FROBENIUS-LIKE STRUCTURE IN GAUDIN MODEL

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Abstract. We introduce a Frobenius-like structure for the $\mathfrak{sl}_2$ Gaudin model. Namely, we introduce potential functions of the first and second kind. We describe the Shapovalov form in terms of derivatives of the potential of the first kind and the action of Gaudin Hamiltonians in terms of derivatives of the potential of the second kind.

Key words: Gaudin Hamiltonians; Shapovalov form; potentials of the first and second kind. *2020*

Mathematics Subject Classification: 81R22 (53D45, 32S22)

1. Introduction

Frobenius manifolds were introduced by B. Dubrovin in the study of topological field theories, [D1]. Frobenius manifolds is an important ingredient of the theory of integrable systems.

A Frobenius algebra is a commutative algebra A with a nondegenerate bilinear form $(,)$ such that $(uv,w) = (v,uw), \forall u,v,w \in A$.

Roughly speaking, a Frobenius manifold is a manifold M with a flat metric $(,)$ and a Frobenius algebra structure on tangent spaces T_xM at points $x \in M$ such that the structure constants of multiplication are given by the third derivatives of a potential function with respect to flat coordinates. More precisely, let $z_1,...,z_n$ be local coordinates on M in which the metric is constant, then

$$\left(\frac{\partial}{\partial z_i}\cdot\frac{\partial}{\partial z_j},\ \frac{\partial}{\partial z_k}\right)_x \ =\ \frac{\partial^3 L}{\partial z_i \partial z_j \partial z_k}(x) \tag{1.1}$$

for a suitable potential function L on the manifold. Formula (1.1) is a remarkable way to pack all information about this family of Frobenius algebras into one function.

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A source of families of Frobenius algebras is quantum cohomology algebras of algebraic varieties. Algebras of such a family depend on quantum parameters and form a Frobenius manifold, in which the bilinear form (,) is the intersection form on the corresponding variety and the potential function is defined in terms of enumerative geometry of curves on the variety, see [D1].

Another source of families of Frobenius algebras is quantum integrable models related to representation theory. In this case, one starts with a tensor product of evaluation representations of some algebra (like the universal enveloping algebra of a current algebra, or a Yangian, or a quantum affine algebra), which has a large commutative subalgebra called the Bethe subalgebra. Then the representation itself depends on the corresponding evaluation parameters, while the image of the Bethe subalgebra in the representation is often a Frobenius algebra with respect to the corresponding Shapovalov form, see for example [MTV1, L]. In this note we discuss the problem, posed in [PV], if this family of Frobenius algebras depending on evaluation parameters has glimpses of a Frobenius structure.

We study the simplest example of the $\mathfrak{sl}_2$ Gaudin model on a tensor product of vector representations. The Bethe algebra acts in the space $(\mathbb{C}^2)^{\otimes n}$ by operators depending on evaluation parameters z_m, $m = 1,...,n$. The operators are symmetric with respect to the Shapovalov form and commute with the diagonal action of $\mathfrak{sl}_2$. The Bethe algebra is generated by the Gaudin Hamiltonians $H_m(z_1,...,z_n)$, $m = 1,...,n,$ and the identity operator. We concentrate on the space $\mathrm{Sing}(\mathbb{C}^2)^{\otimes n}[n-2k]$ of singular vectors of weight $n-2k$, which is invariant with respect to the Bethe algebra. This space is known to be cyclic with respect to the action of Gaudin Hamiltonians, and the algebra of Gaudin Hamiltonians is Frobenius, see [MTV1].

For $k > 1$, one cannot expect formula (1.1) to be literally satisfied, as the number of evaluation parameters in our family of algebras is smaller than the dimension of the algebra, but it turns out that there is an analogous formula which looks as follows.

We have a natural spanning set of vectors $\{v_I\} \subset \mathrm{Sing}(\mathbb{C}^2)^{\otimes n}[n-2k]$ labeled by k-element subsets $I \subset \{1,...,n\}$. The vectors $\{v_I\}$ are orthogonal projections of the standard tensor basis of $(\mathbb{C}^2)^{\otimes n}[n-2k]$. We present two potential functions P and Q depending on nk variables $z_i^{(j)}$, $i = 1, \dots, n$, $j = 1, \dots, k$. We also introduce differential operators ∂_I of order k, which involve derivatives with respect to the variables $z_i^{(j)}$, $i \in I$. Then the following holds.

- The function P is a polynomial of degree $2k$ written as a sum, where up to a common constant, each term is a product of k factors of the form $(z_i^{(j)} - z_l^{(j)})^2$, see (4.2). It has the property

$$(v_I, v_J) = \partial_I \partial_J P \qquad \forall I, J. \tag{1.2}$$

- The function Q is a sum of terms of the form $\ln(z_i^{(1)} - z_s^{(1)})\,(z_i^{(1)} - z_l^{(1)})^2\, p$, where $(z_i^{(1)} - z_l^{(1)})^2\, p$ is a term of P, see (4.6). It has the property

$$(1.3)\qquad (H_m(z_1^{(1)},\dots,z_n^{(1)})v_I,v_J)=\frac{\partial}{\partial z_m^{(1)}}\partial_I\partial_J Q \qquad \forall I,J \text{ and } m=1,\dots,n.$$

We call the functions P and Q the potentials of the first and second kind, respectively. Formula (1.2) describes the Shapovalov form in terms of derivatives of the potential of the first kind, and formula (1.3) describes the action of Gaudin Hamiltonians in terms of derivatives of the potential of the second kind.

The existence of a polynomial P satisfying (1.2) is obvious, but the form of the answer seems to be interesting. The reason for existence of Q is not clear. It looks interesting that the potential of the first kind describing only the Shapovalov form is so closely related to the potential of the second kind which describes the action of the Gaudin Hamiltonians.

