges to a meromorphic function on $\mathbb{C} \times \mathbb{C}^n$, regular on $\mathbb{C} \times \Omega$.

Proof. Let us look for a formal gauge transformation

$$Y = \mathcal{D}^{-1}G\tilde{Y}, \quad G(q,z) = \sum_{k=0}^{\infty} G_k(z)q^k,$$

which puts the system (6.1) into the simplest normal form

$$\frac{d\tilde{Y}}{dq} = \frac{1}{q}\mathcal{Z}\tilde{Y}.$$

This requirement implies the following equation for G:

$$\mathcal{D}\mathcal{A}_0\mathcal{D}^{-1}G - \frac{dG}{dq} + \frac{1}{q}[\mathcal{Z}, G] = 0,$$

which reduces to the following recurrence equations for the coefficients G_k's:

$$ZG_0 = G_0Z, \tag{6.6}$$

$$DA_0D^{-1}G_k + [Z,G_{k+1}] - (k+1)G_{k+1} = 0. \tag{6.7}$$

Equation (6.6) is satisfied if and only if G_0 is diagonal. So we may choose G_0 to be .

For $k \geq 1$ equation (6.7) uniquely determines G_{k+1} in terms of G_k. Indeed, the linear operator

$$\varphi_k \colon M_n(\mathbb{C}) \to M_n(\mathbb{C}), \quad X \mapsto [\mathcal{Z}, X] - (k+1)X,$$

has eigenvalues $z_i - z_j - (k+1)$, $i,j = 1,\dots,n$, which are nonzero, since $z \in \Omega$. Hence, ϕ_k is invertible, and we deduce

$$G_{k+1} = \phi_{-k-1}(-DA_0D^{-1}G_k).$$

The power series $G(q,z)$ is convergent. This follows from the regularity of the singularity $q = 0$ of (6.1). The proof is standard, e.g. see [Was65,Sib90,MS16].

Corollary 6.2. For $(G_k(z))_{k=1}^{\infty}$ as in Theorem 6.1, the matrix valued function

$$Y_o(q,z) = \mathcal{D}^{-1}\left(\mathbb{1} + \sum_{k=1}^{\infty} G_k(z)q^k\right)q^{\mathcal{Z}}, \tag{6.8}$$

is a solution of system (6.1). For each fixed $z \in \Omega$, the function Y_o is a fundamental system of solutions.

We call the fundamental solution Y_o the Levelt fundamental solution, following the terminology of [AB94, Chapter 2].

Fix (q,z) and increase the argument of q by 2π. The analytic continuation of the solutions of (6.1) along this curve produces the monodromy operator $M_0(z)$ on the space of solutions.

Corollary 6.3. The Levelt fundamental solution $Y_o(q,z)$ is an eigenbasis for the the monodromy operator $M_0(z)$. The matrix of the monodromy operator $M_0(z)$ wrt the solution $Y_o(q,z)$ is

$$M_0(\boldsymbol{z}) = \exp\left(2\pi\sqrt{-1}\,\mathcal{Z}\right). \tag{6.9}$$

Proof. We have

$$Y_o\left(e^{2\pi\sqrt{-1}}q, \boldsymbol{z}\right) = Y_o(q,\boldsymbol{z})M_0(\boldsymbol{z}), \quad M_0(\boldsymbol{z}) = \exp\left(2\pi\sqrt{-1}\,\mathcal{Z}\right).$$

6.3. Topological-enumerative solution. Recall the topological-enumerative morphism $S(t,\kappa)$ of Section 5.4, where $t = (t^0,t^1,\dots,t^{n-1})$. Denote

$$S^o(q) := S(0,\log q,0,\dots,0,1), \tag{6.10}$$

where the last argument is $\kappa = 1$. We call $S^o(q)$ the restriction of $S(t,\kappa)$ to the small equivariant quantum locus.

Define the equivariant cohomology valued functions $\Psi_{top,1}(q,z),\dots,\Psi_{top,n}(q,z)$ by the formula:

$$\Psi_{top,m}(q,z) := S^o(q)x_{m-1}, \qquad m = 1,\dots,n. \tag{6.11}$$

By Theorem 5.4, these functions are solutions of the equivariant quantum differential equation (5.19). Let $Y_{top}(q,z)$ be the matrix of the operator $S^o(q)$ wrt the basis $(x_\alpha)_{\alpha=0}^{n-1}$:

$$\Psi_{\mathrm{top},m}(q,\boldsymbol{z}) = \sum_{\alpha=0}^{n-1}[Y_{\mathrm{top}}(q,\boldsymbol{z})]_m^\alpha x_\alpha, \quad m = 1,\dots,n. \tag{6.12}$$

The matrix $Y_{top}(q,z)$ is a solution of the matrix differential system (6.1). We call it the topological-enumerative solution of (6.1).

Theorem 6.4. The topological-enumerative solution is the unique solution of (6.1) of the form

$$Y_{top}(q,z) = \Phi(q,z)q^{A_1(z)}, \tag{6.13}$$

where

$$\Phi(q,\boldsymbol{z}) = \mathbb{1} + \sum_{j=1}^{\infty}\Phi_j(\boldsymbol{z})q^j.$$

The coefficients Φ_j are holomorphic on Ω, they are related to descendant Gromov-Witten invariants through the equation

$$\Phi_j(\boldsymbol{z})^\lambda_\alpha = \left\langle x_\mu, \frac{x_\alpha}{1-\psi} \right\rangle^{\mathbb{P}^{n-1},\mathbb{T}}_{0,2,j} \eta^{\mu\lambda}, \quad j \in \mathbb{N}_{>0}, \quad \alpha,\lambda = 0,\dots,n-1 \tag{6.14}$$

Here ψ is the first Chern class of the universal cotangent line bundle L_2 on the moduli space $\overline{M}_{0,2}(\mathbb{P}^{n-1},j)$ at the second marking. Furthermore, we have

$$Y_{\text{top}}(q,z) = Y_o(q,z) \cdot D,$$

where Y_o is the Levelt fundamental solution of (6.1) described in Corollary 6.2 and D is (6.3). In particular, for each fixed $z \in \Omega$, the matrix $Y_{\text{top}}(q,z)$ is a fundamental system of solutions of (6.1).

Proof. For $a \in H_T^\bullet(\mathbb{P}^{n-1},\mathbb{C})$ let $S^o(q)a$ be the corresponding solution of the qDE. We obtain

$$\mathcal{S}^o(q)a = q^{x_1}a + \sum_{d=1}^{\infty} q^d \sum_{\lambda,\mu=0}^{n-1} \left\langle x_\mu, \frac{q^{x_1}a}{1-\psi} \right\rangle^{\mathbb{P}^{n-1},\mathbb{T}}_{0,2,d} \eta^{\mu\lambda} x_\lambda$$

by using the divisor axiom for descendant Gromov-Witten invariants, see [CK99, Chapter 10]. Notice that

$$\begin{aligned} Y_{\text{top}}(q,z) &= \Big(1 + \sum_{j=1}^{\infty} \Phi_j(z) q^j\Big) q^{A_1(z)} \\ &= \Big(1 + \sum_{j=1}^{\infty} \Phi_j(z) q^j\Big) \cdot D^{-1} \cdot D \cdot q^{A_1(z)} \cdot D^{-1} \cdot D \\ &= \underbrace{\Big(D^{-1} + \sum_{j=1}^{\infty} \Phi_j(z) D^{-1} q^j\Big) q^{D \cdot A_1(z) \cdot D^{-1}}}_{Y_o(q,z)} \cdot D, \end{aligned}$$

where in the last line we used (6.3). The uniqueness of a solution of the form (6.13) thus follows from the uniqueness of the solution Y_o in Corollary 6.2.

6.4. Scalar equivariant quantum differential equation. Let Y be a fundamental solution of the differential system (6.1). Then, the matrix $\hat{Y} := \eta \cdot Y \cdot \eta^{-1}$ is a solution of the differential system

$$\frac{d\hat{Y}}{dq} = \mathcal{A}(q,z)^T \hat{Y}. \tag{6.15}$$

This follows from the Frobenius algebra property (5.4).

Equation (6.15) can be reduced to the scalar differential equation

$$\vartheta_q^n\phi = \big(q + (-1)^{n-1}s_n(\boldsymbol{z})\big)\,\phi + \sum_{j=1}^{n-1}(-1)^{n-j-1}s_{n-j}(\boldsymbol{z})\vartheta_q^j\phi, \quad \vartheta_q := q\frac{d}{dq}, \tag{6.16}$$

that will be called the scalar equivariant quantum differential equation of P^{n-1}.

Given n linearly independent solutions $(\varphi_1,\dots,\varphi_n)$ of (6.16) one can reconstruct a fundamental matrix solution $\hat{Y}$ of system (6.15) by setting

$$(\hat{Y})_k^h := \vartheta_q^h\phi_k, \quad h = 0,\dots,n-1, \quad k = 1,\dots,n.$$

Remark 6.5. In the non-equivariant limit $z_1 = \cdots = z_n = 0$, equation (6.16) reduces to the equation $\vartheta^n{}_q\,\varphi = q\varphi$,

which coincides with the scalar quantum differential equation of P^{n-1},

$$\vartheta_s^n\phi = (ns)^n\phi, \quad \vartheta_s := s\frac{d}{ds},$$

under the change of variables $q = s^n$. The monodromy of this equation has been studied in [Guz99,CDG18].

Theorem 6.6. The matrix

$$\hat{Y}(q,\boldsymbol{z}) := \begin{pmatrix} a_1 & a_2 & \dots & a_n \\ \vartheta_q a_1 & \vartheta_q a_2 & \dots & \vartheta_q a_n \\ \vdots & & & \vdots \\ \vartheta_q^{n-1}a_1 & \vartheta_q^{n-1}a_2 & \dots & \vartheta_q^{n-1}a_n \end{pmatrix} (\mathcal{D}^{-1})^T, \tag{6.17}$$

where

$$a_j(q,\boldsymbol{z}) := q^{z_j}\left(1 + \sum_{d=1}^{\infty} q^d \frac{1}{\prod_{i=1}^n \prod_{m=1}^d (z_j - z_i + m)}\right), \quad j = 1,\dots,n, \tag{6.18}$$

is a fundamental matrix solution of the differential system (6.15). The corresponding solution $\eta^{-1}\cdot\hat{Y}\cdot\eta$ of the equivariant differential system (6.1) is the topologicalenumerative solution,

$$Y_{\text{top}}(q,z) = \eta^{-1}\cdot\hat{Y}\cdot\eta.$$

Proof. Equation (5.8) implies that the components, wrt to the standard basis $(x_\alpha)_{\alpha=1}^n$, of the J-function, restricted to the small equivariant quantum locus, are solutions of the scalar equivariant quantum differential equation (6.16). The small equivariant J-function of P^{n-1}, computed by A.Givental [Giv96], B.J.Lian, K.Liu, S.-T.Yau [LLY97] is given by the formula

$$J(q,\boldsymbol{z}) = q^x\left(1 + \sum_{d=1}^{\infty} q^d \frac{1}{\prod_{i=1}^n \prod_{m=1}^d (x - z_i + m)}\right).$$

We have

$$x = \sum_{j=1}^n z_j\Delta_j, \quad 1 = \sum_{j=1}^n \Delta_j, \quad \frac{1}{\sum_i \alpha_i\Delta_i} = \sum_i \frac{1}{\alpha_i}\Delta_i,$$

for any $\alpha_i \in \mathbb{C}^*$. We deduce

$$
\begin{aligned}
J(q,\boldsymbol{z}) &= q^x \left(1 + \sum_{d=1}^{\infty} q^d \frac{1}{\prod_{i=1}^n \prod_{m=1}^d \sum_{j=1}^n (z_j - z_i + m)\Delta_j}\right) \\
&= q^x \left(1 + \sum_{d=1}^{\infty} q^d \frac{1}{\sum_{j=1}^n \prod_{i=1}^n \prod_{m=1}^d (z_j - z_i + m)\Delta_j}\right) \\
&= q^x \left(1 + \sum_{d=1}^{\infty} \sum_{j=1}^n q^d \frac{\Delta_j}{\prod_{i=1}^n \prod_{m=1}^d (z_j - z_i + m)}\right) \\
&= q^x \sum_{j=1}^n \left(1 + \sum_{d=1}^{\infty} q^d \frac{1}{\prod_{i=1}^n \prod_{m=1}^d (z_j - z_i + m)}\right) \Delta_j \\
&= \sum_{j=1}^n q^{z_j} \left(1 + \sum_{d=1}^{\infty} q^d \frac{1}{\prod_{i=1}^n \prod_{m=1}^d (z_j - z_i + m)}\right) \Delta_j \\
&= \sum_{\alpha=0}^{n-1} \left\{ \sum_{j=1}^n q^{z_j} \left(1 + \sum_{d=1}^{\infty} q^d \frac{1}{\prod_{i=1}^n \prod_{m=1}^d (z_j - z_i + m)}\right) (\mathcal{D}^{-1})_{\alpha j} \right\} x_\alpha
\end{aligned}
$$

If we define

$$
a_j(q,\boldsymbol{z}) := q^{z_j} \left(1 + \sum_{d=1}^{\infty} q^d \frac{1}{\prod_{i=1}^n \prod_{m=1}^d (z_j - z_i + m)}\right), \quad j = 1, \dots, n,
$$

then the matrix

$$
\hat{Y}(q,\boldsymbol{z}) := \begin{pmatrix} a_1 & a_2 & \dots & a_n \\ \vartheta_q a_1 & \vartheta_q a_2 & \dots & \vartheta_q a_n \\ \vdots & & & \vdots \\ \vartheta_q^{n-1} a_1 & \vartheta_q^{n-1} a_2 & \dots & \vartheta_q^{n-1} a_n \end{pmatrix} (\mathcal{D}^{-1})^T
$$

is a solution of the differential system (6.15), and the corresponding solution

$$
\eta^{-1} \cdot Y \cdot \eta
$$

is the topological solution of system (6.1).

7. Solutions of the equivariant *qDE* and *qKZ* difference equations

7.1. *q*-Hypergeometric Solutions. In this section we define a fundamental system of solutions of the joint system of equations (5.19) and (5.26) described in [TV19a].

Definition 7.1 (Master and Weight function). Define the master function Φ and the $H^\bullet_{\mathbb{T}}(\mathbb{P}^{n-1},\mathbb{C})$-valued weight function W by the formulae

(7.1)

$$
\Phi(t,q,\boldsymbol{z}) := e^{\pi\sqrt{-1}\sum_{i=1}^n z_i} \left(e^{-\pi\sqrt{-1}n} q\right)^t \prod_{a=1}^n \Gamma(z_a - t), \quad W(t,\boldsymbol{y}) := \prod_{j=1}^{n-1} (y_j - t)
$$

Recall that $y_1,\dots,y_{n-1}$ denote the equivariant Chern roots of the natural quotient bundle Q on P_{n-1}.

Remark 7.2. Notice the difference in the definition (7.1) of the master function Φ with respect to [TV19b] and [TV19a]. In [TV19a, Section 4.1] the master function is defined as

$$\Phi(t,q,\boldsymbol{z}) := \left(e^{\pi\sqrt{-1}(2-n)}q\right)^t \prod_{a=1}^{n}\Gamma(z_a-t),$$

i.e. differing from (7.1) by the extra factor $\exp(\pi\sqrt{-1}(2t+\sum_{i=1}^{n} z_i))$. In [TV19b] the general case of partial flag varieties is considered. The master function in [TV19b, Section 11.4] (see formula (11.16)), specialized to the case of projective spaces, is

$$\Phi(t,q,\boldsymbol{z}) := e^{-\pi\sqrt{-1}\sum_{i=1}^{n} z_i}\left(e^{-\pi\sqrt{-1}n}q\right)^t \prod_{a=1}^{n}\Gamma(z_a-t).$$

Thus it differs from the function Φ in (7.1) by the factor $\exp(2\pi\sqrt{-1}\sum_{i=1}^{n} z_i)$.

Definition 7.3 (Jackson Integrals). Define the Jackson integrals Ψ_J, $J = 1,\ldots,n$, to be the $H^\bullet_{\mathbb{T}}(\mathbb{P}^{n-1},\mathbb{C})$-valued functions defined on $\mathsf{C}/_* \times \Omega$ by the formula

$$\Psi_J(q,\boldsymbol{y},\boldsymbol{z}) := -\sum_{r=0}^{\infty}\operatorname*{Res}_{t=z_J+r}\Phi(t,q,\boldsymbol{z})W(t,\boldsymbol{y}). \tag{7.2}$$

Here $\mathsf{C}/_*$ is the universal cover of. C^*.

Theorem 7.4 ([TV19b]) The functions $\Psi_J(q,y,z)$ with $J = 1,\ldots,n$ are differential equation/ (5.19) and of the qKZ difference equations (5.26). These funcholomorphic on $\mathsf{C}_* \times \Omega$. Each of them is a solution of the equivariant quantum tions form a basis of solutions of this joint system of equations.

We will call the solutions Ψ_J the q-hypergeometric solutions.

Remark 7.5. Notice that in [TV19a] the system of *qKZ* equations differs from the one considered in [TV19b] and in this article by a sign in the rhs of (5.26), due to the different normalization of the master function Φ.

Corollary 7.6 ([TV19b, Formula (11.19)]). The q-hypergeometric solutions Ψ_J admit the following expansion

$$\Psi_J(q,\boldsymbol{y},\boldsymbol{z}) = e^{\pi\sqrt{-1}\sum_{i=1}^{n} z_i}\left(e^{-\pi\sqrt{-1}n}q\right)^{z_J}\prod_{a\neq J}\Gamma(1+z_a-z_J)\left(\Delta_J+\sum_{k=1}^{\infty}\Psi_{J,k}(\boldsymbol{z})q^k\right),$$

where the classes $\Psi_{J,k}(z)$ are rational functions in $(z_1,\ldots,z_n)$, regular on Ω.

Define the matrix $Y_{q\text{-hyp}} = ([Y_{q\text{-hyp}}]^{\alpha}_{J})_{\alpha,J}$ by the formula

$$\Psi_J = \sum_{\alpha=0}^{n-1}[Y_{q\text{-hyp}}]^{\alpha}_{J}\, x_\alpha, \quad J = 1,\ldots,n,$$

then $Y_{q\text{-hyp}}$ is a fundamental matrix solution of the differential system (6.1).

