# ON THE NUMBER OF $p$-HYPERGEOMETRIC SOLUTIONS OF *KZ* EQUATIONS

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Abstract. It is known that solutions of the *KZ* equations can be written in the form of multidimensional hypergeometric integrals. In 2017 in a joint paper of the author with V.Schechtman the construction of hypergeometric solutions was modified, and solutions of the *KZ* equations modulo a prime number $p$ were constructed. These solutions modulo $p$, called the $p$-hypergeometric solutions, are polynomials with integer coefficients. A general problem is to determine the number of independent $p$-hypergeometric solutions and understand the meaning of that number.

In this paper we consider the *KZ* equations associated with the space of singular vectors of weight $n-2r$ in the tensor power $W^{\otimes n}$ of the vector representation of $\mathfrak{sl}_2$. In this case the hypergeometric solutions of the *KZ* equations are given by $r$-dimensional hypergeometric integrals. We consider the module of the corresponding $p$-hypergeometric solutions, determine its rank, and show that the rank equals the dimension of the space of suitable square integrable differential $r$-forms.

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## 1. Introduction

The Knizhnik-Zamolodchikov (*KZ*) equations is a system of differential equations for conformal blocks in conformal field theory, see [KZ]. Versions of *KZ* equations appear in mathematical physics, representation theory, enumerative geometry, algebraic geometry, theory of special functions, see for example [EFK, MO, B].

It is known that solutions of the *KZ* equations can be written in the form of multidimensional hypergeometric integrals, see [SV1]. Relatively recently, the construction of these hypergeometric solutions was modified, and solutions of the *KZ* equations modulo a prime number $p$ were constructed in [SV2]. These solutions modulo $p$, called the $p$-hypergeometric solutions, are vector-valued polynomials with integer coefficients. The general problem is to understand how these polynomials reflect the remarkable properties of the *KZ* equations and their hypergeometric solutions.

In this paper we address the particular problem to determine the number of independent $p$-hypergeometric solutions and understand the meaning of that number. We consider the *KZ* equations associated with the space of singular vectors of weight $n-2r$ in the tensor power $W^{\otimes n}$ of the vector representation of $\mathfrak{sl}_2$. In this case the hypergeometric solutions of the *KZ* equations are given by $r$-dimensional hypergeometric integrals. We consider the module of the corresponding $p$-hypergeometric solutions, determine its rank, and show that the rank equals the dimension of the space of suitable square integrable differential $r$-forms.

The KZ equations depend on a parameter $q \in \mathbb{C}^\times$. In this paper we assume that $q$ is a prime number less than $p$, and the pair $(p,q)$ satisfies certain conditions (the pair is of type 1).

On $p$-hypergeometric solutions and, more generally, on the solutions of the KZ equations modulo $p^s$ see [SlV, V4, V5, V6, V7, V8, VZ1, VZ2].

In Section 2 we define the KZ equations, recall the construction of solutions in the form of hypergeometric integrals and the construction of $p$-hypergeometric solutions. We also

introduce the module of $p$-hypergeometric solutions. The main result of the paper is Theorem 3.5, formulated in Section 3 and proved in Section 4. Theorem 3.5 determines the rank of the module of $p$-hypergeometric solutions.

In Section 5 we construct a suitable Cartier map, which relates the hypergeometric solutions and $p$-hypergeometric solutions. As a result of this construction, we interpret the rank of the module of $p$-hypergeometric solutions as the dimension of some vector space of square integrable differential $r$-forms on $\mathbb{P}^r$, the $r$-th direct power of the complex projective line.

Previously this result was known for $r = 1$, see [SlV].

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## 2. $\mathfrak{sl}_2$ *KZ* equations

2.1. **Definition of equations.** Consider the complex Lie algebra $\mathfrak{sl}_2$ with generators $e,f,h$ and relations $[e,f] = h$, $[h,e] = 2e$, $[h,f] = -2f$. Consider the complex vector space $W$ with basis $w_1,w_2$ and the $\mathfrak{sl}_2$-action,

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The $\mathfrak{sl}_2$-module $W^{\otimes n}$ has a basis labeled by subsets $J \subset \{1,\dots,n\}$,

$$V_J = w_{j_1} \otimes \cdots \otimes w_{j_n},$$

where $j_i = 1$ if $j_i \notin J$ and $j_i = 2$ if $j_i \in J$.

Consider the weight decomposition of $W^{\otimes n}$ into eigenspaces of $h$, $W^{\otimes n} = \sum_{r=0}^{n} W^{\otimes n}[n-2r]$. The vectors $V_J$ with $|J| = r$ form a basis of $W^{\otimes n}[n-2r]$. Denote $\mathrm{J}_r$ the set of all $r$-element subsets of $\{1,\dots,n\}$.

Define the space of singular vectors of weight $n - 2r$,

$$\mathrm{Sing}W^{\otimes n}[n-2r] = \{w \in W^{\otimes n}[n-2r] \mid ew = 0\}.$$

This space is nonempty if and only if $n > 2r$. We assume that $n > 2r$. Then

$$\dim \mathrm{Sing}W^{\otimes n}[n-2r] = \dim W^{\otimes n}[n-2r] - \dim W^{\otimes n}[n-2r+2] = \binom{n}{r} - \binom{n}{r-1}.$$

Let $w = \sum_{J\in \mathrm{J}_r} c_J V_J \in W^{\otimes n}[n-2r]$. Then $w \in \mathrm{Sing}W^{\otimes n}[n-2r]$, if and only if its coefficients satisfy the system of linear equations labeled by $r-1$-element subsets $K \subset \{1,\dots,n\}$,

$$\sum_{j\notin K} c_{K\cup\{j\}} = 0. \tag{2.1}$$

Define the Casimir element $\Omega = \frac{1}{2}h\otimes h + e\otimes f + f\otimes e \in \mathfrak{sl}_2\otimes\mathfrak{sl}_2$, and the linear operators on $W^{\otimes n}$ depending on parameters $z = (z_1,\dots,z_n)$,

$$H_m(z) = \sum_{\substack{j=1\\ j\neq m}}^{n} \frac{\Omega_{mj}}{z_m - z_j}\,, \qquad m = 1,\dots,n,$$

where $\Omega_{mj}: W^{\otimes n} \to W^{\otimes n}$ is the Casimir operator acting in the $m$th and $j$th tensor factors. The operators $H_m(z)$ are called the Gaudin Hamiltonians. Denote

$$\partial_m = \frac{\partial}{\partial z_m}\,, \qquad m = 1,\dots,n.$$

For any nonzero number $q \in \mathbb{C}^\times$ and $1 \leqslant m,l \leqslant n$, we have

(2.2) $[q\partial_m - H_m(z_1,\dots,z_n), q\partial_l - H_l(z_1,\dots,z_n)] = 0$, and for any $x \in \mathfrak{sl}_2$ and $m = 1,\dots,n$, we have

(2.3) $$[H_m(z_1,\dots,z_n), x\otimes 1\otimes\cdots\otimes 1 + \cdots + 1\otimes\cdots\otimes 1\otimes x] = 0.$$

The system of differential equations

(2.4) $$(q\partial_m - H_m(z_1,\dots,z_n))\tilde{I}(z_1,\dots,z_n), \qquad m = 1,\dots,n,$$

on a $W^{\otimes n}$-valued function $\tilde{I}(z_1,\dots,z_n)$ is called the *KZ* equations, see [KZ, EFK].

By property (2.3) the Gaudin Hamiltonians preserve every space $\mathrm{Sing}W^{\otimes n}[n-2r]$. Hence the system of *KZ* equations can be considered with values in any particular space $\mathrm{Sing}W^{\otimes n}[n-2r]$.