Our construction is motivated by [V3, PV], where a Frobenius-like structure was introduced for a family of weighted hyperplane arrangements, in which every hyperplane independently moves parallelly to itself, see Theorem 2.1 below. It is well-known that the Gaudin model on Sing$W^{\otimes n}[n-2k]$ is related to a certain family of weighted discriminantal hyperplane arrangements in $\mathbb{C}^k$ with hyperplanes depending on n parameters $z_1,\dots,z_n$, see [SV, V1, V2, TV]. These arrangements have $nk+k(k-1)/2$ hyperplanes and are symmetric with respect to permutations of coordinates in $\mathbb{C}^k$. While this family of arrangements does not satisfy the assumptions of Theorem 2.1, we get enough insight to construct the potentials P and Q.

We expect that potentials of the first and second kind exist for spaces of singular vectors in tensor products of $\mathfrak{sl}_N$ vector representations, where the Bethe algebra is still generated by Gaudin Hamiltonians, see [MTV2].

In Section 2 we recall Frobenius-like structures related to arrangements of hyperplanes. In Section 3 we collect preliminary information. In Section 4 we introduce potentials and relate them to the Gaudin model.

The authors thank the referee for the remark in Section 4.5.

2. Crtitical points and arrangements of hyperplanes

Algebras of functions on critical sets of functions produce families of Frobenius algebras as follows. Let $\Phi(t_1,\dots,t_k)$ be a holomorphic function on an open set $D\subset\mathbb{C}^k$ with finitely many critical points $q\in D$,

$$\frac{\partial\Phi}{\partial t_i}(q)=0,\quad i=1,\dots,k.$$

One considers the finite-dimensional algebra of functions on the critical set,

$$A=\mathcal{O}(D)\Big/\Big(\frac{\partial\Phi}{\partial t_j}\,\Big|\, j=1,\dots,k\Big).$$

The Grothendieck residue defines a nondegenerate bilinear form $(\,,\,)$ on A,

$$([f],[g]) = \frac{1}{(2\pi i)^k}\int_\Gamma \frac{fg\ dt_1\wedge\cdots\wedge dt_k}{\prod_{j=1}^k \frac{\partial\Phi}{\partial t_j}}, \quad [f],[g]\in A,$$

where $\Gamma = \{t\in D \mid \ |\frac{\partial\Phi}{\partial t_j}| = \epsilon, j=1,\dots,k\}$. The algebra $(A,(,))$ is a Frobenius algebra. In singularity theory this algebra is called the Milnor algebra.

This algebra of functions is especially interesting in the case when the starting function Φ is the master function of an arrangement of hyperplanes. On the one hand, the arrangements of hyperplanes lead to a simpler, more combinatorial setting. On the other hand, such algebras are known to be related to the algebras coming from quantum integrable systems and quantum cohomology, see for example [SV, V1, V2, MTV1, GRTV].

It turns out that the algebra of functions on the critical set of a master function of an arrangement has a Frobenius-like structure, which is determined by two potentials.

Consider $\mathbb{C}^k$ with coordinates $t_1,\dots,t_k$ and an arrangement $\mathcal{C}(z)$ of n hyperplanes in $\mathbb{C}^k$ depending on parameters $z=(z_1,\dots,z_n)$. The hyperplanes $H_i(z_i)$ of the arrangement are defined by equations, $f_i(t,z)=\sum_{j=1}^k b_i^j t_j + z_i = 0$, where $b_i^j\in\mathbb{C}$ are fixed. If z_i changes, the hyperplane $H_i(z_i)$ moves parallelly to itself.

Fix positive numbers $a=(a_1,\dots,a_n)$ called the weights. The master function of the weighted arrangement $(\mathcal{C}(z),a)$ is the function

$$\Phi(t,z) = \sum_{i=1}^n a_i \log f_i(t,z).$$

Denote $U_z = \mathbb{C}^k - \mathcal{C}(z)$ the complement to $\mathcal{C}(z)$ and $\mathcal{O}(U_z)$ the algebra of regular functions on the complement U_z. Denote

$$A_z = \mathcal{O}(U_z)\Big/\Big(\frac{\partial\Phi}{\partial t_j}\ \Big|\ j=1,\dots,k\Big)$$

the algebra of functions on the critical set of the master function $\Phi(\cdot,z)$ restricted to the complement U_z. Denote $(,)_z$ the Grothiendick residue form on A_z.

In the space $\mathbb{C}^n$ of parameters z there is a hypersurface Σ, called the discriminant, characterized by the property: if $z\in\mathbb{C}^n-\Sigma$, then the arrangement $\mathcal{C}(z)$ has normal crossings only. We may compare the algebras A_z for $z\in\mathbb{C}^n-\Sigma$ as follows.

For $i=1,\dots,n$ the elements $p_i = \lfloor\frac{\partial\Phi}{\partial z_i}\rfloor = \lfloor\frac{a_i}{f_i}\rfloor\in A_z$ generate A_z as an algebra. We say that a subset $\{i_1,\dots,i_k\}\subset\{1,\dots,n\}$ is independent if $d_{i_1,\dots,i_k} = \det(b_{i_\ell}^j)_{\ell,j=1}^k \neq 0$. For $z\in\mathbb{C}^n-\Sigma$ the elements $d_{i_1,\dots,i_k}p_{i_1}\cdots p_{i_k}$ span A_z as a vector space.

The defining linear relations between the elements $d_{i_1,\dots,i_k}p_{i_1}\cdots p_{i_k}$ are labeled by $(k-1)$element subsets $\{i_1,\dots,i_{k-1}\}\subset\{1,\dots,n\}$,

$$\sum_{m=1}^n d_{i_1,\dots,i_{k-1},m}\, p_m\cdot p_{i_1}\cdots p_{i_{k-1}} = 0.$$

Hence the dimension $\dim_{\mathbb{C}} A_z$ does not depend on $z \in \mathbb{C}^n - \Sigma$. Consider the complex vector bundle

$$A = \cup_{z\in\mathbb{C}^n-\Sigma} A_z \to \mathbb{C}^n - \Sigma$$

whose fiber over a point $z \in \mathbb{C}^n - \Sigma$ is the Frobenius algebra A_z. Identifying the elements $d_{i_1,\dots,i_k} p_{i_1} \dots p_{i_k}$ in all fibers we trivialize the bundle.

