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A NOTE ON THE ERDŐS-HAJNAL HYPERGRAPH RAMSEY PROBLEM

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Abstract. We show that there is an absolute constant $c > 0$ such that the following holds. For every $n > 1$, there is a 5-uniform hypergraph on at least $2^{2^{cn^{1/4}}}$ vertices with independence number at most n, where every set of 6 vertices induces at most 3 edges. The double exponential growth rate for the number of vertices is sharp. By applying a stepping-up lemma established by the first two authors, analogous sharp results are proved for k-uniform hypergraphs. This answers the penultimate open case of a conjecture in Ramsey theory posed by Erdős and Hajnal in 1972.

1. Introduction

The Ramsey number $r_k(s, n)$ is the minimum integer N such that for any red/blue coloring of the k-tuples of $[N] = \{1, 2, \dots, N\}$, there is either a set of s integers with all of its k-tuples colored red, or a set of n integers with all of its k-tuples colored blue. Estimating $r_k(s, n)$ is a fundamental problem in combinatorics and has been extensively studied since 1935. For graphs, classical results of Erdős [7] and Erdős and Szekeres [12] imply that $2^{n/2} < r_2(n, n) < 2^{2n}$. While small improvements have been made in both the upper and lower bounds for $r_2(n, n)$ (see [4, 15]), the constant factors in the exponents have not changed over the last 75 years.

Unfortunately for 3-uniform hypergraphs, there is an exponential gap between the best known upper and lower bounds for $r_3(n, n)$. Namely, Erdős, Hajnal, and Rado [10, 11] showed that

$$2^{cn^2} < r_3(n, n) < 2^{2^{c'n}},$$

where c and c' are absolute constants. For $k \geq 4$, their results also imply an exponential gap between the lower and upper bounds for $r_k(n, n)$,

$$\mathrm{twr}_{k-1}(cn^2) < r_k(n, n) < \mathrm{twr}_k(c'n),$$

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where the *tower function* is defined recursively as $\mathrm{twr}_1(x) = x$ and $\mathrm{twr}_{i+1} = 2^{\mathrm{twr}_i(x)}$. Determining the tower growth rate of $r_k(n, n)$ is one of the most central problems in extremal combinatorics. Erdős, Hajnal, and Rado conjectured that the upper bound is closer to the truth, namely $r_k(n, n) = \mathrm{twr}_k(\Theta(n))$, and Erdős offered a \$500 reward for a proof (see [5]).

Off-diagonal Ramsey numbers $r_k(s, n)$ have also been extensively studied. Here, k and s are fixed constants and n tends to infinity. It follows from well-known results that $r_2(s, n) = n^{\Theta(1)}$ (see [1, 2, 3, 11] for the best known bounds), and for 3-uniform hypergraphs, $r_3(s, n) = 2^{n^{\Theta(1)}}$ (see [6] for the best known bounds).

For $k > 3$, Erdős, Hajnal, and Rado showed that $r_k(s, n) \leq \mathrm{twr}_{k-1}(n^c)$ where $c = c(k, s)$, and Erdős and Hajnal conjectured that this bound is the correct tower growth rate. In [13], the first two authors verified the conjecture for $s \geq k+2$, and for the last case $s = k+1$, they showed that $r_k(k+1, n) \geq \mathrm{twr}_{k-2}(n^{c \log n})$. Hence, there remains an exponential gap between the best known lower and upper bounds for $r_k(k+1, n)$ for $k \geq 4$.

Due to our lack of understanding of $r_k(k+1, n)$, Erdős and Hajnal in [9] introduced the following more general function (their notation was different).

Definition 1.1. For integers $2 \leq k < n$ and $2 \leq t \leq k+1$, let $r_k(k+1, t; n)$ be the minimum N such that for every red/blue coloring of the k-tuples of $[N]$, there is a set of $k+1$ integers with at least t of its k-tuples colored red, or a set of n integers with all of its k-tuples colored blue.

Clearly $r_k(k+1, 1; n) = n$ and $r_k(k+1, k+1; n) = r_k(k+1, n)$. For each $t \in \{2, \ldots, k\}$, Erdős and Hajnal [9] showed that $r_k(k+1, t; n) < \mathrm{twr}_{t-1}(n^{\Theta(1)})$ and conjectured that

$$r_k(k+1, t; n) = \mathrm{twr}_{t-1}(n^{\Theta(1)}). \tag{1.1}$$

This is known to be true for $k \leq 3$ and for $t \leq 3$ [9]. When $k \geq 5$, the first two authors [14] verified (1.1) for all $3 \leq t \leq k-2$. Our main result verifies (1.1) for $t = k-1$, which is one of the last two remaining cases.

Theorem 1.2. *For $k \geq 4$, we have $r_k(k+1, k-1; n) = \mathrm{twr}_{k-2}(n^{\Theta(1)})$.*

This significantly improves the previous best known lower bound for $r_k(k+1, k-1; n)$, which was one exponential less than above (see [14]). This also immediately implies the following new lower bound for $r_k(k+1, k; n)$, which is now one exponential off from the upper bound obtained by Erdős and Hajnal.

Corollary 1.3. *For $k \geq 4$, we have $r_k(k+1, k; n) > \mathrm{twr}_{k-2}(n^{\Theta(1)})$.*

Finally, let us point out that Erdős and Hajnal conjectured that the tower growth rate for both $r_k(k+1, k; n)$ and the classical Ramsey number $r_k(k+1, n)$ are the same. Thus, verifying (1.1) for $r_k(k+1, k; n)$ would determine the tower height for $r_k(k+1, n)$.

We develop several crucial new ingredients to the stepping up method in our construction, for example, part (1) of Lemma 2.3, and on page 8, analyzing sequences of local maxima. It is plausible that these new ideas can be further enhanced to determine the tower height of $r_k(k+1, n)$.