2.2. **Gauge transformation.** If $\tilde{I}(z)$ satisfies the *KZ* equations (2.4), then the function $I(z)$, defined by

(2.5) $$\tilde{I}(z) = I(z)\prod_{1\leqslant i<j\leqslant n}(z_i - z_j)^{1/2q},$$

satisfies the equations

(2.6) $$\Big(q\partial_m - \sum_{\substack{j=1\\ j\neq m}}^{n} \frac{\Omega_{mj} - 1/2}{z_m - z_j}\Big) I(z_1,\dots,z_n), \qquad m = 1,\dots,n,$$

which we also call the *KZ* equations.

The linear operator $\Omega^{-\frac{1}{2}}: W^{\otimes 2} \to W^{\otimes 2}$ acts as follows:

(2.7) $$w_1\otimes w_1 \mapsto 0, \qquad w_2\otimes w_2 \mapsto 0,$$
$$w_1\otimes w_2 \mapsto -w_1\otimes w_2 + w_2\otimes w_1, \qquad w_2\otimes w_1 \mapsto w_1\otimes w_2 - w_2\otimes w_1.$$

We shall consider the *KZ* equations (2.6) with values in Sing$W^{\otimes n}[n-2r]$ over the field of complex numbers, and we shall also consider the *KZ* equations (2.6) modulo $p$, where $p$ is a prime number.

2.3. **Solutions of *KZ* equations over** C**.** Denote $t = (t_1,...,t_k)$. Define the master function

$$\Phi(t,z) = \prod_{1\le i\le j\le r} (t_i - t_j)^{2/q} \prod_{s=1}^{n}\prod_{i=1}^{r}(t_i - z_s)^{-1/q}. \tag{2.8}$$

For any function $F(t_1,...,t_r)$, denote $\mathrm{Sym}_t[F(t_1,...,t_r)] = \sum_{\sigma\in S_r} F(t_{\sigma_1},...,t_{\sigma_r})$ . For $J = \{j_1,...,j_r\} \in \mathrm{J}_r$ define the weight function

$$W_J(t,z) = \mathrm{Sym}_t \left[\prod_{i=1}^{r} \frac{1}{t_i - z_{j_i}}\right]$$

For example,

$$W_{\{3\}} = \frac{1}{t_1 - z_3}, \qquad W_{\{4,5\}} = \frac{1}{t_1 - z_4}\frac{1}{t_2 - z_5} + \frac{1}{t_2 - z_4}\frac{1}{t_1 - z_5}.$$

The function

$$W(t,z) = \sum_{J\in \mathrm{J}_r} W_J(t,z) V_J \tag{2.9}$$

is called the $W^{\otimes n}[n-2r]$-weight vector-function.

Consider the $W^{\otimes n}[n-2r]$-valued function

$$I^{(\gamma)}(z_1,...,z_n) = \int_{\gamma(z)} \Phi(t,z) W(t,z) dt_1 \wedge \cdots \wedge dt_r, \tag{2.10}$$

where $\gamma(z)$ in $\{z\}\times \mathrm{C}^r_t$ is a horizontal family of $r$-dimensional cycles of the twisted homology defined by the multivalued function $\Phi(t,z)$, see [CF, SV1, DJMM, V1, V3]. The cycles $\gamma(z)$ are $r$-dimensional analogs of Pochhammer double loops.

**Theorem 2.1.** *The function* $I^{(\gamma)}(z)$ *takes values in* Sing$W^{\otimes n}[n-2r]$ *and satisfies the KZ equations* (2.6)*.*

This theorem and its generalizations can be found, for example, in [CF, DJMM, SV1].

The solutions in Theorem 2.1 are called the hypergeometric solutions of the *KZ* equations.

**Theorem 2.2** ([V1, Theorem 12.5.5])**.** *If* $q \in \mathrm{C}^\times$ *is generic, then all solutions of the KZ equations* (2.6) *have this form.*

For special values of the parameter $q$ the space of the hypergeometric solutions of the *KZ* equations may span only a proper subspace of the space of all solutions, see in [FSV1, FSV2] the discussion of the relations of this subspace of solutions and conformal blocks in conformal field theory.

2.4. ***p*-integrals.** Let $p$ be a positive integer. Let $f(t_1,\dots,t_k) = \sum_{d_1,\dots,d_k} c_{d_1,\dots,d_k} t_1^{d_1}\dots t_k^{d_k}$ be a polynomial. Let $l = (l_1,\dots, l_r) \in \mathbb{Z}^r_{>0}$. The coefficient $c_{l_1p-1,\dots,l_rp-1}$ will be called the $p$-integral of $f(t_1,\dots,t_k)$ over the cycle $\{l_1,\dots,l_r\}_p$ and is denoted by

$$\int_{\{l_1,\dots,l_r\}_p} f(t_1,\dots,t_k)dt_1\dots dt_r.$$

2.5. ***KZ* equations modulo *p*.** Let $p$ and $q$ be prime numbers, $p > q$. We consider the system of *KZ* equations (2.6) with parameter $q$ modulo $p$. Namely, we look for $W^{\otimes n}[n-2r]$valued polynomials in $z_1,\dots,z_n$ with integer coefficients, which satisfy system (2.6) modulo $p$ and with values in the subspace $\mathrm{Sing}W^{\otimes n}[n-2r]$ modulo $p$. More precisely, let $I(z_1,\dots,z_n) = \sum_{J\in\mathbb{J}_r} I_J(z_1,\dots,z_n)V_J$ with $I_J(z_1,\dots,z_n) \in \mathbb{Z}[z_1,\dots,z_n]$. We request that

- for any $m = 1,\dots,n$, the rational function $(q\partial_m - H_m(z))I(z)$ can be written as a ratio of two polynomials with integer coefficients such that the denominator is nonzero modulo $p$, while the numerator is zero modulo $p$;
- the coefficients $I_J(z_1,\dots,z_n)$ of the polynomial $I(z)$ satisfy equations (2.1) modulo $p$.

A construction of such solutions was presented in [SV2]. Let

$M$, $c$ be the least positive integers such that

$$M \equiv -\frac{1}{q}, \qquad c \equiv \frac{2}{q} \pmod p. \tag{2.11}$$

Let $t = (t_1,\dots,t_r)$, $z = (z_1,\dots,z_n)$. Define the master polynomial,

$$\Phi_p(t,z) = \prod_{1\le i<j\le r} (t_i - t_j)^c \prod_{i=1}^{r}\prod_{s=1}^{n} (t_i - z_s)^M. \tag{2.12}$$

Recall the weight vector-function $W(t,z)$ in (2.9). The function $\Phi_p(t,z)W(t,z)$ is a $W^{\otimes n}[n-2r]$-valued polynomial in $t,z$. Let $(l_1,\dots,l_r) \in \mathbb{Z}^r_{>0}$. Denote

$$I^{(l_1,\dots,l_r)}(z) = \int_{\{l_1,\dots,l_r\}_p} \Phi_p(t,z)W(t,z)dt_1\dots dt_r. \tag{2.13}$$

This is a $W^{\otimes n}[n-2r]$-valued polynomial in $z$.

**Theorem 2.3** ([SV2]). *For any positive integers $(l_1,\dots,l_r)$, the polynomial $I^{(l_1,\dots,l_r)}(z)$ is a solution of the KZ equations modulo p with values in* $\mathrm{Sing}W^{\otimes n}[n-2r]$ *modulo p.*

We call the polynomials $I^{(l_1,\dots,l_r)}(z)$ the *p-hypergeometric* solutions.

In this paper we determine the number of independent $p$-hypergeometric solutions under the assumption that $(p,q)$ is a pair of type 1.

**Remark.** The symmetric group $S_n$ acts on the polynomials in $z_1,\dots,z_n$ with values in $W^{\otimes n}$ by permuting the tensor factors and the variables simultaneously, $\sigma(p(z_1,\dots,z_n)v_1 \otimes \cdots \otimes v_n) = (p(z_{\sigma(1)},\dots,z_{\sigma(n)})v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}$, $\sigma \in S_n$.