Theorem 2.1 ([V3, PV]). *There exist two functions P, Q on $\mathbb{C}^n - \Sigma$, called the potentials of the first and second kind, with the following properties. For any two independent sets $\{i_1,\dots,i_k\}$,$\{j_1,\dots,j_k\}$ and any index $m = 1,\dots,n$, we have*

$$(p_{i_1}\cdots p_{i_k}, p_{j_1}\cdots p_{j_k})_z = \frac{\partial^{2k} P}{\partial z_{i_1}\dots\partial z_{i_k}\partial z_{j_1}\dots\partial z_{j_k}}(z),$$

$$(p_m \cdot p_{i_1}\cdots p_{i_k}, p_{j_1}\cdots p_{j_k})_z = \frac{\partial^{2k+1} Q}{\partial z_m \partial z_{i_1}\dots\partial z_{i_k}\partial z_{j_1}\dots\partial z_{j_k}}(z).$$

The potentials are given by some combinatorial formulas.

The potential P of the first kind is a polynomial of degree $2k$ and hence all $(p_{i_1}\cdots p_{i_k}, p_{j_1}\cdots p_{j_k})_z$ are constants.

The first formula determines the Grothendieck residue bilinear form $(,)_z$ in terms of the potential of the first kind. The second formula determines the operators of multiplication by multiplicative generators $\{p_j\}$, $j = 1,\dots,n$, in terms of the potential of the second kind.

This pair of potentials is called in [V3, PV] a Frobenius-like structure associated with the family $(\mathcal{C}(z),a)$ of weighted arrangements in $\mathbb{C}^k$.

Example 2.2 ([V3]). *For $k = 1$ consider the arrangement of n points on line defined by equations $t + z_i = 0$, $i = 1,\dots,n$, with weights $a_1,\dots,a_n$. Then*

$$P(z) = \tfrac{1}{a_1+\dots+a_n} \sum_{1\leqslant i<j\leqslant n} a_i a_j \tfrac{(z_i-z_j)^2}{2}, \qquad Q(z) = \sum_{1\leqslant i<j\leqslant n} a_i a_j \ln(z_i - z_j)\, \tfrac{(z_i-z_j)^2}{2}$$

$$(p_i, p_j)_z = \tfrac{\partial^2 P}{\partial z_i \partial z_j}(z), \qquad (p_m \cdot p_i,\, p_j)_z = \tfrac{\partial^3 P}{\partial z_h \partial z_i \partial z_j}(z).$$

If $a_1 = \dots = a_n$, then this Frobenius-like structure is the almost dual Frobenius structure associated with the Weyl group $W(A_{n-1})$ in [D2].

Example 2.3 ([PV]). *For the arrangement of four lines on plane given by equations $t_2+z_1 = 0$, $t_2 + z_2 = 0$, $t_1 + z_3 = 0$, $t_1 + t_2 + z_4 = 0$ we have*

$$P = \frac{1}{a_1+a_2+a_3+a_4}\Big(a_1a_3a_4\frac{(z_1+z_3-z_4)^4}{4!} + a_2a_3a_4\frac{(z_2+z_3-z_4)^4}{4!} + \frac{a_1a_2a_3a_4}{a_3+a_4}\frac{(z_1-z_2)^2}{2!}\frac{(z_1+z_3-z_4)^2}{2!}\Big),$$

$$Q = a_1a_3a_4 \ln(z_1 + z_3 - z_4)\frac{(z_1 + z_3 - z_4)^4}{4!} + a_2a_3a_4 \ln(z_2 + z_3 - z_4)\frac{(z_2 + z_3 - z_4)^4}{4!} + \frac{a_1a_2a_3a_4}{a_3 + a_4} \ln(z_1 - z_2)\frac{(z_1 - z_2)^2}{2!}\frac{(z_1 + z_3 - z_4)^2}{2!}.$$

Theorem 2.1 in particular says that $(p_1p_3, p_2p_4)_z = \frac{a_1a_2a_3a_4}{(a_1+a_2+a_3+a_4)(a_3+a_4)}$ *and this does not depend on* $z \in \mathbb{C}^n - \Delta$, *and* $(p_4p_1p_3, p_3p_4)_z = \frac{a_1a_3a_4}{z_1+z_3-z_4}$.

In this example the potentials are sums of terms corresponding to subarrangements consisting of three or four lines, corresponding to triangles and trapezoids. It turns out that this is the general case. One introduces the notion of an elementary arrangement in $\mathbb{C}^k$. The elementary subarrangements in $\mathbb{C}^2$ are triangles and trapezoids. An elementary arrangement in $\mathbb{C}^k$ has at most $2k$ hyperplanes. The potentials are sums, over all elementary subarrangements, of some explicit prepotentials of the elementary subarrangements, see [V3, PV]

The fact that the potentials are sums of contributions from elementary subarrangements indicates a phenomenon of *locality* of Grothendieck residue bilinear form and multiplication on the algebra A_z. We observe a similar locality property in the Gaudin model potentials, see formulas (4.2) and (4.6).

On Frobenius-like structures see also [HV, V4].

3. Shapovalov form

Let n,k be positive integers.

3.1. **Space of singular vectors.** Consider the complex Lie algebra $\mathfrak{sl}_2$ with generators e,f,h and relations $[e,f] = h$, $[h,e] = 2e$, $[h,f] = -2f$. Consider the complex vector space W with basis w_1,w_2 and the $\mathfrak{sl}_2$-action,

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The $\mathfrak{sl}_2$-module $W^{\otimes n}$ has a basis labeled by subsets $I \subset \{1,...,n\}$,

$$V_I = w_{i_1} \otimes \cdots \otimes w_{i_n},$$

where $i_j = 1$ if $i_j \notin I$ and $i_j = 2$ if $i_j \in I$.

The Shapovalov form $(,)$ on $W^{\otimes n}$ is the symmetric bilinear form such that $(V_I,V_J) = \delta_{IJ}$. It has the properties: $(hx,y) = (x,hy)$, $(ex,y) = (x,fy)$ for all $x,y \in W$.

