

RIGIDITY PROPERTIES OF THE COTANGENT COMPLEX

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INTRODUCTION

In this work we discover new rigidity properties of the cotangent complex associated to a map of commutative noetherian rings, or of locally noetherian schemes. The phenomena we consider are all local, so we focus in the introduction on a map of commutative noetherian rings $\varphi: R \rightarrow S$; its cotangent complex is denoted $L(\varphi)$. One rigidity property can be stated using the André-Quillen homology functors $D_n(S/R; -) = H_n(L(\varphi) \otimes_S^L -)$ on the category of S -modules.

Theorem A. *Let $\varphi: R \rightarrow S$ be a map of commutative noetherian rings, that is locally of finite flat dimension. If $D_n(S/R; -) = 0$ for some $n \geq 1$, then φ is locally complete intersection.*

Hitherto this result was known if R contains $\mathbb{Q}$ as a subring, and also when $D_n(S/R; -) = 0$ for all $n \gg 0$. Both results are due to Avramov [7]; for the former see also Halperin [22]. The hypothesis of the latter result is equivalent to the finiteness of the flat dimension of $L(\varphi)$, and in this form it settled a longstanding conjecture of Quillen [27]. In low degrees the vanishing of individual $D_n(S/R; -)$ has classically been used to characterise formally étale, formally smooth, and locally complete intersection maps, and each of these conditions imply that $D_n(S/R; -) = 0$ for all $n \geq 2$; this was proved André [2] and Quillen [27] when φ is essentially of finite type, and in general by Avramov [7].

The cotangent complex is well-known to be significantly more complicated in positive and mixed characteristic than it is in characteristic zero. Avramov's proof of Quillen's conjecture used sophisticated positive characteristic methods to connect simplicial and differential graded invariants. No other proofs or simplifications of Avramov's argument have appeared before this work. Our proof of Theorem A is independent of [7, 22], is significantly shorter, and pays no regard to the characteristic of the rings involved. Our arguments highlight a new feature controlling the behaviour of André-Quillen cohomology, namely its torsion with respect to the action of Hochschild cohomology; see Theorem C.

We deduce Theorem A from a more general result on the *higher cotangent modules* $C_n(\varphi)$ introduced in [11]. Assuming that $L(\varphi)$ is represented by a bounded below complex of projectives, these are defined to be the S -modules

$$C_n(\varphi) := H_n(L(\varphi)_{\geq n}) \quad \text{for } n \geq 0.$$

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Each $C_n(\varphi)$ is finitely generated when φ is essentially of finite type—that is to say, S is a localisation of a finitely generated R -algebra. The cotangent complex is well-defined up to quasi-isomorphism, and the higher cotangent modules are correspondingly well-defined objects in the singularity category of S in sense of Buchweitz and Orlov; see Section 7.10.

The $C_n(\varphi)$ can be thought of as higher analogues of the module $C_0(\varphi) = \Omega_{S/R}^1$ of Kähler differentials, and therefore our main result is a higher analogue of the Jacobian criterion:

Theorem B. *Let $\varphi: R \rightarrow S$ be a map of commutative noetherian rings, essentially of finite type and locally of finite flat dimension. If for some integer $n \geq 1$ the S -module $C_n(\varphi)$ has finite flat dimension, then φ is locally complete intersection.*

This answers a question posed by Avramov and Herzog [11]. It also subsumes Theorem A, since if $D_n(S/R; -) = 0$ then both $C_n(\varphi)$ and $C_{n-1}(\varphi)$ are flat; this is explained in Section 7.

The missing case $n = 0$ is related to a still-open conjecture of Eisenbud and Mazur [17]; see [13, 3.4] for discussion. The case $n = 1$ is a recent result of the first author [13], confirming a conjecture of Vasconcelos on the conormal module of a surjective homomorphism. Our arguments share an ingredient with [13], but are for the most part completely different.

The fundamental new input that makes our proof possible is a morphism

$$\mathrm{At}^\varphi: S \longrightarrow \Sigma L(\varphi)$$

in the derived category of the enveloping algebra of S over R . This map was introduced by Illusie in [23], and called the *universal Atiyah class* of φ by Buchweitz and Flenner [15]. It has played an important role in applications of the cotangent complex to deformation theory, among other things. As far as we are aware, it has not been used before to study rigidity properties of the cotangent complex.

Here we focus on a new aspect of the universal Atiyah class: For any S -module ℓ , Yoneda composition with At^φ yields a degree one map

$$\mathrm{At}_\varphi(\ell): D^*(S/R; \ell) \longrightarrow \mathrm{HH}^{*+1}(S/R; \ell),$$

from the André-Quillen cohomology with coefficients in ℓ , to the Hochschild cohomology with coefficients in ℓ viewed as a symmetric bimodule.