2. Proof of Theorem 1.2

In [13], the first two authors proved the following.

Theorem 2.1 (Theorem 7 in [13]). *For $k \geq 6$ and $t \geq 5$, we have $r_k(k+1, t; 2kn) > 2^{r_{k-1}(k,t-1;n)-1}$.*

In what follows, we will prove the following theorem. Together with Theorem 2.1, Theorem 1.2 quickly follows.

Theorem 2.2. *There is an absolute constant $c > 0$ such that $r_5(6, 4; n) > 2^{2^{cn^{1/4}}}$.*

2.1. **A double exponential lower bound for $r_5(6, 4; n)$.** In this section, we begin with a graph coloring with certain properties which we will later use to define a two-coloring of the edges of a 5-uniform hypergraph.

Lemma 2.3. *For $n \geq 6$, there is an absolute constant $c > 0$ such that the following holds. There exists a red/blue coloring ϕ of the pairs of $\{0, 1, \ldots, \lfloor 2^{cn} \rfloor - 1\}$ such that:*

1. *There are no 3 disjoint n-sets $A, B, C \subset \{0, 1, \ldots, \lfloor 2^{cn} \rfloor - 1\}$ with the property that there is a bijection $f : B \to C$ such that for any $a \in A, b \in B$, at least one of $\phi(a, b) = $ red or $\phi(a, f(b)) = $ blue occurs.*
2. *There is no n-set $A \subset \{0, 1, \ldots, \lfloor 2^{cn} \rfloor - 1\}$ with the property that every 4-tuple $a_i, a_j, a_k, a_\ell \in A$ with $a_i < a_j < a_k < a_\ell$ avoids the pattern:*

$$\phi(a_i, a_j) = \phi(a_j, a_k) = \phi(a_j, a_\ell) = \text{red}, \qquad \phi(a_i, a_k) = \phi(a_i, a_\ell) = \phi(a_k, a_\ell) = \text{blue}$$

Proof. Set $D = \lfloor 2^{cn} \rfloor$, where c is a sufficiently small constant that will be determined later. Consider a random 2-coloring of the unordered pairs of $\{0, 1, \ldots, D-1\}$ where each pair is assigned red or blue with equal probability independent of all other pairs. Then, the expected number of A, B, C as in part 1 is at most

$$\binom{D}{n}^3 n! \left(\frac{3}{4}\right)^{n^2} < \frac{1}{3},$$

where the inequality holds by taking c sufficiently small. This is since we pick each of the n-sets, one of $n!$ possible bijections from B to C, and then there is a $\frac{3}{4}$ probability that we have the desired color pattern for each pair of $a \in A, b \in B$.

We call a 4-tuple $a_i, a_j, a_k, a_\ell \in \{0, 1, \ldots, D-1\}$ with $a_i < a_j < a_k < a_\ell$ *bad* if

$$\phi(a_i, a_j) = \phi(a_j, a_k) = \phi(a_j, a_\ell) = \text{red}, \qquad \phi(a_i, a_k) = \phi(a_i, a_\ell) = \phi(a_k, a_\ell) = \text{blue}$$

and *good* otherwise. The probability that such a fixed 4-tuple is bad is $\frac{1}{2^6} = \frac{1}{64}$ and thus the probability that such a fixed 4-tuple is good is $\frac{63}{64}$. Now consider some fixed n-set $A \subset \{0, 1, \ldots, D-1\}$. We estimate the probability that A contains no bad 4-tuple. Note that there exists a partial Steiner $(n, 4, 2)$-system S on A, i.e. a 4-uniform hypergraph on the n-vertex set A with the property that every pair of vertices is contained in at most one 4-tuple, with at least $c'n^2$ edges where $c' > 0$ is some constant (e.g. see [8]). Then, the probability that a 4-tuple in A is good is at most the probability that every 4-tuple in S is good. Since 4-tuples in S are independent as no two 4-tuples have more than one vertex is common, the probability that every 4-tuple in S is a good 4-tuple is at most $\left(\frac{63}{64}\right)^{c'n^2}$. Therefore, the expected number of n-sets A with only good 4-tuples is at most

$$\binom{D}{n} \left(\frac{63}{64}\right)^{c'n^2} < \frac{1}{3},$$

again where we take c sufficiently small. Thus, by Markov's inequality and the union bound, we conclude that there is a 2-coloring ϕ with the desired properties. $\square$

We will use this lemma to produce a coloring of a 5-uniform hypergraph. Given some natural number D, let $V = \{0, 1, \dots, 2^D - 1\}$. Then for $v \in V$, we write $v = \sum_{i=0}^{D-1} v(i)2^i$ where $v(i) \in \{0, 1\}$ for each i. For any $u \neq v$, we then let $\delta(u, v)$ denote the largest $i \in \{0, 1, \dots, D-1\}$ such that $u(i) \neq v(i)$. We then have the following properties.

Property I: For every triple $u < v < w, \delta(u, v) \neq \delta(v, w)$.

Property II: For $v_1 < \cdots < v_r, \delta(v_1, v_r) = \max_{1 \leq j \leq r-1} \delta(v_j, v_{j+1})$.

From Properties I and II, we also derive the following.

Property III: For every 4-tuple $v_1 < \cdots < v_4$, if $\delta(v_1, v_2) > \delta(v_2, v_3)$, then $\delta(v_1, v_2) \neq \delta(v_3, v_4)$. Note that if $\delta(v_1, v_2) < \delta(v_2, v_3)$, it is possible that $\delta(v_1, v_2) = \delta(v_3, v_4)$.