Consider the weight decomposition of $W^{\otimes n}$ into eigenspaces of h, $W^{\otimes n} = \sum_{k=0}^n W^{\otimes n}[n - 2k]$. The vectors V_I with $|I| = k$ form a basis of $W^{\otimes n}[n - 2k]$. Define the space of singular vectors of weight $n - 2k$,

$$\mathrm{Sing}W^{\otimes n}[n - 2k] = \{w \in W^{\otimes n}[n - 2k] \mid ew = 0\}.$$

This space is nonempty if and only if $n > 2k$. We assume that $n > 2k$.

3.2. **Orthogonal projection.** Let

$$\pi : W^{\otimes n}[n-2k] \to \mathrm{Sing}\,W^{\otimes n}[n-2k] \tag{3.1}$$

be the orthogonal projection with respect to the Shapovalov form. The kernel of the projection is the image of the operator

$$f : W^{\otimes n}[n-2(k-1)] \to W^{\otimes n}[n-2k].$$

For a k-element subset $I \subset \{1,\dots,n\}$ denote

$$v_I = \pi(V_I) \in \mathrm{Sing}\,W^{\otimes n}[n-2k]. \tag{3.2}$$

There are $\binom{n}{k}$ vectors v_I in the $\binom{n}{k} - \binom{n}{k-1}$-dimensional space $\mathrm{Sing}\,W^{\otimes n}[n-2k]$. The defining linear relations between these vectors are labeled by $(k-1)$-element subsets K and have the form

$$\sum_{m\notin K} v_{K\cup\{m\}} = 0.$$

Clearly the vector v_I has the form

$$v_I = V_I + b_{k-1}\sum_{k-1} fV_K + b_{k-2}\sum_{k-2} fV_K + \dots + b_0\sum_0 fV_K,$$

where b_ℓ are suitable numbers, and $\sum_\ell$ is the sum over all $(k-1)$-element subsets K such that $|I \cap K| = \ell$.

Lemma 3.1. *We have*

$$b_\ell = (-1)^{k-\ell}\,\frac{(k-\ell-1)!}{\prod_{i=\ell+1}^{k}(n-2k+1+i)} = (-1)^{k-\ell}\,\frac{(k-\ell-1)!(n-2k+\ell+1)!}{(n-k+1)!}, \tag{3.3}$$

$\ell = 0,\dots,k-1.$

Proof. The property $ev_I = 0$ produces the following system of equations:

$$1 + b_{k-1}n + b_{k-2}(k-1)(n-k) = 0,$$

$$b_{\ell+1}(k-\ell)(k-\ell-1) + b_\ell(k-\ell)(n-2k+2\ell+2) + b_{\ell-1}\ell(n-2k+\ell+1) = 0$$

for $\ell = 0,\dots,k-2$. This system implies

$$b_r = b_0\,(-1)^r \prod_{i=1}^{r} \frac{n-2k+1+i}{k-i}, \qquad b_0 = (-1)^k\,\frac{(k-1)!}{\prod_{i=1}^{k}(n-2k+1+i)},$$

and hence (3.3).

Clearly v_I has the form:

$$v_I = a_k V_I + a_{k-1} \sum_{k-1} V_J + a_{k-2} \sum_{k-2} V_J + \dots + a_0 \sum_0 V_J,$$

where a_ℓ are suitable numbers, and $\sum_\ell$ is the sum over all k-element subsets J such that $|I \cap J| = \ell$.

Lemma 3.2. *We have*

$$a_\ell = (-1)^{k+\ell} \frac{(n-2k+1)\,(k-\ell)!}{\prod_{i=\ell}^{k}(n-2k+1+i)} = (-1)^{k+\ell} \frac{(n-2k+1)\,(k-\ell)!\,(n-2k+\ell)!}{(n-k+1)!} \tag{3.4}$$

$\ell = 0, \dots, k.$

Proof. Let K be a $(k-1)$-element subset. The condition $(fV_K, v_I) = 0$ produces the following system of equations. If $|K \cap I| = \ell$, where $\ell = 0,1,\dots,k-1$, then

$$(k-\ell)a_{\ell+1} + (n-2k+\ell+1)a_\ell = 0.$$

This system has the solution

$$a_\ell = (-1)^\ell \frac{(n-2k+1)(n-2k+2)\dots(n-2k+\ell)}{k(k-1)\dots(k-\ell+1)}\, a_0 \tag{3.5}$$

It is easy to see that $a_0 = kb_0$. Thus

$$a_0 = (-1)^k \frac{k!}{\prod_{i=1}^{k}(n-2k+1+i)},$$

and formula (3.4) for any ℓ follows from formula (3.5). **Example 3.3.** *For $k = 1$ we have* $b_0 = -\frac{1}{n}$, $a_0 = -\frac{1}{n}$, $a_1 = \frac{n-1}{n}$. *For $k = 2$ we have*

$$b_0 = \frac{1}{(n-1)(n-2)}, \qquad b_1 = -\frac{1}{n-1},$$
$$a_0 = \frac{2}{(n-2)(n-1)}, \qquad a_1 = -\frac{n-3}{(n-2)(n-1)}, \qquad a_2 = \frac{n-3}{n-1}.$$

Lemma 3.4. *Let I,J be two k-element subsets such that $|I \cap J| = \ell$. Then*

$$(v_I, v_J) = (v_I, V_J) = a_\ell. \tag{3.6}$$

The following formulas are useful:

$$\begin{aligned} a_{\ell-1} - a_\ell &= (-1)^{k-\ell+1} \frac{(n-2k+1)\,(k-\ell)!}{\prod_{i=\ell-1}^{k-1}(n-2k+1+i)} \\ &= (-1)^{k-\ell+1} \frac{(n-2k+1)\,(k-\ell)!\,(n-2k+\ell-1)!}{(n-k)!} \end{aligned} \tag{3.7}$$

$\ell = 1,\dots,k.$

3.3. **Remark.** One may show that the orthogonal projection $\pi : W^{\otimes n}[n-2k] \to \mathrm{Sing}\, W^{\otimes n}[n-2k]$ has the following locality property with respect to the number n of tensor factors.