It is clear that for any positive integers $(l_1,\dots,l_r)$, the polynomial $I^{(l_1,\dots,l_r)}(z)$ is symmetric with respect to the $S_n$-action.

2.6. **Module of $p$-hypergeometric solutions.** Denote $\mathbb{Z}[z^p] = \mathbb{Z}[z_1^p,\dots,z_n^p]$. Let

$$\mathrm{M} = \sum_{l_1,\dots,l_r} f_{l_1,\dots,l_r}(z) I^{(l_1,\dots,l_r)}(z) \mid f_{l_1,\dots,l_r}(z) \in \mathrm{Z}[z_p]\}, \tag{2.14}$$

be the $\mathrm{Z}[z^p]$-module generated by the $p$-hypergeometric solutions $I^{(l_1,\dots,l_r)}(z)$ of Theorem 2.3. Every element of M is a solution of the *KZ* equations modulo $p$ with values in $\mathrm{Sing}W^{\otimes n}[n-2r]$ modulo $p$. Indeed the *KZ* equations (2.6) are linear and $\frac{\partial z_i^p}{\partial z_j} \equiv 0 \pmod p$ for all $i,j$.

We say that two elements $I, I^0 \in \mathrm{M}$ are equivalent and write $I \sim I^0$, if $I - I^0$ is divisible by $p$. Then the set $\mathrm{M}/(\sim)$ of equivalence classes is an $\mathrm{F}_p[z^p]$-module.

2.7. **More general choice of the numbers $M$ and $c$.** For $s = 1,\dots,n$ fix a positive integer $M_s$ such that

$$M_s \equiv -\frac{1}{q} \pmod p.$$

Fix a positive integer $c^0$ such that

$$c' \equiv \frac{2}{q} \pmod p.$$

Define the master polynomial,

$$\Phi_p(t,z;M,c^{\sim 0}) = \prod_{1\le i<j\le r} (t_i - t_j)^{c^0} \prod_{i=1}^{r}\prod_{s=1}^{n} (t_i - z_s)^{M_s}.$$

Recall the weight vector-function $W(t,z)$ in (2.9). The function $\Phi_p(t,z;M,c^{\sim 0})W(t,z)$ is a $W^{\otimes n}[n-2r]$-valued polynomial in $t,z$ with integer coefficients. Let $l = (l_1,\dots,l_r) \in \mathbb{Z}^r_{>0}$. Denote

$$I_{(l_1,\dots,l_r)}(z;M,c\tilde{}\ {}_0) = \int_{\{l_1,\dots,l_r\}_p} \Phi_p(t,z;M,c\tilde{}\ {}_0)W(t,z)dt_1\dots dt_r.$$

This is a $W^{\otimes n}[n-2r]$-valued polynomial in $z$ with integer coefficients.

**Theorem 2.4** ([SV2]). *For any positive integers $(l_1,\dots,l_r)$, the polynomial $I^{(l_1,\dots,l_r)}(z;M,c\tilde{}\ {}^0)$ is a solution of the KZ equations modulo $p$ with values in* $\mathrm{Sing}W^{\otimes n}[n-2r]$ *modulo $p$.*

**Theorem 2.5.** *For any positive integers $(l_1,\dots,l_r)$, the polynomial $I^{(l_1,\dots,l_r)}(z;M,c\tilde{}\ {}^0)$ belongs to the module* M *of $p$-hypergeometric solutions.*

For $l = 1$ this statement follows from [SlV, Theorem 3.1].

*Proof.* Let $M,c$ be defined in (2.11). Then $c^0 = c + d_0p$, $M_s = M + d_sp$, $s = 1,\dots,n$, where $d_0,\dots,d_n$ are nonnegative integers. Then we have modulo $p$,

$$\Phi_p(t,z;\vec{M},c') \equiv \Phi_p(t,z) \prod_{1\leqslant i<j\leqslant r} (t_i^p - t_j^p)^{d_0} \prod_{i=1}^{r}\prod_{s=1}^{n}(t_i^p - z_s^p)^{d_s} \pmod p.$$

This formula implies the theorem.

## 3. Pairs $(p,q)$ of type 1

3.1. **Numbers $M$ and $c$.** Let $p,q$ be prime numbers, $p > q$. Let $k$ be the minimal positive integer such that $q|(kp-1)$. We have $1 \leqslant k < q$.

We say that the pair $(p,q)$ is of type 1 or type 2 if

$$1 \leqslant k \leqslant q/2 \qquad \text{or} \qquad q/2 < k < q, \tag{3.1}$$

respectively.

The pairs $(p,2)$ are all of type 1.

The pairs $(p,3)$ are of type 1 if $p = 3m+1$ and of type 2 if $p = 3m+2$.

Let $p = mq+s$, $s \in \{1,\dots,q-1\}$. Then the minimal positive integer $k$ such that $q|(kp-1)$ belongs to $\{1,\dots,q-1\}$ and is determined by the property $ks \equiv 1 \pmod q$. Hence half of the values of $s$ gives pairs $(p,q)$ of type 1 and half of the values of $s$ gives pairs $(p,q)$ of type 2.

In the rest of the paper *we always assume that $(p,q)$ is of type 1.*

**Lemma 3.1.** *The integer $q-2k$ is the least positive integer $m$ such that $q|(mp+2)$.*

*Proof.* We have $q|(kp-1)$. Hence $q|(qp-2(kp-1))$, and $qp-2(kp-1) = (q-2k)p+2$. We also have $0 \leqslant q-2k \leqslant q-2$.

**Lemma 3.2.** *The integers*

$$M = \frac{kp-1}{q}, \qquad c = \frac{(q-2k)p+2}{q} \tag{3.2}$$

*are the least positive integers satisfying the congruences* (2.11)*. The integer c is odd and*

$$2M + c = p. \tag{3.3}$$

**Example.** If $q = 2$, then $M = \frac{p-1}{2}$, $c = 1$. The case $q = 2$ is the only case in which $c$ does not depend on $p$.

**Lemma 3.3.** *For positive integers* $(l_1,\dots,l_r)$ *the polynomial* $I^{(l_1,\dots,l_r)}(z)$ *equals zero, if* $l_1,\dots,l_r$ *are not pairwise distinct.*

*Proof.* The integer $c$ is odd. Hence the polynomial $\Phi_p(t,z)W(t,z)$ is skew-symmetric with respect to permutations of $t_1,\dots,t_r$.

In what follows *we always assume*

(3.4) $n = qg + 2r - 1$ for some positive integer $g$.

**Lemma 3.4.** *Let q,k,g be fixed. Let p be large enough so that*

$$M - g = \frac{kp-1}{q} - g = \frac{kp-1-gq}{q} \geqslant 0. \tag{3.5}$$

*Let* $l_1 > \cdots > l_r > 1$*. Then the inequality*

$$kg + r - 1 > l_1 \tag{3.6}$$

*is necessary for* $I^{(l_1,\dots,l_r)}(z)$ *to be nonzero. Hence there are at most* $\binom{kg+r-1}{r}$ *tuples* $(l_1 > \cdots > l_r > 1)$ *such that* $I^{(l_1,\dots,l_r)}(z)$ *is nonzero.*

*Proof.* For $i = 1,\dots,r$ we have

$$\begin{aligned} \deg_{t_i} \Phi_p(t,z)W(t,z) &= nM + (r-1)c - 1 \\ &= (gq + 2r - 1)\frac{kp-1}{q} + (r-1)\frac{(q-2k)p+2}{q} - 1 \\ &= (kg + r - 1)p - 1 + \frac{kp-1-gq}{q}. \end{aligned} \tag{3.7}$$

This proves the lemma.

