Determinant of $\mathbb{F}_p$-hypergeometric solutions under ample reduction

Alexander Varchenko[1]

Department of Mathematics, University of North Carolina at Chapel Hill Chapel Hill, NC 27599-3250, USA

Faculty of Mathematics and Mechanics, Lomonosov Moscow State University Leninskiye Gory 1, 119991 Moscow GSP-1, Russia

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Abstract. We consider the KZ differential equations over $\mathbb{C}$ in the case, when the hypergeometric solutions are one-dimensional integrals. We also consider the same differential equations over a finite field $\mathbb{F}_p$. We study the polynomial solutions of these differential equations over $\mathbb{F}_p$, constructed in a previous work joint with V.Schechtman and called the $\mathbb{F}_p$-hypergeometric solutions.

The dimension of the space of $\mathbb{F}_p$-hypergeometric solutions depends on the prime number p. We say that the KZ equations have ample reduction for a prime p, if the dimension of the space of $\mathbb{F}_p$-hypergeometric solutions is maximal possible, that is, equal to the dimension of the space of solutions of the corresponding KZ equations over $\mathbb{C}$. Under the assumption of ample reduction, we prove a determinant formula for the matrix of coordinates of basis $\mathbb{F}_p$-hypergeometric solutions. The formula is analogous to the corresponding formula for the determinant of the matrix of coordinates of basis complex hypergeometric solutions, in which binomials $(z_i-z_j)^{M_i+M_j}$ are replaced with $(z_i-z_j)^{M_i+M_j-p}$ and the Euler gamma function $\Gamma(x)$ is replaced with a suitable $\mathbb{F}_p$-analog $\Gamma_{\mathbb{F}_p}(x)$ defined on $\mathbb{F}_p$.

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[1] *E-mail*: anv@email.unc.edu, supported in part by NSF grant DMS-1954266

1. Introduction

The KZ equations were introduced in [**KZ**] as the differential equations satisfied by conformal blocks on sphere in the Wess-Zumino-Witten model of conformal field theory. The hypergeometric solutions of the KZ equations were constructed more than 30 years ago, see [**SV1**, **SV2**]. The KZ equations and the hypergeometric solutions are related to many subjects in algebra, representation theory, theory of integrable systems, enumerative geometry, to name a few.

The polynomial solutions of the KZ equations over the finite field $\mathbb{F}_p$ with a prime number p of elements were constructed recently in [**SV3**], see also [**V4**, **V5**, **V6**, **V7**]. We call these solutions the $\mathbb{F}_p$*-hypergeometric solutions*. The general problem is to understand relations between the hypergeometric solutions of the KZ equations over $\mathbb{C}$ and the $\mathbb{F}_p$-hypergeometric solutions and observe how the remarkable properties of hypergeometric solutions are reflected in the properties of the $\mathbb{F}_p$-hypergeometric solutions. This program is in the first stages, where we consider essential examples and study the corresponding $\mathbb{F}_p$-hypergeometric solutions by direct methods.

In this paper we consider the KZ differential equations in the case, when the hypergeometric solutions over $\mathbb{C}$ are one-dimensional integrals.

The dimension of the space of $\mathbb{F}_p$-hypergeometric solutions depends on the prime number p. We say that the KZ equations have ample reduction for a prime p, if the dimension of the space of $\mathbb{F}_p$-hypergeometric solutions is maximal possible, that is, equal to the dimension of the space of solutions of the corresponding KZ equations over $\mathbb{C}$. Under the assumption of ample reduction, we prove a determinant formula for the matrix of coordinates of basis $\mathbb{F}_p$-hypergeometric solutions. The formula is analogous to the corresponding formula for the determinant of the matrix of coordinates of basis complex hypergeometric solutions, see [**V1**], in which binomials $(z_i - z_j)^{M_i+M_j}$ are replaced with $(z_i - z_j)^{M_i+M_j-p}$ and the Euler gamma function $\Gamma(x)$ is replaced with a suitable $\mathbb{F}_p$-analog $\Gamma_{\mathbb{F}_p}(x)$ defined on $\mathbb{F}_p$.

In Section 2 we describe our KZ equations and their reduction modulo p. We define the hypergeometric solutions over $\mathbb{C}$ and $\mathbb{F}_p$-hypergeometric solutions. The ample reduction is defined in Section 2.5.

As mentioned earlier, the $\mathbb{F}_p$-hypergeometric solutions are polynomials. In Section 3 we give a formula for their coefficients.

In Section 4, we consider the particular case of our KZ equations, whose space of solutions over $\mathbb{C}$ is one-dimensional, with the basis solution given by the Euler beta integral. We describe the corresponding $\mathbb{F}_p$-hypergeometric solution, which we call the $\mathbb{F}_p$*-beta integral*.

In Section 5 we consider an arbitrary polynomial solution (not necessarily $\mathbb{F}_p$-hypergeometric) of our KZ equations over $\mathbb{F}_p$ and describe its leading term with respect to the lexicographical ordering of monomials, see Theorem 5.3. It turns out that the notion of leading term and the formula for the leading term in Theorem 5.3 are useful in studying polynomial solutions of the KZ equations over $\mathbb{F}_p$. The notion of leading term replaces, to some extend, the notion of initial condition, when the differential equations are over $\mathbb{F}_p$.

The module of $\mathbb{F}_p$-hypergeometric solutions has a natural basis. In Section 6 we describe the leading terms of the basis $\mathbb{F}_p$-hypergeometric solutions. Section 7 contains our main result, Theorem 7.2, describing the determinant of coordinates of the basis $\mathbb{F}_p$-hypergeometric solutions, under assumption of ample reduction.

In Section 2.6 we give an example of KZ equations and a prime p, such that the space of complex solutions is one-dimensional, the space of polynomial solutions over $\mathbb{F}_p$ is one-dimensional, and the KZ equations have no

$\mathbb{F}_p$-hypergeometric solutions. In Section 8 we show that if the reduction of our KZ equations is ample for a prime p, then all polynomial solutions are $\mathbb{F}_p$-hypergeometric.

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2. KZ equations

2.1. Description of equations. In this paper the numbers p, *q are prime numbers, n is a positive integer, $p > n > 2$, $p > q$.* We fix *a vector* $(m_1,\dots,m_n) \in \mathbb{Z}^n_{>0}$*, such that $m_i < q$ for all $i = 1,\dots,n$,* and study the system of equations for a column vector $I(z) = (I_1(z), \dots, I_n(z))$:

$$\frac{\partial I}{\partial z_i} \;=\; \frac{1}{q}\sum_{j\neq i}\frac{\Omega_{ij}}{z_i - z_j} I, \quad i = 1,\dots,n, \qquad m_1 I_1(z) + \dots + m_n I_n(z) = 0 \tag{2.1}$$

where $z = (z_1,\dots,z_n)$, the $n \times n$-matrices Ω_{ij} have the form:

$$\Omega_{ij} = \begin{array}{c} \\ i \\ \\ j \\ \\ \end{array}\begin{pmatrix} & \vdots^{\,i} & & \vdots^{\,j} & \\ \cdots & -m_j & \cdots & m_j & \cdots \\ & \vdots & & \vdots & \\ \cdots & m_i & \cdots & -m_i & \cdots \\ & \vdots & & \vdots & \end{pmatrix}, \tag{2.2}$$

and all other entries are zero. This joint system of *differential and algebraic equations* is called the *system of KZ equations* in this paper.

Remark. System of equations (2.1) is the system of standard KZ differential equations with parameter q, associated with the Lie algebra $\mathfrak{sl}_2$ and the subspace of singular vectors of weight $\sum_{i=1}^n m_i - 2$ of the tensor product $V_{m_1} \otimes \dots \otimes V_{m_n}$, where V_{m_i} is the irreducible $m_i + 1$ dimensional $\mathfrak{sl}_2$-module, up to a gauge transformation, see this example in [**V3**, Section 1.1].

We consider system (2.1) over the field $\mathbb{C}$ and over the field $\mathbb{F}_p$ with p elements.

2.2. Solutions over $\mathbb{C}$. Consider the *master function*

$$\Phi(t, z_1,\dots,z_n) = \prod_{a=1}^{n} (t - z_a)^{-m_a/q} \tag{2.3}$$

and the column n-vector of hypergeometric integrals

$$I^{(\gamma)}(z) = (I_1(z),\dots,I_n(z)), \tag{2.4}$$

where

$$I_j = \int \Phi(t, z_1, \dots, z_n)\,\frac{dt}{t - z_j}, \qquad j = 1,\dots,n\,. \tag{2.5}$$

The integrals I_j, $j = 1,\dots,n$, are over an element γ of the first homology group of the algebraic curve with affine equation

$$y^q = (t - z_1)^{m_1} \dots (t - z_n)^{m_n}.$$

Starting from such γ, chosen for given values $\{z_1,\dots,z_n\}$, the vector $I^{(\gamma)}(z)$ can be analytically continued as a multivalued holomorphic function of z to the complement in $\mathbb{C}^n$ of the union of the diagonal hyperplanes $z_i = z_j$, $i \neq j$.