Consider the orthogonal projection $\pi_{2k} : W^{\otimes 2k}[0] \to \mathrm{Sing}\, W^{\otimes 2k}[0]$ and the image $\pi_{2k} V^{2k}_{\{1,\dots,k\}}$ of the vector

$$V^{2k}_{\{1,\dots,k\}} = w_2 \otimes \cdots \otimes w_2 \otimes w_1 \otimes \cdots \otimes w_1 \in W^{\otimes 2k}[0].$$

For any k-element subset $J \subset \{k+1,\dots,n\}$, let $(\pi_{2k} V^{2k}_{\{1,\dots,k\}})^{(J)} \in \mathrm{Sing}\, W^{\otimes n}[n-2k]$ be the vector $\pi_{2k} V^{2k}_{\{1,\dots,k\}}$ placed in the tensor factors of $W^{\otimes n}$ with indices $\{1,\dots,k\} \cup J \subset \{1,\dots,n\}$, and with the vectors w_1 staying in other factors of $W^{\otimes n}$. Then the vector $v_{\{1,\dots,k\}} \in \mathrm{Sing}\, W^{\otimes n}[n-2k]$, defined in (3.2), equals the sum $\sum_J (\pi_{2k} V^{2k}_{\{1,\dots,k\}})^{(J)}$ up to a multiplicative constant.

For example, for $k = 1$ we have

$$v_{\{1\}} = \frac{n-1}{n} V_{\{1\}} - \sum_{j=2}^{n} \frac{1}{n} V_{\{j\}}, \qquad \pi_2 V^2_{\{1\}} = \frac{1}{2} w_2 \otimes w_1 - \frac{1}{2} w_1 \otimes w_2$$

$$v_{\{1\}} = \frac{2}{n} \sum_{j=2}^{n} \Big(\frac{1}{2} V_{\{1\}} - \frac{1}{2} V_{\{j\}} \Big).$$

The locality property of potential functions in formulas (4.2) and (4.6) is of similar flavor.

4. Potentials

4.1. **Potential of the first kind.** Potentials of the first and second kind are function of nk variables

$$z = (z^{(1)}_1, \dots, z^{(1)}_n; z^{(2)}_1, \dots, z^{(2)}_n; \dots; z^{(k)}_1, \dots, z^{(k)}_n).$$

For every k-element subset $I = (i_1,\dots,i_k) \subset \{1,\dots,n\}$ define the differential operator

$$\partial_I = \sum_{\sigma \in S_k} \frac{\partial^k}{\partial z^{(1)}_{i_{\sigma_1}} \dots \partial z^{(k)}_{i_{\sigma_k}}}.$$

Recall that $n > 2k$. Let) be $\alpha = (\{p_1, q_1\}, \dots, \{p_k, q_k\})$ a sequence of nonintersecting unordered two-element subsets of $\{1,\dots,n\}$. Denote by A the set of all such sequences. The number of elements in A is given by the formula:

$$|\mathcal{A}| = \binom{n}{2}\binom{n-2}{2}\cdots\binom{n-2k+2}{2} = \frac{n!}{2^k (n-2k)!}.$$

For every $\alpha = (\{p_1,q_1\},\dots,\{p_k,q_k\}) \in$ A define

$$P_\alpha(z) = \prod_{i=1}^{k} (z_{p(i_i)} - z_{q(i_i)})^2. \tag{4.1}$$

Define the *potential of the first kind* $P(z)$ by the formula

$$(4.2)\quad P(z) = c_1 \sum_{\alpha\in\mathcal{A}} P_\alpha(z)\,, \qquad c_1 = \frac{1}{2^k\,k!\,\prod_{i=1}^k (n-2k+1+i)} = \frac{(n-2k+1)!}{2^k\,k!\,(n-k+1)!}\,.$$

Theorem 4.1. *For any two k-element subsets I and J we have*

$$(v_I, v_J) = \partial_I \partial_J P(z). \tag{4.3}$$

Proof. It is enough to prove this formula for $I = \{1,2,\dots,k\}$ and $J = \{1,2,\dots,\ell,k+1,k+2,\dots,2k-\ell\}$, where $\ell = 0,1,\dots,k$. It is easy to see that in this case

$$\partial_I\partial_J P \;=\; k!\,\frac{\partial^k}{\partial z_1^{(1)}\partial z_2^{(2)}\dots\partial z_k^{(k)}}\cdot \partial_J P\,.$$

The number $\frac{\partial^k}{\partial z_1^{(1)}\partial z_2^{(2)}\dots\partial z_k^{(k)}}\cdot \partial_J P_\alpha$ is nonzero only if α has the following form,

$$\alpha = (\{1,q_1\},\{2,q_2\},\dots,\{\ell,q_\ell\},\{\ell+1,\sigma_{k+1}\},\{\ell+2,\sigma_{k+2}\},\dots,\{k,\sigma_{2k-\ell}\}),$$

where $(q_1,q_2,\dots,q_\ell)$ is an ordered subset of $\{2k-\ell+1,2k-\ell+2,\dots,n\}$ and $(\sigma_{k+1},\sigma_{k+2},\dots,\sigma_{2k-\ell})$ is a permutation of $(k+1,\dots,2k-\ell)$.

There are $(n-2k+\ell)(n-2k+\ell-1)\dots(n-2k+1)(k-\ell)!$ such sequences α. For every such α, we have $\frac{\partial^k}{\partial z_1^{(1)}\partial z_2^{(2)}\dots\partial z_k^{(k)}}\cdot \partial_J P_\alpha = (-1)^{k-\ell}2^k$. To prove the theorem it is enough to check that

$$a_\ell = c_1\, k!(-1)^{k-\ell}\, 2^k\, (k-\ell)! \prod_{i=0}^{\ell-1}(n-2k+1+i).$$

This follows from formulas (3.4) and (4.2).