3.2. **Main Theorem.** For a polynomial $F$ in some variables with integer coefficients, denote by $[F]_p$ the polynomial $F$ whose integer coefficients are projected to $\mathbb{F}_p$. Let

$$f(z) = \sum_{d_1,\dots,d_n} a_{d_1,\dots,d_n} z_1^{d_1} \dots z_n^{d_n}$$

be a $W^{\otimes n}[n-2r]$-valued polynomial in $z$ with integer coefficients. Here each $a_{d_1,\dots,d_n}$ is a linear combination of basis vectors $\{V_J \,|\, J \in \mathrm{J}_r\}$ with integer coefficients. Denote by $[f(z)]_p$ the polynomial

$$[f(z)]_p = \sum_{d_1,\dots,d_n} [a_{d_1,\dots,d_n}]_p \, z_1^{d_1} \dots z_n^{d_n},$$

where each $[a_{d_1,\dots,d_n}]_p$ is the linear combination $a_{d_1,\dots,d_n}$ whose integer coefficients are projected to $\mathrm{F}_p$.

Denote by $W^{\otimes n}[n-2r]_p$ the vector space over $\mathrm{F}_p$ of linear combinations of symbols $\{V_J \,|\, J \in \mathrm{J}_r\}$ with coefficients in $\mathrm{F}_p$. Define the subspace $\mathrm{Sing} W^{\otimes n}[n-2r]_p \subset W^{\otimes n}[n-2r]_p$ as the subspace of all vectors $\sum_{J\in \mathrm{J}_r} c_J V_J$ whose coefficients satisfy equations (2.1).

**Theorem 3.5.** *Let $(p,q)$ be of type 1. Let inequality (3.5) hold. Then for any $(l_1,\dots,l_r)$ such that $kg + r - 1 > l_1 > \dots > l_r > 1$, the polynomial $[I^{(l_1,\dots,l_r)}(z)]_p$ is nonzero. The polynomials*

$$\{[I_{(l_1,\dots,l_r)}(z)]_p \mid kg + r - 1 > l_1 > \dots > l_r > 1\}$$

*are linear independent over $\mathrm{F}_p[z]$, that is, if*

$$\sum_{kg+r-1>l_1>\dots>l_r>1} f_{(l_1,\dots,l_r)} f_{l_1,\dots,l_r}(z) [I \qquad (z)]_p = 0$$

*for some $f_{l_1,\dots,l_r}(z) \in \mathrm{F}_p[z]$, then all $f_{l_1,\dots,l_r}(z)$ are equal to zero.*

The theorem is proved in Section 4.5.

**Corollary 3.6.** *Let $(p,q)$ be of type 1. Let inequality (3.5) hold. Then the $\mathrm{F}_p[z^p]$-module $\mathrm{M}/(\sim)$ is free of rank $\binom{kg+r-1}{r}$ with a basis $[I^{(l_1,\dots,l_r)}(z)]_p$, where $kg + r - 1 > l_1 > \dots > l_r > 1$.*

## 4. Leading term and leading index

4.1. **Leading term of a polynomial.** Denote by $\succ$ the lexicographical ordering of monomials $z_1^{d_1} \dots z_n^{d_n}$, where $d_1,\dots,d_n \in \mathrm{Z}_{>0}$. Thus $z_1 \succ z_2 \succ \dots \succ z_{n-1} \succ z_n$ and so on. For example, $z_1^2 z_2^2 z_3^2 \succ z_1^2 z_2 z_4^5$.

For a nonzero polynomial

$$f(z) = \sum_{d_1,\dots,d_n} a_{d_1,\dots,d_n} z_1^{d_1} \dots z_n^{d_n}$$

consider the summand $a_{d_1,\dots,d_n} z_1^{d_1} \cdots z_n^{d_n}$ corresponding to the lexicographically largest monomial $z_1^{d_1} \cdots z_n^{d_n}$ entering $f(z)$ with a nonzero coefficient $a_{d_1,\dots,d_n}$. We call this summand the leading term of $f(z)$, the corresponding $a_{d_1,\dots,d_n}$ – the leading coefficient, and $z_1^{d_1} \cdots z_n^{d_n}$ – the leading monomial.

For example, let $I^{(l_1,\dots,l_r)}(z)$ be one of the $p$-hypergeometric solutions of Theorem 2.3. Then

$$[I_{(l_1,\dots,l_r)}(z)]_p = \sum_{d_1,\dots,d_n} [a_{d_1,\dots,d_n}]_p \, z_1^{d_1} \dots z_n^{d_n},$$

with $[a_{d_1,\dots,d_n}]_p \in \mathrm{Sing} W^{\otimes n}[n-2r]_p$. If $[I^{(l_1,\dots,l_r)}(z)]_p$ is nonzero, then it has the leading term, monomial, and coefficient. The leading coefficient is a nonzero vector of $\mathrm{Sing} W^{\otimes n}[n-2r]_p$.

4.2. **Leading index of a vector of $W^{\otimes n}[n-2r]_p$.** We order $\{V_J \mid J \in \mathrm{J}_r\}$, the basis vectors of $W^{\otimes n}[n-2r]_p$, lexicographically with the largest of them being $V_{\{1,2,\dots,r\}} = w_2 \otimes \cdots \otimes w_2 \otimes w_1 \otimes \cdots \otimes w_1$ and the smallest of them being $V_{\{n-r+1,n-r+2,\dots,n\}} = w_1 \otimes \cdots \otimes w_1 \otimes w_2 \otimes \cdots \otimes w_2$.

Every nonzero vector $w \in W^{\otimes n}[n-2r]_p$ is a linear combination of the basis vectors. Let $V_{\{m_1,\dots,m_r\}}$ be the largest of the basis vectors entering $w$ with a nonzero coefficient. We call the index $\{m_1,\dots,m_r\}$ the leading index of $w$.

4.3. **Index $\{m_1,\dots,m_r\}$.** Given integers $(l_1,\dots,l_r)$, consider the system of inequalities for integers $m_1,\dots,m_r$,

$$(m_i - 1)M \le nM + (r-i)c - l_i p < m_i M, \qquad i = 1,\dots,r. \tag{4.1}$$

Clearly, these inequalities uniquely determine the integers $m_1,\dots,m_r$.

**Lemma 4.1.** *Let inequality (3.5) hold. Let $kg+r-1 > l_1 > \cdots > l_r > 1$. Then $m_i+2 \le m_{i+1}$ for $i = 1,\dots,r-1$, and $1 \le m_1, m_r < n$.*

*Proof.* For $i = 1,\dots,r-1$ we have $(m_i - 1)M \le nM + (r-i)c - l_i p = nM + (r-i-1)c + c - l_{i+1}p + (l_{i+1} - l_i)p \le nM + (r-i-1)c + c - l_{i+1}p - p = nM + (r-i-1)c - l_{i+1}p - 2M < (m_{i+1} - 2)M$. Hence $m_{i+1} > m_i + 1$. This implies that $m_i + 2 \le m_{i+1}$.

The inequality $1 \le m_1$ follows from the inequality $l_1 p \le nM + (r-1)c$, which is true since

$$l_1 p \leqslant (kg + r - 1)p \leqslant (kg + r - 1)p + \frac{kp - 1 - gq}{q} = nM + (r-1)c,$$

see (3.7). We also have $M + 2c = p \le l_r p \le (n - m_r + 1)M$. Hence $m_r < n$.

**Lemma 4.2.** *Let $(l_1,\dots,l_r)$ and $(l'_1,\dots,l'_r)$ be two distinct tuples of integers such that $kg+r-1 > l_1 > \cdots > l_r > 1$ and $kg+r-1 \geqslant l'_1 > \cdots > l'_r \geqslant 1$. Let $\{m_1,\dots,m_r\}$ and $\{m'_1,\dots,m'_r\}$ be the corresponding sets defined by (4.1). Then $\{m_1,\dots,m_r\} \neq \{m'_1,\dots,m'_r\}$.*

*Proof.* Let $l_i > l_i^0$ for some $i$. Then $(m_i - 1)M \leqslant nM + (r-i)c - l_i p = nM + (r-i)c - l_i' p + (l_i^0 - l_i)p \leqslant nM + (r-i)c - l_i^0 p - p = nM + (r-i)c - l_i p - 2M - c < nM + (r-i)c - l_i p - 2M < (m_i' - 2)M$. Hence $m_i' > m_i + 1$.