Property IV: For $v_1 < \cdots < v_r$, set $\delta_j = \delta(v_j, v_{j+1})$ for $j \in [r-1]$ and suppose that $\delta_1, \dots, \delta_{r-1}$ forms a monotone sequence. Then for every subset of k vertices $v_{i_1}, v_{i_2}, \dots, v_{i_k}$ where $v_{i_1} < \cdots < v_{i_k}$, $\delta(v_{i_1}, v_{i_2}), \delta(v_{i_2}, v_{i_3}), \dots, \delta(v_{i_{k-1}}, v_{i_k})$ forms a monotone sequence. Moreover for every subset of $k-1$ such δ_j's, i.e. $\delta_{j_1}, \delta_{j_2}, \dots, \delta_{j_{k-1}}$, there are k vertices $v_{i_1}, \dots, v_{i_k}$ such that $\delta(v_{i_t}, v_{i_{t+1}}) = \delta_{j_t}$.

We now turn to the coloring of a 5-uniform hypergraph. Let $c > 0$ be the constant given by Lemma 2.3 and let $U = \{0, 1, \dots, \lfloor 2^{cn} \rfloor - 1\}$ and $\phi : \binom{U}{2} \to \{\text{red}, \text{blue}\}$ be a 2-coloring of the pairs of U satisfying the properties given in the lemma. Now let $N = 2^{\lfloor 2^{cn} \rfloor}$ and let $V = \{0, 1, \dots, N-1\}$. In the following, we will use the coloring ϕ to define a red/blue coloring $\chi : \binom{V}{5} \to \{\text{red}, \text{blue}\}$ of the 5-tuples of V such that χ produces at most 3 red edges among any 6 vertices and χ does not produce a blue copy of $K^{(5)}_{128n^4}$. This would imply that $r_5(6, 4; n) > 2^{2^{c' n^{1/4}}}$ for some constant $c' > 0$.

For $v_1, \dots, v_5 \in V$ with $v_1 < v_2 < \cdots < v_5$, let $\delta_i = \delta(v_i, v_{i+1})$. We set $\chi(v_1, \dots, v_5) = \text{red}$ if:

(1) We have that $\delta_1, \delta_2, \delta_3, \delta_4$ are monotone and form a bad 4-tuple, that is, if $\delta_1 < \delta_2 < \delta_3 < \delta_4$ then:

$$\phi(\delta_1, \delta_2) = \phi(\delta_2, \delta_3) = \phi(\delta_2, \delta_4) = \text{red}, \qquad \phi(\delta_1, \delta_3) = \phi(\delta_1, \delta_4) = \phi(\delta_3, \delta_4) = \text{blue},$$

and if $\delta_1 > \delta_2 > \delta_3 > \delta_4$ then:

$$\phi(\delta_4, \delta_3) = \phi(\delta_3, \delta_2) = \phi(\delta_3, \delta_1) = \text{red}, \qquad \phi(\delta_4, \delta_2) = \phi(\delta_4, \delta_1) = \phi(\delta_2, \delta_1) = \text{blue}.$$

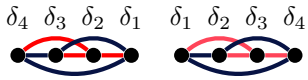
(2) We have that $\delta_1 > \delta_2 < \delta_3 > \delta_4$, where $\delta_1, \delta_2, \delta_3, \delta_4$ are all distinct with $\delta_1 < \delta_3, \delta_2 > \delta_4$ and $\phi(\delta_1, \delta_4) = \text{red}$, $\phi(\delta_2, \delta_4) = \text{blue}$. The ordering can also be expressed as $\delta_3 > \delta_1 > \delta_2 > \delta_4$.

(3) We have that $\delta_1 < \delta_2 > \delta_3 < \delta_4$, where $\delta_1, \delta_2, \delta_3, \delta_4$ are all distinct with $\delta_1 < \delta_3, \delta_2 > \delta_4$ and $\phi(\delta_1, \delta_4) = \text{red}$, $\phi(\delta_1, \delta_3) = \text{blue}$. The ordering can also be expressed as $\delta_2 > \delta_4 > \delta_3 > \delta_1$.

(4) We have that $\delta_1 < \delta_2 > \delta_3 < \delta_4$ and $\delta_1 = \delta_4$. In other words, $\delta_2 > \delta_1 = \delta_4 > \delta_3$.

Otherwise $\chi(v_1, \dots, v_5) = \text{blue}$.

Assume for the sake of contradiction that there are at least 4 red edges among some 6 vertices. Let these vertices be $v_1, \dots, v_6$ where $v_1 < v_2 < \cdots < v_6$ and let $\delta_i = \delta(v_i, v_{i+1})$. Let $e_i = \{v_1, \dots, v_6\} \setminus \{v_i\}$. Let $\delta(e_i)$ be the resulting sequence of δ's. In particular, for $i = 1$, $\delta(e_1) = (\delta_2, \delta_3, \delta_4, \delta_5)$. For $2 \leq i \leq 5$, $\delta(e_i) =$



	δ_4	δ_3	δ_2	δ_1	δ_1	δ_2	δ_3	δ_4
v_1:	0	0	0	0	0	0	0	0
v_2:	0	0	0	1	1	0	0	0
v_3:	0	0	1	1	1	1	0	0
v_4:	0	1	1	1	1	1	1	0
v_5:	1	1	1	1	1	1	1	1

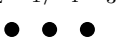
(A) Monotone

	δ_3	δ_1	δ_2	δ_4
v_1:	0	0	0	0
v_2:	0	0	0	1
v_3:	0	0	1	1
v_4:	0	1	1	1
v_5:	1	1	1	1

(B) $\delta_3 > \delta_1 > \delta_2 > \delta_4$

	δ_2	δ_4	δ_3	δ_1
v_1:	0	0	0	0
v_2:	0	0	0	1
v_3:	1	0	0	0
v_4:	1	0	1	0
v_5:	1	1	0	0

(C) $\delta_2 > \delta_4 > \delta_3 > \delta_1$



	δ_2	δ_1/δ_4	δ_3
v_1:	0	0	0
v_2:	0	1	0
v_3:	1	0	0
v_4:	1	0	1
v_5:	1	1	0

(D) $\delta_2 > \delta_1 = \delta_4 > \delta_3$

FIGURE 1. Examples of $v_1 < v_2 < v_3 < v_4 < v_5$ and $\delta_i = \delta(v_i, v_{i+1})$ for $i \in [4]$ such that $\chi(v_1, \ldots, v_5)$ is red. Each v_i is represented in binary with the left-most entry corresponding to the most significant bit.