The complex vector space of such integral solutions is the $n-1$-dimensional vector space of all solutions of system (2.1). See these statements in the example in [**V3**, Section 1.1], also in [**SliV1**], see also the determinant formula (7.1) below.

2.3. $\mathbb{F}_p$-Integrals. Let $P(x_1,\dots,x_k)$ be a polynomial with coefficients in an $\mathbb{F}_p$-module,

$$P(x_1,\dots,x_k) = \sum_{d_1,\dots,d_k} c_{d_1,\dots,d_k} x_1^{d_1} \dots x_k^{d_k}.$$

Let $(l_1,\dots,l_k) \in \mathbb{Z}^k_{>0}$. The coefficient $c_{l_1p-1,\dots,l_kp-1}$ is called the $\mathbb{F}_p$-*integral over cycle* $[l_1,\dots,l_k]_p$ and denoted by

$$\int_{[l_1,\dots,l_k]_p} P(x_1,\dots,x_k)\, dx_1 \dots dx_k.$$

We have an analog of Stokes' Theorem:

$$\int_{[l_1,\dots,l_k]_p} \frac{\partial P}{\partial x_i}(x_1,\dots,x_k)\, dx_1 \dots dx_k \;=\; 0$$

for any $[l_1,\dots,l_k]_p$.

2.4. Solutions over $\mathbb{F}_p$. Polynomial solutions of system (2.1), considered over the field $\mathbb{F}_p$, were constructed in [**SV3**].

For $i = 1,\dots,n$, choose the least positive integers M_i such that

$$M_i \equiv -\frac{m_i}{q} \pmod{p}. \tag{2.6}$$

Let

$$\Phi_p(t,z,M) \;:=\; \prod_{i=1}^n (t - z_i)^{M_i}, \tag{2.7}$$

$$P(t,z) \;:=\; \Big(\frac{\Phi_p(t,z)}{t - z_1},\dots,\frac{\Phi_p(t,z)}{t - z_n}\Big) \;=\; \sum_i P^i(z)\, t^i \tag{2.8}$$

where $P(t,z)$ is considered as a column n-vector of polynomials in $t, z_1,\dots,z_n$ and $P^i(z)$ as column n-vectors of polynomials in $z_1,\dots,z_n$ with coefficients in $\mathbb{F}_p$. For a positive integer l, denote

$$I^{[l]}(z) \;:=\; \int_{[l]_p} \Big(\frac{\Phi_p(t,z)}{t - z_1},\dots,\frac{\Phi_p(t,z)}{t - z_n}\Big)\, dt. \tag{2.9}$$

Theorem 2.1 ([**SV3**, Theorem 1.2]). *For any positive integer l, the vector of polynomials $I^{[l]}(z)$ is a solution of KZ system* (2.1).

The solutions $I^{[l]}(z)$ given by this construction are called the $\mathbb{F}_p$-*hypergeometric solutions* of equations

(2.1).

Remark. The polynomial $\Phi_p(t,z)$ is an $\mathbb{F}_p$-analog of the master function $\Phi(t,z)$. The polynomial $P(t,z)$ is an analog of the integrand of integral (2.4). The transformation $P(t,z) \to I^{[l]}(z)$ is an analog of the integral and the index $[l]_p$ is an analog of the integration cycle.

Denote $\mathbb{F}_p[z^p] := \mathbb{F}_p[z_1^p,\dots,z_n^p]$. The set of all polynomial solutions of system (2.1) with coefficients in $\mathbb{F}_p$ is a module over the ring $\mathbb{F}_p[z^p]$ since equations (2.1) are linear and $\frac{\partial z_i^p}{\partial z_j} = 0$ in $\mathbb{F}_p[z]$ for all i,j. The $\mathbb{F}_p[z^p]$-module

$$\mathcal{M} = \Big\{ \sum_l c_l(z) I^{[l]}(z) \mid c_l(z) \in \mathbb{F}_p[z^p] \Big\}, \tag{2.10}$$

spanned by $\mathbb{F}_p$-hypergeometric solutions, is called the *module of* $\mathbb{F}_p$*-hypergeometric solutions*.

The range for the index l is defined by the inequalities $0 < lp - 1 \leqslant \sum_{i=1}^n M_i - 1$. Hence $l = 1,\dots,r$, where

$$r := \Big[\sum_{i=1}^n M_i/p \Big], \tag{2.11}$$

the integer part of the number $\sum_{i=1}^n M_i/p$.

Theorem 2.2. *The $\mathbb{F}_p$-hypergeometric solutions $I^{[l]}(z)$, $l = 1,\dots,r$, are linearly independent over $\mathbb{F}_p[z^p]$.*

Proof. The proof coincides with the proof of [**V5**, Theorem 3.1], see also the proof of [**SliV1**, Theorem 3.2]. Other proofs see in [**V6**, Section 4.1] and in Section 6.3 below.

Lemma 2.3. *We have $r < n$.*

2.5. Ample reduction. We say that system (2.1) has *ample reduction* for a prime p if

$$\Big[\sum_{i=1}^n M_i/p \Big] = n - 1, \tag{2.12}$$

that is, the rank $r = \left[\sum_{i=1}^n M_i/p\right]$ of the module $\mathcal{M}$ of $\mathbb{F}_p$-hypergeometric solutions takes the possible maximum value $n-1$.

Example 2.4. *On the one hand, if $q > n$, $p = lq + q - 1$ for some $l \in \mathbb{Z}_{>0}$ and $m_i = 1$, $i = 1,\dots,n$. Then $M_i = ((q-1)p - 1)/q = p - l - 1$ and*

$$\Big[\sum_{i=1}^n \frac{M_i}{p}\Big] = \Big[n\frac{(q-1)p-1}{pq}\Big] = \Big[n - \frac{n}{q} - \frac{n}{pq}\Big] = n-1 .$$

Hence under these assumptions system (2.1) has ample reduction.

On the one hand, if $q > n$, $p = lq+1$ for some $l \in \mathbb{Z}_{>0}$ and $m_i = 1$, $i = 1,\dots,n$. Then $M_i = (p-1)/q = l$ and

$$\Big[\sum_{i=1}^n \frac{M_i}{p}\Big] = \Big[n\frac{p-1}{pq}\Big] = 0 .$$

Hence under these assumptions system (2.1) does not have ample reduction.

In this example and in general the ampleness depends on the residue of p modulo q.

Lemma 2.5. *If system* (2.1) *has ample reduction for a prime p, then for any $l = 1,\dots,n-1$ and any subset $I \subset \{1,\dots,n\}$ with $|I| = l$, we have*

$$(l-1)p < \sum_{i\in I} M_i < lp. \tag{2.13}$$

Proof. The second inequality holds since $0 < M_i < p$ for any i. Assume that the first inequality is not true and $\sum_{i\in I} M_i \leqslant (l-1)p$ for some l and I. Then

$$\sum_{i=1}^{n} M_i \leqslant (l-1)p + \sum_{i\in \bar I} M_i < (l-1)p + (n-l)p = (n-1)p,$$

where $\bar I$ is the complement of I. That contradicts to the ampleness of the reduction.

2.6. Example. Let $n = 2$, $p = 3$, $q = 2$, $m_1 = m_2 = 1$. Then $M_1 = M_2 = 1$. The KZ equations take the form

$$\frac{\partial I}{\partial z_1} = \frac{\Omega}{z_1 - z_2} I, \qquad \frac{\partial I}{\partial z_2} = \frac{\Omega}{z_2 - z_1} I, \qquad \Omega = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

The polynomial $(z_1 - z_2)^2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is a solution. At the same time $r = [(M_1 + M_2)/p] = 0$ and there are no $\mathbb{F}_p$-hypergeometric solutions.

