A note on visible islands

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Abstract

Given a finite point set P in the plane, a subset $S \subseteq P$ is called an *island* in P if $\text{conv}(S) \cap P = S$. We say that $S \subset P$ is a *visible island* if the points in S are pairwise visible and S is an island in P. The famous Big-line Big-clique Conjecture states that for any $k \geq 3$ and $\ell \geq 4$, there is an integer $n = n(k, \ell)$, such that every finite set of at least n points in the plane contains ℓ collinear points or k pairwise visible points. In this paper, we show that this conjecture is false for visible islands, by replacing each point in a Horton set by a triple of collinear points. Hence, there are arbitrarily large finite point sets in the plane with no 4 collinear members and no visible island of size 13.

1 Introduction

Given a finite point set P in the plane, two points $p, q \in P$ are *visible* in P if no other point in P lies in the interior of the segment $\overline{pq}$. The famous Big-line Big-clique Conjecture, introduced by Kára, Pór, and Wood [3], states that for any $k \geq 3$ and $\ell \geq 3$, there is an integer $n = n(k, \ell)$, such that every finite set of at least n points in the plane contains either ℓ collinear points or k pairwise visible points. Clearly, the conjecture holds when $k = 3$ or $\ell = 3$. Kára, Pór, and Wood showed that the conjecture holds when $k = 4$ and $\ell \geq 4$, and Abel et al. [1] verified the conjecture for $k = 5$ and $\ell \geq 4$. The conjecture remains open for all $k \geq 6$ and $\ell \geq 4$. See [6, 4] for more related results.

A natural approach to the Big-line Big-clique Conjecture is to find *holes* in planar point sets. Given a finite point set P in the plane, a k-subset $Q \subset P$ is called a *k-hole* in P if Q is in convex position and $\text{conv}(Q) \cap P = Q$. Clearly, if Q is a k-hole in P, then Q consists of k pairwise visible points in P. Does every sufficiently large finite point set P in the plane contain ℓ collinear points or a k-hole? In [1], Abel et al. proved this to be true when $k = 5$, and conjectured it to be true when $k = 6$. However, a famous construction due to Horton [2] shows that this is false for $k \geq 7$ (See also Chapter 3 in [5]). In this paper, we study a relaxed version of this question by replacing holes with *visible islands*.

Given a finite point set P in the plane, a subset $S \subseteq P$ is called an *island* in P if $\text{conv}(S) \cap P = S$. We say that $S \subset P$ is a *visible island* if the points in S are pairwise visible and S is an island in P.

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14 Feb 2022 06:42:20 PST
220117-AndrewSuk Version 2 - Submitted to Stud. Sci. Math. Hung.

Problem 1. *Given integers $k, \ell \geq 4$, is there an integer $n = n(k,\ell)$ such that every n-element planar point set P contains either ℓ collinear points or a visible island of size k?*

For $\ell \leq 3$, clearly we have $n(k,\ell) = k$. For $\ell \geq 4$ and $k \leq 5$, $n(k,\ell)$ exists by the result of Abel et al. [1] stated above. Our main result shows that by modifying Horton's construction [2], $n(k,\ell)$ does not exist for $\ell = 4$ and $k \geq 13$.

Theorem 2. *There exist arbitrarily large, finite point sets in the plane with no 4 collinear points and no visible island of size 13.*

When $\ell \geq 4$ and $6 \leq k \leq 12$, Problem 1 remains open.

2 Proof of Theorem 2

Let us begin by recalling the definition of Horton sets. Given finite point sets P and Q in the plane, We say that P is *high above* Q (or, equivalently, Q is *deep below* P) if each line determined by two points of P lies above all the points of Q, and each line determined by two points of Q lies below all of the points of P. Finally, given a point p in the plane, we denote $x(p)$ to be the x-coordinate of p. Throughout the proof, we will only consider point sets whose members have distinct x-coordinates.

For $n \geq 0$, a *Horton set* H_n is a set of 2^n points in the plane with no three collinear members, defined recursively as follows. Set H_0 to be a single point in the plane. Having constructed $H_{n-1} = \{p_1, p_2, \ldots, p_{2^{n-1}}\}$, whose elements are ordered by increasing x-coordinate, set

$$H^{(1)}_{n-1} = \{p_1, p_2, \ldots, p_{2^{n-1}}\},$$

$$H^{(2)}_{n-1} = H^{(1)}_{n-1} + (\varepsilon, K),$$

where K is a sufficiently large number such that $H^{(1)}_{n-1}$ lies deep below $H^{(2)}_{n-1}$. Likewise, we set $\varepsilon > 0$ to be sufficiently small such that for each i,

$$x(p_i) < x(p_i) + \varepsilon < x(p_{i+1}).$$

Then we set $H_n = H^{(1)}_{n-1} \cup H^{(2)}_{n-1}$. It is known that H_n has the following property (see Chapter 3 in [5]).

Lemma 3 ([2, 5]). *If $S \subset H_n$ such that $|S| = 7$, then the interior of* $\mathrm{conv}(S)$ *contains a point from H_n.*

For each $p_i \in H_n$, replace p_i with three collinear points q_i, u_i, v_i that are very close together, while avoiding the creation of four collinear points. The points q_i, u_i, v_i are called *triplets* of each other and p_i is the *parent* of them. Let $\hat{H}_n$ be the resulting set. Then $\hat{H}_n$ contains no visible island on 13 points. Indeed, for sake of contradiction, suppose $\hat{S} \subset \hat{H}_n$ is a visible island in $\hat{H}_n$ and $|\hat{S}| = 13$. Since at most two members of a triplet can belong to $\hat{S}$, by the pigeonhole principle, there are 7 points $p_{i_1}, \ldots, p_{i_7} \in H_n$ that are parents of points in $\hat{S}$. By setting $S = \{p_{i_1}, \ldots, p_{i_7}\}$, Lemma 3 implies that there is a point $p_j \in H_n$ that lies in the interior of $\mathrm{conv}(S)$. However, this implies that the triplet $q_j, u_j, v_j \in \mathrm{conv}(\hat{S})$, a contradiction. □

14 Feb 2022 06:42:20 PST
220117-AndrewSuk Version 2 - Submitted to Stud. Sci. Math. Hung.

3 Concluding remarks

Our initial goal was to find arbitrarily large visible islands in point sets with no four collinear members. Unfortunately, Theorem 2 shows that this is not possible. However, the following conjecture remains open, which would imply the Big-line Big-clique Conjecture for $\ell = 4$ by applying an induction on k and setting n sufficiently large. Given a finite point set P, the *neighborhood* of $p \in P$ is the set of points in P that are visible to p.

Conjecture 4. *Every n-element planar point set P with no four collinear members, contains a point $p \in P$ such that its neighborhood contains an island of size $f(n)$, where $f(n)$ tends to infinity as n tends to infinity.*

Acknowledgement. The first author would like to thank her sponsors for their generous support during the 2021 Undergraduate Summer Research Program at UCSD.

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14 Feb 2022 06:42:20 PST
220117-AndrewSuk Version 2 - Submitted to Stud. Sci. Math. Hung.