$(\delta_1, \ldots, \delta(v_{i-1}, v_{i+1}), \ldots, \delta_5)$. For $i = 6$, $\delta(e_6) = (\delta_1, \delta_2, \delta_3, \delta_4)$. In the following we will often use that if $2 \leq i \leq 5$, then $\delta(v_{i-1}, v_{i+1}) = \max(\delta_{i-1}, \delta_i)$ by Property II.

For convenience, if inequalities are known between consecutive δ's, this will be indicated in the sequence by replacing the comma with the respective sign. For instance, assume that $\delta_1 < \delta_2 > \delta_3 < \delta_4 > \delta_5$. Then since $\delta(e_1) = (\delta_2, \delta_3, \delta_4, \delta_5)$ has $\delta_2 > \delta_3 < \delta_4 > \delta_5$, we will write

$$\delta(e_1) = (\delta_2 > \delta_3 < \delta_4 > \delta_5).$$

Similarly, if not all inequalities are known, as in $\delta(e_3)$, we write,

$$\delta(e_3) = (\delta_1 < \delta_2 \ , \ \delta_4 > \delta_5).$$

Now we will consider cases depending on the sequence of inequalities between $\delta_1, \ldots, \delta_5$, and we will further split into subcases by taking a sequence of inequalities and reversing it. There are 16 possible sequences so we will have 8 cases in what follows.

Case 1a: Suppose $\delta_1 > \delta_2 < \delta_3 > \delta_4 < \delta_5$. This implies that

$$\begin{aligned} \delta(e_1) &= (\delta_2 < \delta_3 > \delta_4 < \delta_5), \\ \delta(e_2) = \delta(e_3) &= (\delta_1 \ , \ \delta_3 > \delta_4 < \delta_5), \\ \delta(e_4) = \delta(e_5) &= (\delta_1 > \delta_2 < \delta_3 \ , \ \delta_5), \\ \delta(e_6) &= (\delta_1 > \delta_2 < \delta_3 > \delta_4). \end{aligned}$$

In particular, note that at least one of e_4, e_5, e_6 must be red so we must have that $\delta_1 < \delta_3$ and $\delta_2 > \delta_4$. However, since $\delta_1 > \delta_2 > \delta_4$, note that e_1 is only red if

$\delta_2 = \delta_5$ and similarly e_2, e_3 are only red if $\delta_1 = \delta_5$. Since these cannot happen simultaneously, there is at least one blue edge among these three edges. Thus, we must have that e_4, e_5 are also red to avoid having three blue edges, making $\delta_2 > \delta_5$ (and $\delta_3 > \delta_5$). However, then $\delta_1 > \delta_2 > \delta_5$ so none of e_1, e_2, e_3 are red and thus there are at most 3 red edges.

Case 1b: Suppose $\delta_1 < \delta_2 > \delta_3 < \delta_4 > \delta_5$. This implies that

$$\begin{aligned}\delta(e_1) = \delta(e_2) &= (\delta_2 > \delta_3 < \delta_4 > \delta_5),\\ \delta(e_3) = \delta(e_4) &= (\delta_1 < \delta_2 \ , \ \delta_4 > \delta_5),\\ \delta(e_5) = \delta(e_6) &= (\delta_1 < \delta_2 > \delta_3 < \delta_4).\end{aligned}$$

Note that e_3, e_4 are blue so we must have that all of e_1, e_2, e_5, e_6 are red. If e_5, e_6 are red, then regardless of which rule applies, $\delta_2 > \delta_4$ and thus e_1, e_2 are blue, so there are at most 2 red edges.

Case 2a: Suppose $\delta_1 > \delta_2 > \delta_3 < \delta_4 > \delta_5$. This implies that

$$\begin{aligned}\delta(e_1) &= (\delta_2 > \delta_3 < \delta_4 > \delta_5),\\ \delta(e_2) &= (\delta_1 > \delta_3 < \delta_4 > \delta_5),\\ \delta(e_3) = \delta(e_4) &= (\delta_1 > \delta_2 \ , \ \delta_4 > \delta_5),\\ \delta(e_5) = \delta(e_6) &= (\delta_1 > \delta_2 > \delta_3 < \delta_4).\end{aligned}$$

Note that e_5, e_6 are blue so that all of $e_1, \ldots, e_4$ are red. Since e_1 is red, we must have that $\delta_2 < \delta_4$, so $\delta(e_i)$ are ordered as in the second condition for red edges for all $i \in [4]$. Thus, e_1 implies that $\phi(\delta_2, \delta_5) =$ red while e_3 implies that $\phi(\delta_2, \delta_5) =$ blue, a contradiction.