3. Coefficients of polynomials

3.1. Coefficients of $\mathbb{F}_p$-hypergeometric solutions. For $l = 1,\dots,r$, the coordinates of the column vector $I^{[l]}(z) = (I_1^{[l]}(z),\dots,I_1^{[l]}(z))$ are homogeneous polynomials in $z_1,\dots,z_n$ of degree

$$\delta_l := \sum_{j=1}^{n} M_j - lp\,. \tag{3.1}$$

Let

$$I^{[l]}(z) = \sum_{d_1,\dots,d_n} I^{[l]}_{d_1,\dots,d_n} z_1^{d_1}\cdots z_n^{d_n}, \qquad I^{[l]}_{d_1,\dots,d_n} \in \mathbb{F}_p^n\,.$$

Lemma 3.1. *The coefficient $I^{[l]}_{d_1,\dots,d_n} \in \mathbb{F}_p^n$ is nonzero if and only if*

$$\sum_{i=1}^{n} d_i = \delta_l, \quad \text{and} \quad d_i \leqslant M_i \quad \text{for} \quad i = 1,\dots,n, \tag{3.2}$$

moreover,

$$I^{[l]}_{d_1,\dots,d_n} = (-1)^{\delta_l} \prod_{j=1}^{n} \binom{M_j}{d_j} \Big(1 - \frac{d_1}{M_1}, \dots, 1 - \frac{d_n}{M_n}\Big). \tag{3.3}$$

If $(I^{[l]}_{d_1,\dots,d_n;1},\dots,I^{[l]}_{d_1,\dots,d_n;n})$ are coordinates of $I^{[l]}_{d_1,\dots,d_n}$, then

$$\sum_{i=1}^{n} m_i\, I^{[l]}_{d_1,\dots,d_n;i} = \sum_{i=1}^{n} M_i\, I^{[l]}_{d_1,\dots,d_n;i} = 0. \tag{3.4}$$

Proof. The first statements follow from formulas (2.7) and (2.8). Formula (3.4) follows from formulas (3.3), (2.6).

3.2. Coefficients and singular vectors. Consider the Lie algebra $\mathfrak{sl}_2$ over the field $\mathbb{F}_p$ with standard generators e,f,h and relations $[e,f] = h$, $[h,e] = 2e$, $[h,f] = -2f$.

For $m \in \mathbb{Z}_{>0}$, $m < p$, let V_m be the irreducible $\mathfrak{sl}_2$-module over $\mathbb{F}_p$ with highest weight m, basis $f^j v_m$, $j = 0,\dots,m$, and standard $\mathfrak{sl}_2$-action.

Consider the $\mathfrak{sl}_2$-module $\otimes_{j=1}^n V_{m_j}$. For $i = 1,\dots,n$, let

$$f^{(i)}v := v_{m_1}\otimes\cdots\otimes v_{m_{i-1}}\otimes f v_{m_i}\otimes v_{m_{i+1}}\otimes\cdots\otimes v_{m_n} \ \in\ \otimes_{j=1}^n V_{m_j}\,. \tag{3.5}$$

Then

$$hf^{(i)}v = \Big(\sum_{j=1}^n m_j - 2\Big) f^{(i)}v\,, \qquad ef^{(i)}v = m_i\, v_{m_1}\otimes\cdots\otimes v_{m_i}\otimes\cdots\otimes v_{m_n}\,.$$

Denote by $V[-2]$ the n-dimensional subspace of $\otimes_{j=1}^n V_{m_j}$ generated by $f^{(i)}v$, $i = 1,\dots,n$. Denote

$$\mathrm{Sing}V[-2] = \Big\{ \sum_{i=1}^n c_i f^{(i)}v \ \Big|\ \sum_{i=1}^n c_i m_i = 0\Big\} \subset V[-2]\,.$$

The $n-1$-dimensional subspace $\mathrm{Sing}V[-2] \subset V[-2]$ is the kernel of the restriction to $V[-2]$ of the operator $e : \otimes_{j=1}^n V_{m_j} \to \otimes_{j=1}^n V_{m_j}$. Denote

$$\mathrm{Sing}V[-2][z] := \mathrm{Sing}V[-2] \otimes_{\mathbb{F}_p} \mathbb{F}_p[z].$$

Define an isomorphism of vector spaces

$$\iota : \mathbb{F}_p^n \to V[-2], \qquad (c_1,\dots,c_n) \mapsto \sum_{i=1}^n c_i f^{(i)}v\,. \tag{3.6}$$

Then an $\mathbb{F}_p$-hypergeometric solution $I^{[l]}(z)$ is identified with the polynomial

$$\iota I^{[l]}(z) := \sum_{d_1,\dots,d_n} \iota I^{[l]}_{d_1,\dots,d_n} z_1^{d_1}\cdots z_n^{d_n} \in \mathrm{Sing}V[-2][z].$$

3.3. Operators $\Omega^{\mathfrak{sl}_2}_{ij}$. The isomorphism ι identifies a linear operator $\Omega^{ij} : \mathbb{F}_p^n \to \mathbb{F}_p^n$, appearing in system (2.1), with a linear operator on $V[-2]$, which we denote by $\iota\Omega_{ij}$. Namely, the linear operator $\iota\Omega_{ij}$ is the restriction to $V[-2]$ of the Casimir operator on $\otimes_{j=1}^n V_{m_j}$ defined by the formula

$$\Omega^{\mathfrak{sl}_2}_{ij} := \frac12 h^{(i)}h^{(j)} + e^{(i)}f^{(j)} + f^{(i)}e^{(j)} - \frac{m_i m_j}{2}\,\mathrm{Id},$$

where for $x \in \mathfrak{sl}_2$ we define the operator $x^{(i)}$ on $\otimes_{j=1}^n V_{m_j}$ by

$$x^{(i)} := 1\otimes\cdots\otimes 1\otimes x\otimes 1\otimes\cdots\otimes 1$$

with x at the ith position. Notice that each $\Omega^{\mathfrak{sl}_2}_{ij}$ preserves $\mathrm{Sing}V[-2]$.

4. $\mathbb{F}_p$-Beta integral and KZ equations for $n = 2$

4.1. Solutions over $\mathbb{C}$. Consider the system of KZ equations (2.1) over $\mathbb{C}$ for $n = 2$. Then the master function is

$$\Phi(t,z_1,z_2) = (t-z_1)^{-m_1/q}(t-z_2)^{-m_2/q}\,, \tag{4.1}$$

and the one-dimensional space of solutions is generated by the 2-column vector

$$I(z_1,z_2) = \int_{z_1}^{z_2} \Phi(t,z)\Big(\frac{f^{(1)}v}{t-z_1} + \frac{f^{(2)}v}{t-z_2}\Big)dt. \tag{4.2}$$

To determine this integral we assume that z_1,z_2 are real, $z_1 < z_2$, and fix a univalued branch on $[z_1,z_2]$ of each of the factors $(t-z_1)^{-m_1/q}$, $(t-z_2)^{-m_2/q}$. Then

(4.3)
$$I(z_1,z_2) = (z_2-z_1)^{-m_1/q}(z_1-z_2)^{-m_2/q}\frac{\Gamma(-m_1/q+1)\Gamma(-m_1/q+1)}{\Gamma(-m_1/q-m_2/q+1)}\Big(\frac{f^{(1)}v}{-m_1/q}-\frac{f^{(2)}v}{-m_2/q}\Big),$$

where $(z_k - z_l)^{-m_l/q}$ is the value of the chosen branch of the function $(t - z_l)^{-m_l/q}$ at $t = z_k$.

Remark. Calculation of each coordinate of the vector $I(z_1,z_2)$ is reduced to the beta integral,

$$\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \qquad (4.4)$$

after change of variables. Formula (4.3) shows how the beta integral appears in hypergeometric solutions of KZ equations.

If $(-m_1/q,-m_2/q) = (M_1,M_2)$, where M_1,M_2 are positive integers, then

$$I(z_1,z_2) = (z_2-z_1)^{M_1}(z_1-z_2)^{M_2}\frac{\Gamma(M_1+1)\,\Gamma(M_2+1)}{\Gamma(M_1+M_2+1)}\Big(\frac{f^{(1)}v}{M_1}-\frac{f^{(2)}v}{M_2}\Big). \qquad (4.5)$$

4.2. Factorial and gamma functions. Recall that p is an odd prime number. The p-adic factorial function is defined on positive integers by

$$(x!)_p = \prod_{1\leqslant j\leqslant x,\ (j,p)=1} j\,.$$

The Morita p-adic gamma function is the unique continuous function of a p-adic integer x (with values in $\mathbb{Z}_p$) such that

$$\Gamma_p(x) = (-1)^x \prod_{1\leqslant j<x,\ (j,p)=1} j\,,$$

for positive integers x. Thus $\Gamma_p(x + 1) = (-1)^x(x!)_p$ for positive integers x. Define the function

$$\Gamma_{\mathbb{F}_p} : \mathbb{Z}_{>0} \to \mathbb{F}_p$$

by setting $\Gamma_{\mathbb{F}_p}(0) = 1$, $\Gamma_{\mathbb{F}_p}(1) = -1$ and mapping an integer $x > 1$ to the image of the integer $\Gamma_p(x)$ in $\mathbb{F}_p$.

Lemma 4.1. *We have* $\Gamma_{\mathbb{F}_p}(x + p) = \Gamma_{\mathbb{F}_p}(x)$ *for all* x.