Taking ℓ to be a residue field of S , it turns out that $\mathrm{HH}^*(S/R; \ell)$ is a Hopf algebra over ℓ . The crucial result in this context is that the $\mathrm{At}_\varphi(\ell)$ is equivariant with respect to the characteristic action of $\mathrm{HH}^*(S/R; \ell)$ on André-Quillen cohomology, and the adjoint action of $\mathrm{HH}^*(S/R; \ell)$ on itself, which arises from its Hopf algebra structure. For this reason we call $\mathrm{At}_\varphi(\ell)$ the *Atiyah character* of φ . Seen through the looking glass [9] it is analogous to the Hurewicz map from the rational homotopy of a loop space to its rational homology.

The equivariance statement is Theorem 5.3. It is deduced from Theorem 4.7, which is a general statement about the interplay between symmetric bimodules and all bimodules. Combining the equivariance theorem with an earlier result of Avramov and Halperin [10] on the structure of $\mathrm{HH}^*(S/R; \ell)$ leads to the next result concerning torsion in André-Quillen cohomology with respect to the characteristic action of Hochschild cohomology.

Theorem C. *Let $\varphi: R \rightarrow S$ be a surjective local map of finite projective dimension, and ℓ the residue field of S . If $\mathrm{HH}^{\geq n}(S/R; \ell) \cdot D^1(S/R; \ell) = 0$ for some integer n , then φ is complete intersection.*

This exploits the fact that $\mathrm{HH}^*(S/R; \ell)$ is the universal envelope of the homotopy Lie algebra of φ , introduced by Avramov [4]. Typically, the ℓ -algebra $\mathrm{HH}^*(S/R; \ell)$ is not finitely generated, much less commutative, and it is hard to verify that elements in $D^1(S/R; \ell)$ are torsion. One scenario where it is easy to do so is when the natural augmentation $\varepsilon: L(\varphi) \rightarrow \Sigma D^1(S/R; \ell)$ factors through a perfect complex of S -modules. With this observation, Theorem B is a simple corollary of Theorem C; this is explained in Section 7. The condition that ε factors through a perfect complex is tantamount to the condition that it induces the zero map in the singularity category of S . This perspective leads to extensions of Theorems A and B to locally noetherian schemes, recorded in 7.10.

1. GRADED HOPF ALGEBRAS

In this section we collect some basic notions concerning graded Hopf algebras over fields. Everything we need is already in the paper of Milnor and Moore [25]; see also a pre-publication of the same, reprinted in [24, pp. 7]. Throughout this section we work over a field ℓ and in the category of graded ℓ -vector spaces; in particular tensor products and the module of homomorphisms are taken in this category.

1.1. Let $(A, \mu, \eta, \Delta, \varepsilon)$ be a graded Hopf algebra over ℓ , with product μ , unit η , coproduct Δ , and counit ε . We only consider positively graded (either upper or lower) and connected algebras: $A^0 = \ell$. Every such Hopf algebra has an *antipode* $\sigma: A \rightarrow A$, namely, an inverse to the identity map on A for the convolution product on $\mathrm{Hom}_{\ell}(A, A)$; see [25, §8], where this is called the “conjugation”. Being an inverse, the antipode is uniquely defined by that property.

Under suitable finiteness hypotheses the dual of a Hopf algebra is also a Hopf algebra. This is explained below.

1.2. Let $(A, \mu, \eta, \Delta, \varepsilon)$ be a graded Hopf algebra over ℓ of *finite type*: the ℓ -vector space A_i is finite dimensional for each i . Set $A^{\vee} := \mathrm{Hom}_{\ell}(A, \ell)$, the dual graded vector space. The finiteness hypothesis implies that the natural map

$$\begin{aligned} \delta: A^{\vee} \otimes_{\ell} A^{\vee} &\longrightarrow \mathrm{Hom}_{\ell}(A \otimes_{\ell} A, \ell), \quad \text{where} \\ \delta(\alpha \otimes \beta)(a \otimes b) &= (-1)^{|a||\beta|} \alpha(a) \beta(b), \end{aligned}$$

is an isomorphism. It follows that $A^{\vee}$ has a structure of a Hopf algebra over ℓ , with product and coproduct defined by the compositions

$$\begin{aligned} A^{\vee} \otimes_{\ell} A^{\vee} &\xrightarrow{\delta} \mathrm{Hom}_{\ell}(A \otimes_{\ell} A, \ell) \xrightarrow{\Delta^{\vee}} \mathrm{Hom}_{\ell}(A, \ell) = A^{\vee} \\ A^{\vee} = \mathrm{Hom}_{\ell}(A, \ell) &\xrightarrow{\mu^{\vee}} \mathrm{Hom}_{\ell}(A \otimes_{\ell} A, \ell) \xrightarrow{\delta^{-1}} A^{\vee} \otimes_{\ell} A^{\vee}. \end{aligned}$$

If σ is the antipode on A then $\sigma^{\vee}$ is the antipode on $A^{\vee}$.