Case 2b: Suppose $\delta_1 < \delta_2 < \delta_3 > \delta_4 < \delta_5$. This implies that

$$\begin{aligned}\delta(e_1) = \delta(e_2) &= (\delta_2 < \delta_3 > \delta_4 < \delta_5),\\ \delta(e_3) &= (\delta_1 < \delta_3 > \delta_4 < \delta_5),\\ \delta(e_4) = \delta(e_5) &= (\delta_1 < \delta_2 < \delta_3 \ , \ \delta_5),\\ \delta(e_6) &= (\delta_1 < \delta_2 < \delta_3 > \delta_4).\end{aligned}$$

Since e_6 is blue, in order to have at least 4 red edges, we must have that e_4, e_5 are red. Thus $\delta_3 < \delta_5$. However, then for e_1, e_2 to be red, we must have that $\delta_2 = \delta_5$, which is impossible since $\delta_2 < \delta_5$. Thus, there are at most 3 red edges here.

Case 3a: Suppose $\delta_1 > \delta_2 < \delta_3 > \delta_4 > \delta_5$. This implies that

$$\begin{aligned}\delta(e_1) &= (\delta_2 < \delta_3 > \delta_4 > \delta_5),\\ \delta(e_2) = \delta(e_3) &= (\delta_1 \ , \ \delta_3 > \delta_4 > \delta_5),\\ \delta(e_4) &= (\delta_1 > \delta_2 < \delta_3 > \delta_5),\\ \delta(e_5) = \delta(e_6) &= (\delta_1 > \delta_2 < \delta_3 > \delta_4).\end{aligned}$$

Since e_1 is blue, we must have that e_5, e_6 are red and thus $\delta_1 < \delta_3$. However, we also must have e_2, e_3 are red and thus $\delta_1 > \delta_3$, a contradiction.

Case 3b: Suppose $\delta_1 < \delta_2 > \delta_3 < \delta_4 < \delta_5$. This implies that

$$\begin{aligned}\delta(e_1) = \delta(e_2) &= (\delta_2 > \delta_3 < \delta_4 < \delta_5),\\ \delta(e_3) = \delta(e_4) &= (\delta_1 < \delta_2 \ , \ \delta_4 < \delta_5),\\ \delta(e_5) &= (\delta_1 < \delta_2 > \delta_3 < \delta_5),\\ \delta(e_6) &= (\delta_1 < \delta_2 > \delta_3 < \delta_4).\end{aligned}$$

Since e_1, e_2 are blue, we must have that the remaining edges are red. If $\delta_2 < \delta_4$, then e_6 is blue. Otherwise $\delta_2 > \delta_4$. First if $\delta_1 = \delta_4$ then e_3, e_4 are blue. Thus, for e_6 to be red, we have that $\delta_1 < \delta_3$, which implies that $\delta_1 < \delta_4 < \delta_5$. From e_3 being red, we find that $\delta_2 > \delta_5$ as well. We then have that $\phi(\delta_1, \delta_4) =$ red from e_6 while $\phi(\delta_1, \delta_4) =$ blue from e_3, a contradiction.

Case 4a: Suppose $\delta_1 > \delta_2 < \delta_3 < \delta_4 > \delta_5$. This implies that

$$\begin{aligned} \delta(e_1) &= (\delta_2 < \delta_3 < \delta_4 > \delta_5), \\ \delta(e_5) = \delta(e_6) &= (\delta_1 > \delta_2 < \delta_3 < \delta_4). \end{aligned}$$

so we have at least 3 blue edges.

Case 4b: Suppose $\delta_1 < \delta_2 > \delta_3 > \delta_4 < \delta_5$. This implies that

$$\begin{aligned} \delta(e_1) = \delta(e_2) &= (\delta_2 > \delta_3 > \delta_4 < \delta_5), \\ \delta(e_6) &= (\delta_1 < \delta_2 > \delta_3 > \delta_4). \end{aligned}$$

so we have at least 3 blue edges.

Case 5: Suppose $\delta_1 > \delta_2 < \delta_3 < \delta_4 < \delta_5$ or $\delta_1 < \delta_2 > \delta_3 > \delta_4 > \delta_5$. In the first case, each of $\delta(e_4), \delta(e_5), \delta(e_6)$ is in the form $\delta_1 > \delta_2 < \delta_i < \delta_j$ where $i, j \in \{3, 4, 5\}$, so these are blue. In the second case, each of $\delta(e_4), \delta(e_5), \delta(e_6)$ is in the form $\delta_1 < \delta_2 > \delta_i > \delta_j$ where $i, j \in \{3, 4, 5\}$, so these are blue.

Case 6: Suppose $\delta_1 > \delta_2 > \delta_3 < \delta_4 < \delta_5$ or $\delta_1 < \delta_2 < \delta_3 > \delta_4 > \delta_5$. In the first case,

$$\begin{aligned} \delta(e_1) &= (\delta_2 > \delta_3 < \delta_4 < \delta_5), \\ \delta(e_2) &= (\delta_1 > \delta_3 < \delta_4 < \delta_5), \\ \delta(e_6) &= (\delta_1 > \delta_2 > \delta_3 < \delta_4). \end{aligned}$$

so there are at least 3 blue edges. In the second case, $\delta(e_1), \delta(e_2)$ are both $\delta_2 < \delta_3 > \delta_4 > \delta_5$ and thus blue. Similarly, $\delta(e_6) = \delta_1 < \delta_2 < \delta_3 > \delta_4$, so there are at least 3 blue edges.

Case 7: Suppose $\delta_1 > \delta_2 > \delta_3 > \delta_4 < \delta_5$ or $\delta_1 < \delta_2 < \delta_3 < \delta_4 > \delta_5$. In the first case, each of $\delta(e_1), \delta(e_2), \delta(e_3)$ is in the form $\delta_i > \delta_j > \delta_4 < \delta_5$ for $i, j \in [3]$ and thus blue. In the second case, each of $\delta(e_1), \delta(e_2), \delta(e_3)$ is in the form $\delta_i < \delta_j < \delta_4 > \delta_5$ for $i, j \in [3]$ and thus blue.