Theorem 7.7. The connection matrix C relating the topological-enumerative solution with the q-hypergeometric solution,

$$Y_{q\text{-hyp}}(q,z) = Y_{\text{top}}(q,z)\cdot C,$$

is given by the formula

$$C = \mathrm{D}^{-1} \cdot \operatorname{diag}\left(e^{\pi\sqrt{-1}(-nz_j+\sum_{i=1}^n z_i)} \prod_{a\neq j} \Gamma(1+z_a-z_j) \right)_{j=1}^{n}. \tag{7.3}$$

This matric C is the matrix attached to the morphism

$$\rho\colon H^\bullet_{\mathbb{T}}(\mathbb{P}^{n-1},\mathbb{C}) \to H^\Omega_{\mathbb{T}}(\mathbb{P}^{n-1}), \quad v \mapsto e^{\pi\sqrt{-1}c_1(\mathbb{P}^{n-1})} \cdot \widehat{\Gamma}^+_{\mathbb{P}^{n-1}} \cdot v, \tag{7.4}$$

where we fix

- the basis $(\Delta_j)_{j=1}^n$ in the domain of ρ,
- the basis $(x_\alpha)_{\alpha=0}^{n-1}$ in the target space of ρ.

Proof. The proof follows from Theorem 6.4 and Corollary 7.6. Notice that

$$c_1(\mathbb{P}^{n-1}) = \sum_{i=1}^n z_i - nx, \qquad \Gamma^+_{\mathbb{P}^{n-1}} = \prod_{a=1}^{n-1} \Gamma(1+y_a-x).$$

Each term of the entries of the diagonal matrix in (7.3) can be indentified with the multiplication by these classes wrt the basis $(\Delta_i)_i$.

Remark 7.8. The functions $\Psi_{\mathrm{top},m}(q,z)$ defined in (6.11) are not solutions of the *qKZ* difference equations (5.26), since the matrix C given by (7.3) is not 1-periodic in the equivariant parameters $z_1,\dots,z_n$.

7.2. Identification of solutions with K-theoretical classes. Following [TV19a], we introduce the symbols

$$\acute{T} := \exp(2\pi\sqrt{-1}t), \quad \acute{Z}_J := \exp(2\pi\sqrt{-1}z_J), \quad J=1,\dots,n. \tag{7.5}$$

Definition 7.9. Let $Q(X,Z) \in \mathbb{C}[X^{\pm1},Z^{\pm1}]$ be a Laurent polynomial. Define

$$\Psi_Q(q,\boldsymbol{y},\boldsymbol{z}) := \sum_{J=1}^n Q(\acute{Z}_J,\acute{\boldsymbol{Z}})\Psi_J(q,\boldsymbol{y},\boldsymbol{z}).$$

The function Ψ_Q is a solution of the joint system of equations (5.19) and (5.26). If $Q(X,Z) = X^m$, we denote the corresponding solution Ψ_Q by Ψ^m, i.e.

$$\Psi^m = \sum_{J=1}^n \acute{Z}_J^m \Psi_J.$$

Remark 7.10. Notice that Ψ^m in [TV19a] equals of $\exp\left(-\pi\sqrt{-1}\sum_{i=1}^n z_i\right)\Psi^{m-2}$ this paper. This is due to the difference of normalizations of the master function, see Remark 7.2.

Theorem 7.11 ([TV19a]). There is a well-defined morphism from $K_0^{\mathbb{T}}(\mathbb{P}^{n-1})_{\mathbb{C}}$ to the space of solutions of the joint system of equations (5.19) and (5.26), defined by the association

$$Q \mapsto \Psi_Q,$$

under the isomorphism (3.4).

Corollary 7.12 ([TV19a, Corollary 4.4]). For any $k \in \mathsf{Z}$, we have

$$\sum_{i=0}^{n}(-1)^{n-i}s_{n-i}(\acute{\boldsymbol{Z}})\Psi^{k+i}(q,\boldsymbol{y},\boldsymbol{z})=0 \tag{7.6}$$

where $s_i(\mathsf{Z}')$ are the elementary symmetric polynomials in Z'.

Theorem 7.13 ([TV19b, Theorem 11.3]). For any k, $(\Psi^{k+i}(q,\boldsymbol{y},\boldsymbol{z}))_{i=0}^{n-1}$ is a basis of the space of solutions of the joint system (5.19) and (5.26).

Remark 7.14. The idea that space of solutions of the *qDE* and *qKZ* equations is naturally identified with the space of the K-algebra can be observed in [TV97a, TV97b] and was implicitly discussed there.

7.3. Module S_n of solutions.

Definition 7.15. Define the space S_n of solutions to the joint system (5.19) and (5.26) of the form

$$\sum_{m=1}^{n}Q_m(\acute{\boldsymbol{Z}})\Psi^m(q,\boldsymbol{y},\boldsymbol{z}),\quad Q_m\in\mathbb{C}[\boldsymbol{Z}^{\pm1}] \tag{7.7}$$

The space S_n admits a structure of a $\mathsf{C}[\mathsf{Z}^{\pm1}]$-module, the multiplication by $Q(\mathsf{Z})$ being defines as the multiplication by $Q(\mathsf{Z}')$.

By Corollary 7.12, the module S_n contains all the solutions $\Psi^m(q,\mathsf{y},\mathsf{z})$, $m \in \mathsf{Z}$.

Corollary 7.16 ([TV19a, Corollary 4.6]). The module S_n contains a basis of solutions of the joint system of equations (5.19), (5.26). Moreover, the map $\theta\colon K_0^{\mathbb{T}}(\mathbb{P}^{n-1})_{\mathbb{C}} \to \mathcal{S}_n$ defined by the formula

(7.8) $\theta(X^m) := \Psi^m(q,\mathsf{y},\mathsf{z})$, $m \in \mathsf{Z}$, defines an isomorphism of $\mathsf{C}[\mathsf{Z}^{\pm1}]$-modules.

Using the isomorphism $^{\mathsf{T}}\theta$ we define a sesquilinear form on the module S_n as the image of the χ-form on $K_0^{\mathbb{T}}(\mathbb{P}^{n-1})_{\mathbb{C}}$. The notions of exceptional bases and the action of the braid group on them can be lifted to S_n.

7.4. Integral representations for solutions. For $p \in \mathsf{C}$, let us denote by $C(p)$ the parabola in C defined by the equation

$$C(p) := \{p + t^2 + t\sqrt{-1}\colon t \in \mathbb{R}\}.$$

Given a point $\mathsf{z} \in \Omega$, take p such that all the points $z_1,\dots,z_n$ line inside $C(p)$. The value of the integral (7.9) below does not depend on a particular choice of p, so we will simply denote $C(p)$ by $C(\mathsf{z})$.

Lemma 7.17 ([TV19b, Lemma 11.5]). For any Laurent polynomial $Q(X,\mathsf{Z})$ we have

$$\Psi_Q(q,\boldsymbol{y},\boldsymbol{z}) = \frac{1}{2\pi\sqrt{-1}}\int_{C(\boldsymbol{z})} Q(\acute{T},\acute{\boldsymbol{Z}})\Phi(t,q,\boldsymbol{z})W(t,\boldsymbol{y})\,dt, \tag{7.9}$$

(7.10)

$$\Psi^m(q,\boldsymbol{y},\boldsymbol{z}) = \frac{e^{\pi\sqrt{-1}\sum_{i=1}^n z_i}}{2\pi\sqrt{-1}}\int_{C(\boldsymbol{z})} e^{2\pi\sqrt{-1}} \quad e^{-} \quad {}^{-1nt}q^t\prod_{a=1}^{n}\Gamma(z_a-t)\prod_{j=1}^{n-1}(y_j-t)\,dt.$$

$$(q,\boldsymbol{z})\in\widetilde{\mathbb{C}^*}\times\Omega$$

where the integral converges for any $\overline{\quad}_{mt}$ $\pi\sqrt{\quad}$. In particular, we have

Remark 7.18. These formulae differ from the corresponding ones in [TV19a]. See also Remarks 7.2 and 7.10.

7.5. Coxeter element, and elements $\gamma_n, \delta_{n,\text{odd}}, \delta_{n,\text{even}} \in B_n$. The Coxeter element of B_n is the braid

$$C := \tau_1\tau_2\ldots\tau_{n-1} \in B_n. \tag{7.11}$$

For any $n \geq 3$, let

$$\ell_n := \begin{cases} n-1, & \text{for } n \text{ odd,} \\ n-2, & \text{for } n \text{ even.} \end{cases}$$

Set $\gamma_2 := 1$, and for $n \geq 3$,

$$\beta_k := \tau_k\tau_{k+1}\ldots\tau_{n-1}, \qquad \gamma_n := \beta_{\ell_n}\beta_{\ell_n-2}\ldots\beta_2.$$

Define also

$$\delta_{n,\text{odd}} = \tau_1\tau_3\ldots\tau_{n-2}, \qquad \delta_{n,\text{even}} = \tau_2\tau_4\ldots\tau_{n-1}, \qquad \text{for } n \text{ odd,}$$

$$\delta_{n,\text{odd}} = \tau_1\tau_3\ldots\tau_{n-1}, \qquad \delta_{n,\text{even}} = \tau_2\tau_4\ldots\tau_{n-2}, \qquad \text{for } n \text{ even.}$$

The elements C, γ_n, $\delta_{n,\text{odd}}$, $\delta_{n,\text{even}}$ satisfy the following relation.

Lemma 7.19 ([TV19a, Lemma 6.3]). We have the following identity in B_n:

$$\delta_{n,\text{even}}\,\delta_{n,\text{odd}}\,\gamma_n = \gamma_n\,C. \tag{7.12}$$

7.6. Exceptional bases $Q_k, Q'_k, Q''_k, \widetilde{Q}_k, \widetilde{Q}'_k, \widetilde{Q}''_k$. For any $k \in \mathbb{Z}$, we define the basis Q_k of solutions of the joint system (5.19), (5.26) to be the basis

$$Q_k := (\Psi^{k+n-1},\ldots,\Psi^{k+1},\Psi^k). \tag{7.13}$$

Lemma 7.20. The basis Q_k is an exceptional basis of $\mathcal{S}_n$. Via the isomorphism $\theta\colon K_0^{\mathbb{T}}(\mathbb{P}^{n-1})_{\mathbb{C}} \to \mathcal{S}_n$, it is identified with the exceptional basis $([\mathcal{O}(\bar{k}\quad \bar{n}+1)],\ldots,[\mathcal{O}(\bar{k}$ $\;1)],[\mathcal{O}(\bar{k})])$, *obtained from the Beilinson basis* $([\mathcal{O}(i)])_{i=0}^{n-1}$

of $K_0^{\mathrm{T}}(\mathrm{P}^{n-1})_{\mathrm{C}}$by twisting it with

$\otimes[\mathrm{O}(-k - n + 1)]$.

Proof. It follows from Corollary 7.16.

For any $k \in \mathrm{Z}$, we define the exceptional basesQ'_k and Q''_k through the mutations

(7.14) $$Q'_k := \gamma_n Q_k, \quad Q''_k := \delta_{n,\mathrm{odd}} Q'_k.$$

Proposition 7.21 ([TV19a, Lemma 6.6, Corollary 7.2]). The basis Q_k and

Qobtained from$_{k-1}$ are related by the so-called$CQ$by multiplying its last element bymodified

Coxeter map$(-:1)$this means that$_{n+1}S_n(\mathrm{Z}_{-1})$. Moreover,$Q_{k-1}$ is

k

the basis$_n$ Q'_{k-1} is obtained from the basis$\delta_{n,\mathrm{even}}Q''_k$ by multiplying its last vector by (-1) $_{+1}S_n(\mathrm{Z}_{-1})$.

Remark 7.22. Our bases Q_k, Q'_k, Q''_k have the same elements as the bases $Q_{k-1}, Q'_{k-1}, Q''_{k-1}$ of [TV19a], but ordered in the opposite way, see Remark 3.3.

Introduce three more families of exceptional bases of S_n, denoted by$\widetilde{Q}_k, \widetilde{Q}'_k, \widetilde{Q}''_k$. For $k \in \mathrm{Z}$ define

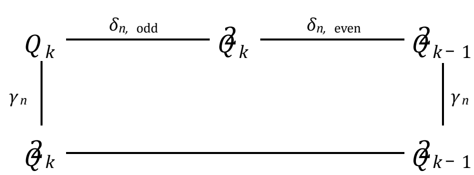
(7.15) $$\widetilde{Q}_k := C^{-k}\widetilde{Q}_0\,, \qquad \widetilde{Q}'_k := \gamma_n \widetilde{Q}_k, \qquad \widetilde{Q}''_k := \delta_{n,\mathrm{odd}}\widetilde{Q}'_k.$$

We have by formula (7.12). The $\widetilde{Q}'_{k-1} = \delta_{n,\mathrm{even}}\widetilde{Q}''_k$

diagram

2

(7.16)



is commutative by Lemma 7.19.

C

Remark 7.23 Notice that the pre-image of a basis Q_k via the isomorphism

$\theta\colon K_0^{\mathbb{T}}(\mathbb{P}^{n-1})_{\mathbb{C}} \to \mathcal{S}_n$.is a foundationO − OE$_k$−of the helix generated by the BeilinsonO 2

exceptional collection ((n+1),..., (1),). In particular, E$_n$ is the adjacent foundation to the left of E$_0$: by Proposition 2.24, the objects of E$_n$ are obtained by applying the Serre functor to objects of E$_0$.

For any$\ell, m \in \mathbb{Z}$such that $0 \le m - \ell \le n$, denote

(7.17) $$\Psi^m(\ell) := \Psi^m \quad s_1(\boldsymbol{Z})\Psi^{m-1} + \quad + (\ -1)^{m-\ell} s_{m-\ell}(\boldsymbol{Z})\Psi^{\ell} \_ \quad \ldots \quad \_$$

.

The explicit formulae for Q'_k and Q''_k follow from [TV19a, Section 6.3], Remarks 7.10 and 7.22. If $n = 2h + 1$,

we have

- the basis Q'_k is the basis in which the solutions $\Psi^k,\ldots,\Psi^{k+h}$ stay at the positions $2h{+}1,2h{-}1,\ldots,1$, and the solutions $\Psi^{k+2h}(k{+}1),\Psi^{k+2h-1}(k{+}2),\ldots,\Psi^{k+h+1}(k+h)$ stay at the positions $2h,2h-2,\ldots,6,4,2$, respectively;
- the basis Q''_k is the basis in which the solutions $\Psi^k,\ldots,\Psi^{k+h}$ stay at the positions $2h,2h{-}2,\ldots,6,4,2,1$, and the solutions $\Psi^{k+2h}(k),\Psi^{k+2h-1}(k{+}1),\ldots,\Psi^{k+h+1}(k+h-1)$ stay at the positions $2h+1,\ldots,5,3$.

If $n = 2h$, we have

- the basis Q'_k is the basis in which the solutions $\Psi^k,\ldots,\Psi^{k+h}$ stay at the positions $2h,2h{-}2,\ldots,4,2,1$, and the solutions $\Psi^{k+2h-1}(k{+}1),\Psi^{k+2h-2}(k+2),\ldots,\Psi^{k+h+1}(k+h-1)$ stay at the positions $2h-1,\ldots,7,5,3$;
- the basis Q''_k is the basis in which the solutions $\Psi^k,\ldots,\Psi^{k+h-1}$ stay at the positions $2h-1,\ldots,5,3,1$, and the solutions $\Psi^{k+2h-1}(k),\Psi^{k+2h-2}(k+1),\ldots,\Psi^{k+h}(k+h-1)$ stay at the positions $2h,\ldots,6,4,2$.

The bases $\widetilde{Q}'_k, \widetilde{Q}''_k$ can be obtained from the bases Q'_k, Q''_k by application of the following rule.

Rule 7.24 The basis $Q_{k\ m\ \ell}{}^{\prime}$ (resp., Q'') is obtained from the basis Q'_k (resp., Q''_k) by substituting any solution $2\,\Psi(\,)$ with 2

$$\left((-1)^{n+1} s_n(\boldsymbol{Z})\right)^a \Psi^m(\ell),$$

where $a \in \mathbb{Z}$ is such that

$$0 \le m + an \le n - 1.$$

Example 7.25. Let $n = 5$ and $k = -1$. We have

$$\widetilde{}''_{-1} = (\Psi^1, \Psi^0, \Psi^2(0), s_5(\boldsymbol{Z})\Psi^{-1}, \Psi^3(-$$

. *Via the isomorphism* θ: $K_0^{\mathbb{T}}(\mathbb{P}^{n-1})_{\mathbb{C}} \to \mathcal{S}_n$

$$Q'_{-1} = (\Psi^1, \Psi^2(1), \Psi^0, \Psi^3(0), \Psi^{-1}),$$
$$\widetilde{Q}'_{-1} = (\Psi^1, \Psi^2(1), \Psi^0, \Psi^3(0), s_5(\boldsymbol{Z})\Psi^{-1}),$$
$$Q''_{-1} = (\Psi^1, \Psi^0, \Psi^2(0), \Psi^{-1}, \Psi^3(-1)),$$
$$Q \qquad\qquad 1)).$$

Theorem 7.26 , the solution $\Psi^m(\ell)$ corresponds to the *K*-class of the exceptional object

$$\bigwedge^{m-\ell} \mathcal{T}(-m) := \left(\bigwedge^{m-\ell} \mathcal{T}\right) \otimes \mathcal{O}(-m),$$

placed in $\ell - m$ degree. Here T denotes the tangent sheaf of $\mathbb{P}^{n-1}$ with its natural T-equivariant structure.