Proof. The lemma follows from Wilson's theorem, $(p - 1)! \equiv -1 \pmod p$.

We extend the function $\Gamma_{\mathbb{F}_p}$ to the set $\mathbb{Z}$ by periodicity, $\Gamma_{\mathbb{F}_p}(x + p) = \Gamma_{\mathbb{F}_p}(x)$. Then we get

$$\Gamma_{\mathbb{F}_p}(x)\Gamma_{\mathbb{F}_p}(1-x) = (-1)^x$$

also by Wilson's theorem.

Lemma 4.2. *Let* A,B *be positive integers such that* $A < p$, $B < p$, $p \leqslant A+B$. *Then we have an identity in* $\mathbb{F}_p$,

$$B\binom{B-1}{A+B-p} = B\binom{B-1}{p-A-1} = (-1)^{A+1}\,\frac{A!\,B!}{(A+B-p)!} = (-1)^A\,\frac{\Gamma_{\mathbb{F}_p}(A+1)\Gamma_{\mathbb{F}_p}(B+1)}{\Gamma_{\mathbb{F}_p}(A+B-p+1)} \tag{4.6}$$

Proof. We have

$$B\binom{B-1}{p-A-1} = \frac{B(B-1)\cdots(A+B-p+1)}{1\cdots(p-A-1)} = \frac{B\cdots(A+B-p+1)(A+B-p)!A!}{(-1)^{p-A-1}(p-1)(p-2)\cdots(A+1)\,A!(A+B-p)!} = (-1)^{A+1}\,\frac{A!\,B!}{(A+B-p)!} = (-1)^A\,\frac{\Gamma_{\mathbb{F}_p}(A+1)\,\Gamma_{\mathbb{F}_p}(B+1)}{\Gamma_{\mathbb{F}_p}(A+B-p+1)}.$$

4.3. $\mathbb{F}_p$-hypergeometric solutions. Consider the system of KZ equations (2.1) over $\mathbb{F}_p$ for $n = 2$. Assume that system (2.1) has ample reduction for a prime p. Then the integers M_1, M_2, introduced in (2.6), satisfy the inequalities

$$0 < M_1,\ M_2,\ M_1 + M_2 - p + 1 < p. \tag{4.7}$$

In this case the module M of $\mathbb{F}_p$-hypergeometric solutions is of rank one and generated by $I^{[1]}(z_1,z_2)$. The solution $I^{[1]}(z_1,z_2)$ is the coefficient of t^{p-1} in the Taylor expansion of the polynomial 2-vector

$$P(t,z_1,z_2) = (t-z_1)^{M_1}(t-z_2)^{M_2}\Big(\frac{f^{(1)}v}{t-z_1} + \frac{f^{(2)}v}{t-z_2}\Big), \tag{4.8}$$

cf. (4.2)

Theorem 4.3. *We have*

$$I^{[1]}(z_1,z_2) = (-1)^{M_2}(z_2-z_1)^{M_1+M_2-p}\frac{\Gamma_{\mathbb{F}_p}(M_1+1)\,\Gamma_{\mathbb{F}_p}(M_2+1)}{\Gamma_{\mathbb{F}_p}(M_1+M_2-p+1)}\Big(\frac{f^{(1)}v}{M_1} - \frac{f^{(2)}v}{M_2}\Big) = (-1)^{M_1}(z_1-z_2)^{M_1+M_2-p}\frac{\Gamma_{\mathbb{F}_p}(M_1+1)\,\Gamma_{\mathbb{F}_p}(M_2+1)}{\Gamma_{\mathbb{F}_p}(M_1+M_2-p+1)}\Big(\frac{f^{(2)}v}{M_2} - \frac{f^{(1)}v}{M_1}\Big), \tag{4.9}$$

cf. (4.5)*.*

Proof. Make the transformation

$$P(t,z_1,z_2) \ \mapsto\ P(t+z_1,z_1,z_2) = t^{M_1}(t+z_1-z_2)^{M_2}\Big(\frac{f^{(1)}v}{t} + \frac{f^{(2)}v}{t+z_1-z_2}\Big).$$

This change of variables does not change the coefficient of t^{p-1} in the Taylor expansion by Lucas theorem, see [**Lu**] and the proof of [**V5**, Lemma 5.2]. Hence

$$I^{[1]}(z_1,z_2) = (z_1-z_2)^{M_1+M_2-p}\Big(\binom{M_2}{p-M_1}f^{(1)}v + \binom{M_2-1}{p-M_1-1}f^{(2)}v\Big).$$

Then

$$\binom{M_2}{p-M_1} f^{(1)}v + \binom{M_2-1}{p-M_1-1} f^{(2)}v = M_2\binom{M_2-1}{p-M_1-1}\Big(\frac{f^{(1)}v}{p-M_1}+\frac{f^{(2)}v}{M_2}\Big). \tag{4.10}$$

Now the theorem follows from Lemma 4.2.

Remark. For positive integers a,b satisfying the inequalities $a < p$, $b < p$, $p-1 \leqslant a+b$, we have the $\mathbb{F}_p$*-beta integral formula*

$$\int_{[1]_p} t^a(1-t)^b dt \;=\; -\,\frac{a!\,b!}{(a+b-p+1)!}, \tag{4.11}$$

which follows from Lemma 4.2.

Formula (4.4) for the beta integral is the one-dimensional case of the n-dimensional Selberg integral formula, see [**Se**]. In [**RV1**, **RV2**] we develop $\mathbb{F}_p$-analogs of the n-dimensional Selberg integral formulas.

5. Leading term of a polynomial solution

5.1. Lexicographical ordering. For a permutation $\sigma = (\sigma_1,\dots,\sigma_n) \in S_n$ denote by $>_\sigma$ the *lexicographical ordering* of monomials

$$z_1^{d_1}\dots z_n^{d_n}\,, \qquad d_1,\dots,d_n \in \mathbb{Z}_{\geqslant 0},$$

relative to the ordering $(\sigma_1,\dots,\sigma_n)$ of the integers $(1,\dots,n)$. So $z_{\sigma_1} >_\sigma z_{\sigma_2} >_\sigma \dots >_\sigma z_{\sigma_{n-1}} >_\sigma z_{\sigma_n}$ and so on.

For a nonzero polynomial

$$f(z) = \sum_{d_1,\dots,d_n} a_{d_1,\dots,d_n} z_1^{d_1}\dots z_n^{d_n}$$

let $f_\sigma(z)$ be the summand $a_{d_1,\dots,d_n} z_1^{d_1}\cdots z_n^{d_n}$ corresponding to the largest monomial entering $f(z)$ with a nonzero coefficient. We call $f_\sigma(z)$ the σ*-leading term* of $f(z)$, the corresponding $a_{d_1,\dots,d_n}$ the σ*-leading coefficient*, the corresponding $z_1^{d_1}\cdots z_n^{d_n}$ the σ*-leading monomial*.

In particular, consider elements $f(z) = (f_1(z),\dots,f_n(z)) \in (\mathbb{F}_p[z])^n$ as polynomials in z with coefficients in $\mathbb{F}_p^n$. Then for any $\sigma \in S_n$ a nonzero element $f(z)$ has σ-leading term $f_\sigma(z) = a_{d_1,\dots,d_n} z_1^{d_1}\dots z_n^{d_n}$ and σ-leading coefficient $a_{d_1,\dots,d_n} \in \mathbb{F}_p^n$.

5.2. Leading term of a polynomial solution. Consider the lexicographical ordering $>_{\mathrm{id}}$ of monomials corresponding to the identity permutation $\mathrm{id} \in S_n$. Hence $z_1 >_{\mathrm{id}} z_2 >_{\mathrm{id}} \dots >_{\mathrm{id}} z_n$ and so on.

Lemma 5.1. *Let $I(z) = (I_1(z),\dots,I_n(z))$ be a polynomial solution of system* (2.1) *over $\mathbb{F}_p$ (not necessarily an $\mathbb{F}_p$-hypergeometric solution). Let $Cz_1^{d_1}\dots z_n^{d_n}$, $C = (C_1,\dots,C_n) \in \mathbb{F}_p^n$, be the* id*-leading term of $I(z)$. Then*

$$\sum_{j=1}^n m_jC_j = 0, \qquad \sum_{l=j+1}^n \Omega_{jl}\, C \,=\, qd_j\, C\,, \quad j=1,\dots,n-1\,, \quad d_n \equiv 0 \pmod p, \tag{5.1}$$

where C is considered as a column vector.

Proof. Rewrite the KZ equations as

$$\Big(\prod_{k\neq j}(z_j-z_k)\Big)\frac{\partial I}{\partial z_j} - \frac{1}{q}\sum_{l\neq j}\Big(\prod_{k\notin\{l,j\}}(z_j-z_k)\Big)\Omega_{jl} I \;=\; 0, \tag{5.2}$$

$j=1,\dots,n$. Now the lemma follows from calculating the leading term of the left-hand side in (5.2) and equating it to zero.