Case 8a: Suppose $\delta_1 > \delta_2 > \delta_3 > \delta_4 > \delta_5$. This implies that

$$\begin{aligned} \delta(e_1) &= (\delta_2 > \delta_3 > \delta_4 > \delta_5), \\ \delta(e_2) &= (\delta_1 > \delta_3 > \delta_4 > \delta_5), \\ \delta(e_3) &= (\delta_1 > \delta_2 > \delta_4 > \delta_5), \\ \delta(e_4) &= (\delta_1 > \delta_2 > \delta_3 > \delta_5), \\ \delta(e_5) = \delta(e_6) &= (\delta_1 > \delta_2 > \delta_3 > \delta_4). \end{aligned}$$

First if e_5, e_6 are red, then $\phi(\delta_4, \delta_1) =$ blue implies that e_2, e_3 are blue, and $\phi(\delta_4, \delta_2) =$ blue implies that e_1 is blue, a contradiction. Thus, e_5, e_6 are blue and e_1 must be red but then $\phi(\delta_5, \delta_3) =$ blue implies that e_4 is blue, a contradiction.

Case 8b: Suppose $\delta_1 < \delta_2 < \delta_3 < \delta_4 < \delta_5$. This implies that

$$\begin{aligned}\delta(e_1) = \delta(e_2) &= (\delta_2 < \delta_3 < \delta_4 < \delta_5),\\ \delta(e_3) &= (\delta_1 < \delta_3 < \delta_4 < \delta_5),\\ \delta(e_4) &= (\delta_1 < \delta_2 < \delta_4 < \delta_5),\\ \delta(e_5) &= (\delta_1 < \delta_2 < \delta_3 < \delta_5),\\ \delta(e_6) &= (\delta_1 < \delta_2 < \delta_3 < \delta_4).\end{aligned}$$

If e_1, e_2 are red, then $\phi(\delta_2, \delta_5) =$ blue implies that e_4, e_5 are blue and $\phi(\delta_2, \delta_4) =$ blue implies that e_6 is blue, a contradiction. Thus, e_1, e_2 are blue and e_6 must be red but then $\phi(\delta_1, \delta_3) =$ blue implies that e_3 is blue, a contradiction.

Thus, for every 6 vertices in $V = \{0, 1, \ldots, 2^{\lfloor 2^{cn} \rfloor} - 1\}$, χ produces at most 3 red edges among them.

Now, we show that there is no blue $K^{(5)}_{128n^4}$ in coloring χ. We first make the following definitions. Given a sequence $\{a_i\}_{i=1}^r \subseteq \mathbb{R}$ and $j \in \{2, \ldots, r-1\}$, we say that a_j is a *local minimum* if $a_{j-1} > a_j < a_{j+1}$, a *local maximum* if $a_{j-1} < a_j > a_{j+1}$, and a *local extremum* if it is either a local minimum or local maximum. In particular, when looking at some set of vertices $\{v_1, \ldots, v_s\}$ where $v_1 < v_2 < \cdots < v_s$ and considering the sequence $\{\delta(v_i, v_{i+1})\}_{i=1}^{s-1}$, by Property I, $\delta(v_j, v_{j+1}) \neq \delta(v_{j+1}, v_{j+2})$ for every j, so every nonmonotone sequence will have local extrema.

Set $m = 128n^4$ and consider vertices $v_1, \ldots, v_m \in V$ such that $v_1 < v_2 < \cdots < v_m$. Assume for the sake of contradiction that these m vertices correspond to a blue clique in the coloring χ. Again, let $\delta_i = \delta(v_i, v_{i+1})$. We first note the following lemma.

Lemma 2.4. *There is no monotone subsequence $\{\delta_{k_\ell}\}_{\ell=1}^n \subset \{\delta_i\}_{i=1}^{m-1}$ such that for any $a, b, c, d \in [n]$ with $a < b < c < d$, there exists $u_1, u_2, u_3, u_4, u_5 \subset \{v_1, \ldots, v_m\}$ such that $\delta(u_1, \ldots, u_5) = \{\delta_{k_a}, \delta_{k_b}, \delta_{k_c}, \delta_{k_d}\}$.*

Proof. Indeed, if such a monotone subsequence existed, then as $\chi(u_1, \ldots, u_5) =$ blue, we have that $\{\delta_{k_\ell}\}_{\ell=1}^n$ would form an n-set with no bad 4-tuple in the graph coloring ϕ, a contradiction. □

From this, we note that there is no integer $j \in [m-n+1]$ such that the sequence $\{\delta_i\}_{i=j}^{j+n-1}$ is monotone. Otherwise, by Property IV, we have that for any length 4 subsequence $\{\delta_{i_1}, \delta_{i_2}, \delta_{i_3}, \delta_{i_4}\} \subset \{\delta_i\}_{i=j}^{j+n-1}$, there is a 5-tuple $e \subset \{v_1, \ldots, v_m\}$ such that $\delta(e)$ corresponds to this monotone sequence. From here, we apply Lemma 2.4 to get a contradiction. Thus, we can find a sequence of consecutive local extrema and from this extract a sequence of local maxima $\delta_{i_1}, \ldots, \delta_{i_{32n^3}}$.