Proof. Let $V = \mathbb{C}^n$ be the diagonal representation on T described in Section 3.2. Consider the Euler exact sequence, together with its exterior powers

(7.18) $$0 \longrightarrow \mathcal{O} \longrightarrow V \otimes \mathcal{O}(1) \longrightarrow \mathcal{T} \longrightarrow 0,$$

$$0 \longrightarrow \mathcal{T} \longrightarrow \bigwedge^2 V \otimes \mathcal{O}(2) \longrightarrow \bigwedge^2 \mathcal{T} \longrightarrow 0$$
$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$
$$0 \longrightarrow \bigwedge^{h-1} \mathcal{T} \longrightarrow \bigwedge^h V \otimes \mathcal{O}(h) \longrightarrow \bigwedge^h \mathcal{T} \longrightarrow 0$$
$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$
$$0 \longrightarrow \bigwedge^{n-2} \mathcal{T} \longrightarrow \bigwedge^{n-1} V \otimes \mathcal{O}(n-1) \longrightarrow \mathcal{O}(n) \longrightarrow 0$$

Each morphism in (7.18) is T-equivariant. In equivariant *K*-theory we have

$$43_h\,\mathrm{T}\,5 = 43_h\,V \otimes \mathrm{O}(h)5 - 43_{h-1}\,\mathrm{T}\,5 = s_h(\mathrm{Z})[\mathrm{O}(h)] - 43_{h-1}\,\mathrm{T}\,5,$$

for $h = 1,\ldots,n$. By induction, we obtain

$$\left[\bigwedge^h \mathcal{T}\right] = \pm \sum_{j=0}^{h} (-1)^j s_j(\boldsymbol{Z})\,[\mathcal{O}(j)],$$

where the sign is + for even *h*, and − for odd *h*. The result follows from identity

(7.17).

$$\theta\colon K_0^{\mathbb{T}}(\mathbb{P}^{n-1})_{\mathbb{C}} \to \mathcal{S}_n$$

the basesCorollaryQ'_k, Q''_k 7.27correspond to the. For any $k \in \mathbb{Z}K$, via the isomorphism-

classes of the following T-full exceptional, collections:

(1) If n is odd, the basisQ'_k corresponds to

$$\mathcal{O}\left(-k-\frac{n-1}{2}\right),\ \mathcal{T}\left(-k-\frac{n-1}{2}-1\right),\ \mathcal{O}\left(-k-\frac{n-1}{2}+1\right),\ \bigwedge^3\mathcal{T}\left(-k-\frac{n-1}{2}-2\right),\dots,$$
$$\dots,\ \bigwedge^{n-4}\mathcal{T}(-k-n+2),\ \mathcal{O}(-k-1),\ \bigwedge^{n-2}\mathcal{T}(-k-n+1),\ \mathcal{O}(-k),$$

and the basisQ''_k corresponds to

$$\mathcal{O}\left(-k-\frac{n-1}{2}\right),\ \mathcal{O}\left(-k-\frac{n-1}{2}+1\right),\ \bigwedge^2\mathcal{T}\left(-k-\frac{n-1}{2}-1\right),\ \mathcal{O}\left(-k-\frac{n-1}{2}+2\right),\dots,$$
$$\dots,\ \mathcal{O}(-k-1),\ \bigwedge^{n-3}\mathcal{T}(-k-n+2),\ \mathcal{O}(-k),\ \bigwedge^{n-1}\mathcal{T}(-k-n+1).$$

(2) If n is even, the basis Q'_k corresponds to

$$\mathcal{O}\left(-k-\frac{n}{2}\right),\ \mathcal{O}\left(-k-\frac{n}{2}+1\right),\ \bigwedge^2\mathcal{T}\left(-k-\frac{n}{2}-1\right),\ \mathcal{O}\left(-k-\frac{n}{2}+2\right),\dots,$$
$$\dots,\ \bigwedge^{n-4}\mathcal{T}(-k-n+2),\ \mathcal{O}(-k-1),\ \bigwedge^{n-2}\mathcal{T}(-k-n+1),\ \mathcal{O}(-k),$$

and the basis Q''_k corresponds to

$$\mathcal{O}\left(-k-\frac{n}{2}+1\right),\ \mathcal{T}\left(-k-\frac{n}{2}\right),\ \mathcal{O}\left(-k-\frac{n}{2}+2\right),\ \bigwedge^3\mathcal{T}\left(-k-\frac{n}{2}-1\right),\dots,$$
$$\dots,\ \mathcal{O}(-k-1),\ \bigwedge^{n-3}\mathcal{T}(-k-n+2),\ \mathcal{O}(-k),\ \bigwedge^{n-1}\mathcal{T}(-k-n+1).$$

In these exceptional collections, each of the objects $\mathrm{O}(m)$ sits in degree 0 and each of the objects $\bigwedge^h\mathcal{T}(m)$ sits in degree $-h$.

Proof. It follows from Theorem 7.26 and the description of the bases given Q'_k, Q''_k above. □

Corollary 7.28. The objects corresponding to the elements of the bases $\tilde{Q}'_k, \tilde{Q}''_k$ are obtained from the objects corresponding to the elements of the bases Q'_k, Q''_k by twisting their T-equivariant structures.

More precisely, for $a \in \mathbb{Z}$ define the T-characters

$$\underbrace{\bigwedge^n V \otimes \cdots \otimes \bigwedge^n V}_{a\ times}, \qquad \text{if } a \geq 0, \tag{7.19}$$

$$\underbrace{\bigwedge^n V^* \otimes \cdots \otimes \bigwedge^n V^*}_{-a\ times}, \qquad \text{if } a < 0, \tag{7.20}$$

where $V = \sim \mathbb{C}^n$ is the diagonal representation of T. Given $m \in \mathbb{Z}$ define $a \in \mathbb{Z}$ from

$$0 \leq m + an \leq n - 1. \tag{7.21}$$

Then the T-equivariant structure of any object $\mathcal{O}(-m)$ or $\bigwedge^{m-\ell}\mathcal{T}(-m)$ corresponding to bases Q'_k, Q''_k must be tensored with the corresponding character defined above.

Proof. It follows from Corollary 7.27 and Rule 7.24.

7.7. Asymptotic expansion of bases Q'_k and Q''_k in sectors $\mathcal{V}'_k$ and $\widetilde{\mathcal{V}''_k}$.

Following [TV19a], introduce the coordinates (r,φ) on the universal cover $\mathbb{C}^*$ of the punctured s-plane $\mathbb{C}^*$:

$$q = s^n, \quad s = re^{-2\pi\sqrt{-1}\phi}, \quad r > 0, \ \phi \in \mathbb{R}. \tag{7.22}$$

Lemma 7.29 ([TV19a, Lemma 5.1]). For $m \in \mathbb{Z}$ and $\varphi \in \mathbb{R}$ such that

$$\frac{m}{n} - 1 < \phi < \frac{m}{n}, \tag{7.23}$$

we have the asymptotic expansion as $s \to \infty$

$$\Psi^m(s^n, \boldsymbol{y}, \boldsymbol{z}) = \frac{(2\pi)^{\frac{n-1}{2}}}{\sqrt{n}} e^{\pi\sqrt{-1}\sum_{i=1}^n z_i} \left(e^{-\sqrt{-1}\pi}\zeta_n^m s\right)^{\sum_{i=1}^n z_i + \frac{n-1}{2}} e^{ns\zeta_n^m} \left(1 + O\left(\frac{1}{s}\right)\right), \tag{7.24}$$

$$\zeta_n := \exp\left(\frac{2\pi\sqrt{-1}}{n}\right),$$

where $\arg\left(e^{-\sqrt{-1}\pi}\zeta_n^m s\right) = 2\pi\frac{m}{n} - \pi - 2\pi\phi$, so $\left|\arg\left(e^{-\sqrt{-1}\pi}\zeta_n^m s\right)\right| < \pi$ that).

Consider the sectors

$$\mathcal{V}'_k := \left\{ s \in \widetilde{\mathbb{C}^*} : \frac{k}{n} - \frac{1}{2} - \frac{1}{2n} < \phi < \frac{k}{n} \right\}, \tag{7.25}$$

$$\mathcal{V}''_k := \left\{ s \in \widetilde{\mathbb{C}^*} : \frac{k}{n} - \frac{1}{2} - \frac{1}{n} < \phi < \frac{k}{n} - \frac{1}{2n} \right\} \tag{7.26}$$

for $k \in \mathbb{Z}$.

Let us recall the main result of [TV19a] concerning the asymptotic expansion of the bases Q'_k, Q''_k.

Theorem 7.30 ([TV19a, Theorem 7.1]). The elements of the basis of solutions Q'_k (resp. Q''_k) can be reordered to a basis $(I_m(s^n, \boldsymbol{y}, \boldsymbol{z}))_{m=0}^{n-1}$ with asymptotic expansion

$$I_m(s^n, \boldsymbol{y}, \boldsymbol{z}) = \frac{(2\pi)^{\frac{n-1}{2}}}{\sqrt{n}} e^{\pi\sqrt{-1}\sum_{i=1}^n z_i} \left(e^{-\sqrt{-1}\pi}\zeta_n^m s\right)^{\sum_{i=1}^n z_i + \frac{n-1}{2}} e^{ns\zeta_n^m} \left(1 + O\left(\frac{1}{s}\right)\right), \tag{7.27}$$

for $s \to \infty$ *and* $s \in \mathcal{V}'_k$ *(resp* $s \in \mathcal{V}''_k$). Here, for defining $\left(e^{-\sqrt{-1}\pi}\zeta_n^m s\right)^{\sum_{i=1}^n z_i + \frac{n-1}{2}}$, the following choice of the branch of $\log\left(e^{-\sqrt{-1}\pi}\zeta_n^m s\right)$ is done: for every m the argument of $\left(e^{-\sqrt{-1}\pi}\zeta_n^m s\right)$ is chosen so that $|\arg(e^{-\sqrt{-1}\pi}\zeta_n^m s)| < \pi$ when φ tends

(1) *to* $\frac{k}{n}$ inside $\left(\frac{2k-n-1}{2n}, \frac{k}{n}\right)$, for the case of Q'_k,

(2) *to* $\frac{2k-1}{2n}$ inside $\left(\frac{2k-n-2}{2n}, \frac{2k-1}{2n}\right)$, for the case of Q''_k.

In both cases, the argument of $\left(e^{-\sqrt{-1}\pi}\zeta_n^m s\right)$ is continuous for φ in the intervals above.

In terms of the bases Q'_k and Q''_k22 we can recast this result as follows.

Corollary 7.31. The elements of the basis of solutions $\widetilde{Q}'_k$ (resp. $\widetilde{Q}''_k$) can be reordered to a basis $(I_m(s^n, \boldsymbol{y}, \boldsymbol{z}))_{m=0}^{n-1}$ with asymptotic expansion

$$I_m(s^n, \boldsymbol{y}, \boldsymbol{z}) = \frac{(2\pi)^{\frac{n-1}{2}}}{\sqrt{n}} e^{-\pi\sqrt{-1}\frac{n-1}{2}} (\zeta_n^m s)^{\sum_{i=1}^n z_i + \frac{n-1}{2}} e^{ns\zeta_n^m} \left(1 + O\left(\frac{1}{s}\right)\right), \tag{7.28}$$

for $s \to \infty$ and $s \in \mathcal{V}'_k$ (resp $s \in \mathcal{V}''_k$). Here for defining $(\zeta_n^m s)^{\sum_{i=1}^n z_i + \frac{n-1}{2}}$, the principal determination of the argument of $(\zeta_n^m s)$ is chosen, i.e. $\arg(\zeta_n^m s) = 2\pi\frac{m}{n} - 2\pi\varphi$.

Proof. From Lemma 7.29 and Theorem 7.30, we have that, for any $m \in \mathbb{Z}$, the element $\Psi^m(\ell)$ of Q'_k admits the following expansion on $\mathcal{V}'_k$:

$$\begin{aligned}\Psi^m(\ell)(s^n, \boldsymbol{y}, \boldsymbol{z}) &= \frac{(2\pi)^{\frac{n-1}{2}}}{\sqrt{n}} e^{\pi\sqrt{-1}s_1(\boldsymbol{z})} \left(e^{-\sqrt{-1}\pi}\zeta_n^m s\right)^{s_1(\boldsymbol{z})+\frac{n-1}{2}} e^{ns\zeta_n^m} \left(1 + O\left(\frac{1}{s}\right)\right) \\ &= \frac{(2\pi)^{\frac{n-1}{2}}}{\sqrt{n}} e^{\pi\sqrt{-1}s_1(\boldsymbol{z})} \left(e^{-\sqrt{-1}\pi}\zeta_n^{m+an} s\right)^{s_1(\boldsymbol{z})+\frac{n-1}{2}} \cdot \\ &\quad \cdot (\zeta_n^{an})^{-s_1(\boldsymbol{z})-\frac{n-1}{2}} e^{ns\zeta_n^m} \left(1 + O\left(\frac{1}{s}\right)\right),\end{aligned}$$

where $a \in \mathbb{Z}$ is such that

$$0 \le m + an \le n - 1.$$

Thus, we have the following asymptotic expansion for $s \to \infty$ and $s \in \mathcal{V}'_k$:

$$\begin{aligned}&(\zeta_n^{an})^{s_1(\boldsymbol{z})+\frac{n-1}{2}} \Psi^m(\ell)(s^n, \boldsymbol{y}, \boldsymbol{z}) = \\ &\qquad \frac{(2\pi)^{\frac{n-1}{2}}}{\sqrt{n}} e^{-\pi\sqrt{-1}\frac{n-1}{2}} (\zeta_n^{m+an} s)^{s_1(\boldsymbol{z})+\frac{n-1}{2}} e^{ns\zeta_n^m} \left(1 + O\left(\frac{1}{s}\right)\right).\end{aligned}$$

Notice that

$$(\zeta_n^{an})^{s_1(\boldsymbol{z})+\frac{n-1}{2}} \Psi^m(\ell) = \left((-1)^{n-1} s_n(\boldsymbol{\hat{Z}})\right)^a \Psi^m(\ell)$$

is the element of $\widetilde{Q}'_k$ obtained the element $\Psi^m(\ell)$ of Q'_k by applying the Rule 7.24. This proves the statement for $\widetilde{Q}'_k$. The same argument applies for $\widetilde{Q}''_k$.

8. B-classes and B-Theorem

In this Section we prove the B-Theorem, one of the main results of this paper.

8.1. Morphism B.

Definition 8.1. Define the morphism of complex vector spaces B:

$$K_0^{\mathbb{T}}(\mathbb{P}^{n-1})_{\mathbb{C}} \to H_{\mathbb{T}}^{\Omega}(\mathbb{P}^{n-1})$$

by

$$\mathrm{B}(F) := \widehat{\Gamma}^+_{\mathbb{P}^{n-1}} \cdot \exp\left(\pi\sqrt{-1}c_1(\mathbb{P}^{n-1})\right) \cdot \mathrm{Ch}_{\mathbb{T}}(F). \tag{8.1}$$

See Section 4.4 for the definition of the characteristic classes in the r.h.s. of (8.1).

8.2. B-Theorem.

Consider the space S_n of solutions $I(q,z)$ of the equivariant quantum differential equation (5.19) that are holomorphic wrt z in Ω. The space S_n is a

module over 0_Ω. Since elements of S_n can be seen as element of S_n, the isomorphism $\theta\colon K_0^{\mathbb{T}}(\mathbb{P}^{n-1}) \to \mathcal{S}_n$ of Corollary 7.16 induces a map

$$\theta\colon K_0^{\mathbb{T}}(\mathbb{P}^{n-1}) \to \mathscr{S}_n.$$

The restriction S^o of the topological-enumerative morphism, defined in (6.10), allows us to associate an element of S_n to any element of $H_{\mathbb{T}}^{\bullet}(\mathbb{P}^{n-1},\mathbb{C})$. By extension of scalars, there is an induced morphism

$$\mathcal{S}^o\colon H_{\mathbb{T}}^{\Omega}(\mathbb{P}^{n-1}) \to \mathscr{S}_n, \quad \alpha \mapsto \mathcal{S}^o(q)\alpha.$$

Theorem 8.2 (B-Theorem). The following diagram is commutative:

$$\begin{array}{ccc} K_0^{\mathbb{T}}(\mathbb{P}^{n-1})_{\mathbb{C}} & \xrightarrow{\quad \text{Б} \quad} & H_{\mathbb{T}}^{\Omega}(\mathbb{P}^{n-1}) \\ & S_o\,\theta & \\ & S_n & \end{array}$$

In other words, if $F \in K_0^{\mathbb{T}}(\mathbb{P}^{n-1})_{\mathbb{C}}$ and $\theta(F) \in S_n$ is the corresponding solution to the joint system of equations (5.19) and (5.26), then the meromorphic functions $h_{X,j}(z)$, with $j = 1,\dots,n$, defined by the identity

$$\theta(F) = \sum_{j=1}^{n} h_{X,j}(\boldsymbol{z})\Psi_{\mathrm{top},j}, \tag{8.2}$$

are the components of the equivariant cohomology class B(F) wrt the basis $(x_\alpha)_{\alpha=0}^{n-1}$, i.e.

$$(F) = \sum_{j=1}^{n} h_{X,j}(\boldsymbol{z})x_{j-1}. \tag{8.3}$$
B

Remark 8.3. The relation between $\theta(F)$ and the equivariant-topological solution is the equivariant version of part 3.b of [CDG18, Conjecture 5.2] for P^{n-1}, see also [Dub98,KKP08,GGI16]. Notice that this is also a refinement of Gamma Theorem of [TV19b].

Proof. We prove the statement of the theorem for a basis in $K_0^{\mathbb{T}}(\mathbb{P}^{n-1})_{\mathbb{C}}$. Then the result for an arbitrary element $F \in K_0^{\mathbb{T}}(\mathbb{P}^{n-1})_{\mathbb{C}}$ follows by linearity and Lemma 4.5.

For $k \in \mathsf{Z}$ consider the basis $([\mathsf{O}(-k-n+1)],\dots,[\mathsf{O}(-k-1)],[\mathsf{O}(-k)])$ in $K_0^{\mathbb{T}}(\mathbb{P}^{n-1})_{\mathbb{C}}$. Its θ-image in S_n is the basis $\left(\Psi \quad - - \quad\right)_{m=0^{k+n\ 1\ m\ n}}^{-1}$. Let $Y_{\mathrm{TV},k} = \left([Y_{\mathrm{TV},k}]_m^{\lambda}\right)_{\lambda,m}$ be the matrix defined by

$$\Psi^{k+n-1-m} = \sum_{\lambda} [Y_{\mathrm{TV},k}]_m^{\lambda}\, x_\lambda, \quad m = 0,\dots,n-1.$$

For $z \in \Omega$, the matrix $Y_{\mathrm{TV},k}$ is a fundamental system of solutions of system (6.1). The matrix $C_{\mathrm{TV},k}$ connecting the basis $Y_{\mathrm{TV},k}$ with the topological-enumerative solution Y_{top},

$$Y_{\mathrm{TV},k} = Y_{\mathrm{top}} \cdot C_{\mathrm{TV},k},$$

equals

$C_{\mathrm{TV},k} = C \cdot \mathrm{diag}\left(\exp(2\pi\sqrt{-1}(k+n-1-m)z_{m+1})\right)_{m=0}^{n-1}$, where C is given by (7.3). This shows that $C_{\mathrm{TV},k}$ is the matrix of the morphism

B with respect the bases $([\mathrm{O}(-k-n+1)],\dots,[\mathrm{O}(-k-1)],[\mathrm{O}(-k)])$ and $(x_\alpha)_{\alpha=0}^{n-1}$.