For $j \neq l$ introduce the $n\times n$-matrices

$$\Omega^M_{jl} = \begin{array}{c} \\ j \\ \\ l \\ \\ \end{array} \begin{array}{c} \begin{array}{ccccccc} & & j & & l & & \end{array} \\ \begin{pmatrix} & \vdots & & \vdots & \\ \cdots & M_l & \cdots & -M_l & \cdots \\ & \vdots & & \vdots & \\ \cdots & -M_j & \cdots & M_j & \cdots \\ & \vdots & & \vdots & \end{pmatrix} \end{array} \tag{5.3}$$

with all other entries equal to zero. For $j = 1,\dots,n-1$, denote $\Omega^M_j = \sum_{l=j+1}^n \Omega^M_{jl}$.

Corollary 5.2. *Let $I(z)$ be a polynomial solution of system* (2.1) *over $\mathbb{F}_p$. Then the* id*-leading term $Cz_1^{d_1}\dots z_n^{d_n}$ of $I(z)$ satisfies the system of equations:*

$$\sum_{j=1}^n M_jC_j = 0, \qquad \Omega^M_j C = d_j C, \qquad j = 1,\dots,n-1, \qquad d_n \equiv 0 \pmod p. \tag{5.4}$$

Theorem 5.3. *Let a pair $C = (C_1,\dots,C_n)$, $(d_1,\dots,d_n) \in \mathbb{F}_p^n$ be a solution of system*

$$\sum_{j=1}^n M_jC_j = 0, \quad \Omega^M_j C = d_j C, \quad j = 1,\dots,n-1, \quad d_n = 0 . \tag{5.5}$$

Let the index i be such that

$$C_i \neq 0 \quad \text{and} \quad C_j = 0, \qquad j = 1,\dots,i-1. \tag{5.6}$$

Then

$$\begin{aligned} d_j &= M_j, \qquad j = 1,\dots,i-1, \\ d_i &= \sum_{j=i}^n M_j, \qquad d_i \neq M_i, \\ d_j &= 0, \qquad j = i+1,\dots,n, \end{aligned} \tag{5.7}$$

$$\sum_{l=i+1}^n M_l \neq 0, \qquad C_j = -\frac{M_i}{\sum_{l=i+1}^n M_l} C_i, \qquad j = i+1,\dots,n. \tag{5.8}$$

Conversely, if a pair $C = (C_1,\dots,C_n)$, $(d_1,\dots,d_n) \in \mathbb{F}_p^n$ has properties (5.6), (5.7), (5.8), *then it is a solution of system* (5.5).

Proof. For any $j = 1,\dots,n-1$, we have

$$\begin{aligned} \Omega^M_j(C_1,\dots,C_n) = (0,\dots,0,\ & M_{j+1}(C_j - C_{j+1}) + \dots + M_n(C_j - C_n), M_j(C_{j+1} - C_j),\dots,M_j(C_n - C_j)), \end{aligned} \tag{5.9}$$

where in the right-hand side 0 is repeated $j-1$-times,

Assume (5.5) and (5.6). First we check that $d_j = M_j$, $j = 1,\dots,i-1$. Indeed,

$$\Omega^M_j C = \Big(0,\dots,0, -\sum_{l=j+1}^n M_lC_l,\ M_jC_{j+1},\dots,M_jC_n\Big). \tag{5.10}$$

$$= \Big(0,\, \dots,\, 0, \qquad 0 \qquad,\, M_jC_{j+1}\,,\, \dots,\, M_jC_n\Big) \,=\, M_j\, C\,. \tag{5.11}$$

Here we used that $\sum_{l=j+1}^n M_lC_l = \sum_{l=1}^n M_lC_l = 0$. Hence if $\Omega^{M}_j C = d_jC$, the $d_j = M_j$.

We also have

$$\begin{aligned} \Omega_i^M C &= (0,\dots,0,\, \sum_{l=i+1}^{n} M_l(C_i - C_l),\, M_i(C_{i+1}-C_i),\dots, M_j(C_n - C_i)) \\ &= (0,\dots,0, \quad C_i\sum_{l=i}^{n} M_l \quad, M_i(C_{i+1}-C_i),\, \dots,\, M_j(C_n-C_i)) = d_iC\,. \end{aligned} \tag{5.12}$$

Hence

$$d_i = \sum_{l=i}^{n} M_l, \qquad M_i(C_j - C_i) = C_j \sum_{l=i}^{n} M_l, \qquad j = i+1,\dots,n. \tag{5.13}$$

The second equality in (5.13) implies $-M_iC_i = C_j \sum_{l=i+1}^n M_l$. Hence $\sum_{l=i+1}^n M_l \neq 0$. This inequality and the first equality in (5.13) imply $d_i \neq M_i$. Also the second equality in (5.13) implies that $C_{i+1} = \dots = C_n$ and, therefore, $\Omega^{M}_j C = 0$ for $j > i$. Hence $d_j = 0$ for $j = i+1,\dots,n-1$. Thus we deduced (5.7) and (5.8).

The proof that (5.6), (5.7), (5.8) imply (5.5) is straightforward.

Corollary 5.4. *Let a pair $C = (C_1,\dots,C_n),\, (d_1,\dots,d_n) \in \mathbb{F}_p^n$ be a solution of system (5.4) such that $C \neq 0$. Then $\sum_{j=1}^n d_j = \sum_{j=1}^n M_j$.*

Corollary 5.5. *Let $I(z) = (I_1(z),\dots,I_n(z))$ be a homogeneous polynomial solution of system (2.1) over $\mathbb{F}_p$ of degree d, then*

$$d \,\equiv\, \sum_{j=1}^{n} M_j \pmod p\,, \tag{5.14}$$

cf. Example 2.6.

Corollary 5.6. *Let a pair $C = (C_1,\dots,C_n)$, $(d_1,\dots,d_n) \in \mathbb{F}^n_p$ be a solution of system (5.4) such that $C \neq 0$. Then $(d_1,\dots,d_n)$ is uniquely determined by C. If $c \in \mathbb{F}_p^\times$, then the pair $cC,(d_1,\dots,d_n)$ is also a solution of system (5.4). The number of equivalence classes of solutions $C,(d_1,\dots,d_n)$ of system (5.4), modulo the equivalence relation $C \mapsto cC$, equals $n-1$ decreased by the number of indices i, $1 \leqslant i \leqslant n-2$, such that $\sum_{j=i+1} M_j = 0$.*

5.3. Example. Let $n = 2$, $p = 3$, $q = 2$, $m_1 = m_2 = 1$, $M_1 = M_2 = 1$. Then system (2.1) has a (non-$\mathbb{F}_p$-hypergeometric) polynomial solution $I(z) = (z_1 - z_2)^2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, see Example 2.6. The leading id-term of $I(z)$ equals $z_1^2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ in agreement with Theorem 5.3.

In this example the module of all polynomial solutions is one-dimensional and generated by $I(z)$. This follows from the fact that the id-leading term of any polynomial solution $J(z)$ has the form $cz_1^{a_1}z_2^{a_2}\cdot z_1^2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, where $c \in \mathbb{F}_p^\times,\ a_1, a_2 \in p\mathbb{Z}$, by Theorem 5.3.

6. Leading term of an $\mathbb{F}_p$-hypergeometric solution

6.1. Leading term of $I^{[l]}(z)$. Using the isomorphism ι, defined in (3.6), we consider $\mathbb{F}_p$-hypergeometric solutions as polynomials in z with coefficients in $\operatorname{Sing} V\,[-2]$.

Recall that for $l = 1,\dots,r$, the $\mathbb{F}_p$-hypergeometric solution $I^{[l]}(z)$ is a homogeneous polynomial in z of degree $\delta_l = \sum_{j=1}^n M_j - lp$.

We describe the leading term $I^{[l]}_{\mathrm{id}}(z)$ of $I^{[l]}(z)$ with respect to the lexicographical ordering $>_{\mathrm{id}}$.

Theorem 6.1. *Given $l = 1,\dots,r$, denote by $i = i(l)$ the unique positive integer such that*

$$0 \leqslant \sum_{j=i}^n M_j - lp < M_i \qquad (6.1)$$

Then

$$I^{[l]}_{\mathrm{id}}(z) = (-1)^{\sum_{j=1}^i M_j} \frac{\Gamma_{\mathbb{F}_p}\left(\sum_{j=i}^n M_j - lp + 1\right)}{\Gamma_{\mathbb{F}_p}(M_i+1)\,\Gamma_{\mathbb{F}_p}\left(\sum_{j=i+1}^n M_j - (l-1)p + 1\right)} \qquad (6.2)$$

$$\times \left(\frac{\sum_{j=i+1}^n f^{(j)}v}{\sum_{j=i+1}^n M_j} - \frac{f^{(i)}v}{M_i}\right) z_1^{M_1} z_2^{M_2} \dots z_{i-1}^{M_{i-1}} z_i^{\sum_{j=i}^n M_j - lp},$$

where $f^{(k)}v$ are introduced in (3.5).