We now restrict our attention to this sequence of local maxima $(\delta_{i_1}, \ldots, \delta_{i_{32n^3}})$. Note that any two local maxima are distinct: assume for the sake of contradiction that we have maxima $\delta_{i_j} = \delta_{i_k}$ where $j < k$. First consider if there is no δ_ℓ for $i_j < \ell < i_k$ such that $\delta_\ell > \delta_{i_j} = \delta_{i_k}$. Then, $\delta(v_{i_j}, v_{i_k}) = \delta_{i_j} = \delta_{i_k} = \delta(v_{i_k}, v_{i_k+1})$, a contradiction of Property I. Otherwise, there exists $i_j < \ell < i_k$ such that $\delta_\ell > \delta_{i_j} = \delta_{i_k}$. By letting ℓ correspond to the maximum δ_ℓ in this range, we have

$$\delta(v_{i_j}, v_{i_j+1}, v_{i_k-1}, v_{i_k}, v_{i_k+1}) = (\delta_{i_j} < \delta_\ell > \delta_{i_k-1} < \delta_{i_k}),$$

which implies that $\chi(v_{i_j}, v_{i_j+1}, v_{i_k-1}, v_{i_k}, v_{i_k+1}) =$ red as $\delta_{i_j} = \delta_{i_k}$, contradiction.

Moreover, there is no $j \in [32n^3 - n + 1]$ such that the sequence $\{\delta_{i_k}\}_{k=j}^{j+n-1}$ is monotone. If there is such j and the sequence is increasing, for any $a, b, c, d \in$

$\{j, j+1, \ldots, j+n-1\}$ with $a < b < c < d$, then

$$\delta(v_{i_a}, v_{i_a+1}, v_{i_b+1}, v_{i_c+1}, v_{i_d+1}) = (\delta_{i_a} < \delta_{i_b} < \delta_{i_c} < \delta_{i_d}).$$

This follows by Property II; in particular, if there exists ℓ such that $i_a+1 \leq \ell < i_b+1$ and $\delta_\ell > \delta_{i_b}$, then there must exist some greater local maxima between δ_{i_a} and δ_{i_b}, a contradiction of the monotonicity of $\{\delta_{i_k}\}_{k=j}^{j+n-1}$, as these are consecutive local maxima. Thus, by Lemma 2.4, we have a contradiction.

Similarly, if the sequence is decreasing, consider any $a, b, c, d \in \{j, j+1, \ldots, j+n-1\}$ with $a < b < c < d$. Then

$$\delta(v_{i_a}, v_{i_b}, v_{i_c}, v_{i_d}, v_{i_d+1}) = (\delta_{i_a} > \delta_{i_b} > \delta_{i_c} > \delta_{i_d}).$$

As with the above, we apply Lemma 2.4 to derive a contradiction.

Thus, within the sequence $(\delta_{i_1}, \delta_{i_2}, \ldots, \delta_{i_{32n^3}})$, we can find a subsequence of consecutive local extrema $\delta_{j_1}, \ldots, \delta_{j_{16n^2}}$, where $\delta_{j_1}, \delta_{j_3}, \ldots, \delta_{j_{16n^2-1}}$ are local maxima and $\delta_{j_2}, \delta_{j_4}, \ldots, \delta_{j_{16n^2}}$ are local minima (within the sequence $\delta_{i_1}, \delta_{i_2}, \ldots, \delta_{i_{32n^3}}$).

We now claim that there exists $k \in \{4n+1, 4n+2, \ldots, 16n^2 - 4n\}$ such that $\delta_{j_\ell} < \delta_{j_k}$ if $k - 4n \leq \ell \leq k + 4n$ and $\ell \neq k$. Assume for the sake of contradiction that this is not the case. We then recursively build the following sets S_r, T_r. Start with $S_0 = T_0 = \varnothing, \sigma_0 = 0, \tau_0 = 16n^2 + 1$. At each step r,

(1) $\sigma_r = 0$ if S_r is empty and $\sigma_r = \max(S_r)$ otherwise. Similarly, $\tau_r = 16n^2+1$ if T_r is empty and $\tau_r = \min(T_r)$ otherwise.
(2) If $s \in S_r$ and $s < \ell < \tau_r$, then $\delta_{j_s} > \delta_{j_\ell}$. Similarly if $t \in T_r$ and $\sigma_r < \ell < t$, then $\delta_{j_t} > \delta_{j_\ell}$.
(3) $|S_r| + |T_r| = r$ and $\tau_r - \sigma_r \geq 16n^2 - 4nr$.

Note that these properties hold for $r = 0$ by definition. Now assume that we have $S_r, T_r, \sigma_r, \tau_r$ satisfying the desired properties for some $r < 2n$. Note that by the properties, we have that

$$\tau_r - \sigma_r \geq 16n^2 - 4nr > 16n^2 - 8n^2 \geq 8n^2 > 0.$$

Consider $\sigma_r < k < \tau_r$ such that $\delta_{j_k} = \max_{\sigma_r < \ell < \tau_r} \delta_{j_\ell}$. If $k - \sigma_r > 4n$ and $\tau_r - k > 4n$, then k would satisfy that $\delta_{j_\ell} < \delta_{j_k}$ if $k-4n \leq \ell \leq k+4n$ and $\ell \neq k$, a contradiction. Now if $k - \sigma_r \leq 4n$, set

$$S_{r+1} = S_r \cup \{k\}, \quad T_{r+1} = T_r, \quad \sigma_{r+1} = k, \quad \tau_{r+1} = \tau_r.$$

Then, the first property holds by definition. The second property holds for every $s \in S_r, t \in T_r$ by assumption, and it holds for $k \in S_{r+1}$ since $\delta_{j_k} = \max_{\sigma_r < \ell < \tau_r} \delta_{j_\ell}$. The first part of the third property clearly holds and

$$\tau_{r+1} - \sigma_{r+1} = \tau_r - k \geq \tau_r - \sigma_r - 4n \geq 16n^2 - 4n(r+1).$$

Otherwise if $\tau_r - k \leq 4n$, set

$$S_{r+1} = S_r, \quad T_{r+1} = T_r \cup \{k\}, \quad \sigma_{r+1} = \sigma_r, \quad \tau_{r+1} = k.$$

By the same reasoning, the three properties hold as desired. Thus, we can construct these sets while $r \leq 2n$.