This concludes the proof.

9. Formal solutions of the system of *qDE* and *qKZ* equations

9.1. Matrix form of *qDE* and *qKZ* difference equations.

The sections

$$(q,\boldsymbol{z}) \mapsto x_\alpha|_{q,\boldsymbol{z}}, \qquad \alpha = 0,\dots,n-1,$$

of the bundle pr^*H, introduced in Section 5.7, define a trivialization of pr^*H. This trivialization, allows us write the joint system of the *qDE* and *qKZ* difference equations (5.19), (5.26) in matrix form.

For a basis $I_1(q,\mathrm{z}),\dots,I_n(q,\mathrm{z})$ of solutions to the joint system (5.19), (5.26), introduce a matrix $Y(q,\mathrm{z}) = (Y_m{}^\alpha(q,\mathrm{z}))_{\alpha,m}$, with $\alpha = 0,\dots,n-1$, $m = 1,\dots,n$, by the formula:

$$I_m(q,\boldsymbol{z}) = \sum_{\alpha=0}^{n-1} Y_m^\alpha(q,\boldsymbol{z}) x_\alpha|_{q,\boldsymbol{z}}, \quad m = 1,\dots,n. \tag{9.1}$$

Then $Y(q,\mathrm{z})$ is a fundamental system of solutions of the joint system of equations

$$\frac{d}{dq} Y(q,\boldsymbol{z}) = \mathcal{A}(q,\boldsymbol{z}) Y(q,\boldsymbol{z}), \tag{9.2}$$

$$Y(q,z_1,\dots,z_i-1,\dots,z_n) = \mathrm{K}_i(q,\mathrm{z}) Y(q,\mathrm{z}), \tag{9.3}$$

where $\mathrm{A}(q,\mathrm{z})$ is the matrix (6.1) attached to the operator $x*_{q,\mathrm{z}}$ wrt the basis $(x_\alpha)_\alpha$, and the matrix $\mathrm{K}_i(q,\mathrm{z})$ is the matrix attached to the isomorphism (5.25) wrt the basis $(x_\alpha)_\alpha$.

Remark 9.1. The sections

$$(q,\boldsymbol{z}) \mapsto g_j|_{q,\boldsymbol{z}}, \qquad j = 1,\dots,n,$$

define another trivialization of pr^*H. In this trivialization, the *qKZ* difference equations are

$$\widehat{Y}(q,z_1,\dots,z_i-1,\dots,z_n) = K_i(q,\boldsymbol{z})\widehat{Y}(q,\boldsymbol{z}), \quad i = 1,\dots,n, \tag{9.4}$$

where $K_i(q,\boldsymbol{z})$ are the matrices of *qKZ*-operators (5.16) wrt the basis $(g_j)_j$ and the matrix $Y(q,\mathrm{z})$ is defined by

$$I_m(q,\boldsymbol{z}) := \sum_{j=1}^{n} \widehat{Y}_m^j(q,\boldsymbol{z}) g_j|_{q,\boldsymbol{z}}, \quad m = 1,\dots,n.$$

Notice the difference between (9.3) and (9.4).

In Section 6 we studied equation (9.2) only. Now we will study the joint system of equations (9.2) and (9.3). As a result of this Section and Section 10 we will deduce the following theorem. (For its precise statement see in Theorem 9.6 and Corollary 10.14.)

Theorem 9.2. Consider the joint system (9.2), (9.3) of the *qDE* and *qKZ* equations for P^{n-1}. This system is equivalent at $q = \infty$, up to change of variable $q = s^n$, to the system

(9.5)
$$\frac{dZ}{ds} = UZ,$$

(9.6) where
$$Z(s, z_1, \dots, z_j - 1, \dots, z_n) = \mathcal{K}_j Z(s, \boldsymbol{z}), \quad j = 1, \dots, n,$$
$$U := \operatorname{diag}(n\zeta_n^0, \dots, n\zeta_n^{n-1}),$$
$$\mathcal{K}_j := \operatorname{diag}\left(\zeta_n^0, \zeta_n^{-1}, \dots, \zeta_n^{-(n-1)}\right), \quad j = 1, \dots, n,$$
$$\zeta_n := \exp\left(\frac{2\pi\sqrt{-1}}{n}\right).$$

The theorem says that after a formal transformation, the system of *qDE* and *qKZ* equations becomes a system with constant coefficients and separated variables. Moreover, the system splits into the direct sum of systems of rank one.

System (9.5), (9.6) admits the basis of solutions

$$Z_i(s, \boldsymbol{z}) = \exp\left(n\zeta_n^{i-1}s + \frac{2(i-1)\pi\sqrt{-1}}{n}\sum_{a=1}^{n} z_a\right)\begin{pmatrix}0\\ \vdots\\ 1_i\\ \vdots\\ 0\end{pmatrix} \qquad i = 1, \dots, n.$$

All solutions of system (9.5), (9.6) are linear combinations of these basis solutions with coefficients 1-periodic in $z_1,\dots,z_n$.

The formal transformation which realizes the reduction to system (9.5), (9.6), will be described in the following subsections.

9.2. Shearing transformation. The singularity at $q = \infty$ of the differential system (6.1) is irregular of Poincaré rank 1. It is known [Was65,BJL79a,MS16, LR16] that (6.1) admits a formal solution of the form

(9.7)
$$Y_{\text{form}}(q, \boldsymbol{z}) = \Phi\left(q^{\frac{1}{\nu}}, \boldsymbol{z}\right) q^{\Lambda(\boldsymbol{z})} \exp\left(P\left(q^{\frac{1}{\nu}}, \boldsymbol{z}\right)\right),$$

where

- $\nu \in \mathbb{N}$ is the degree of ramification of the singularity,
- Φ is an $n \times n$ matrix-valued formal power series in $q^{-\frac{1}{\nu}}$ of the form
$$\Phi\left(q^{\frac{1}{\nu}}, \boldsymbol{z}\right) = \sum_{j=0}^{\infty} \Phi_j(\boldsymbol{z}) q^{-\frac{j}{\nu}}, \quad \det \Phi_0(\boldsymbol{z}) \neq 0,$$
- Λ is an $n \times n$-matrix depending only on $\boldsymbol{z}$ (the exponent of formal monodromy),
- $P = \operatorname{diag}(p_1,\dots,p_n)$ where each $p_j\left(q^{\frac{1}{\nu}}, \boldsymbol{z}\right)$ is a polynomial in $q^{\frac{1}{\nu}}$ of the form
$$p_j(q^{\frac{1}{\nu}}, \boldsymbol{z}) = \sum_{\ell=1}^{N_j} p_{j\ell}(\boldsymbol{z}) q^{\frac{\ell}{\nu}}, \quad N_j \geq 1.$$

To find the formal solution Y_{form}, we perform the gauge transformation of (6.1) defined by

(9.8)
$$Y(q, \boldsymbol{z}) = \mathcal{H}(q) \cdot \widetilde{T}(q, \boldsymbol{z}), \quad \mathcal{H}(q) := \operatorname{diag}\left(1, q^{-\frac{1}{n}}, \dots, q^{-\frac{n-1}{n}}\right),$$

called the shearing transformation, see [Was65, Section 19]. The function T satisfies the differential equation

$$\frac{d\widetilde{T}}{dq} = \mathcal{A}_{sh}(q,\boldsymbol{z})\widetilde{T}, \quad \mathcal{A}_{sh} = \mathcal{H}^{-1}\cdot\mathcal{A}\cdot\mathcal{H} - \mathcal{H}^{-1}\frac{d\mathcal{H}}{dq}. \tag{9.9}$$

Explicitly, the entries of $\mathcal{A}_{sh}$ are given by

$$(\mathcal{A}_{sh})^{\alpha}_{\beta} = q^{\frac{1-n}{n}}\delta_{\alpha-\beta,1} + q^{\frac{1-n}{n}}\delta_{\alpha,1}\delta_{\beta,n} + \sum_{j=1}^{n}(-1)^{j+1}s_j(\boldsymbol{z})\delta_{\alpha+j,n+1}\delta_{\beta,n}q^{\frac{1-n-j}{n}} + \delta_{\alpha\beta}\frac{\beta-1}{n}q^{-1},$$

for $\alpha,\beta = 1,\dots,n$. With the change of variable $q = s^n$, the function $T(s,z) :=$ $\widetilde{T}(s^n,\boldsymbol{z})$ is a solution of the equation

$$\frac{d}{ds}T(s,\boldsymbol{z}) = \mathcal{B}(s,\boldsymbol{z})T(s,\boldsymbol{z}), \quad \mathcal{B}(s,\boldsymbol{z}) := ns^{n-1}\mathcal{A}_{sh}(s^n,\boldsymbol{z}). \tag{9.10}$$

Lemma 9.3. We have the following expansion for the coefficient $\mathcal{B}(s,z)$:

$$\mathcal{B}(s,\boldsymbol{z}) = \mathcal{B}_0 + \frac{1}{s}\mathcal{B}_1(\boldsymbol{z}) + \sum_{j=2}^{n}\frac{1}{s^j}\mathcal{B}_j(\boldsymbol{z}), \tag{9.11}$$

where

$$\mathcal{B}_0 := \begin{pmatrix} 0 & & \dots & 0 & n \\ n & 0 & \dots & 0 & 0 \\ 0 & n & \dots & 0 & 0 \\ & & \ddots & & \vdots \\ & & & n & 0 \end{pmatrix}, \tag{9.12}$$

$$\mathcal{B}_1(\boldsymbol{z}) := \begin{pmatrix} 0 & & & & & \\ & 1 & & & & \\ & & 2 & & & \\ & & & \ddots & & \\ & & & & n-2 & \\ & & & & & n-1+ns_1(\boldsymbol{z}) \end{pmatrix}, \tag{9.13}$$

0 0 0 0 0 0

j ... ... $j+1$... j

(9.14) B (z) := 00 00 (−1) 0ns (z) , $j = 2,\dots,n$.

... ...

0 0 0

By the shearing transformation (9.8) and the change of variable $q = s^n$, we have reduced the equivariant quantum differential equation of $\mathbb{P}^{n-1}$ to the equation

$$\frac{dT}{ds} = \mathcal{B}(s, \boldsymbol{z})T. \tag{9.15}$$

Equation (9.15) has an irregular singularity at $s = \infty$ of Poincaré rank 1. The essential difference between differential systems (6.1) and (9.15): the matrix $B(s,z)$ has the leading term B_0 with distinct eigenvalues

$$u_k := n\zeta_n^{k-1}, \quad \zeta_n := \exp\left(\frac{2\pi\sqrt{-1}}{n}\right), \quad k = 1, \dots, n, \tag{9.16}$$

while the matrix $A(q,z)$ has the nilpotent leading term $A_0(z)$.

9.3. The E-matrix. Let $(e_1,\dots,e_n)$ be the standard basis of C^n. Let[4] η_{cl} be the bilinear form on C_n with matrix

(9.17) $(\eta_{cl})_{\alpha\beta} = \delta_{n+1,\alpha+\beta}$ wrt the standard basis.

For fixed $z \in C_n$, consider the C-linear endomorphisms $B_0, B_1(z) \in \mathrm{End}(C_n)$

(9.13)defined, in the standard basis, by the matrices. Introduce the matrix $E \in GL(n,C)$, B_0 and $B_1(z)$ of equations (9.12),

$$(\mathcal{E})_{i\alpha} := \frac{1}{\sqrt{n}} \exp\left(\frac{(i-1)(2\alpha-1)\sqrt{-1}\pi}{n}\right), \quad \alpha, i = 1, \dots, n, \tag{9.18}$$

with inverse

$$(\mathcal{E}^{-1})_{\alpha i} = \frac{1}{\sqrt{n}} \exp\left(\frac{(i-1)(1-2\alpha)\sqrt{-1}\pi}{n}\right), \quad \alpha, i = 1, \dots, n. \tag{9.19}$$

Lemma 9.4. Define the basis $(f_1,\dots,f_n)$ of C^n by

$$f_j := \sum_{\alpha=1}^{n} \left(\mathcal{E}^{-1}\right)_{\alpha j} e_\alpha, \quad j = 1, \dots, n,$$

then

(1) The basis $(f_1,\dots,f_n)$ is orthonormal wrt the bilinear form η_{cl}.
(2) The basis $(f_1,\dots,f_{nn})$ is an eigenbasis of the operator B_0.
(3) For any fixed $z \in C$,

$$\eta_{cl}\left(\mathcal{B}_1(\boldsymbol{z})f_i, f_i\right) = s_1(\boldsymbol{z}) + \frac{n-1}{2}, \quad i = 1, \dots, n.$$

Proof. The statements are equivalent to the identities

(9.21) $(E^{-1})^T \eta_{cl} E^{-1} =$,

[4] Here the subscript "cl" stands for classical. The matrix η_{cl}, indeed, appears in the study of the quantum cohomology of P^n as the classical Poincaré metric. See Remark 9.5.

(9.22) $$\mathrm{E}\,\mathrm{B}_0\,\mathrm{E}^{-1} = \mathrm{diag}(u_1,\dots,u_n),$$

(9.23) $$\mathcal{E}\ \mathcal{B}_1(\boldsymbol{z})\ \mathcal{E}^{-1} = \left(s_1(\boldsymbol{z}) + \frac{n-1}{2}\right)\cdot \mathbb{1} + B^{\mathrm{od}}(\boldsymbol{z})$$

,

where B^{od} is an off-diagonal matrix, i.e. $(B^{\mathrm{od}})_{ii} = 0$. A straightforward computation shows the validity of these identities.

Remark 9.5. The matrices B_0, η_{cl} and E appear in the study of the quantum cohomology of P^{n-1} seen as a Frobenius manifold, see Appendix B for details.

9.4. Formal reduction of the system of *qDE* and *qKZ* equations.

Theorem 9.6. There exists a unique $n \times n$-matrix $G(s,\mathrm{z})$, of the form

(9.24) $$G(s,\mathrm{z}) = \mathrm{H}(s^n)\mathrm{E}^{-1}F(s,\mathrm{z})s^{\Lambda(\mathrm{z})},$$

with

(9.25) $$\mathrm{H}(s^n) = \mathrm{diag}(1,s^{-1},\dots,s^{-(n-1)}),$$

(9.26) $$F(s,\boldsymbol{z}) = \mathbb{1} + \sum_{k=1}^{\infty}\frac{\widetilde{F}_k(\boldsymbol{z})}{s^k}, \quad \widetilde{F}_k(\boldsymbol{z})$$

polynomials,

(9.27) $$\Lambda(\boldsymbol{z}) = \left(s_1(\boldsymbol{z}) + \frac{n-1}{2}\right)\cdot \mathbb{1},$$ such that the transformation

(9.27) $$Y(s^n,\boldsymbol{z}) = G(s,\boldsymbol{z})Z(s,\boldsymbol{z})$$

transforms the joint system (9.2), (9.3) of qDE and qKZ equations to the system

(9.28) $$\frac{dZ}{ds} = UZ, \quad U = \mathrm{diag}(n\zeta_n{}^0,\dots,n\zeta_n{}^{n-1}),$$

(9.29) $$Z(s,z_1,\dots,z_j-1,\dots,z_n) = \mathrm{K}_j(\mathrm{z})Z(s,\mathrm{z}), \qquad j = 1,\dots,n,$$

where the matrices $\mathrm{K}_j(\mathrm{z})$ are diagonal and polynomial in z.

Proof. The theorem follows from Theorem A.1 of Appendix A, after shearing transformation (9.8) and change of variables $q = s^n$. Notice that Assumption (1)(4) of Theorem A.1 are satisfied: see Lemma 9.3, Lemma 9.4, the expression of *qKZ*-operators K_i's in the g-bases (5.16) and Remark 9.1. in the proof of Theorem A.1 and the fact that2

The functions F_k are polynomial in z: this follows from the procedure described

- the matrices $\mathrm{B}_0(\mathrm{z}),\dots,\mathrm{B}_n(\mathrm{z})$ are polynomial in z,
- the matrices U and E do not depend on z.

The matrices K_j's and K_j's are related by the identity

(9.30)

$K_j(z) := sF(s,z_1,\dots,z_j-1,\dots,z_n)^{-1}EH(s^n)^{-1}\mathrm{K}_j(s^n,z)H(s^n)E^{-1}F(s,z)$, for $j = 1,\dots,n$. This implies

that the matrices $K_j(z)$ are polynomial in z. Corollary 9.7. The following identity holds true

$$\mathcal{K}_j(\boldsymbol{z}) = \operatorname*{Res}_{s=0}\left(\mathcal{E}\ \mathcal{H}(s)^{-1}\mathscr{K}_j(s^n,\boldsymbol{z})\mathcal{H}(s)\ \mathcal{E}^{-1}\right) \quad (9.\qquad 31),$$

for any $j = 1,\dots,n.$

Proof. Since the series $F(s,z)$ has the form (9.25), from (9.30) we deduce that

$$\begin{aligned}\mathcal{K}_j(\boldsymbol{z}) =& s\left(\mathbb{1}+O\left(\frac{1}{s}\right)\right)\left(\mathcal{E}\ \mathcal{H}(s)^{-1}\mathscr{K}_j(s^n,\boldsymbol{z})\mathcal{H}(s)\ \mathcal{E}^{-1}\right)\left(\mathbb{1}+O\left(\frac{1}{s}\right)\right)\\ =& s\left(\mathcal{E}\ \mathcal{H}(s)^{-1}\mathscr{K}_j(s^n,\boldsymbol{z})\mathcal{H}(s)\ \mathcal{E}^{-1}\right)\left(\mathbb{1}+O\left(\frac{1}{s}\right)\right).\end{aligned}$$

Hence,

$$\mathcal{E}\ \mathcal{H}(s)^{-1}\mathscr{K}_j(s^n,\boldsymbol{z})\mathcal{H}(s)\ \mathcal{E}^{-1} = \frac{\mathcal{K}_j(\boldsymbol{z})}{s}+O\left(\frac{1}{s^2}\right).$$

9.5. Formal solutions of the system of *qDE* and *qKZ* equations at $q = \infty$. Consider the system

$$\frac{dZ}{ds} = UZ, \tag{9.32}$$

(9.33) $Z(s,z_1,\dots,z_j-1,\dots,z_n) = K_j(z)Z(s,z)$, $j = 1,\dots,n$, in Theorem 9.6.