Proof. The theorem follows from formula (3.3), Corollary 5.2, Theorem 5.3 and Lemma 4.2, where Lemma 4.2 is applied with $A = M_i$ and $B = \sum_{j=i+1}^n M_j - (l-1)p$.

Remark. Let $\sigma = (\sigma_1,\dots,\sigma_n) \in S_n$. The leading term $I^{[l]}_\sigma(z)$ of $I^{[l]}(z)$ with respect to the lexicographical ordering $>_\sigma$ is obtained from the right-hand side of formula (6.2) by simultaneous reordering of variables $z_1,\dots,z_n$ and parameters $M_1,\dots,M_n$.

6.2. Indices $i(l)$. In Theorem 6.1 we define the numbers $i(l)$, $l = 1,\dots,r$. The leading term $I^{[l]}_{\mathrm{id}}(z)$ of the $\mathbb{F}_p$-hypergeometric solution $I^{[l]}(z)$ is a monomial in $z_1,\dots,z_{i(l)}$.

Lemma 6.2. *We have*

$$1 \leqslant i(r) < i(r-1) < \dots < i(1) < n. \qquad (6.3)$$

Corollary 6.3. *If system* (2.1) *has ample reduction for a prime p, then*

$$i(l) = n - l, \qquad l = 1,\dots,n-1. \qquad (6.4)$$

6.3. Corollary of Theorem 6.1. The $\mathbb{F}_p$-hypergeometric solutions $I^{[l]}(z)$, $l = 1,\dots,r$, are linearly independent over the field $\mathbb{F}_p(z)$ of rational functions in z with coefficients in $\mathbb{F}_p$.

This follows from the fact that the leading coefficients of $I^{[l]}(z)$, $l = 1,\dots,r$, are linear independent over the field $\mathbb{F}_p$.

6.4. Example. Let $q = 3$, $p = 13$, $(m_1,\dots,m_6) = (2,2,2,1,1,1)$. Then $(M_1,\dots,M_6) = (8,8,8,4,4,4)$, $r = \left[\sum_j M_j/p\right] = 2$. We have two $\mathbb{F}_p$-hypergeometric solutions $I^{[l]}(z)$, $l = 1,2$, of degrees 23 and 10, respectively.

Consider the identity element id $= (1,2,3,4,5,6)$ of S_6 and the elements $s_{3,4} = (1,2,4,3,5,6)$, $\sigma = (6,5,4,3,2,1) \in S_6$. Then the leading terms are

$$I^{[1]}_{\mathrm{id}}(z) = -\binom{8}{7}\Big((1-\frac{7}{8})f^{(3)}v + f^{(4)}v + f^{(5)}v + f^{(6)}v\Big) z_1^8 z_2^8 z_3^7\,,$$
$$I^{[1]}_{s_{3,4}}(z) = -\binom{4}{3}\Big((1-\frac{3}{4})f^{(3)}v + f^{(5)}v + f^{(6)}v\Big) z_1^8 z_2^8 z_4^4 z_3^3\,,$$
$$I^{[1]}_{\sigma}(z) = -\binom{8}{3}\Big((1-\frac{3}{8})f^{(2)}v + f^{(1)}v\Big) z_6^4 z_5^4 z_4^4 z_3^8 z_2^3\,,$$

$$I^{[2]}_{\mathrm{id}}(z) = \binom{8}{2}\Big((1-\frac{2}{8})f^{(2)}v + f^{(3)}v + f^{(4)}v + f^{(5)}v + f^{(6)}v\Big) z_1^8 z_2^2$$
$$I^{[2]}_{s_{3,4}}(z) = \binom{8}{2}\Big((1-\frac{2}{8})f^{(2)}v + f^{(3)}v + f^{(4)}v + f^{(5)}v + f^{(6)}v\Big) z_1^8 z_2^2\,,$$
$$I^{[2]}_{\sigma}(z) = \binom{4}{2}\Big((1-\frac{2}{4})f^{(4)}v + f^{(3)}v + f^{(2)}v + f^{(1)}v\Big) z_6^4 z_5^4 z_4^2\,.$$

7. Determinant of $\mathbb{F}_p$-hypergeometric solutions

7.1. Determinant over $\mathbb{C}$. Consider the system of KZ equations (2.1) over $\mathbb{C}$. Then the space of solutions is $n-1$-dimensional.

Recall the master function $\Phi(t,z)$ introduced in (2.4). Consider the hypergeometric integrals

$$I^{(j)}(z) = \int_{z_j}^{z_{j+1}} \Phi(t,z)\sum_{j=1}^{n}\frac{f^{(j)}v}{t-z_j}\,dt, \qquad j=1,\dots,n-1\,.$$

To determine the integrals we assume that $z_1,\dots,z_n$ are real, $z_1<\dots<z_n$, and for every $l=1,\dots,n$, we fix a univalued branch of the function $(t-z_l)^{-m_l/q}$ on the rays $\{t\in\mathbb{R}\mid t<z_l\}$ and $\{t\in\mathbb{R}\mid t>z_l\}$.

The integrals $I^{(j)}$, $j=1,\dots,n-1$, form a basis of solutions of system (2.1) due to the following theorem.

Theorem 7.1 ([**V1**, **V2**, **V3**]). *We have*

$$\det_{j,l=1}^{n-1}\Big(\frac{-m_{l+1}}{q}\int_{z_j}^{z_{j+1}}\Phi(t,z)\frac{dt}{t-z_{l+1}}\Big) = \frac{\Gamma(-m_1/q+1)\cdots\Gamma(-m_n/q+1)}{\Gamma(-m_1/q-\dots-m_n/q+1)}\prod_{1\leqslant j,\,l\leqslant n,\ j\neq l}(z_j-z_l)^{-m_l/q}\,. \tag{7.1}$$

Below we prove an analog of this theorem for $\mathbb{F}_p$-hypergeometric solutions.

7.2. Determinant over $\mathbb{F}_p$. Introduce a basis of Sing$V[-2]$,

$$w_j = \frac{f^{(j)}v}{M_j} - \frac{f^{(j+1)}v}{M_{j+1}}, \qquad j=1,\dots,n-1\,. \tag{7.2}$$

Expand each of the $\mathbb{F}_p$-hypergeometric solutions $I^{[l]}(z)$ with respect to this basis,

$$I^{[l]}(z) = \sum_{j=1}^{n-1} c^l_j(z)\,w_j, \tag{7.3}$$

where $c^l_j(z)$ are scalar homogeneous polynomials of degree $\sum_{j=1}^n M_j - lp$.

Theorem 7.2. *Assume that system* (2.1) *has ample reduction for a prime p. Then we have the square matrix* $c(z) = (c^l_j(z))_{l,j=1,\dots,n-1}$ *with nonzero determinant, given by the following formula*

$$\det c(z) \;=\; \frac{\Gamma_{\mathbb{F}_p}(M_1+1)\cdots\Gamma_{\mathbb{F}_p}(M_n+1)}{\Gamma_{\mathbb{F}_p}(M_1+\dots+M_n-(n-1)p+1)} \prod_{1\leqslant i<j\leqslant n} (-1)^{M_j}(z_j-z_i)^{M_i+M_j-p} \, . \tag{7.4}$$

Notice that

$$\frac{\Gamma_{\mathbb{F}_p}(M_1+1)\cdots\Gamma_{\mathbb{F}_p}(M_n+1)}{\Gamma_{\mathbb{F}_p}(M_1+\dots+M_n-(n-1)p+1)} \;=\; (-1)^{n-1}\frac{M_1!\cdots M_n!}{(M_1+\dots+M_n-(n-1)p)!} \, . \tag{7.5}$$

The theorem is proved in Sections 7.3, 7.4.