4.2. **Gaudin model.** Define the Casimir element

$$\Omega = \frac{1}{2} h\otimes h + e\otimes f + f\otimes e \quad\in\quad \mathfrak{sl}_2\otimes\mathfrak{sl}_2\,.$$

Define the linear operators on $W^{\otimes n}$ depending on parameters $u = (u_1,\dots,u_n)$,

$$H_m(u) = \sum_{\substack{j=1\\ j\neq m}}^{n} \frac{\Omega_{mj}}{u_m-u_j}\,, \qquad m = 1,\dots,n,$$

where $\Omega_{mj} : W^{\otimes n}\to W^{\otimes n}$ is the Casimir operator acting in the mth and jth tensor factors. The operators $H_m(u)$ are called the Gaudin Hamiltonians. The operators are symmetric and commute,

$$(H_m(u)x,y) = (x,H_m(u)y)\ \forall x,y\in W^{\otimes n}, \qquad [H_m(u),H_j(u)] = 0\ \forall\, m,j.$$

The operators commute with the $\mathfrak{sl}_2$-action on $W^{\otimes n}$ and hence preserve every $W^{\otimes n}[n-2k]$ and $\mathrm{Sing}\,W^{\otimes n}[n-2k]$. The operators commute with the orthogonal projection π defined in (3.1), $\pi H_m(u) = H_m(u)\pi$.

The algebra of endomorphisms of $\mathrm{Sing}W^{\otimes n}[n-2k]$ generated by the scalar operators and the Gaudin Hamiltonians is called the Bethe algebra of $\mathrm{Sing}W^{\otimes n}[n-2k]$.

Introduce the reduced Casimir element and reduced Gaudin Hamiltonians,

$$\bar\Omega = \frac{1}{2}h\otimes h + e\otimes f + f\otimes e - \frac{1}{2}, \qquad \bar H_m(u) = \sum_{\substack{j=1\\ j\neq m}}^{n} \frac{\bar\Omega_{mj}}{u_m-u_j}, \quad m=1,\dots,n.$$

The reduced Gaudin Hamiltonians are symmetric, commute, and generate together with scalar operators the same Bethe algebra as the Gaudin Hamiltonians.

Lemma 4.2. *Consider $\bar\Omega$ as a linear operator on $W^{\otimes 2}$. Then $\bar\Omega = P - 1$, where P is the permutation of tensor factors and* 1 *is the identity operator.*

Hence

$$(4.4)\qquad \bar H_m(u)V_I = \sum_{j\notin I}\frac{-V_I + V_{I\cup\{j\}-\{m\}}}{u_m-u_j}, \quad \text{if } m \in I,$$
$$\bar H_m(u)V_I = \sum_{j\in I}\frac{-V_I + V_{I\cup\{m\}-\{j\}}}{u_m-u_j}, \quad \text{if } m \notin I.$$

In (4.4) the vectors $V_I, V_{I\cup\{j\}-\{m\}}, V_{I\cup\{m\}-\{j\}}$ can be replaced with v_I, $v_{I\cup\{j\}-\{m\}}$, $v_{I\cup\{m\}-\{j\}}$ since the operators $\bar H_m(u)$ commute with the orthogonal projection π.

The reduced Gaudin Hamiltonians are governed by a potential function of the second kind.

4.3. **Potential of the second kind.** For every $\alpha = (\{p_1,q_1\},\dots,\{p_k,q_k\}) \in \mathcal{A}$ define

$$(4.5)\qquad Q_\alpha(z) = \ln(z_{p(1)_1} - z_{q(1)_1})\prod_{i=1}^{k}(z_{p(i_i)} - z_{q(i_i)})^2.$$

Define the *potential of the second kind* $Q(z)$ by the formula

$$(4.6)\qquad Q(z) = c_2\sum_{\alpha\in\mathcal{A}} Q_\alpha(z),$$
$$c_2 = \frac{-1}{2^k\,(k-1)!\,\prod_{i=1}^{k-1}(n-2k+1+i)} = \frac{-(n-2k+1)!}{2^k\,(k-1)!\,(n-k)!}.$$

The following theorem describes the action of the reduced Gaudin Hamiltonians on the space $\mathrm{Sing}W^{\otimes n}[n-2k]$ in terms of derivatives of the potential of the second kind.

Theorem 4.3. *Let I,J be two k-element subsets of* $\{1,\dots,n\}$ *and $m \in \{1,\dots,n\}$. Then*

$$(\bar H_m(z_1^{(1)},\dots,z_n^{(1)})v_I,v_J) \;=\; \frac{\partial}{\partial z_m^{(1)}}\partial_I\partial_J Q(z) \tag{4.7}$$

Proof. We prove the theorem by an explicit computation based on the simple fact that any derivative of order $2k$ of any term Q_α in (4.5) with respect to various variables $z_i^{(j)}$ equals a constant multiple of $\ln(z_{p_1}^{(1)} - z_{q_1}^{(1)})$ (which can be zero) plus a constant.

Assume that $m \notin I \cup J$ and $|I \cap J| = \ell$, $\ell = 0,\dots,k$. Under this assumption, without loss of generality we may assume that $I = \{1,\dots,k\}$, $J = \{1,\dots,\ell,k+1,\dots,2k-\ell\}$, and $m = n$. Then

$$\begin{aligned}(\bar H_n(z_1^{(1)},\dots,z_n^{(1)})v_I,v_J) \;&=\; \sum_{i=1}^{k}\frac{(-v_{1,\dots,k}+v_{1,\dots,\widehat{i},\dots,k,n},\,v_{1,\dots,\ell,k+1,\dots,2k-\ell})}{z_n^{(1)}-z_i^{(1)}}\\ &=\; \sum_{i=1}^{\ell}\frac{-a_\ell+a_{\ell-1}}{z_n^{(1)}-z_i^{(1)}}+\sum_{i=\ell+1}^{k}\frac{-a_\ell+a_\ell}{z_n^{(1)}-z_i^{(1)}} = \sum_{i=1}^{\ell}\frac{-a_\ell+a_{\ell-1}}{z_n^{(1)}-z_i^{(1)}}.\end{aligned}$$