Lemma 9.8. Let $C(z)$ be a meromorphic $n \times n$-matrix-valued function on C^n, regular on Ω and with non-vanishing determinant. The following conditions are equivalent:

(1) The matrix $C(z)$ is a fundamental system of solutions of equation (9.33) only, over the ring of 1-periodic functions in z;

(2) The matrix $Z(s,z) := \exp(sU)C(z)$ is a fundamental system of solutions of the joint system of equations (9.32), (9.33), over the ring of 1-periodic functions in z.

Proof. We have

(9.34) $[K_j(z),\exp(sU)] = 0$, since both $\exp(sU)$ and $K_j(z)$ are diagonal.

(1) imples (2): Let $v(s,z)$ be a column vector satisfying both (9.32) and (9.33). Then there exists a unique column vector $c_1(z)$ such that $v(s,z) = \exp(sU)c_1(z)$, since the columns of $\exp(sU)$ give a C-basis of solutions of (9.32). The vector $c_1(z)$ is a solution of (9.33) by (9.34). Hence $c_1(z) = C(z)c_2(z)$ for a unique column vector $c_2(z)$, which is 1-periodic with respect ot $z_1,\dots,z_n$. This shows that $\exp(sU)C(z)$ is a system of fundamental solutions.

(2) implies (1): Equation (9.34) easily implies that $C(z)$ is a solution of the equation (9.33) only, and moreover $C(z)$ is a fundamental solution.

Theorem 9.9. Let $C(z)$ be an $n \times n$-diagonal-matrix-valued function, meromorphic on C^n, regular on Ω and with non-vanishing entries on Ω. The following conditions are equivalent:

(1) The matrix $C(z)$ is a fundamental system of solutions of the difference equations (9.33), over the ring of 1-periodic functions in z;

(2) There exist meromorphic $n \times n$-matrix valued functions $(F_k(z))_{k=1}^{\infty}$ regular on Ω, such that the matrix

$$Y_{\text{form}}(s,z) = H(s)E^{-1}F(s,z)s^{\Lambda(z)}e^{Us}, \tag{9.35}$$

$$F(s,z) = C(z) + \sum_{j=1}^{\infty} \frac{F_j(z)}{s^j}, \tag{9.36}$$

is a formal solution of the joint system of the qDE and qKZ equations for P_{n-1},

$$\frac{d}{ds}Y(s^n,z) = ns^{n-1}\mathcal{A}(s^n,z)Y(s^n,z), \tag{9.37}$$

$$Y(s^n,z_1,\dots,z_j-1,\dots,z_n) = \mathscr{K}_j(s^n,z)Y(s^n,z) \tag{9.38}$$

Moreover, if such a formal solution exists, then it is unique.

Proof. We have

$$[s^{\Lambda(z)}e^{Us},C(z)] = 0, \tag{9.39}$$

since both $s^{\Lambda(z)}e^{Us}$ and $C(z)$ are diagonal. Let

$$G(s,z) = \mathcal{H}(s^n)\mathcal{E}^{-1}\left(\mathbb{1} + \sum_{k=1}^{\infty} \frac{\widetilde{F}_k(z)}{s^k}\right)s^{\Lambda(z)} \tag{9.40}$$

be as in Theorem 9.6.

(1) implies (2): By Lemma 9.8, the matrix $Z(s,z) := \exp(sU)C(z)$ is a solution of the joint system (9.32), (9.33). By Theorem 9.6, the matrix

$$Y_{\text{form}}(s^n,z) := G(s,z)Z(s,z), \tag{9.41}$$

is a formal solution of the joint system (9.37), (9.38). By (9.39), $Y_{\text{form}}(s^n,z)$ can be re-written in the form (9.35), with $F_k(z) := F_k(z)C(z)$ for $k \in \mathbb{N}^*$.

(2) implies (1): By (9.39), we have $2 \in$

$$Y_{\text{form}}(s^n,z) = G2(s,z)e^{Us}C(z), \tag{9.42}$$

$$G(s,\boldsymbol{z}) := \mathcal{H}(s^n)\mathcal{E}^{-1}\left(\mathbb{1} + \sum_{k=1}^{\infty} \frac{F_k(\boldsymbol{z})C(\boldsymbol{z})^{-1}}{s^k}\right) s^{\Lambda(\boldsymbol{z})} \tag{9.43}$$

2 .

(9.37) into (9.32). Hence, it automatically transforms the joint system2 (9.37), (9.38) Thus, the gauge transformation $Y(s^n,z) = G(s,z)Z(s,z)$ transforms the *qDE* into (9.32), (9.33), see the proof of Theorem A.1.

The function $Z(s,z) := \exp(sU)C(z)$ is a solution of the joint system (9.32), (9.33). By Lemma 9.8, one concludes.

The uniqueness of the formal solution follows from Theorem 9.6: we have $G2(s,z) = G(s,z)$. Example 9.10. Let us consider the case of P^1. The original system of *qDE* and *qKZ* equations is the following

$$\frac{d}{dq}Y(q,\boldsymbol{z}) = \frac{1}{q}\begin{pmatrix} 0 & q - z_1z_2 \\ 1 & z_1 + z_2 \end{pmatrix} Y(q,\boldsymbol{z}),$$

$$Y(q, z_1 - 1, z_2) = \frac{1}{q}\begin{pmatrix} -z_2 & q - z_1z_2 \\ 1 & z_1 \end{pmatrix} Y(q,\boldsymbol{z})$$

$$Y(q, z_1, z_2 - 1) = \frac{1}{q}\begin{pmatrix} -z_1 & q - z_1z_2 \\ 1 & z_2 \end{pmatrix} Y(q,\boldsymbol{z}).$$

Through a formal gauge transformation $Y(s^2,z) = G(s,z)Z(s,z)$, the system above can be reduced to the system

$$\frac{d}{ds}Z(s,\boldsymbol{z}) = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} Z(s,\boldsymbol{z})$$

$$Z(s, z_1 - 1, z_2) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} Z(s,\boldsymbol{z}),$$

$$Z(s, z_1, z_2 - 1) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} Z(s,\boldsymbol{z}).$$

The formal gauge $G(s,z)$ is given by

$$G(s,\boldsymbol{z}) = \mathcal{H}(s^2)\mathcal{E}^{-1}\left(\mathbb{1} + \frac{F_1(\boldsymbol{z})}{s} + \frac{F_2(\boldsymbol{z})}{s^2} + \dots\right) s^{\Lambda(\boldsymbol{z})},$$

where

$$\mathrm{H}(s^2) = \mathrm{diag}\left(1, \frac{1}{s}\right), \quad \mathcal{E}^{-1} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix}, \quad \Lambda(\boldsymbol{z}) = \left(\frac{1}{2} + z_1 + z_2\right)\mathbb{1},$$

and the coefficients can be computed recursively as in the proof of Theorem A.1. Here we give just the first coefficient F_1:

$$F_1 = \begin{pmatrix} F_{1,1} & F_{1,2} \\ F_{1,2} & -F_{1,1} \end{pmatrix},$$

where

$$F_{1,1} = s_2(\boldsymbol{z}) - \frac{1}{16}\left(2s_1(\boldsymbol{z}) + 1\right)^2, \quad F_{1,2} = -\frac{\sqrt{-1}}{8}\left(2s_1(\boldsymbol{z}) + 1\right).$$

Notice that Corollary 9.7 allow us to compute directly the coefficients K_1,K_2:

$$\mathcal{E}\,\mathcal{H}(s)^{-1}\mathscr{K}_1(s^2,\boldsymbol{z})\,\mathcal{H}(s)\,\mathcal{E}^{-1} = \begin{pmatrix} \frac{z_1-z_2}{2s^2} + \frac{1}{s} - \frac{z_1z_2}{2s^3} & \frac{\sqrt{-1}(z_1+z_2)}{2s^2} - \frac{\sqrt{-1}z_1z_2}{2s^3} \\ -\frac{\sqrt{-1}z_1z_2}{2s^3} - \frac{\sqrt{-1}(z_1+z_2)}{2s^2} & \frac{z_1-z_2}{2s^2} + \frac{z_1z_2}{2s^3} - \frac{1}{s} \end{pmatrix},$$

$$\mathcal{E}\ \mathcal{H}(s)^{-1}\,\mathscr{K}_2(s^2,\boldsymbol{z})\ \mathcal{H}(s)\ \mathcal{E}^{-1} = \\ = \begin{pmatrix} \frac{-z_1+z_2}{2s^2}+\frac{1}{s}-\frac{z_1z_2}{2s^3} & \frac{\sqrt{-1}(z_1+z_2)}{2s^2}-\frac{\sqrt{-1}z_1z_2}{2s^3} \\ -\frac{\sqrt{-1}z_1z_2}{2s^3}-\frac{\sqrt{-1}(z_1+z_2)}{2s^2} & \frac{-z_1+z_2}{2s^2}+\frac{z_1z_2}{2s^3}-\frac{1}{s} \end{pmatrix}.$$

By Corollary 9.7, we obtain that

$$\mathcal{K}_1(\boldsymbol{z}) = \mathcal{K}_2(\boldsymbol{z}) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Notice, in particular, that both K_1 and K_2 are equal constant matrices: in Corollary 10.14 we will prove that this is the general property valid for all projective spaces P^{n-1}.

10. Stokes bases of the system of *qDE* and *qKZ* equations

10.1. Stokes rays, Stokes sectors. The solution $Y_{\mathrm{form}}(s^n,\boldsymbol{z})$ described in Theorem 9.9 is formal: the series $F(s,z)$ is typically divergent. Nevertheless, $Y_{\mathrm{form}}(s^n,\boldsymbol{z})$ contains information about genuine solutions of the differential system (6.1). The formal solution prescribes indeed the asymptotics of genuine fundamental solutions of (6.1).

As in Section 7.7, we use the coordinates (r,φ) for the universal cover C^* of the punctured s-plane C^*, see equation (7.22). /

Definition 10.1 We call a Stokes ray any ray in the universal cover C^* of the s-plane, defined by the equation. /

$$\phi = \frac{k}{2n}, \quad k \in \mathbb{Z} \tag{10.1}$$

We will denote the ray in (10.1) by R_k.

The meaning of Stokes rays is explained by the following lemma.

Lemma 10.2. A number $\varphi \in R$ is of the form $\varphi = k/2n$ for some $k \in Z$, if and only if there are integers m_1, m_2 such that $\mathrm{Re}(\zeta_n^{m_1} s) = \mathrm{Re}(\zeta_n^{m_2} s)$ *and* $m_1 \not\equiv m_2 \pmod{n}$. □

Definition 10.3 We call Stokes sector any open sector in C^* which contains exactly n consecutive Stokes rays. $R_k,\dots,R_{k+n}$ for some $k \in Z$. /

Lemma 10.4. *Any open sector* $\mathcal{V} \subset$ ~ *of width* $\pi +$ C^* δ, i.e. of the form

$$\mathcal{V} = \left\{ s \in \mathbb{C}^* \colon a - \frac{1}{2} - \delta < \phi < a \right\} \tag{10.2}$$

/ , $a \in R$,

is a Stokes sector for $\delta > 0$ sufficiently small.

The following theorem follows from the general theory of differential equations.

Theorem 10.5 ([Was65,BJL79b,Fed87,Sib90,FIKN06]). Let V ⊆ C/* be a

Stokes sector, and let $Y_{\text{form}}(s^n,z)$ denote the unique solution described in Theorem 9.9. There exists a unique fundamental solution $Y(s^n,z)$ of the differential system (6.1) satisfying the asymptotic condition

$$Y(s^n,\boldsymbol{z}) \sim Y_{\text{form}}(s^n,\boldsymbol{z}), \quad s\to\infty, \quad s\in\mathcal{V}, \tag{10.3}$$

uniformly on compact subsets of Ω. The asymptotic expansion (10.3) can actually be extended to a sector wider than V, up to the nearest Stokes rays.

Remark 10.6. In the notations of Theorem 9.9, the precise meaning of the asymptotic relation (10.3) is the following:

$$\forall K \Subset \Omega,\ \forall h\in\mathbb{N},\ \forall\overline{\mathcal{V}}\subsetneq\mathcal{V},\ \exists C_{K,h,\overline{\mathcal{V}}}>0:\ \text{if } s\in\overline{\mathcal{V}}\setminus\{0\}$$

then

$$\sup_{z\in K}\left\|\mathcal{E}\cdot\mathcal{H}(s)^{-1}\cdot Y(s,\boldsymbol{z})\cdot\exp(-sU)s^{-\Lambda(\boldsymbol{z})}-\sum_{m=0}^{h-1}\frac{F_m(\boldsymbol{z})}{s^m}\right\|<\frac{C_{K,h,\overline{\mathcal{V}}}}{|s|^h}.$$

Here $\overline{V}$ denotes any unbounded closed sector of C/* with vertex at 0, and $F_0(z) = C(z)$. Here, for defining $s^{\Lambda(z)}$, the principal branch of log *s* is chosen.

Lemma 10.7. The sectors $\mathcal{V}'_k, \mathcal{V}''_k$, defined by (7.25)-(7.26), are maximal Stokes sectors wrt to the inclusion, i.e.

(1) they are Stokes sectors,
(2) any Stokes sector V is contained in one (and only one) $\mathcal{V}'_k$ or $\mathcal{V}''_k$.

10.2. Stokes bases and Stokes matrices. Consider a basis $(I_i(s^n,\boldsymbol{y},\boldsymbol{z}))_{i=1}^n$ of solutions of the joint system (5.19) and (5.26), and denote by $Y(s^n,\boldsymbol{z})$ the corresponding matrix-valued function defined by

$$I_\beta(s^n,\boldsymbol{y},\boldsymbol{z})=\sum_{\alpha=0}^{n-1}Y(s^n,\boldsymbol{z})^\alpha_\beta\, x_\alpha, \quad \beta=1,\dots,n. \tag{10.4}$$

The function $Y(s^n,z)$ is a fundamental system of solution of the joint system (9.2) and (9.3). Denote by $Y_{\text{form}}(s^n,z)$ the unique formal solution associated to a diagonal matrix $C(z)$ described in Theorem 9.9.

Definition 10.8. We say that a basis $(I_i(s^n, \boldsymbol{y}, \boldsymbol{z}))_{i=1}^n$ of solutions of the joint system (5.19) and (5.26) is a Stokes basis with normalization $C(\mathsf{z})$ on a sector V if it can be reordered in such a way that the corresponding matrix-valued solution $Y(s^n,\mathsf{z})$ satisfies the asymptotic expansion

$$Y(s^n, \boldsymbol{z}) \sim Y_{\text{form}}(s^n, \boldsymbol{z}), \quad s \to \infty, \quad s \in \mathcal{V}, \tag{10.5}$$

uniformly on compact subsets of Ω. The matrix $Y(s^n,\mathsf{z})$ is called[5] the Stokes fundamental solution with normalization $C(\mathsf{z})$ on V of the joint system (9.2) and

(9.3).

Remark 10.9. By Theorem 10.5, if V is a Stokes sector, and $C(\mathsf{z})$ is a fixed normalization, two Stokes bases on V differ only for the order of their objects. Thus, by abuse of language, we will refer to the S_n-orbit of Stokes bases on V as the Stokes basis on V. Furthermore, if $\mathcal{V} \subseteq \mathcal{V}'_k$ (or $\mathcal{V}''_k$, resp.) then the Stokes basis on V is actually the Stokes basis on $\mathcal{V}'_k$ (or $\mathcal{V}''_k$, resp.), by Theorem 10.5.

Notice that if V is a Stokes sector, then also $e^{\pi\sqrt{-1}}\mathcal{V}$ and $e^{2\pi\sqrt{-1}}\mathcal{V}$ are Stokes sectors.

Definition 10.10. Let $Y(s^n, \boldsymbol{z})$ be the Stokes fundamental solution (with normalization $C(\mathsf{z})$) of system (6.1) on the Stokes sector V. Let $Y_1(s^n, \boldsymbol{z})$, and $Y_2(s^n,\mathsf{z})$, be the Stokes solutions on $e^{\pi\sqrt{-1}}\mathcal{V}$ and on $e^{2\pi\sqrt{-1}}\mathcal{V}$, respectively. Define the Stokes matrices attached to V and $C(\mathsf{z})$ as the matrices S_1,S_2 (depending on $\mathsf{z} \in \Omega$) for which we have

$$Y_1(s^n, \boldsymbol{z}) = Y(s^n, \boldsymbol{z})\mathbb{S}_1, \quad Y_2(s^n, \boldsymbol{z}) = Y_1(s^n, \boldsymbol{z})\mathbb{S}_2, \quad s \in \widetilde{\mathbb{C}^*}, \quad \boldsymbol{z} \in \Omega. \tag{10.6}$$

10.3. Properties of Stokes matrices and lexicographical order. Let V be a Stokes sector, and let $(I_i(s^n, \boldsymbol{y}, \boldsymbol{z}))_{i=1}^n$ be the Stokes basis of the joint system (5.19) and (9.3) on V with normalization $C(\mathsf{z})$. Each element I_i corresponds to one eigenvalue $u_j = n\zeta_n{}^j$, $j = 0,\dots,n-1$. Any ordering of the eigenvalues u_j's (i.e. any permutation of the diagonal entries of U) corresponds to an ordering of the elements I_i's. Correspondingly, the Stokes matrices S_1 and S_2 attached to V transform by conjugation by a permutation matrix.

Proposition 10.11. Denote by S_1,S_2 the Stokes matrices computed wrt the Stokes sector V. There exists a unique order of the entries of U such that for all $\mathsf{z} \in \Omega$ the matrix S_1 (resp. S_2) is upper triangular (resp. lower triangular) with ones along the diagonal.

Proof. The reader may consult [Was65,BJL79b,FIKN06,MS16,LR16].
See also [CDG19,CDG18].