4.4. **Special summand.** We have

$$\prod_{1\leqslant i<j\leqslant r}(t_i - t_j)^c = \prod_{1\leqslant i<j\leqslant r}\Big(\sum_{a_{ij}+a_{ji}=c}\binom{c}{a_{ji}}(-1)^{a_{ji}}t_i^{a_{ij}}t_j^{a_{ji}}\Big)$$
$$= \sum_{(a_{ij})\in A} t_1^{\sum_{j\neq 1}a_{1j}}\dots t_r^{\sum_{j\neq r}a_{rj}}\prod_{i<j}(-1)^{a_{ji}}\binom{c}{a_{ji}},$$

where $A = \{(a_{ij})_{1\leqslant i,j\leqslant r, i\neq j} \mid a_{ij} \in \mathbb{Z}_{>0},\ a_{ij} + a_{ji} = c \text{ for every } i \neq j\}$. Hence

$$[I^{(l_1,\dots,l_r)}(z)]_p = [\int_{\{l_1,\dots,l_r\}_p}\Phi_p(t,z)W(t,z)dt_1\dots dt_r]_p = \tag{4.2}$$
$$= \sum_{(a_{ij})\in A}\Big(\prod_{i<j}(-1)^{a_{ji}}\binom{c}{a_{ji}}\Big)[\int_{\{l_1,\dots,l_r\}_p}W(t,z)\prod_{i=1}^r t_i^{\sum_{j\neq i}a_{ij}}\prod_{s=1}^n(t_i - z_s)^M dt_1\dots dt_r]_p.$$

This sum contains the special summand

$$S(z) := [\int_{\{l_1,\dots,l_r\}_p}W(t,z)\prod_{i=1}^r t_i^{(r-i)c}\prod_{s=1}^n(t_i - z_s)^M dt_1\dots dt_r]_p$$

corresponding to the collection $(a_{ij})$ such that $a_{ij} = c$ for $i < j$ and $a_{ij} = 0$ for $i > j$. For $1 \leqslant u \leqslant nM$, denote

$$z(u) := z_s, \qquad \text{if} \qquad (s-1)M < u \leqslant sM. \tag{4.3}$$

Thus $z(1) = z(2) = \dots = z(M) = z_1$, $z(M+1) = z_2$ and so on.

**Lemma 4.3.** *The vector-polynomial $S(z)$ is nonzero. The leading monomial of $S(z)$ equals*

$$z_{l_1,\dots,l_r} := \prod_{i=1}^{r}\prod_{u=1}^{nM+(r-i)c-l_i p} z(u). \tag{4.4}$$

*Denote by $C_{l_1,\dots,l_r}$ the leading coefficient of $S(z)$. Then the leading index of the vector $C_{l_1,\dots,l_r}$ equals $\{m_1,\dots,m_r\}$ determined by inequalities (4.1). Moreover, the leading monomial of any other nonzero summand in (4.2) is lexicographically smaller than $z^{l_1,\dots,l_r}$. Thus $C_{l_1,\dots,l_r}z^{l_1,\dots,l_r}$ is the leading term of $[I^{(l_1,\dots,l_r)}(z)]_p$.*

*Proof.* We rewrite $z^{l_1,\dots,l_r}$ using the integers $m_1,\dots,m_r$,

(4.5) $$z^{l_1,\dots,l_r} = \prod_{i=1}^{r} z_{1M}\cdots z_{mM_i-1}\, z_{m(n_i-m^i+1)M+(r-i)c-l^ip}.$$

Let $z^{l_1,\dots,l_r} = z_1^{d_1}\cdots z_n^{d_n}$ for some $d_1,\dots,d_n$. Then

(4.6)
$$\begin{aligned} d_{m_i} &= (n-m_i+1)M + (r-i)(c+M) - l_ip, && i = 1,\dots,r;\\ d_j &= rM, && j = 1,\dots,m_1-1;\\ d_j &= (r-i)M, \qquad j = m_i+1,\dots,m_{i+1}-1, && i = 1,\dots,r-1;\\ d_j &= 0 && j = m_r+1,\dots,n. \end{aligned}$$

It is clear that $z^{l_1,\dots,l_r}$ is the lexicographically maximal monomial which can be produced by $S(z)$. Let $C_{l_1,\dots,l_r}$ be the coefficient of $z_{l_1,\dots,l_r}$ in $S(z)$. Then $C_{l_1,\dots,l_r} = \sum_{J\in J_r} c_J V_J$ with $c_J \in \mathbb{F}_p$. It is clear that $\{m_1,\dots,m_r\}$ is the leading index of $C_{l_1,\dots,l_r}$, if $c_{\{m_1,\dots,m_r\}}$ is nonzero, but

$$c_{\{m_1,\dots,m_r\}} = \prod_{i=1}^{r}(-1)^{(n-m_i+1)M+(r-i)c-l_ip}\binom{M-1}{(n-m_i+1)M+(r-i)c-l_ip}, \tag{4.7}$$

which is nonzero due to (3.3) and (4.1). Thus $S(z)$ is nonzero, its leading term equals $C_{l_1,\dots,l_r}z^{l_1,\dots,l_r}$, and the leading index of $C_{l_1,\dots,l_r}$ is $\{m_1,\dots,m_r\}$. It remains to prove that the leading monomial of any other nonzero summand in (4.2) is lexicographically smaller than $z_{l_1,\dots,l_r}$.

Let $\tilde{S}(z)$ be any other summand in (4.2),

$$\tilde{S}(z) := \Big(\prod_{i<j}(-1)^{a_{ji}}\binom{c}{a_{ji}}\Big)\Big[\int_{\{l_1,\dots,l_r\}_p} W(t,z)\prod_{i=1}^{r} t_i^{\sum_{j\neq i}a_{ij}}\prod_{s=1}^{n}(t_i-z_s)^M dt_1\dots dt_r\Big]_p.$$

It is clear that the lexicographically maximal monomial which can be produced by $\tilde{S}(z)$ equals

(4.8) $$\tilde{z}^{l_1,\dots,l_r} := \prod_{i=1}^{r}\ \prod_{u=1}^{nM-l_ip+\sum_{j\neq i}a_{ij}} z(u).$$

Assume that $\sum_{j\neq 1} a_{1j} < (r-1)c$. Let $s$ be the least index such that the maximal power of $z_s$ dividing $\prod_{u=1}^{nM-l_1p+\sum_{j\neq 1}a_{1j}} z(u)$ is strictly smaller than the maximal power of $z_s$ (which we denote by $b$) dividing $\prod_{u=1}^{nM+(r-1)c-l_1p} z(u)$. Then the maximal power of $z_s$ dividing $\prod_{i=1}^{r}\prod_{u=1}^{nM-l_ip+\sum_{j\neq i}a_{ij}} z(u)$ is strictly smaller than $b+(r-1)M$. This implies that $\tilde{z}^{l_1,\dots,l_r}$ is lexicographically smaller than $z^{l_1,\dots,l_r}$.

Thus, a summand $\tilde{S}(z)$ must have $\sum_{j\neq 1} a_{ij} = (r-1)c$ in order to have a monomial as large lexicographically as $z^{l_1,\dots,l_r}$. This means that $a_{1j} = c$ and $a_{j1} = 0$ for $j \neq 1$.

Now take any summand $\tilde{S}(z)$ in (4.2) with $\sum_{j=1}^{6} a_{1j} = (r-1)c$. In a similar way we show that $\tilde{S}(z)$ must have $\sum_{j=2}^{6} a_{2j} = (r-2)c$ in order to have a monomial as large lexicographically as $z^{l_1,\dots,l_r}$. Repeating this reasoning we conclude that the special summand $S(z)$ is the only summand in (4.2) which may have the monomial $z^{l_1,\dots,l_r}$ with a nonzero coefficient; and no summands in (4.2) may have a monomial larger than $z^{l_1,\dots,l_r}$. Lemma 4.3 is proved.