If $I = \{1,\dots,k\}$, $J = \{1,\dots,\ell,k+1,\dots,2k-\ell\}$, $m = n$, then $\frac{\partial}{\partial z_n^{(1)}}\partial_{\{1,\dots,k\}}\partial_{\{1,\dots,\ell,k+1,\dots,2k-\ell\}}Q_\alpha(z)$ is nonzero only if α has the following form, $\alpha = (\{p_1,n\},\{p_2,q_2\},\dots,\{p_k,q_k\})$, where

(i) $p_1 \in \{1,\dots,\ell\}$;

(ii) the sequence $(p_2,\dots,p_k)$ is a permutation of the sequence $(1,\dots,\widehat{p_1},\dots,k)$ (the sequence $(p_2,\dots,p_k)$ determines a partition of $(2,\dots,k)$ into two subsequences: $(i_1 < \dots < i_{\ell-1})$ such that all $p_{i_r} \leq \ell$ and $(j_1 < \dots < j_{k-\ell})$ such that all $p_{j_r} > \ell$, these subsequences are used below);

(iii) the sequence $(q_{i_1},\dots,q_{i_{\ell-1}})$ is an ordered $(\ell-1)$-element subset of $\{2k-\ell+1,\dots,n-1\}$;

(iv) the sequence $(q_{j_1},\dots,q_{j_{k-\ell}})$ is a permutation of the sequence $(k+1,\dots,2k-\ell)$.

The number of such α with fixed p_1 equals $(k-1)!\,(k-\ell)!\,\prod_{i=0}^{\ell-2}(n-2k+1+i)$. For every such α,

$$\frac{\partial}{\partial z_n^{(1)}}\partial_{\{1,\dots,k\}}\partial_{\{1,\dots,\ell,k+1,\dots,2k-\ell\}}Q_\alpha(z) = \frac{(-1)^{k-\ell}2^k}{z_n^{(1)}-z_{p_1}^{(1)}}.$$

It follows from (3.4) and (4.6) that

$$\frac{a_{\ell-1}-a_\ell}{z_n^{(1)}-z_{p_1}^{(1)}} \;=\; \frac{(-1)^{k-\ell}2^k}{z_n^{(1)}-z_{p_1}^{(1)}}\,c_2\,(k-1)!\,(k-\ell)!\,\prod_{i=0}^{\ell-2}(n-2k+1+i), \tag{4.8}$$

and (4.7) holds in this case.

Assume that m belongs to I but not to J and $|I \cap J| = \ell - 1$, $\ell = 1,\dots,k$. Under this assumption, without loss of generality we may assume that $I = \{1,\dots,k\}$, $J = \{2,\dots,\ell,k+1,\dots,2k-\ell+1\}$, and $m = 1$. Then

$$(\bar{H}_1(z_1^{(1)},\dots,z_n^{(1)})v_I,v_J)=\sum_{i=k+1}^{n}\frac{(-v_{1,\dots,k}+v_{2,\dots,k,i},\,v_{2,\dots,\ell,k+1,\dots,2k-\ell+1})}{z_1^{(1)}-z_i^{(1)}}=\sum_{i=k+1}^{2k-\ell+1}\frac{-a_{\ell-1}+a_\ell}{z_1^{(1)}-z_i^{(1)}}.$$

If $I=\{1,\dots,k\}$, $J=\{2,\dots,\ell,k+1,\dots,2k-\ell+1\}$, $m=1$, then $\frac{\partial}{\partial z_1^{(1)}}\partial_{\{1,\dots,k\}}\partial_{\{2,\dots,\ell,k+1,\dots,2k-\ell+1\}}Q_\alpha(z)$ is nonzero only if α has the following form, $\alpha=(\{1,q_1\},\{p_2,q_2\},\dots,\{p_k,q_k\})$, where

(i) $q_1\in\{k+1,\dots,2k-\ell+1\}$;

(ii) the sequence $(p_2,\dots,p_k)$ is a permutation of the sequence $(2,\dots,k)$ (the sequence $(p_2,\dots,p_k)$ determines a partition of $(2,\dots,k)$ into two subsequences: $(i_1<\dots<i_{\ell-1})$ such that all $p_{i_r}\leqslant\ell$ and $(j_1<\dots<j_{k-\ell})$ such that all $p_{j_r}>\ell$, these subsequences are used below);

(iii) the sequence $(q_{i_1},\dots,q_{i_{\ell-1}})$ is an ordered $(\ell-1)$-element subset of $\{2k-\ell+2,\dots,n\}$; (iv) the sequence $(q_{j_1},\dots,q_{j_{k-\ell}})$ is a permutation of the sequence $(k+1,\dots,\widehat{q_1},\dots,2k-\ell+1)$.

The number of such α with fixed p_1 equals $(k-1)!\,(k-\ell)!\,\prod_{i=0}^{\ell-2}(n-2k+1+i)$. For every such α,

$$\frac{\partial}{\partial z_1^{(1)}}\partial_{\{1,\dots,k\}}\partial_{\{2,\dots,\ell,k+1,\dots,2k-\ell+1\}}Q_\alpha(z)=\frac{(-1)^{k-\ell}2^k}{z_{q_1}^{(1)}-z_1^{(1)}}.$$

It follows from (3.4) and (4.6) that

$$\frac{-a_{\ell-1}+a_\ell}{z_1^{(1)}-z_{q_1}^{(1)}}=\frac{(-1)^{k-\ell}2^k}{z_{q_1}^{(1)}-z_1^{(1)}}\,c_2\,(k-1)!\,(k-\ell)!\prod_{i=0}^{\ell-2}(n-2k+1+i),$$

and (4.7) holds in this case.