[5] Here we introduce this convenient terminology, though not standard in the literature of ordinary differential equations.

The order which realizes the upper triangular form of S_1 (and consequently the lower triangular form of S_2) is unique, since $u_i \neq u_j$ for $i \neq j$, and it will be called the lexicographical order wrt the Stokes sector V.

tities hold true for any $s \in \mathbb{C}^*$ and $z \in \Omega$:

(1) $Y_2\ (e^{2\pi\sqrt{-1}}s)^n, z\ = Y(s^n, z) \cdot \exp\ 2\pi\sqrt{-1}\Lambda(z)$

(2) $Y_2(s^n, z) = Y(s^n, z) \cdot S_1 S_2$

(3) $Y\ (e^{2\pi\sqrt{-1}}s)^n, z\ = Y(s^n, z) \cdot \exp\ 2\pi\sqrt{-1}\Lambda(z)\ \cdot (S_1 S_2)^{-1}$.

Proposition 10.12. In the notations of Definition 10.10, the following iden-

,

,

Here $\Lambda(z)$ is the exponent of formal monodromy, i.e.

$$\Lambda(\boldsymbol{z}) = \Lambda(\boldsymbol{z}) := \left(s_1(\boldsymbol{z}) + \frac{n-1}{2}\right) \cdot \mathbb{1}$$

.

Proof. For (1), notice that

$$Y_2\left((e^{2\pi\sqrt{-1}}s)^n, \boldsymbol{z}\right) \cdot \exp\left(-2\pi\sqrt{-1}\Lambda(\boldsymbol{z})\right)$$

is a solution of (6.1) with asymptotic expansion $Y_{\text{form}}(s^n, z)$ on the Stokes sector V. Hence it must coincide with $Y(s^n, z)$. Point (2) is a direct consequence of the definition of Stokes matrices. Point (3) $\widetilde{Q}'_k$ $\sim$ follows from points (1) and (2). 10.4. Stokes bases and $\sim$ Q''_k. Q'_k

Proposition 10.13. The basis (resp. $\widetilde{Q}''_k$) is a Stokes basis on $\mathcal{V}'_k$ (resp. $\mathcal{V}''_k$) with normalization

$$C(\boldsymbol{z}) = (2\pi)^{\frac{n-1}{2}} e^{-\pi\sqrt{-1}\frac{n-1}{2}} \operatorname{diag}\left(e^{\frac{m\pi\sqrt{-1}}{n}} (\zeta_n^m)^{s_1(\boldsymbol{z})+\frac{n-1}{2}}\right)_{m=0}^{n-1}. \tag{10.7}$$

Proof. It follows from Corollary 7.31, and formula (9.19) for E^{-1}.

Corollary 10.14. The operators $K_j(z)$, with $j = 1,\ldots,n$ are all equal and independent of z. Indeed, we have

$$K_j = \operatorname{diag}\left(\zeta_n^{-m}\right)_{m=0}^{n-1}, \quad j = 1, \ldots, n.$$

Proof. It follows from Theorem 9.9 and the explicit computation

$$C(z_1,\ldots,z_j - 1,\ldots,z_n)C(z)^{-1}.$$

10.5. Stokes bases as T-full exceptional collections.

Theorem 10.15. Via the isomorphism $\theta\colon K_0^{\mathbb{T}}(\mathbb{P}^{n-1})_{\mathbb{C}} \to \mathcal{S}_n$, Stokes bases on Stokes sectors of the *qDE* of P^{n-1} correspond to K-classes of T-full exceptional collections in $\mathcal{D}^b_{\mathbb{T}}(\mathbb{P}^{n-1})$.

Proof. Stokes bases correspond to T-full exceptional collections of Corollary 7.27.

In particular, the Stokes basis $\widetilde{Q}''_{1-n}$ corresponds (up to shifts) to the exceptional collection

(10.8)
$$\left(\mathcal{O}\left(\frac{n}{2}\right), \bigwedge^1 \mathcal{T}\left(\frac{n}{2}-1\right), \mathcal{O}\left(\frac{n}{2}+1\right), \bigwedge^3 \mathcal{T}\left(\frac{n}{2}-2\right), \dots, \mathcal{O}(n-1), \bigwedge^{n-1} \mathcal{T}\right)$$

for n even, and

(10.9)
$$\left(\mathcal{O}\left(\frac{n-1}{2}\right), \mathcal{O}\left(\frac{n+1}{2}\right), \bigwedge^2 \mathcal{T}\left(\frac{n-3}{2}\right), \mathcal{O}\left(\frac{n+3}{2}\right), \bigwedge^4 \mathcal{T}\left(\frac{n-5}{2}\right), \dots, \mathcal{O}(n-1), \bigwedge^{n-1} \mathcal{T}\right)$$

for n odd. All other Stokes bases, and corresponding exceptional collections, are obtained by application of a braid of the form

$$\dots \delta_{n,\mathrm{odd}} \delta_{n,\mathrm{even}} \delta_{n,\mathrm{odd}} \delta_{n,\mathrm{even}},$$

or

$$\dots \delta^{-1}_{n,\mathrm{even}} \delta^{-1}_{n,\mathrm{odd}} \delta^{-1}_{n,\mathrm{even}} \delta^{-1}_{n,\mathrm{odd}}.$$

Remark 10.16. Exceptional collections (10.8) and (10.9) are the natural equivariant lift in $\mathrm{D}^b_{\mathsf{T}}(\mathsf{P}^{n-1})$ of the exceptional collections of [CDG18, Corollary 6.11]. Also in this non-equivariant case, these collections are identified with Stokes bases of the *qDE* of P^{n-1} in suitable Stokes sectors, see [CDG18, Section 6] for details.

Remark 10.17. All the objects on the T-full exceptional collections attached to Stokes bases are equipped with their natural T-equivariant structure, restriction of the natural $GL(n,\mathsf{C})$-equivariant structure. Under the presentation (3.4), their K-theoretical classes in $K_0^{\mathbb{T}}(\mathbb{P}^{n-1})$ are symmetric polynomials wrt the equivariant parameters Z.

11. Stokes matrices as Gram matrices of exceptional collections

11.1. Musical notation for braids. We introduce a notation for braids in B_n. Elements of B_n will be represented as notes on a musical $(n-1)$-line staff. The lines are enumerated from the bottom (1-st line) to the top ($(n-1)$-th line). The generator τ_i is represented as a hollow oval note head on the i-th line. The relations defining the braid group B_n translate into the diagrammatic rules described in Figure 11.1.

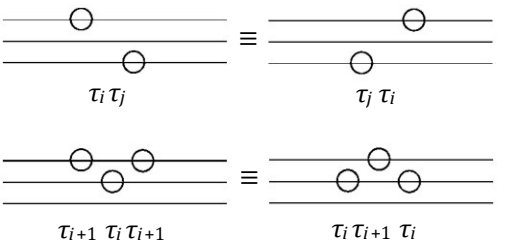


Figure 11.1. Braid relations in musical notation.

11.2. An identity in B_n. For $n \geq 2$, define the braids

(11.1) $$\sigma_{n,\mathrm{even}} := \underbrace{\delta_{n,\,\mathrm{even}}\,\delta_{n,\,\mathrm{odd}}\,\delta_{n,\,\mathrm{even}}\,\delta_{n,\,\mathrm{odd}}\,\ldots}_{n\ \text{factors}}$$

(11.2), $$\sigma_{n,\,\mathrm{odd}} := \underbrace{\delta_{n,\,\mathrm{odd}}\,\delta_{n,\,\mathrm{even}}\,\delta_{n,\,\mathrm{odd}}\,\delta_{n,\,\mathrm{even}}\,\ldots}_{n\ \text{factors}}$$

where $\delta_{n,\mathrm{even}}$ and $\delta_{n,\mathrm{odd}}$ are defined in Section 7.5. Set $\sigma_{1,\mathrm{even}} := 1$, $\sigma_{1,\mathrm{odd}} = 1$.

Lemma 11.1. For any $n \geq 2$, the following identities hold true in B_n:

(11.3) $$\sigma_{n,\mathrm{odd}} = \sigma_{n-1,\mathrm{odd}}(\tau_{n-1}\tau_{n-2}\ldots\tau_1),$$

(11.4) $$\sigma_{n,\mathrm{even}} = \sigma_{n-1,\mathrm{even}}(\tau_{n-1}\tau_{n-2}\ldots\tau_1).$$

Proof. We prove (11.3) by induction on n. For $n = 2$, the statement is obvious, being

$$\delta_{2,\mathrm{odd}} = \tau_1, \qquad \delta_{2,\mathrm{even}} = 1.$$

The musical diagram corresponding to the braid $\delta_{n,\mathrm{odd}}\delta_{n,\mathrm{even}}\delta_{n,\mathrm{odd}}\delta_{n,\mathrm{even}}\ldots$ is the following, according to the parity of n.

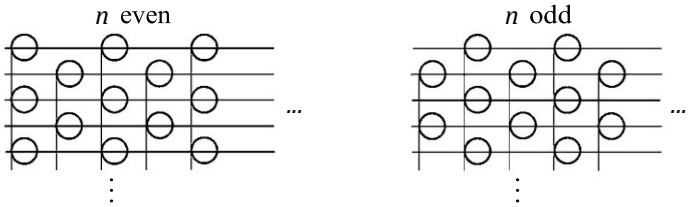


Figure 11.2. Diagrammatic notation for the braid $\delta_{n,\mathrm{odd}}\delta_{n,\mathrm{even}}\delta_{n,\mathrm{odd}}\ldots$ according to the parity of n.

We collected with a stem the notes corresponding to a same factor $\delta_{n,\mathrm{odd}}$ (or $\delta_{n,\mathrm{even}}$). By commutativity, the order of the notes in any factor $\delta_{n,\mathrm{odd}}$ (or $\delta_{n,\mathrm{even}}$) can be modified at will, and for this reason we simply collect them with a vertical stem. In the top line we have

- $\frac{n}{2}$ notes, if n is even,
- $\frac{n-1}{2}$ notes, if n is odd.

We call top factors those factors $\delta_{n,\mathrm{odd}}$'s (or $\delta_{n,\mathrm{even}}$'s) which contains the notes on the top line. In other words, the top factors are

(1) the factors $\delta_{n,\text{odd}}$'s for n even,
(2) the factors $\delta_{n,\text{even}}$'s for n odd.

The factorization (11.3) can be reached by filling the empty spaces between two notes in the last factor $\delta_{n,\text{odd}}$ (or $\delta_{n,\text{even}}$), from the bottom to the top line. We perform this in several steps:

(1) Label by A_0 the first (from the left) elementary braid on the $(n-1)$-th line. By a chain of elementary moves, the braid A_0 can be moved on the $(n-2)$-th line, towards the right, and can be collected with the next top factor, as described in the following Figure 11.3.

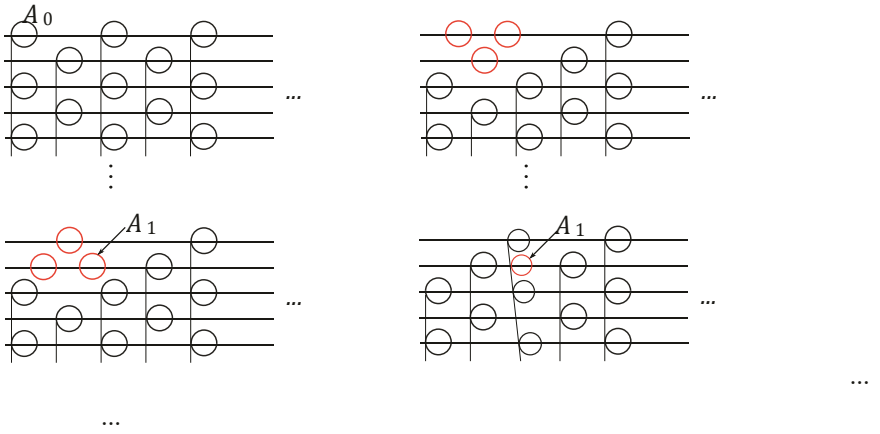
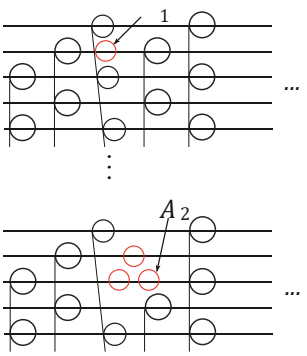


Figure 11.3

We call A_1 the new note obtained from A_0. In this way, this factor is "overcharged" of notes (i.e. it contains notes τ_i and τ_{i+1} for some i), and we have an inclination of the stem, the order of the elementary braids being not anymore arbitrary.

(2) By the braid relations, the braid A_1 can be moved on the $(n-3)$-th line, towards the right, and can be collected with the next factor (not a top factor), as described in the following picture.



A A_1

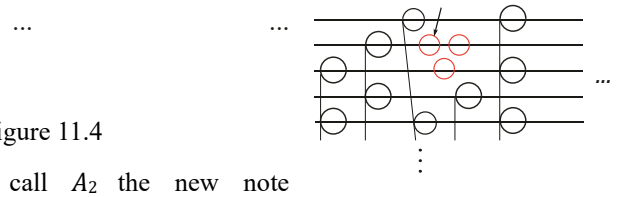
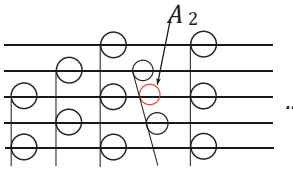


Figure 11.4

We call A_2 the new note obtained from A_1. Also in this case, we have an inclination of the stem.



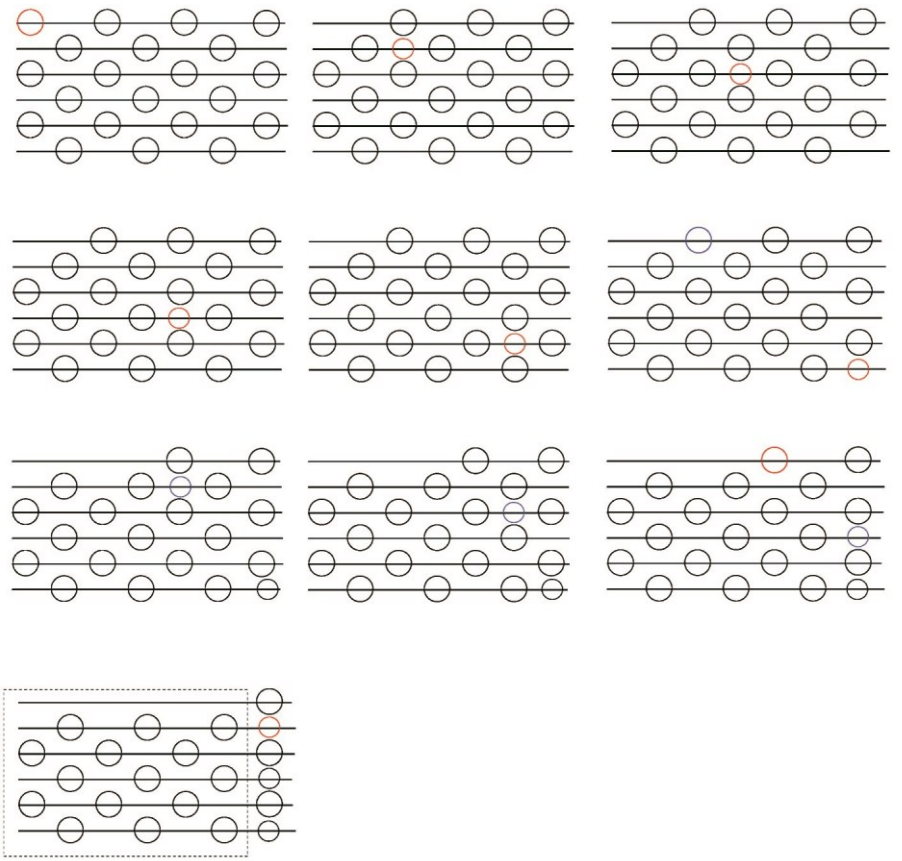
(3) Starting from A_j on the $(n-1-j)$-th line, iterate the procedure of point (2) in order to produce a new braid A_{j+1} in the line $(n-2-j)$-th line, by overcharging the next factor.

(4) Stop when the final braid A_{j+1} fills the empty space on the
• 1-st line if n is even, • 2-nd line if n is odd.

(5) Iterate points (1),(2),(3),(4) and stop when the final braid A_{j+1} fills the first empty space from the bottom line to the top.

By applying the procedure above, the factorization (11.3) is reached. The argument for (11.4) is similar.

Example 11.2. Consider $n = 7$. The factorization (11.4) is obtained with the moves described in Figure 11.5. For simplicity, we remove all the stems of the notes.



$\sigma_{6,\text{even}}$

Figure 11.5. These are the moves described in the proof of Lemma 11.1 in order to obtain the factorization (11.4) for $n = 7$.

Corollary 11.3. For $n \geq 2$, we have

(11.5) $\sigma_{n,\text{even}} = \beta$, $\sigma_{n,\text{odd}} = \beta$, where β is given by equation (2.22),

i.e.

$$\beta := \tau_1(\tau_2\tau_1)\dots(\tau_{n-2}\dots\tau_1)(\tau_{n-1}\tau_{n-2}\dots\tau_1).$$

11.3. Stokes matrices as Gram matrices.

Lemma 11.4. The following identities among Stokes sectors hold true:

$$e^{\pi\sqrt{-1}}\mathcal{V}'_k = \begin{cases} \mathcal{V}'_{k-\frac{n}{2}} & \\ , & \text{if } n \equiv 0 \pmod 2, \\ \mathcal{V}''_{k-\frac{n-1}{2}}, & \text{if } n \equiv 1 \pmod 2 \end{cases},$$

$$e^{\pi\sqrt{-1}}\mathcal{V}''_k = \begin{cases} \mathcal{V}''_{k-\frac{n}{2}} & \\ , & \text{if } n \equiv 0 \pmod 2, \end{cases}$$

$$\begin{cases} \mathcal{V}'_{k-\frac{n-1}{2}}, & \text{if } n \equiv 1 \pmod 2 \end{cases}.$$

Proof. It is readily obtained from the definition of $\mathcal{V}'_k$ and $\mathcal{V}''_k$.