4.5. **Proof of Theorem 3.5.** Let $[I^{(l_1,\dots,l_r)}(z)]_p$ be one of the vector-polynomials of Theorem 3.5. Then $[I^{(l_1,\dots,l_r)}(z)]_p$ has the leading term $C_{l_1,\dots,l_r} z^{l_1,\dots,l_r}$ described in Lemma 4.3. Also the leading index $\{m_1,\dots,m_r\}$ of $C_{l_1,\dots,l_r}$ is determined by $l_1,\dots,l_r$ in (4.1). Let $f_{l_1,\dots,l_r}(z) \in \mathsf{F}_p[z]$. Then the leading term of the vector-polynomial $f_{l_1,\dots,l_r}(z)[I^{(l_1,\dots,l_r)}(z)]_p$ equals the product of leading terms of $f_{l_1,\dots,l_r}(z)$ and $[I^{(l_1,\dots,l_r)}(z)]_p$. Moreover, the leading index of the leading coefficient of $f_{l_1,\dots,l_r}(z)[I^{(l_1,\dots,l_r)}(z)]_p$ equals the leading index $\{m_1,\dots,m_r\}$ of the leading coefficient of $[I^{(l_1,\dots,l_r)}(z)]_p$.

Consider a sum $\sum_{kg+r-1>l_1>\dots>l_r>1} f_{l_1,\dots,l_r}(z)[I_{(l_1,\dots,l_r)}(z)]_p$ as in Theorem 3.5. Then all nonzero summands have different leading indices due to the previous remark and Lemma 4.2. This implies that such a sum is not zero if it has nonzero summands. Theorem 3.5 is proved.

**Example.** Let $q = 2$, $n = 5$, $r = 2$. Then Theorem 3.5 says that for any odd prime $p$ there is just one $p$-hypergeometric solution $[I^{(2,1)}(z_1,\dots,z_5)]_p$. This polynomial is homogeneous of degree $2p - 4$ and takes values in $\mathrm{Sing}W^{\otimes 5}[1]_p$. The leading term of $[I^{(2,1)}(z_1,\dots,z_5)]_p$ is $C_{2,1} z_1^{p-2} z_2^{2(p-1)/2} z_3^{(p-3)/2}$ where the leading coefficient $C_{2,1} \in W^{\otimes 5}[1]_p$ has the leading index $\{m_1,m_2\} = \{1,3\}$.

Notice that the space $\mathrm{Sing}W^{\otimes 5}[1]$ has dimension 5.

4.6. **Leading terms and eigenvectors.** Let $I(z) \in \mathrm{Sing}W^{\otimes n}[n-2r]_p \otimes \mathsf{F}_p[z]$ be a polynomial solution of the KZ equations (2.6) with a positive integer parameter $q$ (not necessarily a $p$-hypergeometric solution). Let $C z_1^{d_1} \cdots z_n^{d_n}$ be its leading term, $C \in \mathrm{Sing}W^{\otimes n}[n-2r]_p$. Then we have modulo $p$,

$$\sum_{\ell=j+1}^{n} \bar{\Omega}_{j,\ell} C \equiv q d_j C, \qquad j = 1,\dots,n-1, \qquad d_n \equiv 0, \tag{4.9}$$

see [V7, Lemma 5.1]. Thus the leading coefficient $C$ is an eigenvector of the linear operators $\bar{\Omega}_j = \sum_{\ell=j+1}^{n} \bar{\Omega}_{j,\ell}$, $j = 1,\dots,n-1$, with prescribed eigenvalues.

An eigenbasis of the operators $\bar{\Omega}_j$, $j = 1,\dots,n-1$, on $\mathrm{Sing}W^{\otimes n}[n-2r]_p$ is formed by the so-called iterated singular vectors, for example see [V2]. Such an iterated vector is determined by its leading index.

If $p$ is large enough with respect to $n$, then the vectors of that eigenbasis are separated by eigenvalues.

Let $[I^{(l_1,\dots,l_r)}(z)]_p$ be one of the $p$-hypergeometric solutions of Theorem 3.5. Let $C_{l_1,\dots,l_r} z_1^{d_1}\cdots z_n^{d_n}$ be its leading term, where $d_j$ are defined in (4.6). Let $\{m_1,\dots,m_r\}$ be the leading index of $C_{l_1,\dots,l_r}$. If $p$ is large enough with respect to $n$, then $C_{l_1,\dots,l_r}$ is the iterated singular vector with leading index $\{m_1,\dots,m_r\}$. It is the eigenvector of the operators $\bar{\Omega}_j$, $j = 1,\dots,n-1$, with eigenvalues defined by formula (4.9).

## 5. Solutions and a Cartier map

In this section we discuss how the two objects:

- the set of indices $(l_1,\dots,l_r)$ with $kg+r-1 > l_1 > \cdots > l_r > 1$, appearing in Theorem 3.5,
- and the number of such indices $\binom{kg+r-1}{r}$, appearing in Corollary 3.6, are related to the integrand $\Phi(t,z)W(t,z)dt_1\wedge\cdots\wedge dt_r$ of the integral representation of complex hypergeometric solutions of the KZ equations, appearing in Theorems 2.1 and 2.2.

Recall that the complex hypergeometric solutions with values in $\mathrm{Sing} W^{\otimes n}[n-2r]$ are given by the formulas:

$$I^{(\gamma)}(z) = \mathrm{R}_{\gamma(z)}\,\Phi(t,z)W(t,z)dt_1\wedge\cdots\wedge dt_r,$$

$$\Phi(t,z) = \mathrm{Q}_{1\le i<j\le r}(t_i-t_j)^{2/q}\,\mathrm{Q}_{ri=1}\,\mathrm{Q}_{ns=1}(t_i-z_s)^{-1/q},$$

$$W(t,z) = \mathrm{P}_{1\le i_1<\cdots<i_r\le n} W_{i_1,\dots,i_r}(t,z)V_{i_1,\dots,i_r}, \qquad W_{i_1,\dots,i_r}(t,z) = \mathrm{Sym}^{t_1,\dots,t_r}\prod_{j=1}^{r}\frac{1}{t_j-z_{i_j}},$$

see Section 2.3, while the corresponding $p$-hypergeometric solutions are given by the formulas:

$$I_{(l_1,\dots,l_r)})(z) = \mathrm{R}_{\{l_1,\dots,l_r\}p}\,\Phi_p(t,z)W(t,z)dt_1\dots dt_r,$$

$\Phi_p(t,z) = \mathrm{Q}_{1\le i<j\le r}(t_i-t_j)_c\,\mathrm{Q}_{ri=1}\,\mathrm{Q}_{ns=1}(t_i-z_s)_M$, $W(t,z) = \mathrm{P}_{J\in J_r} W_J(t,z)V_J$, see Section 2.5.

5.1. **Square integrability criterion.** Let $M$ be a complex manifold of complex dimension $r$. If $f$ is a meromorphic function on $M$ and $S$ is an irreducible subvariety of $M$, then the order of $f$ along $S$, $\mathrm{ord}_S(f)$, is the coefficient of the exceptional divisor of the blow up of $S$ at the divisor of $f$. This notion generalizes to the setting where $f$ has only finitely many determinations, which means that $f$ becomes univalued on a finite (possibly ramified) cover of $M$. Then $f$ has at a generic point of the exceptional divisor a fractional order.

Let $\omega$ be a multivalued meromorphic $r$-form on $M$ with only finitely many determinations and let $S$ be an irreducible subvariety of $M$. Write $\omega$ at some point $s$ of $S$ as $f\omega_0$ where $\omega_0$ is $d$-form on $M$ that is nonzero at $s$ and $f$ is multivalued meromorphic at $s$. The logarithmic order of $\omega$ along $S$ is $\mathrm{codim}(S) + \mathrm{ord}_{S;s}(f)$. This only depends on $\omega$ and $S$.