7.3. Preliminary remarks.

Lemma 7.3. *The function* $\det c(z)$ *is a nonzero homogeneous polynomial of degree*

$$(n-1)\sum_{j=1}^n M_j - \frac{n(n-1)}{2}\,p \tag{7.6}$$

with id*-leading term*

$$\det{}_{\mathrm{id}} c(z) = \mathrm{const}\, z_1^{(n-1)(M_1-p)+\sum_{j=2}^n M_j} z_2^{(n-2)(M_2-p)+\sum_{j=3}^n M_j} \dots z_{n-1}^{M_{n-1}-p+M_n} , \tag{7.7}$$

where

$$\mathrm{const} = (-1)^{n(n-1)/2+\sum_{j=1}^{n-1}(n-j)M_j}\,\frac{\Gamma_{\mathbb{F}_p}(M_1+1)\cdots\Gamma_{\mathbb{F}_p}(M_n+1)}{\Gamma_{\mathbb{F}_p}(M_1+\dots+M_n-(n-1)p+1)} , \tag{7.8}$$

Proof. If $\det c(z)$ is nonzero, then it is of degree

$$\deg(\det c(z)) = \sum_{l=1}^{n-1}\Big(\sum_{j=1}^n M_j - lp\Big) ,$$

which gives (7.6). By Theorem 6.1 we have

$$I^{[l]}_{\mathrm{id}}(z) \;=\; (-1)^{\sum_{j=1}^{n-l} M_j}\, \frac{\Gamma_{\mathbb{F}_p}(M_{n-l}+1)\,\Gamma_{\mathbb{F}_p}\Big(\sum_{j=n-l+1}^n M_j-(l-1)p+1\Big)}{\Gamma_{\mathbb{F}_p}\Big(\sum_{j=n-l}^n M_j - lp+1\Big)} \times \left(\frac{\sum_{j=n-l+1}^n f^{(j)}v}{\sum_{j=n-l+1}^n M_j} - \frac{f^{(n-l)}v}{M_{n-l}}\right) z_1^{M_1} z_2^{M_2}\dots z_{n-l-1}^{M_{n-l-1}} z_{n-l}^{\sum_{j=n-l}^n M_j - lp} , \tag{7.9}$$

Expanding vectors $\left(\frac{\sum_{j=n-l+1}^n f^{(j)}v}{\sum_{j=n-l+1}^n M_j} - \frac{f^{(n-l)}v}{M_{n-l}}\right)$, $l=1,\dots,n-1$, with respect to the basis $w_1,\dots,w_{n-1}$, we obtain a square $(n-1)\times(n-1)$-matrix $C(\mathrm{id}) = (C(\mathrm{id})^l_j)$ of coefficients of the expansion. The matrix is triangular with respect to the main anti-diagonal. The anti-diagonal entries are

$$C(\mathrm{id})^l_{n-l} = (-1)^{\sum_{j=1}^{n-l} M_j+1}\,\frac{\Gamma_{\mathbb{F}_p}(M_{n-l}+1)\,\Gamma_{\mathbb{F}_p}\Big(\sum_{j=n-l+1}^n M_j-(l-1)p+1\Big)}{\Gamma_{\mathbb{F}_p}\Big(\sum_{j=n-l}^n M_j - lp+1\Big)} \, .$$

Hence

$$\det C(\mathrm{id}) = (-1)^{n(n-1)/2+\sum_{j=1}^{n-1}(n-j)M_j}\ \frac{\Gamma_{\mathbb{F}_p}(M_1+1)\cdots\Gamma_{\mathbb{F}_p}(M_n+1)}{\Gamma_{\mathbb{F}_p}(M_1+\dots+M_n-(n-1)p+1)}.$$

These formulas imply the lemma.

7.4. Differential equations for $\det c(z)$**.**

Lemma 7.4. *The polynomial* $\det c(z)$ *satisfies the system of scalar differential equations*

$$\frac{\partial y}{\partial z_i} = \sum_{j\neq i}\frac{M_i+M_j}{z_i-z_j}\,y, \qquad i=1,\dots,n\,. \tag{7.10}$$

Proof. The operators Ω^M_{ij} defined in (5.3) preserve $\mathrm{Sing}V[-2]$ and

$$\mathrm{Tr}\big|_{\mathrm{Sing}[-2]}\Omega^M_{ij} = M_i + M_j. \tag{7.11}$$

The polynomial $\det c(z)$, as the determinant of solutions, satisfies the system of equations (7.12)

$$\frac{\partial \det c(z)}{\partial z_i} = \sum_{j\neq i}\frac{\mathrm{Tr}\,\big|_{\mathrm{Sing}[-2]}\Omega^M_{ij}}{z_i-z_j}\ \det c(z), \qquad i=1,\dots,n\,.$$

This proves the lemma. Lemma 7.5. *The polynomial*

$$y_0(z) = \mathrm{Y}\,(z_i - z_j)_{M_i+M_j-p}$$

$$16i<j6n$$

is a solution of system (7.12)*.*

Lemma 7.6. *For any* i 6= j*, the polynomial* $\det c(z)$ *is divisible by* $(z_i - z_j)^{M_i+M_j-p}$*.*

Proof. We will prove that $\det c(z)$ is divisible by $(z_{n-1} - z_n)^{M_{n-1}+M_n-p}$. The divisibility by $(z_i - z_j)^{M_i+M_j-p}$ for other i 6= j is proved by reordering variables.

Introduce new variables $u_1,\dots,u_n$ by the equaions: $u_n = z_1 + \dots + z_n$,

$$z_{j+1} - z_j = u_1u_2\dots u_j, \qquad j = 1,\dots,n-1. \tag{7.13}$$

The variables $z_1,\dots,z_n$ are polynomials in $u_1,\dots,u_n$. Hence if $I(z)$ is a polynomial solution of system (2.1), then $I(z(u))$ is a polynomial solution of the transformed differential KZ equations, which take the form:

$$\frac{\partial I}{\partial u_i} = \Big(\frac{\sum_{j>i}\Omega^M_{ij}}{u_i} + \mathrm{Reg}^i(u)\Big)I, \quad i=1,\dots,n-1, \qquad \frac{\partial I}{\partial u_n}=0, \tag{7.14}$$

where $\mathrm{Reg}_i(u)$ is an operator-valued function of u regular at the point $u_1 = \dots = u_n = 0$. See this statement in [V7, Proposition 2.2.3]. In particular, we have

$$\frac{\partial I}{\partial u_{n-1}} = \Big(\frac{\Omega^M_{n-1,n}}{u_{n-1}} + \mathrm{Reg}^{n-1}\Big)I. \tag{7.15}$$

Hence the polynomial $\det c(z(u))$ a solution of the system of differential equations;

$$\frac{\partial y}{\partial u_i} = \Big(\frac{\sum_{j>i}(M_i+M_j)}{u_i} + \mathrm{Tr}\mathrm{Reg}^i(u)\Big)y, \quad i=1,\dots,n-1, \qquad \frac{\partial y}{\partial u_n}=0, \tag{7.16}$$

cf. formulas (7.11), (7.12). In particular, we have

$$\frac{\partial I}{\partial u_{n-1}} = \Big(\frac{M_{n-1}+M_n}{u_{n-1}} + \mathrm{Reg}^{n-1}\Big)I. \tag{7.17}$$

Hence the polynomial $\det c(z(u))$ is divisible by a monomial u^d_{n-1}, where d is a nonnegative solution of the congruence

$$d \equiv M_{n-1} + M_n \pmod p.$$

The number $M_{n-1}+M_n-p$ is the smallest nonnegative solution. Hence $\det c(z(u))$ is divisible by $u_{n-1}^{M_{n-1}+M_n-p}$, that is,

$$\det c(z(u)) \quad = \quad F(u_1,\dots,u_n)\, u_{n-1}^{M_{n-1}+M_n-p},$$

where $F(u_1,\dots,u_n)$ is a polynomial. Thus

$$\det c(z) \quad = \quad F\Big(z_2-z_1, \frac{z_3-z_2}{z_2-z_1},\dots,\frac{z_n-z_{n-1}}{z_{n-1}-z_{n-2}}, z_1+\dots+z_n\Big)\Big(\frac{z_n-z_{n-1}}{z_{n-1}-z_{n-2}}\Big)^{M_{n-1}+M_n-p}.$$

Hence $\det c(z)$ is divisible $(z_n - z_{n-1})^{M_{n-1}+M_n-p}$. The lemma is proved.

Proof of Theorem 7.2. The theorem follows from Lemmas 7.6 and 7.3.

7.5. Remark on the initial value problem.

Corollary 7.7. *Assume that system (2.1) has ample reduction for a prime p. For $x = (x_1,\dots,x_n) \in \mathbb{F}_p^n$ with distinct coordinates and $w \in \mathrm{Sing}V[-2]$, there exist a unique vector $(c_1,\dots,c_{n-1}) \in \mathbb{F}_p^{n-1}$ such that*

$$w = \sum_{l=1}^{n-1} c_l I^{[l]}(x). \tag{7.18}$$

Denote $(\mathbb{F}_p^n)^o = \{x \in \mathbb{F}_p^n \mid x \text{ has distinct coordinates}\}$. We have an isomorphism of the two trivial bundles

$$\mathrm{Sing}V[-2] \times (\mathbb{F}_p^n)^o \to (\mathbb{F}_p^n)^o \quad \text{and} \quad \mathbb{F}_p^{n-1} \times (\mathbb{F}_p^n)^o \to (\mathbb{F}_p^n)^o,$$

which sends (w,x) to $((c_1,\dots,c_{n-1}),x)$.