Corollary 11.5. For any $k \in \mathbb{Z}$, the Stokes basis on $e^{\pi\sqrt{-1}}\mathcal{V}'_k$ is obtained by acting (on the left) on the Stokes basis with the braid $\widetilde{Q}'_k$

(11.6).
$$\underbrace{\delta_{n,\text{ odd}}\,\delta_{n,\text{ even}}\,\delta_{n,\text{ odd}}\,\delta_{n,\text{ even}}\,\delta_{n,\text{ odd}}\cdots}_{n \text{ factors}}$$

For any $k \in \mathbb{Z}$, the Stokes basis on $e^{\pi\sqrt{-1}}\mathcal{V}''_k$ is obtained by acting (on the left) on the Stokes basis with the braid $\widetilde{Q}''_k$

(11.7).
$$\underbrace{\delta_{n,\text{ even}}\,\delta_{n,\text{ odd}}\,\delta_{n,\text{ even}}\,\delta_{n,\text{ odd}}\,\delta_{n,\text{ even}}\cdots}_{n \text{ factors}}$$

Proof. It follows from the definition of $\widetilde{Q}'_k$ and $\widetilde{Q}''_k$ (see also Diagram (7.16)), from Proposition 10.13 and Lemma 11.4.

Corollary 11.6. Let V be a Stokes sector of the *qDE* of $\mathbb{P}^{n-1}$, and let E be the T-full exceptional collection corresponding to the Stokes basis on V. The exceptional collection E corresponding to the Stokes sector $e^{2\pi\sqrt{-1}}\mathcal{V}$ is a foundation of the helix generated by E. More precisely, E is the adjacent foundation on the right of E. The collection E is obtained by application of the inverse Serre functor to objects of E:

$$(E_1, \dots, E_n) \mapsto \left(E_1 \otimes (\omega^{\mathbb{T}}_{\mathbb{P}^{n-1}})^{-1}[-n+1], \dots, E_n \otimes (\omega^{\mathbb{T}}_{\mathbb{P}^{n-1}})^{-1}[-n+1]\right), \tag{11.8}$$

where $\omega_{\mathbb{P}^{n-1}}$ denotes the T-equivariant canonical sheaf of $\mathbb{P}^{n-1}$.

Proof. By Corollary 11.5, E is obtained from E by mutation either with the braid

(11.9),
$$\underbrace{\delta_{n,\text{ odd}}\,\delta_{n,\text{ even}}\,\delta_{n,\text{ odd}}\,\delta_{n,\text{ even}}\cdots}_{2n \text{ factors}}$$

or with the braid

(11.10).
$$\underbrace{\delta_{n,\text{ even}}\,\delta_{n,\text{ odd}}\,\delta_{n,\text{ even}}\,\delta_{n,\text{ odd}}\cdots}_{2n \text{ factors}}$$

In both cases, by Corollary 11.3, the resulting braid is β^2, where

$$\beta := \tau_1(\tau_2\tau_1)\dots(\tau_{n-2}\dots\tau_1)(\tau_{n-1}\tau_{n-2}\dots\tau_1).$$

It is well-known that $\beta^2 = (\tau_1\dots\tau_{n-1})^n$ (see e.g. Theorem 1.24 of [KT08]). The result follows from Proposition 2.24.

Theorem 11.7. Let S_1, S_2 the Stokes matrices computed wrt a Stokes sector V in lexicographical order. Let ε be the exceptional basis of $K_0^{\mathbb{T}}(\mathbb{P}^{n-1})_{\mathbb{C}}$ associated with the Stokes basis on V via the isomorphism $\theta\colon K_0^{\mathbb{T}}(\mathbb{P}^{n-1})_{\mathbb{C}} \to \mathcal{S}_n$ defined in (7.8), and let G be the Gram matrix of χ wrt ε. Let J be the anti-diagonal matrix

$$J_{\alpha\beta} := \delta_{\alpha+\beta,n+1}, \qquad \alpha,\beta = 1,\dots,n.$$

(1) The Stokes matrix S_1 is equal to the Gram matrix of χ^{T} wrt the left dual exceptional basis ${}^{\vee}\varepsilon$, i.e.

$$\mathbb{S}_1 = J\left(\mathcal{G}^{\dagger}\right)^{-1} J. \tag{11.11}$$

(2) The matrix JS_2J is equal to the Gram matrix of χ^{T} wrt the exceptional basis ε, i.e.

$$S_2 = JGJ. \tag{11.12}$$

Proof. By Lemma 10.7, V is contained in one (and only one) $\mathcal{V}'_k$ or $\mathcal{V}''_k$. By Corollary 11.5, the Stokes basis on $e^{\sqrt{-1}\pi}\mathcal{V}$ is obtained from the Stokes basis on V by applying either the braid

$$\underbrace{\delta_n, \text{ odd } \delta_n, \text{ even } \delta_n, \text{ odd } \delta_n, \text{ even } \dots}_{n \text{ factors}}$$

or the braid

$$\underbrace{\delta_n, \text{ even } \delta_n, \text{ odd } \delta_n, \text{ even } \delta_n, \text{ odd } \dots}_{n \text{ factors}}.$$

Consequently, by Corollary 11.3, the exceptional basis associated to the Stokes basis on $e^{\sqrt{-1}\pi}\mathcal{V}$ is ${}^{\vee}\varepsilon$. Both points (1) and (2) then follow from Proposition 2.22, more precisely from the second identity (2.25).

Corollary 11.8. Let S_1 and S_2 be the Stokes matrices computed wrt a Stokes sector V. We have

$$\mathbb{S}_2 = \left(\mathbb{S}_1^{\dagger}\right)^{-1}. \tag{11.13}$$

Remark 11.9. In theory of Frobenius manifolds [Dub96, Dub99, Dub98, CDG20], the Stokes matrices of the associated isomonodromic system of differential equations satisfy an analogous identity, in which the †-operator is replaced by transposition, see [Dub99, Theorem 4.3].

Corollary 11.10. Let S_1 and S_2 be the Stokes matrices computed wrt a Stokes sector V. Both S_1 and S_2 have entries in the ring of symmetric Laurent polynomials with integer coefficients, i.e. $\mathbb{S}_1, \mathbb{S}_2 \in M_n\left(\mathbb{Z}[\boldsymbol{Z}^{\pm 1}]^{\mathfrak{S}_n}\right)$.

Proof. By Lemma 3.2, the Gram matrix associated with the Beilinson exceptional collection is with integer symmetric Laurent polynomial entries. The braid group action preserves this property.

Corollary 11.11. Let S be a Stokes matrix of the differential system (6.1) computed wrt a Stokes sector V. Then

$$\det\left(\lambda\cdot\mathbb{1} - \mathbb{S}^{\dagger}\mathbb{S}^{-1}\right) = \sum_{j=0}^{n}(-1)^j\lambda^{n-j}s_j\left((-1)^{n+1}\frac{Z_1^n}{s_n(\boldsymbol{Z})},\dots,(-1)^{n+1}\frac{Z_n^n}{s_n(\boldsymbol{Z})}\right). \tag{11.14}$$

Proof. The Corollary immediately follows from Theorem 3.5 and Theorem 11.7. For an alternative proof (purely analytical), notice that without loss of generality we can assume that S is the matrix S_1 computed wrt V. From point (3) of Proposition 10.12, and equation (11.13) we deduce that

$$\exp(2\pi\sqrt{-1}\Lambda(\boldsymbol{z}))\mathbb{S}^{\dagger}\mathbb{S}^{-1} = M_0(\boldsymbol{z})^n,$$

where $M_0(z)$ is the monodromy operator of differential system (6.1). From Corollary 6.3 we deduce the constraint

$$\det\left(\lambda\cdot\mathbb{1} - (-1)^{n+1}s_n(\boldsymbol{Z})\mathbb{S}^{\dagger}\mathbb{S}^{-1}\right) = \sum_{j=0}^{n}(-1)^j\lambda^{n-j}s_j(\boldsymbol{Z}^n) \tag{11.15}$$

which is easily seen to be equivalent to equation (11.14).

12. Specialization of the *qDE* at roots of unity

12.1. Specialization of equivariant *K*-theory at roots of unity. Fix the equivariant parameters in $K_0^{\mathbb{T}}(\mathbb{P}^{n-1})$ by setting

$$Z_m = \zeta_n^{m-1}, \quad m = 1, \ldots, n. \tag{12.1}$$

Denote by K_ζ this specialization of $K_0^{\mathbb{T}}(\mathbb{P}^{n-1})$.

Theorem 12.1 ([Pol11, Theorem 1.1]).

(1) The Grothendieck-Euler-Poincaré pairing $\chi^{\mathbb{T}}$ specializes to an Hermitian positive definite form χ_ζ on K_ζ.
(2) If E is an exceptional object in $D^b(\mathbb{P}^{n-1})$ equipped with a T-equivariant structure, then the class $[E]$ in K_ζ has length 1 wrt the Hermitian form χ_ζ.
(3) If (E_1,E_2) is an exceptional pair in $D^b(\mathbb{P}^{n-1})$, with both E_1 and E_2 equipped with a T-structure, then the classes $[E_1],[E_2]$ in K_ζ are orthogonal wrt χ_ζ.
(4) If $(E_1,\ldots,E_n)$ is a full exceptional collection in $D^b(\mathbb{P}^{n-1})$, with each E_i equipped with a T-structure, then each unit vector in K_ζ is of the form $\pm\zeta_n^k[E_i]$ for some i and some k.
(5) The action of the braid group on the set of orthonormal exceptional bases of K_ζ reduces to the action by permutations of basis vectors.

12.2. Identities for Stirling numbers. The Stirling numbers of the first kind $\left[{n \atop k}\right]$ are defined recursively by

$$\begin{bmatrix} n+1 \\ k \end{bmatrix} = n\begin{bmatrix} n \\ k \end{bmatrix} + \begin{bmatrix} n \\ k-1 \end{bmatrix}, \tag{12.2}$$

for $k > 0$, with the initial conditions

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = 1, \quad \begin{bmatrix} 0 \\ n \end{bmatrix} = \begin{bmatrix} n \\ 0 \end{bmatrix} = 0, \quad n > 0 \tag{12.3}$$

The Stirling numbers of the second kind $\left\{{n \atop k}\right\}$ are defined recursively by

$$\left\{ {n+1 \atop k} \right\} = k \left\{ {n \atop k} \right\} + \left\{ {n \atop k-1} \right\}, \tag{12.4}$$

for $k > 0$, with the initial conditions

$$\left\{ {0 \atop 0} \right\} = 1, \quad \left\{ {0 \atop n} \right\} = \left\{ {n \atop 0} \right\} = 0, \quad n > 0 \tag{12.5}$$

.

The Stirling numbers of the first and second kind are related by the identity

$$\sum_{j \geq 0} (-1)^{n-j} \left\{ {n \atop j} \right\} \left[{j \atop k} \right] = \delta_{nk} \tag{12.6}$$

.

Lemma 12.2. Let $n \geq 2$ and $1 \leq k \leq n$. We have

$$s_k \left(0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n} \right) = \frac{1}{n^k} \left[{n \atop n-k} \right], \tag{12.7}$$

$$m_k \left(0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n} \right) = \frac{1}{n^k} \left\{ {n+k-1 \atop n-1} \right\} \tag{12.8}.$$

Proof. It is sufficient to prove the identities

4

$$n$$

6 (12.9) $s_k(0,1,\dots,n-1) = -5,$

n k $-$

n. For $n = 2$

$-n + k - 17$

(12.10) $m_k(0,\dots,n \quad 1) =.$

n 1

They are proved by induction on both equations (12.9) and (12.10) hold true. Recall the following recurrence equations: for $k \geq 2$

(12.11) $s_k(z_1,\dots,z_n) = s_k(z_1,\dots,z_{n-1}) + z_n s_{k-1}(z_1,\dots,z_{n-1})$

(12.12) $m_k(z_1,\dots,z_n) = m_k(z_1,\dots,z_{n-1}) + z_n m_{k-1}(z_1,\dots,z_n).$

Now equation (12.9) follows from (12.11) and (12.2), equation (12.10) follows from (12.12) and (12.4).

Lemma 12.3. *If* $\vartheta_s := s\frac{d}{ds}$, then

$$\vartheta_s^n = \sum_{j=1}^{n} \left\{ {n \atop j} \right\} s^j \frac{d^j}{ds^j}, \tag{12.13}$$

$$s^n \frac{d^n}{ds^n} = \sum_{j=1}^{n} (-1)^{n-j} \left[{n \atop j} \right] \vartheta_s^j \tag{12.14}.$$

Proof. Identity (12.13) is easily proved by induction on n. Identity (12.14) follows from (12.13) and (12.6).

12.3. Scalar equivariant quantum differential equation at roots of unity. Consider the specialization of the equivariant parameters z in $H_T^\bullet(\mathbb{P}^{n-1},\mathbb{C})$ defined by the equations

$$\exp(2\pi\sqrt{-1}z_m) = \zeta_n^{m-1}, \quad m = 1, \dots, n. \tag{12.15}$$

These equations define the locus

$$\mathcal{P} := \left\{ \boldsymbol{z} \in \mathbb{C}^n : \boldsymbol{z} = \left(k_1, k_2 + \frac{1}{n}, \dots, k_n + \frac{n-1}{n} \right), \quad \boldsymbol{k} \in \mathbb{Z}^n \right\} \tag{12.16}$$

.

We have a distinguished point $z_o \in P$,

$$z^o := \left(0, \frac{1}{n}, \dots, \frac{n-1}{n} \right). \tag{12.17}$$

Theorem 12.4. At $z = z_o$ the scalar equivariant quantum differential equation (6.16) of P^{n-1} for the function $\varphi(q)$ reduces to the linear differential equation with constant coefficients,

$$\frac{d^n}{ds^n}\varphi(s) = n^n \varphi(s), \quad \varphi(s) := \phi(s^n) \tag{12.18}$$

.

Proof. By the change of variable $q = s^n$, equation (6.16) reduces to

$$\left[\frac{1}{n^n}\vartheta_s^n + \sum_{j=1}^{n-1} (-1)^{n-j} s_{n-j}(\boldsymbol{z}) \frac{1}{n^j}\vartheta_s^j - \left(s^n + (-1)^{n-1} s_n(\boldsymbol{z}) \right) \right] \phi(s^n) = 0 \tag{12.19}$$

.

If $z = z_o$, then the equation reduces to

$$\left[\frac{1}{n^n}\vartheta_s^n + \sum_{j=1}^{n-1} (-1)^{n-j} \frac{1}{n^n} \begin{bmatrix} n \\ j \end{bmatrix} \vartheta_s^j - s^n \right] \phi(s^n) = 0 \tag{12.20}$$

,

by identity (12.7). Using identity (12.14), we obtains the equation

$$\left(s^n \frac{d^n}{ds^n} - n^n s^n \right) \phi(s^n) = 0 \tag{12.21}$$

.

Equation (12.18) has two natural bases of solutions:

(1) the basis $(f_m(s))_{m=0}^{n-1}$,

$$f_m(s) := \exp(n\zeta_n^m s), \quad m = 0, \dots, n-1; \tag{12.22}$$

(2) the basis $(g_m(s))_{m=0}^{n-1}$,

$$g_m(s) := \sum_{k=0}^{\infty} \frac{(ns)^{m+kn}}{(m+kn)!}, \quad m = 0, \dots, n-1 \tag{12.23}$$

.

The functions $g_m(s)$ are real-valued for $s \in R$ and define a partition of the exponential function e^{ns},

$$\sum_{m=0}^{n-1} g_m(s) = e^{ns} \tag{12.24}$$

.

Lemma 12.5. The cyclic group Z/nZ acts on the space of solutions of equation (12.18) via the transformations $T_k : s \mapsto \zeta_n^k s$, $k = 1, \dots, n$. The basis $(f_m(s))_{m=0}^{n-1}$ is cyclically permuted by this action, while the basis $(g_m(s))_{m=0}^{n-1}$ is an eigenbasis.

Introduce the matrices

$$\hat{Y}_f(s^n) := \left(\hat{Y}_f(s^n)^h_m\right)_{h,m=0,\dots,n-1}, \quad \hat{Y}_f(s^n)^h_m := \frac{1}{n^h}\vartheta_s^h f_m(s) \tag{12.25}$$

,

$$\hat{Y}_g(s^n) := \left(\hat{Y}_g(s^n)^h_m\right)_{h,m=0,\dots,n-1}, \quad \hat{Y}_g(s^n)^h_m := \frac{1}{n^h}\vartheta_s^h g_m(s) \tag{12.26}$$

.

Both $\hat{Y}_f(s^n)$ and $\hat{Y}_g(s^n)$ are solutions of the differential equation (6.15), specialized at $z = z_o$.

Proposition 12.6. The matrix-valued function $\eta(z_o)^{-1}\hat{Y}_f(s^n)$ is a fundamental system of solutions of (6.1) of the form

$$\eta(z_o)^{-1}\hat{Y}_f(s^n) = G(s)\exp(sU), \tag{12.27}$$

where $U = \mathrm{diag}(n\zeta_n^0,\dots,n\zeta_n^{n-1})$ and $G(s)$ is a polynomial in s of degree $n-1$.

Hence, $\eta(z_o)^{-1}\hat{Y}_f(s^n)$ is a Stokes basis of the equivariant qDE (6.1) at $z = z_o$ for any Stokes sector. In particular, the corresponding formal series $F(s,z_o)$ of the form (9.36) is actually convergent.

Proof. The matrix $\eta(z_o)^{-1}\hat{Y}_f(s^n)\eta(z_o)$ is a fundamental system of solutions of (6.1) by the discussion in Section 6.4. Hence $\eta(z_o)^{-1}\hat{Y}_f(s^n)$ also is a fundamental system of solutions. The matrix $G(s)$ is given by

$$G(s) = \eta(\boldsymbol{z}_o)^{-1}\cdot L(s), \quad L(s) = \left(L(s)^h_m\right)^{n-1}_{h,m=0}, \quad L(s)^h_m := \zeta_n^h s^h \tag{12.28}$$

.

Thus the series $F(s,z_o)$ is given by

$$F(s,z_o) = \mathrm{D}_q\mathrm{H}(s^n)^{-1}\eta(z_o)^{-1}L(s)s^{1-n}, \tag{12.29}$$

and is convergent. The normalization of the Stokes basis can be readily computed from this formula.