Suppose that $D \subset M$ is a hypersurface which is arrangement-like in the sense that $D$ can be covered by analytic coordinate charts of M on which D is given by a product of linear forms in the coordinates, see [STV]. It is clear that D then comes with a natural partition into connected, locally closed submanifolds, its strata.

We say that a stratum $S$ is decomposable if at a generic point $s \in S$ the germ $D_s$ of the hyperplane arrangement can be decomposed into the disjoint sum $A_1 \cup A_2$ of two germs of hyperplane arrangements and, after a suitable linear coordinate change, defining equations for $A_1$ and $A_2$ have no common variables. We say that a stratum $S$ is dense if it is of codimension 1 or if it is not decomposable, see [V1, STV]. We have the following trivial observation.

**Proposition 5.1** ([LV])**.** *Suppose $M$ is compact and $\omega$ is a meromorphic multivalued rform on $M$ with only finitely many determinations and whose polar set is contained in an arrangement-like hypersurface $D$. Then the following properties are equivalent:*

(i) *the form $\omega$ is square integrable in the sense that $\mathrm{R}_M\, \omega \wedge \bar{\omega}$ converges;*

(ii) *the form $\omega$ has positive logarithmic order along any dense stratum of $D$;*

(iii) *if $\tilde{M} \to M$ is a proper surjective map with $\tilde{M}$ a complex manifold of the same dimension as $M$ and such that $\omega$ becomes a univalued $d$-form on $\tilde{M}$, then the latter form is regular.*

5.2. **Square integrable differential $r$-forms.** Let $t = (t_1,...,t_r)$ be coordinates on $\mathbb{C}^r \subset (\mathbb{P}^1)^r$ and $z = (z_1,...,z_n)$ distinct parameters. Consider a differential $r$-form

$$\begin{aligned} \omega &= P(t)\,\Phi(t,z)^k dt \\ &= P(t)\cdot \prod_{i=1}^{r}\prod_{s=1}^{n}(t_i - z_s)^{-k/q}\cdot \prod_{1\leqslant i<j\leqslant n}(t_i - t_j)^{2k/q}\cdot dt_1\wedge\cdots\wedge dt_r \end{aligned} \tag{5.1}$$

on $\mathbb{C}^r \subset (\mathbb{P}^1)^r$. Here $k$, $0 < k < q$, is a positive integer; $dt = dt_1 \wedge \cdots \wedge dt_r$; and $P(t)$ is a polynomial in $t$.

Let $n = qg + 2r - 1$ where $g$ is some positive integer, cf. (3.4). **Theorem 5.2.**

*The form $\omega$ is square integrable on $(\mathbb{P}^1)^r$ if and only if*

$$\deg_{t_i} P \leqslant kg - 1 \qquad \text{forall} \qquad i = 1,...,r. \tag{5.2}$$

*Proof.* Denote by $D$ the support of the divisor of $\omega$. The support $D$ lies in the union of hypersurfaces defined by $t_i = z_s$, $t_i = \infty$, and $t_i = t_j$ for $i < j$. This union is clearly arrangement-like. Its dense strata of codimension $l$ are:

(i) diagonals in $(\mathbb{P}^1)^r$ defined by letting $l > 2$ coordinates to coalesce;

(ii) loci in $(\mathbb{P}^1)^r$ defined by setting $l > 1$ coordinates equal to $\infty$; (iii) loci in $(\mathbb{P}^1)^r$ defined by setting $l > 1$ coordinates equal to some $z_s$.

Let $S$ be defined by the equations $t_{i1} = \cdots = t_{il}$ for some $1 \leqslant i_1 < \cdots < i_l \leqslant r$. Then the logarithmic order of $\omega$ along $S$ is $\geqslant l - 1 + \binom{l}{2} \cdot 2k/q > 0$.

Let $S$ be defined by the equations $t_{i1} = \cdots = t_{il} = z_s$ for some $1 \leqslant i_1 < \cdots < i_l \leqslant r$. Then the logarithmic order $\omega$ along $S$ is $\geqslant l - lk/q + \binom{l}{2} \cdot 2k/q = l(1 - k/q) + \binom{l}{2} \cdot 2k/q > 0$.

Let $S$ be defined by the equation $t_{i1} = \infty$ for some $i_1$. In the coordinates $u_1,\ldots,u_r$, $u_i = 1/t_i$ for $i = 1,\ldots,r$, we have

$$\omega = (-1)^r \Big( P(1/u_1, \ldots, 1/u_r, z) \cdot \prod_{i=1}^{r} u_r^{\deg_{t_i} P} \Big) \cdot \prod_{i=1}^{r} \prod_{s=1}^{n} (1 - z_s u_i)^{-k/q} \cdot \tag{5.3}$$

$$\cdot \prod_{1 \leqslant i < j \leqslant n} (u_j - u_i)^{2k/q} \cdot u_{-r \deg^{t_i} P + nk/q - (r-1)2k/q - 2} \cdot du_1 \wedge \cdots \wedge du_r$$

Hence the logarithmic order of $\omega$ along $S$ equals

$$1 + nk/q - (r-1)2k/q - 2 - \deg_{t_{i1}} P = (n - 2(r-1))k/q - 1 - \deg_{t_{i1}} P = kg - 1 - \deg_{t_{i1}} P + k/q.$$

Hence the logarithmic order along $S$ is positive if and only if $\deg_{t_{i1}} P \leqslant kg - 1$.

Let $S$ be defined by the equations $t_{i1} = \cdots = t_{il} = \infty$ for some $1 \leqslant i_1 < \cdots < i_l \leqslant r$. It follows from (5.3) that the logarithmic order along $S$ is positive if the logarithmic order is positive along every hyperplpane defined by the equation $t_{ij} = \infty$ for $j = 1,\ldots,l$. The theorem is proved.

5.3. **Schur polynomials.** A sequence of integers $a = (a_1 > a_2 > \cdots > a_r > 0)$ is called a partition. For a partition $a$ the polynomial $m_a(t) = \operatorname{Sym}_t t_1^{a_1} \cdots t_r^{a_r}$ is called a symmetric monomial function. The polynomial

$$s_{(a_1,\ldots,a_r)}(t) = \frac{\det (t_i^{a_j + r - j})_{i,j=1,\ldots,r}}{\prod_{1 \leqslant i < j \leqslant r} (t_i - t_j)}$$

is called a Schur polynomial. It is known that

$$s_a(t) = \sum_{b \leqslant a} K_{a,b}\, m_{b_1,\ldots,b_r}(t), \tag{5.4}$$

where $K_{a,b}$ are nonnegative integers, $K_{a,a} = 1$. The inequality $b \leqslant a$ means $\sum_{j=1}^{i} b_j \leqslant \sum_{j=1}^{i} a_j$ for all $i = 1,\ldots,r$. The numbers $K_{a,b}$ are called the Kostka numbers. For a positive integer $d$, denote

$$A(d) = \{(a_1,\ldots,a_r) \mid d > a_1 > a_2 > \cdots > a_r > 0,\ a_i \in \mathbb{Z}\}.$$

Let $V(d)$ be the free $\mathbb{Z}$-module with basis $\{m_a(t)|a \in A(d)\}$. The module $V(d)$ has rank $\binom{d+r}{r}$. The set $\{s_a(t)|a \in A(d)\}$ of Schur polynomials is a basis of $V(d)$ by formula (5.4).

Let us return to Theorem 5.2. Let W be the vector space of all differential $r$-forms $\omega = P(t)\Phi(t,z)^k dt$ such that $P(t)$ is a polynomial in $t_1,\dots,t_r$ symmetric with respect to permutations of $t_1,\dots,t_r$, and $\omega$ is square integrable on $(\mathbb{P}^1)^r$.