8. Properties of $\mathbb{F}_p$-hypergeometric solutions

In this section we add more properties of $\mathbb{F}_p$-hypergeometric solutions.

8.1. Uniqueness property. Given $l = 1,\dots,r$, the $\mathbb{F}_p$-hypergeometric solution $I^{[l]}(z)$ has degree $\sum_{j=1}^n M_j - lp$ and id-leading term

$$(-1)^{M_{i(l)}+1}\frac{\Gamma_{\mathbb{F}_p}(M_{i(l)}+1)\,\Gamma_{\mathbb{F}_p}\Big(\sum_{j=i(l)+1}^n M_j-(l-1)p+1\Big)}{\Gamma_{\mathbb{F}_p}\Big(\sum_{j=i(l)}^n M_j - lp+1\Big)} \times\Big(\frac{f^{(i(l))}v}{M_{i(l)}} - \frac{\sum_{j=i(l)+1}^n f^{(j)}v}{\sum_{j=i(l)+1}^n M_j}\Big) z_1^{M_1}z_2^{M_2}\dots z_{i(l)-1}^{M_{i(l)-1}} z_{i(l)}^{\sum_{j=i(l)}^n M_j - lp}, \tag{8.1}$$

where the number $i(l)$ is defined in Theorem 6.1.

Theorem 8.1. *If $I(z)$ is a homogeneous polynomial solution of system (2.1) with* id*-leading term (8.1), then $I(z) = I^{[l]}(z)$.*

Proof. By Theorem 5.3 the id-leading term of the difference $I(z) - I^{[l]}(z)$ has the form

$$(0,\dots,0,C_k,\dots,C_n)z_1^{a_1}\dots z_{k-1}^{a_{k-1}}z_k^{a_k}, \tag{8.2}$$

for some k, where $C_k \neq 0$ and

$$a_j \equiv M_j \pmod p, \qquad j = 1,\dots,k-1, \qquad a_k \equiv \sum_{j=k}^n M_j \pmod p, \qquad a_k \not\equiv M_k, \tag{8.3}$$

(8.4) $$\sum_{j=1}^{k} a_j = \sum_{j=1}^{n} M_j - lp,$$

(8.5) $$z_1^{M_1} z_2^{M_2} \dots z_{i(l)-1}^{M_{i(l)-1}} z_{i(l)}^{\sum_{j=i(l)}^{n} M_j - lp} >_{\mathrm{id}} z_1^{a_1} \dots z_{k-1}^{a_{k-1}} z_k^{a_k}.$$

The inequality $k > i(l)$ is impossible due to (8.3), (6.1). The inequality $k < i(l)$ is impossible due to (8.3), (8.5).

8.2. ***L*-admissible solutions and filtration on space of all polynomial solutions.** Let $L = (L_1, \dots, L_n) \in \mathbb{Z}^n_{>0}$. Let $I(z)$ be a polynomial in z with coefficients in $\mathbb{F}_p^n$. We say that $I(z)$ is *L-admissible* if

(8.6) $$\frac{\partial^{L_j+1} I}{\partial z_i^{L_j+1}}(z) = 0, \qquad j = 1, \dots, n\,.$$

Denote by $\mathcal{M}_L$ the $\mathbb{F}_p[z^p]$-module of all L-admissible polynomial solutions of (2.1).

For example, $\mathcal{M}_{(0,\dots,0)}$ consists of polynomial solutions $I(z) = (I_1(z), \dots, I_n(z))$ lying in $(\mathbb{F}_p[z^p])^n$, in other words, it consists of all $I(z)$, such that

$$\sum_{j \neq i} \frac{\Omega_{ij}}{z_i - z_j} I(z) = 0\,, \quad i = 1, \dots, n, \qquad m_1 I_1(z) + \dots + m_n I_n(z) = 0\,.$$

In particular, if system (2.1) has constant solutions then they lie in $\mathcal{M}_{(0,\dots,0)}$.

The modules $\mathcal{M}_L$ form a filtration on the module of all polynomial solutions of system (2.1). Namely, if $L = (L_1, \dots, L_n)$, $L' = (L'_1, \dots, L'_n)$ and $L_j \leq L'_j$ for all j, then $\mathcal{M}_L \subset \mathcal{M}_{L'}$.

Theorem 8.2. *Let $L = (M_1 + 1, \dots, M_n + 1)$. Then $\mathcal{M}_L$ coincides with the module $\mathcal{M}$ of $\mathbb{F}_p$-hypergeometric solutions of system* (2.1).

Proof. Any $\mathbb{F}_p$-hypergeometric solution $I^{[l]}(z)$ lies in $\mathcal{M}_L$ by construction. We show that any element $I(z) \in \mathcal{M}_L$ is a linear combination of the $\mathbb{F}_p$-hypergeometric solutions with coefficients in $\mathbb{F}_p[z^p]$. Since system (2.1) is homogeneous, it is enough to assume that $I(z)$ is a homogeneous polynomial.

Assume that $I(z)$ is a homogeneous polynomial. Let $(0, \dots, 0, C_i, \dots, C_n) z_1^{d_1} \dots z_n^{d_n}$ be the id-leading term of $I(z)$, where $1 \leq i < n$, $C_i \neq 0$. Divide each d_j by p with remainder,

$$d_j = q_j p + r_j, \qquad 0 \leq r_j < p, \qquad j = 1, \dots, n.$$

Then $r_j \leq M_j$, $j = 1, \dots, n$, since $I(z) \in \mathcal{M}_L$, and $(r_1, \dots, r_n)$ has the form $(M_1, \dots, M_{i-1}, r_i, 0, \dots, 0)$, $r_i \neq M_i$, by Theorem 5.3. We have

$$r_i = \sum_{j=i}^{n} M_j - lp$$

for some positive integer l, by Corollary 5.4.

Consider the $\mathbb{F}_p$-hypergeometric solution $I^{[l]}(z)$. By Theorem 5.3 the id-leading term of $I^{[l]}(z)$ is

$$(0, \dots, 0, C_i^l, \dots, C_n^l)\, z_1^{M_1} \dots z_{i-1}^{M_{i-1}} z_i^{r_i},$$

where

(8.7) $$(0, \dots, 0, C_i, \dots, C_n) = c\,(0, \dots, 0, C_i^l, \dots, C_n^l).$$

for some $c \in \mathbb{F}_p^\times$. Both $I(z)$ and $cz_1^{q_1p}\dots z_n^{q_np}I^{[l]}(z)$ belong to M_L and have the same leading term. Hence the leading monomial of the difference $I(z)-cz_1^{q_1p}\dots z_n^{q_np}I^{[l]}(z)$ is lexicographically smaller than the leading monomial $z_1^{d_1}\dots z_n^{d_n}$ of $I(z)$. Notice that the difference is also a homogeneous polynomial.

Iterating this procedure, which decreases the leading monomial, we present $I(z)$ as a linear combination of the $\mathbb{F}_p$-hypergeometric solutions with coefficients in $\mathbb{F}_p[z^p]$.

8.3. Ample reduction.

Theorem 8.3. *Assume that system* (2.1) *has ample reduction for a prime p. Then any polynomial solution $I(z)$ of system* (2.1) *belongs to the module of $\mathbb{F}_p$-hypergeometric solutions.*

Proof. System (2.1) is homogeneous. Hence it is enough to prove the theorem assuming that $I(z)$ is a homogeneous polynomial solution.

Let $I_{\mathrm{id}}(z) = Cz_1^{d_1}\dots z_n^{d_n}$ be the id-leading term of $I(z)$. By Corollary 5.2 the id-leading term has the form described in Theorem 5.3. The ampleness of the reduction implies there exists an $\mathbb{F}_p$-hypergeometric solution $I^{[l]}(z)$, whose id-leading $I_{\mathrm{id}}^{[l]}(z)$ has the property

$$I_{\mathrm{id}}(z) = cz_1^{a_1}\dots z_n^{a_n}I_{\mathrm{id}}^{[l]}(z),$$

where $c \in \mathbb{F}_p$ and $a_1,\dots,a_n \in p\mathbb{Z}_{>0}$. The difference $I(z) - cz_1^{a_1}\dots z_n^{a_n}I^{[l]}(z)$ is a homogeneous polynomial solution of system (2.1) with id-leading term $>_{\mathrm{id}}$-smaller than the id-leading term of $I(z)$. Iteration of this procedure implies the theorem.

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