Proposition 12.7. The matrix-valued function $\eta(z_o)^{-1}\hat{Y}_g(s^n)$ is a fundamental system of solutions of (6.1) of the form

$$\eta(\boldsymbol{z}_o)^{-1}\hat{Y}_g(s^n) = Y_o(s^n,\boldsymbol{z}_o)\cdot C, \tag{12.30}$$

where the matrix $Y_o(s^n,z)$ is the Levelt solution defined in Corollary 6.2 and C is a diagonal matrix.

Proof. The proposition follows from Lemma 12.5 and Corollary 6.3. We leave to the reader the explicit computation of the matrix C.

Theorem 12.8. The following conditions are equivalent:

(1) $\boldsymbol{z}' \in \mathcal{P}$;

(2) the formal gauge transformation $G(s,\boldsymbol{z}')$ of Theorem 9.6 is convergent;

(3) the Stokes phenomenon of the differential system (6.1) specialized at $z = z$ is trivial, i.e. all the Stokes matrices $\mathbb{S}(z')$ for all Stokes sectors are the identity matrix;

(4) the monodromy matrix $M_0(z')$ of the equivariant quantum differential equation (6.1) specialized at $z = z' \in \Omega$ has order n.

Proof. We prove that (1) ⇒ (2) ⇒ (4) ⇒ (1).

Assume $z' = \left(k_1, k_2 + \frac{1}{n}, \dots, k_n + \frac{n-1}{n}\right)$ for some $k \in \mathbb{Z}^n$. Then, we have to show that the series $F(s,z)$ in (9.23) is convergent for $z = z'$. From the identity (9.30) we deduce that

$$F(s,z_1,\dots,z_j - 1,\dots,z_n) = W_j(s,z)F(s,z)K_j^{-1},$$

where

$$W_j(s,z) := sD_qH(s^n)^{-1}K_j(s^n,z)H(s^n)D_q^{-1},\ j = 1,\dots,n.$$ Hence, we have

$$F(s,z') = \left(\prod_{j=1}^{n} W_j(s,z_o)^{-k_j}\right) F(s,z_o) \left(\prod_{j=1}^{n} K_j^{-k_j}\right),$$

and the convergence of $F(s,z')$ follows from the convergence of $F(s,z_o)$.

If (2) holds then $Y(s^n,z') = G(s,z')e^{sU}$ is a solution of system (6.1) at $z = z'$, and the transformation $s \mapsto \zeta_n s$ cyclically permutes the diagonal entries of U. Thus (4) holds true.

If (4) holds true, then from Corollary 6.3 we deduce that $\exp(2\pi\sqrt{-1}z_j')$ is a n-th root of unity, i.e. $z' \in \mathcal{P}$.

The equivalence of (2) and (3) is obvious.

Appendix A. Formal reduction of the joint system

Consider a joint system of differential difference equations

$$\frac{d}{ds}X(s,z) = A(s,z)X(s,z),$$

$$X(s,z_1,\dots,z_i - 1,\dots,z_n) = P_i(s,z)X(s,z), \quad i = 1,\dots,n, \tag{A.1}$$

(A.2) where A,P_i are meromorphic $m \times m$-matrix valued functions of $(s,z) \in \mathbb{C} \times \mathbb{C}^n$.

Assume that equations (A.1), (A.2) are compatible,

(A.3)
$$\frac{d}{ds}P_i(s,z) = A(s,z_1,\dots,z_i - 1,\dots,z_n)P_i(s,z) - P_i(s,z)A(s,z), \quad i = 1,\dots,n,$$

(A.4)
$$P_i(s,z_1,\dots,z_i,\dots,z_j - 1,\dots,z_n)P_j(s,z) = P_j(s,z_1,\dots,z_i \quad 1,\dots,z_j,\dots,z_n)P_i(s,z)_-,$$

for all i,j. Assume that

(1) the matrices $A(s,z),P_i(s,z)$ have the following convergent power series expansions

$$A(s,z)=\sum_{k=0}^{\infty}A_k(z)\frac{1}{s^k},\quad P_i(s,z)=\sum_{k=0}^{\infty}P_k(z)\frac{1}{s^k},\quad |s|>\rho, \tag{A.5}$$

where the matrices $A_k(z)$,$P_k(z)$ are holomorphic functions of z and $\rho > 0$;

(2) the matrix $A_0(z)$ is diagonalizable,

(A.6) $D(z)\cdot A_0(z)\cdot D(z)^{-1} = U(z)$, $U(z) := \mathrm{diag}(u_1(z),\dots,u_n(z))$, with $D(z)$ a holomorphic matrix;

(3) the matrix $U(z)$ is 1-periodic,

$$U(z_1,\dots,z_i-1,\dots,z_n)=U(z),\qquad i=1,\dots,n; \tag{A.7}$$

(4) the eigenvalues $u_i(z)$'s are pairwise distinct for values of z in an open subset $W \subset C^n$.

Introduce the diagonal matrix $\Lambda_1(z)$ by the formula

$$[\Lambda_1(z)]_{ij} := {}^*D(z)\cdot A_1(z\in)\cdot D(z)^{-1}{+}_{ij}\,\delta_{ij}. \qquad \infty \tag{A.8}$$

Assumptions (1-4) imply that for all z W the point s = is an irregular singularity (of Poincaré rank 1) of the differential equation (A.1).

Theorem A.1. If the joint system of equations (A.1), (A.2) satisfies assumptions (1-4), then there exists a unique $m \times m$-matrix $G(s,z)$ of the form

(A.9) $G(s,z) = D(z)^{-1}F(s,z)s^{\Lambda_1(z)}$, where $F(s,z)$ is a formal power series

$$F(s,z)=\mathbb{1}+\sum_{k=1}^{\infty}F_k(z)\frac{1}{s^k}, \tag{A.10}$$

with $F_k(z)$ regular $m \times m$-matrix-valued functions on W, such that the change of variables $X(s,z) = G(s,z)Z(s,z)$ transforms

$$\frac{d}{ds}Z(s,z)=U(z)Z(s,z),$$

$$Z(s,z_1,\dots,z_i-1,\dots,z_n)=\mathcal{P}_i(z)Z(s,z),\quad i=1,\dots,n,$$

system (A.1), (A.2) into the joint system

(A.11)

(A.12) where P_i are diagonal matrices.

Proof. First we prove that there exists a unique formal transformation $X(s,z) = G(s,z)Z(s,z)$, which transforms equation (A.1) into equation (A.11). Then we prove that this transformation automatically transforms equations (A.2) into equations (A.12) with diagonal P_i's.

If a transformation $X(s,z) = G(s,z)Z(s,z)$,

$$G(s,z) = D(z)^{-1}F(s,z)s^{\Lambda_1(z)}, \qquad F(s,z) = \mathbb{1} + \sum_{k=1}^{\infty} F_k(z)\frac{1}{s^k}$$

(A.13)

transforms equation (A.1) to the equation

$$\frac{d}{ds}Z(s,z) = U(z)Z(s,z)$$

(A.14)

then $Z(s,z)$ is a solution of the equation

$$\frac{d}{ds}Z(s,z) = \left(G(s,z)^{-1}A(s,z)G(s,z) - G(s,z)^{-1}\frac{d}{ds}G(s,z)\right)Z(s,z)$$

(A.15)

Thus, $F(s,z)$ satisfies the equation

$$U(z)F(s,z) + \left(\sum_{k=1}^{\infty}\frac{D(z)A_k(z)D(z)^{-1}}{s^k}\right)F(s,z) = \frac{d}{ds}F(s,z) + \frac{1}{s}F(s,z)\Lambda_1(z) + F(s,z)U(z)$$

which gives a system of equations for the coefficients $F_k(z)$. Denote $\hat{A}_j := DA_jD^{-1}$. The matrix $\hat{A}_1^{\mathrm{od}} := \hat{A}_1 - \Lambda_1$ is an off-diagonal matrix. The first equation is

$$UF_1 + \hat{A}_1^{\mathrm{od}} = F_1U.$$

(A.16)

For $\alpha \neq \beta$,

$$F_1(z)_{\alpha\beta} = \frac{1}{u_\beta(z) - u_\alpha(z)}\left(\hat{A}_1^{\mathrm{od}}(z)\right)_{\alpha\beta}.$$

(A.17)

The second equation is

$$UF_2 + \hat{A}_2 + \hat{A}_1F_1 = -F_1 + F_1\Lambda + F_2U.$$

(A.18)

We find the diagonal entries of $F_1(z)$ from the diagonal part of equations (A.18). We compute the off-diagonal entries $F_2(z)_{\alpha\beta}$ with $\alpha \neq \beta$, by the formula:

$$F_2(z)_{\alpha\beta} = \frac{1}{u_\beta(z) \quad u_\alpha(z)}\left(F_1(z) - F_1(z)\Lambda(z) + \hat{A}_2(z) + \hat{A}_1(z)F_1(z)\right)$$

(A.19)

After k steps of this procedure, we will determine all the coefficients $F_1,\ldots,F_{k-1}$ and all the off-diagonal entries of F_k. Then the $k+1$st equation

$$-kF_k + [F_{k+1}, U] = -F_k\Lambda + \hat{A}_{k+1} + \sum_{h+\ell=k+1}\hat{A}_hF_\ell$$

determines uniquely the diagonal entries of F_k and the off-diagonal entries of F_{k+1} and we may continue this procedure.

This procedure shows that the desired series $F(s,z)$ does exist and is unique.

The gauge transformation $X(s,z) = G(s,z)Z(s,z)$ transforms the joint system (A.1), (A.2) into the joint system

$$\frac{dZ}{ds} = U(z)Z,$$

(A.20)

(A.21) $Z(s,z_1,\ldots,z_i-1,\ldots,z_n) = P_i(s,z)Z(s,z)$, $i = 1,\ldots,n$, where we set

(A.22) $P_i(s,z) := G(s,z_1,\dots,z_i-1,\dots,z_n)^{-1}P_i(s,z)G(s,z)$ for $i=1,\dots,n$. We claim that the matrices P_i are diagonal and independent of s. The compatibility conditions of (A.20) and (A.21) imply that

$$\frac{d}{ds}\mathcal{P}_i = [U,\mathcal{P}_i], \quad i=1,\dots,n. \tag{A.23}$$

Thus the entries of P_i are of the form

(A.24) $P_i(s,z)_{\alpha\beta} = f_{\alpha\beta}(z)\exp((u_\alpha(z)-u_\beta(z))s)$, where $\alpha,\beta = 1,\dots,n$, and $f_{\alpha\beta}(z)$ are functions of z.

Also we know that all the entries of the right-hand side of (A.22) are formal power series of the form

$$s^{m(z)}\sum_{n=0}^{\infty}\frac{a_n(z)}{s^n},$$

for suitable functions $m(z), a_n(z)$. This shows that the operator P_i can be of the form (A.24) if and only if

$$f_{\alpha\beta}(z)=0, \quad \alpha\neq\beta.$$

This concludes the proof. Appendix B. Relation of *qDE* to Dubrovin's equation for $QH^\bullet(\mathbb{P}^{n-1})$

Denote by

$$\iota^*\colon H^\bullet_{\mathbb{T}}(\mathbb{P}^{n-1},\mathbb{C})\to H^\bullet(\mathbb{P}^{n-1},\mathbb{C}), \quad f(x,z)\mapsto f(x,0), \tag{B.1}$$

the non-equivariant limit morphism[6]. It maps the $C[z]$-basis $(x_\alpha)_{\alpha=0}^{n-1}$ of $H_T^\bullet(\mathbb{P}^{n-1},C)$ to the C-basis $(\iota^*x_\alpha)_{\alpha=0}^{n-1}$ of $H^\bullet(\mathbb{P}^{n-1},\mathbb{C})$. Denote the dual coordinates on $H^\bullet(\mathbb{P}^{n-1},C)$ by the notation $t := (t^0,\dots,t^{n-1})$. Consider the non-equivariant limit $F_0^{\mathbb{P}^{n-1}} \in \mathbb{C}[\![t]\!]$ of (5.2),

$$F_0^{\mathbb{P}^{n-1}}(t) := \sum_{m=0}^{\infty}\sum_{d=0}^{\infty}\sum_{\alpha_1,\dots\alpha_m=0}^{n-1}\frac{t^{\alpha_1}\dots t^{\alpha_m}}{m!}\langle\iota^*x_{\alpha_1},\dots,\iota^*x_{\alpha_m}\rangle_{0,m,d}^{\mathbb{P}^{n-1}} \tag{B.2}$$

.

It is known that the Gromov-Witten potential $F_0^{\mathbb{P}^{n-1}}(t)$ is convergent. The domain of convergence $M\subseteq H^\bullet(\mathbb{P}^{n-1},C)$ of the Gromov-Witten potential $F_0^{\mathbb{P}^{n-1}}(t)$ carries a Frobenius manifold structure [Dub92,Dub96,Dub98,Dub99,Man99,Her02, Sab08]. Tangent spaces[7] T_pM are equipped with an associative, commutative algebra structure: the product $*_p$: $T_pM \times T_pM \to T_pM$ is compatible with the non-equivariant Poincaré metric $\eta_{cl} := \eta|_{z=0}$,

$$\eta_{cl}(\alpha *_p \beta,\gamma) = \eta_{cl}(\alpha,\beta *_p \gamma), \qquad \alpha,\beta,\gamma\in T_pM. \tag{B.3}$$

The metric η_{cl} is a non-degenerate pseudo-riemannian metric on M, whose LeviCivita connection ∇ is flat. Consider the semisimple part M_{ss} of M, namely the subset of points p

whose corresponding Frobenius algebra T_pM is without nilpotents. Denote by $(\pi_1,\dots,\pi_n)$ the idempotent tangent vectors at $p \in M_{ss}$, and introduce the normalized frame $(f_1,\dots,f_n)$ by

$$f_i := \eta_{\mathrm{cl}}(\pi_i,\pi_i)^{-\frac{1}{2}} \cdot \pi_i, \quad i = 1,\dots,n, \tag{B.4}$$

for arbirary choice of the square roots. Consider the Euler vector field E on M defined by

$$E := c_1(\mathbb{P}^{n-1}) + \sum_{\alpha=0}^{n-1} \left(1 - \frac{1}{2}\deg(\iota^* x_\alpha)\right) t^\alpha \frac{\partial}{\partial t^\alpha} \tag{B. 5).}$$

Let $p \in M_{ss}$, and denote by $U(p)$ and $V(p)$ the matrices, wrt the frame $(f_1,\dots,f_n)$, of the morphisms (B.6) $\mathcal{U}(p)\colon T_pM \to T_pM, \quad v \mapsto E|_p *_p v,$

$$\mu(p)\colon T_pM \to T_pM, \quad v \mapsto \frac{3-n}{2} v - \nabla_v E|_p. \tag{B.7}$$

It is easily seen that they satisfy

$$U(p)^T = U(p), \qquad V(p)^T + V(p) = 0. \tag{B.8}$$

There is a local identification of M_{ss} with the space of parameters of isomonodromic deformations of the ordinary differential equation

$$\frac{d}{d\lambda} Y(\lambda,p) = \left(U(p) + \frac{1}{\lambda} V(p)\right) Y(\lambda,p), \quad \lambda \in \mathbb{C}^*,\ p \in M_{ss} \tag{B.9}$$

$^{n-1}$[6]×Recall that this morphism is induced in cohomology by the inclusion^T ET. See [AB84, Section 2].

$\mathrm{P}^{n-1},\mathrm{C}).$ $\iota\colon \mathrm{P}^{n-1} \to \mathrm{P}^{n}{}_{\mathrm{T}-1} :=$

P

[7]Tangent spaces to M are canonically identified with $H^\bullet($

for a $n \times n$-matrix valued function Y. Equation (B.9) is central in Dubrovin's study of Frobenius manifolds, see [Dub96, Lecture 3], [Dub99, Lectures 3 and 4], [Dub98]. See also [Guz99] and [CDG18, Section 6] for details on the monodromy and Stokes phenomenon of (B.9).

The tangent space $T_0M =\sim H^\bullet(\mathrm{P}^{n-1},\mathrm{C})$ can be identified with C^n by fixing the frame $(1,x,\dots,x^{n-1})$, where x denote the non-equivariant hyperplane class. Under this identification,

- the Gram matrix of the non-equivariant Poincaré metric coincides with the matrix η_{cl} of equation (9.17);
- the matrix of the operator $\mathrm{U}(0)\colon T_0M \to T_0M$ of multiplication by the Euler vector field coincides with the matrix B_0 of equation (9.12);
- the basis $(f_1,\dots,f_n)$ of Lemma 9.4 coincides with the orthonormalized idempotent frame (B.4) at T_0M for suitable choices of the square roots, see [CDG18, Section 6.1].

In the standard notations of Dubrovin's theory of Frobenius manifolds, the matrix E is usually denoted by Ψ. Here we avoid this notation, the symbol Ψ being already used for solutions of the joint systems (5.19), (5.26).

We close this Appendix by commenting the relation between equation (9.15) and the isomonodromic differential equation (B.9).

Proposition B.1. For $z = 0$, equation (9.15) is

$$\frac{d}{ds}T(s) = \left(\mathcal{B}_0 + \frac{1}{s}\mathcal{B}_1(0)\right) T(s) \tag{B.10}$$

If $T(s)$ is a solution of (B.10), then

$$Y(\lambda) := \lambda^{-\frac{n-1}{2}} \mathcal{E}\ T(\lambda)$$

is a solution of the equation (B.9) specialized at $p = 0$.

Proof. if $z = 0$, all the coefficients $B_2,\dots,B_n$ vanish, and the coefficient B_1 takes the form

$$B_1(0) = \mathrm{diag}(0,1,2,\dots,n-1). \tag{B.11}$$

We have

$$\mathcal{B}_1(0) - \frac{n-1}{2}\mathbb{1} = \mu,$$

where μ is the matrix of the grading operator (B.7), written in coordinates wrt the basis $(\iota^* x_\alpha)_{\alpha=0}^{n-1}$. As a consequence, the matrix B^{od} of formula (9.22) in the non-equivariant limit is given by

$$B^{\mathrm{od}}(0) = \mathcal{E}\ \left(\mathcal{B}_1(0) - \frac{n-1}{2}\mathbb{1}\right)\ \mathcal{E}^{-1} = \mathcal{E}\ \mu\ \mathcal{E}^{-1} = V,$$

where the matrix V is the antisymmetric matrix $V(p)$ specialized at $p = 0$. The antisymmetry of B^{od} however is lost for $z = 0$.

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