Now, consider S_{2n}, T_{2n}. Since $|S_{2n}| + |T_{2n}| = 2n$, at least one of these sets has size at least n. If $|S_{2n}| \geq n$, consider $\{s_1, \ldots, s_n\} \subseteq S_{2n}$ where $i < j \Rightarrow s_i < s_j$. Then, since $\min(T_{2n}) > \max(S_{2n})$ by Property 3 and 1, by Property 2 we have

$$\delta_{j_{s_1}} > \delta_{j_{s_2}} > \cdots > \delta_{j_{s_n}}.$$

In particular, Property 2 implies that for $a, b, c, d \in [n]$ and $a < b < c < d$,

$$\delta(v_{j_{s_a}}, v_{j_{s_b}}, v_{j_{s_c}}, v_{j_{s_d}}, v_{j_{s_d}+1}) = (\delta_{j_{s_a}} > \delta_{j_{s_b}} > \delta_{j_{s_c}} > \delta_{j_{s_d}}),$$

and thus, by Lemma 2.4, we have a contradiction. If instead $|T_{2n}| \geq n$, a similar argument shows that we derive a contradiction. Thus, such a k exists and note that in particular k must be odd.

Order the set of local minima $\{\delta_{j_{k-4n+1}}, \delta_{j_{k-4n+3}}, \ldots, \delta_{j_{k+4n-1}}\}$ in increasing order as $\gamma_1, \ldots, \gamma_{4n}$. Let

$$A' = \{\delta_{j_{k-4n+1}}, \delta_{j_{k-4n+3}}, \ldots, \delta_{j_{k-1}}\} \text{ and } B' = \{\delta_{j_{k+1}}, \delta_{j_{k+3}}, \ldots, \delta_{j_{k+4n-1}}\}.$$

Note that since A', B' partition $\{\delta_{j_{k-4n+1}}, \delta_{j_{k-4n+3}}, \ldots, \delta_{j_{k+4n-1}}\}$, either we have that $|A' \cap \{\gamma_1, \ldots, \gamma_{2n}\}| \geq n$ or $|B' \cap \{\gamma_1, \ldots, \gamma_{2n}\}| \geq n$. Without loss of generality, we assume that the former occurs since a symmetric argument would follow otherwise. Then, we also have that $|B' \cap \{\gamma_{2n+1}, \ldots, \gamma_{4n}\}| \geq n$. Set

$$A = A' \cap \{\gamma_1, \ldots, \gamma_{2n}\} \text{ and } B = B' \cap \{\gamma_{2n+1}, \ldots, \gamma_{4n}\}.$$

Let $a \in A$ and $b \in B$. By definition, $\delta_{j_a} < \delta_{j_b}$, and note that $b < k + 4n \Rightarrow b + 1 \leq k + 4n$, so

$$\delta(v_{j_a}, v_{j_a+1}, v_{j_b}, v_{j_b+1}, v_{j_{b+1}+1}) = (\delta_{j_a} < \delta_{j_k} > \delta_{j_b} < \delta_{j_{b+1}}),$$

where $\delta_{j_k} > \delta_{j_{b+1}}$ by definition. Since

$$\chi(v_{j_a}, v_{j_a+1}, v_{j_b}, v_{j_b+1}, v_{j_{b+1}+1}) = \text{blue},$$

we cannot have both $\phi(\delta_{j_a}, \delta_{j_{b+1}}) = \text{red}$ and $\phi(\delta_{j_a}, \delta_{j_b}) = \text{blue}$. Finally, restricting to any n elements of A, B and letting

$$C = \{\delta_{j_{b+1}} : \delta_{j_b} \in B\},$$

and defining $f : B \to C$ via $\delta_{j_b} \mapsto \delta_{j_{b+1}}$, we obtain 3 disjoint n-sets with precisely the structure avoided in the graph coloring ϕ, a contradiction.

Thus, χ does not produce a blue $K^{(5)}_{128n^4}$ on V. □

3. Concluding remarks

We have determined the tower growth rate for $r_k(k+1, k-1; n)$. Thus, the only problem remaining for the Erdős-Hajnal hypergraph Ramsey conjecture, is to determine the tower growth rate for $r_k(k+1, k; n)$.

Let us remark that similar arguments show that $r_5(6, 5; 4n^2) > 2^{r_4(5,4;n)-1}$. To define such a coloring, let $N = r_4(5, 4; n) - 1$ and let φ be a red/blue coloring of the 4-tuples of $\{0, \ldots, N-1\}$ such that there are there are at most 3 red edges among every 5 vertices and there is no blue clique of size n. We then color the 5-tuples of $V = \{0, 1, \ldots, 2^N - 1\}$ so that χ produces at most 4 red edges among any 6 vertices and χ does not produce a blue clique of size $4n^2$. For vertices $v_1, \ldots, v_5$ with $v_1 < v_2 < \cdots < v_5$, let $\delta_i = \delta(v_i, v_{i+1})$. We set $\chi(v_1, \ldots, v_5) = \text{red}$ if:

(1) We have that $\delta_1, \delta_2, \delta_3, \delta_4$ are monotone and $\varphi(\delta_1, \delta_2, \delta_3, \delta_4) = \text{red}$.
(2) We have that $\delta_1 > \delta_2 < \delta_3 > \delta_4$ and $\delta_1 < \delta_3$.

Together with Lemma 2.1, showing that $r_4(5, 4; n)$ grows double exponential in a power of n would thus show that $r_k(k+1, k; n) = \text{twr}_{k-1}(n^{\Theta(1)})$.

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