**Corollary 5.3.** *The set* $\{s_a(t)\Phi(t,z)^k dt|a \in A(kg-1)\}$ *of differential r-forms on* $(\mathbb{P}^1)^r$ *is a basis of* W. *The vector space* W *has dimension* $\binom{kg+r-1}{r}$. Notice that this binomial coefficient equals the rank of the module $M/(\sim)$ in Corollary 3.6.

We introduce the following $\mathbb{F}_p[z]$-variant of the vector space W. Let $W_p[z]$ be the free $\mathbb{F}_p[z]$-module with basis B of formal algebraic differential $r$-forms

$$\omega_a = s_a(t)\Phi(t,z)^k dt, \qquad a \in A(kg-1).$$

Let $B^* = \{\omega^a \,|a \in A(kg-1)\}$ be the collection of $\mathbb{F}_p[z]$-linear functions on $W_p[z]$ such that $\langle\omega^a,\omega_b\rangle = \delta_{a,b}$ for all $a,b \in A(kg-1)$.

5.4. **Cartier map.** Assume that $(p,q)$ is of type 1. Hence $q|(kp-1)$, $0 < k \leq q/2$.

We define a map which sends every differential form $\Phi(t,z)W_J(t,z)dt$, $J \in \mathrm{J}_r$, to $W_p[z]$, that is, to a linear combination of differential $r$-forms $\omega_a$, $a \in A(kg-1)$, with coefficients in $\mathbb{F}_p[z]$. We call this map the Cartier map. We have

$$\begin{aligned}
&\Phi(t,z)W_J(t,z)dt = \frac{\Phi(t,z)^{kp}}{\Phi(t,z)^{kp-1}}W_J(t,z)dt \\
&= \Phi(t,z)^{kp}\prod_{1\leqslant i<j\leqslant r}(t_i-t_j)^{(2-2kp)/q}\prod_{i=1}^{r}\prod_{s=1}^{n}(t_i-z_s)^{(kp-1)/q}W_J(t,z)dt \\
&= \frac{\Phi(t,z)^{kp}}{\prod_{1\leqslant i<j\leqslant r}(t_i-t_j)^p}\prod_{1\leqslant i<j\leqslant r}(t_i-t_j)^{(2+(q-2k)p)/q}\prod_{i=1}^{r}\prod_{s=1}^{n}(t_i-z_s)^{(kp-1)/q}W_J(t,z)dt \\
&= \frac{\Phi(t,z)^{kp}}{\prod_{1\leqslant i<j\leqslant r}(t_i-t_j)^p}\prod_{1\leqslant i<j\leqslant r}(t_i-t_j)^{c}\prod_{i=1}^{r}\prod_{s=1}^{n}(t_i-z_s)^{M}W_J(t,z)dt \\
&= \frac{\Phi(t,z)^{kp}}{\prod_{1\leqslant i<j\leqslant r}(t_i-t_j)^p}\Phi_p(t,z)W_J(t,z)dt
\end{aligned}$$

where $c, M$, $\Phi_p(t,z)$ are defined in (3.2) and (2.12).

Since $(p,q)$ is of type 1, the integer $c$ is odd. Hence the polynomial $\Phi_p(t,z)W_J(t,z)$ is skew-symmetric with respect to permutations of $t_1,\dots,t_r$. We expand $\Phi_p(t,z)W_J(t,z)$ with respect to the $t$-variables as follows:

$$\begin{aligned}\Phi_p(t,z)W_J(t,z) &= \\ &= (t_1\dots t_r)^{p-1}\sum_{(a_1,\dots,a_r)\in A(kg-1)} c_J^{(a_1,\dots,a_r)}(z)\Big(\sum_{\sigma\in S_r}(-1)^{|\sigma|}t_{\sigma(1)}^{(a_1+r-1)p}t_{\sigma(1)}^{(a_2+r-2)p}\dots t_{\sigma(r)}^{a_rp}\Big)+\sum{}' \\ &= (t_1\dots t_r)^{p-1}\sum_{(a_1,\dots,a_r)\in A(kg-1)} c_J^{(a_1,\dots,a_r)}(z)s_{(a_1,\dots,a_r)}(t_1^p,\dots,t_r^p)\prod_{1\leqslant i<j\leqslant r}(t_i^p-t_j^p)+\sum{}',\end{aligned}$$

where $\sum'$ denotes the sum of the monomials $t_1^{d_1}\cdots,t_r^{d_r}$ such that at least one of $d_1,\dots,d_r$ is not of the form $lp-1$ for some positive integer $l$; the coefficients $c_J^{(a_1,\dots,a_r)}(z)$ are suitable polynomials in $z$.

Returning to $\Phi(t,z)W_J(t,z)dt$ we write

$$\begin{aligned}\Phi(t,z)W_J(t,z)dt = \\ = \sum_{(a_1,\dots,a_r)\in A(kg-1)} c_J^{(a_1,\dots,a_r)}(z)s_{(a_1,\dots,a_r)}(t_1^p,\dots,t_r^p)\prod_{1\leqslant i<j\leqslant r}\frac{t_i^p-t_j^p}{(t_i-t_j)^p}\Phi(t,z)^{kp}(t_1\dots t_r)^{p-1}dt \\ +\ \frac{\sum'}{\prod_{1\leqslant i<j\leqslant r}(t_i-t_j)^p}\Phi(t,z)^{kp}dt\,.\end{aligned}$$

Notice that $\frac{t_i^p-t_j^p}{(t_i-t_j)^p}\equiv 1 \pmod p$. We define the *Cartier map* C by the formula

$$\mathrm{C}:\Phi(t,z)W_J(t,z)dt \mapsto \sum_{(a_1,\dots,a_r)\in A(kg-1)} [c_J^{(a_1,\dots,a_r)}(z)]_p\, s_{(a_1,\dots,a_r)}(t_1,\dots,t_r)\Phi(t,z)^k dt, \tag{5.5}$$

cf. [AH].

Recall the collection $\{\omega^{(a_1,\dots,a_r)}\,|\,(a_1,\dots,a_r)\in A(kg-1)\}$ of linear functions on $\mathrm{W}_p[z]$. We have

$$\left\langle \omega^{(a_1,\dots,a_r)},\, \mathcal{C}(\Phi(t,z)W_J(t,z)dt)\right\rangle = [c_J^{(a_1,\dots,a_r)}(z)]_p.$$

**Theorem 5.4.** *For any $p$-hypergeometric solution $[I^{(a_1+r,a_2+r-1,\dots,a_r+1)}(z)]_p$, $kg-1>a_1>a_2>\dots>a_r>0$ we have*

$$[I_{(a_1+r,a_2+r-1,\dots,a_r+1)}(z)]_p = \sum_{J\in \mathrm{J}_r}[c_{J(a_1,\dots,a_r)}(z)]_p V_J. \tag{5.6}$$

*Proof.* The proof follows from the definition of $I^{(a_1+r,a_2+r-1,\dots,a_r+1)}(z)$ in (2.13). □

Formula (5.6) can be reformulates as follows. For any integers $(l_1,\dots,l_r)$, $kg+r-1>l_1>\dots>l_r>1$, the $p$-hypergeometric solution $[I^{(l_1,\dots,l_r)}(z)]_p$ is given by the formula

$$[I^{(l_1,\dots,l_r)}(z)]_p = \left\langle \omega^{(l_1-r,\,l_2-r+1,\,\dots,\,l_r-1)},\, \mathcal{C}(\Phi(t,z)W(t,z)dt)\right\rangle. \tag{5.7}$$

Notice that $\Phi(t,z)W(t,z)dt$ is the integrand of the integral representation of complex hypergeometric solutions of the *KZ* equations, see (2.10), while $[I^{(l_1,\dots,l_r)}(z)]_p$ is a solution of the *KZ* equations over $\mathbb{F}_p$.

Formula (5.7) for $r = 1$ and two prime numbers ($p > q$) not necessarily of type 1 is the subject of [SlV, Theorem 6.2].

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