Assume that m belongs to $I\cap J$ and $|I\cap J|=\ell$, $\ell=1,\dots,k$. Under this assumption, without loss of generality we may assume that $I=\{1,\dots,k\}$, $J=\{1,\dots,\ell,k+1,\dots,2k-\ell\}$, and $m=1$. Then

$$(\bar{H}_1(z_1^{(1)},\dots,z_n^{(1)})v_I,v_J)=\sum_{i=k+1}^{n}\frac{(-v_{1,\dots,k}+v_{2,\dots,k,i},\,v_{1,\dots,\ell,k+1,\dots,2k-\ell})}{z_1^{(1)}-z_i^{(1)}}=\sum_{i=2k-\ell+1}^{n}\frac{-a_\ell+a_{\ell-1}}{z_1^{(1)}-z_i^{(1)}}.$$

If $I=\{1,\dots,k\}$, $J=\{1,\dots,\ell,k+1,\dots,2k-\ell\}$, $m=1$, then $\frac{\partial}{\partial z_1^{(1)}}\partial_{\{1,\dots,k\}}\partial_{\{1,\dots,\ell,k+1,\dots,2k-\ell\}}Q_\alpha(z)$ is nonzero only if α has the following form, $\alpha=(\{1,q_1\},\{p_2,q_2\},\dots,\{p_k,q_k\})$

(i) $q_1\in\{2k-\ell+1,\dots,n\}$;

(ii) the sequence $(p_2,\dots,p_k)$ is a permutation of the sequence $(2,\dots,k)$ (the sequence $(p_2,\dots,p_k)$ determines a partition of $(2,\dots,k)$ into two subsequences: $(i_1<\dots<i_{\ell-1})$ such that all $p_{i_r}\leqslant\ell$ and $(j_1<\dots<j_{k-\ell})$ such that all $p_{j_r}>\ell$, these subsequences are used below);

(iii) the sequence $(q_{i_1},\dots,q_{i_{\ell-1}})$ is an ordered $(\ell-1)$-element subset of $\{2k-\ell+1,\dots,\widehat{q_1},\dots,n\}$;

(iv) the sequence $(q_{j_1},\dots,q_{j_{k-\ell}})$ is a permutation of the sequence $(k+1,\dots,2k-\ell)$.

The number of such α with fixed q_1 equals $(k-1)!\,(k-\ell)!\,\prod_{i=0}^{\ell-2}(n-2k+1+i)$. For every such α, we have

$$\frac{\partial}{\partial z_1^{(1)}}\partial_{\{1,\dots,k\}}\partial_{\{1,\dots,\ell,k+1,\dots,2k-\ell\}}Q_\alpha(z)=\frac{(-1)^{k-\ell}2^k}{z_{q_1}^{(1)}-z_1^{(1)}}.$$

It follows from (3.4) and (4.6) that

$$\frac{-a_\ell+a_{\ell-1}}{z_1^{(1)}-z_{q_1}^{(1)}}=\frac{(-1)^{k-\ell}2^k}{z_{q_1}^{(1)}-z_1^{(1)}}\,c_2\,(k-1)!\,(k-\ell)!\prod_{i=0}^{\ell-2}(n-2k+1+i),$$

and (4.7) holds in this case. Theorem 4.3 is proved.

4.4. **Example.** For $k=1$, we have

$$Q(z_1^{(1)},\dots,z_n^{(1)})=\frac{-1}{2}\sum_{1\leqslant i<j\leqslant n}\ln(z_i^{(1)}-z_j^{(1)})\,(z_i^{(1)}-z_j^{(1)})^2,$$

and Theorem 4.3 takes the following form.

Theorem 4.4. *Let $i,j,m\in\{1,\dots,n\}$. Then*

$$(\bar H_m(z_1^{(1)},\dots,z_n^{(1)})v_i,v_j)=\frac{\partial^3 Q}{\partial z_i^{(1)}\partial z_j^{(1)}\partial z_m^{(1)}}(z_1^{(1)},\dots,z_n^{(1)}). \tag{4.9}$$

Compare these formulas with formulas of Example 2.2.

4.5. **A relation between P and Q.**

Theorem 4.5. *Let I,J be two k-element subsets of $\{1,\dots,n\}$. Then*

$$\frac{1}{c_1}\partial_I\partial_J P(z)\ =\ \frac{1}{c_2}\sum_{m=1}^n z_m^{(1)}\frac{\partial}{\partial z_m^{(1)}}\partial_I\partial_J Q(z). \tag{4.10}$$

Proof. The theorem follows from formulas (4.1) and (4.5) for functions $P_\alpha(z)$ and $Q_\alpha(z)$ and the identity $(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y})\ln(x-y)=1.$

Corollary 4.6. *The operator $\sum_{m=1}^n u_m\bar H_m(u_1,\dots,u_n)$ restricted to $\mathrm{Sing}W^{\otimes n}[n-2k]$ is the scalar operator of multiplication by $-k(n-k+1)$.*

Proof. Equation (4.10) can be written as

$$\frac{1}{c_1}(v_I,v_J)=\frac{1}{c_2}\Big(\sum_{m=1}^n z_m^{(1)}\bar H_m(z_1^{(1)},\dots,z_n^{(1)})v_I,v_J\Big).$$

Notice that $c_2/c_1=-k(n-k+1)$. This implies the corollary.

Remark. Corollary 4.6 also follows from the representation theory. Indeed,

$$\sum_{m=1}^{n} u_m \bar{H}_m(u_1,\dots,u_n) = \sum_{1\leqslant m<j\leqslant n} (\Omega_{mj} - 1/2) = C/2 - \sum_{m=1}^{n} C_m/2 - n(n-1)/4,$$

where

$$C = h^2/2 + ef + fe$$

is the Casimir element acting on the tensor product of representations and

$$C_m = (h_m)^2/2 + e_m f_m + f_m e_m$$

is the Casimir element acting on the m-th tensor factor of the tensor product. On the subspace $\mathrm{Sing}\, W^{\otimes n}[n-2k]$ of singular vectors of weight $n-2k$, the operator C acts as the scalar operator of multiplication by $(n-2k)(n-2k+2)/4$, and each C_m acts as the scalar operator of multiplication by $-3/4$, so that $\sum_{m=1}^{n} u_m \bar{H}_m(u_1,\dots,u_n)$ acts as the scalar operator of multiplication by

$$(n-2k)(n-2k+2)/4 - 3n/4 - n(n-1)/4 = -k(n-k+1).$$

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