We use the following observation: Fix $\alpha \in A^\vee$ and say $(\delta^{-1}\mu^\vee)(\alpha) = \sum \alpha_1 \otimes \alpha_2$ in $A^\vee \otimes_\ell A^\vee$. Then for any a, b in A one has

$$\begin{aligned} \alpha(ab) &= \mu^\vee(\alpha)(a \otimes b) \\ (1.2.1) \quad &= \sum \delta(\alpha_1 \otimes \alpha_2)(a \otimes b) \\ &= \sum (-1)^{|a||\alpha_2|} \alpha_1(a) \alpha_2(b). \end{aligned}$$

The reader may find reassurance in the last lines of [28, pp. 39].

1.3. An element $a \in A_{\geq 1}$ in a Hopf algebra A is *primitive* if its coproduct satisfies

$$\Delta(a) = a \otimes 1 + 1 \otimes a.$$

For such an element a it follows from the definition of the antipode that $\sigma(a) = -a$.

When A is a Hopf algebra, any A -bimodule action can be intertwined into a single left action described below.

1.4. Let A be a Hopf algebra over ℓ and W an A -bimodule. The *adjoint action* of A on W is the (left) A -module structure on W defined as follows: For any $a \in A$ and $w \in W$ one has

$$\mathrm{ad}(a)(w) = \sum (-1)^{|a_2||w|} a_1 w \sigma(a_2), \quad \text{where } \Delta(a) = \sum a_1 \otimes a_2.$$

In these terms the condition that σ is the antipode is precisely the condition that it is an ℓ -linear map such that $\mathrm{ad}(a) = 0$ on A^0 whenever $|a| \neq 0$; that is to say, A^0 is in the socle of A for the adjoint action.

When a is primitive, the adjoint action has a simple description:

$$\mathrm{ad}(a)(w) = [a, w] := aw - (-1)^{|a||w|} wa.$$

This is the (graded) commutator of a and w .

2. COHOMOLOGY OF COMMUTATIVE DG ALGEBRAS

This section too is mostly a recollection of known results, now about the existence of Hopf algebra structures on the homology and cohomology of algebras. Here we take as basic references the books of Avramov [6], Félix, Halperin, and Thomas [18], and Gulliksen and Levin [21]. In what follows we have to use the derived enveloping algebra. For concreteness, we use commutative dg algebras to model this, and other derived rings, but one could as well use simplicial models.

2.1. By a differential graded (abbreviated to “dg”) algebra we shall mean a non-negatively graded complex $F = \{F_i\}_{i \geq 0}$ over some commutative ground ring ($\mathbb{Z}$ is always a choice), equipped with a graded-commutative multiplication, bilinear with respect to the ground ring, so that the differential satisfies the Leibniz rule. In our applications the dg algebras that arise will be strictly graded-commutative, and with divided powers, but these structures will not be used in our arguments.

The derived category of dg F -modules (which we usually speak of as F -modules) will be denoted $D(F)$. We use suitable resolutions as needed, including semi-free dg modules, and semi-free dg algebras which we call *models*; see [6, 18].

Throughout this section F will be a dg algebra equipped with a surjective map $F \rightarrow \ell$, where ℓ is a field. We recall, from [21], the construction of Hopf algebra structures on $\mathrm{Tor}^F(\ell, \ell)$ and its dual $\mathrm{Ext}_F(\ell, \ell)$.

2.2. For $F \rightarrow \ell$ as above, $\mathrm{Tor}^F(\ell, \ell)$ has a natural structure of a graded-commutative Hopf algebra over ℓ . The product is induced by the map

$$(\ell \otimes_F^L \ell) \otimes_\ell^L (\ell \otimes_F^L \ell) \longrightarrow \ell \otimes_F^L \ell$$

obtained by multiplication on the respective factors, and following the Koszul sign rule. In homology this induces the second map below:

$$H(\ell \otimes_F^L \ell) \otimes_\ell H(\ell \otimes_F^L \ell) \xrightarrow{\cong} H((\ell \otimes_F^L \ell) \otimes_\ell^L (\ell \otimes_F^L \ell)) \longrightarrow H(\ell \otimes_F^L \ell).$$

The first one is the Künneth map, which is an isomorphism since ℓ is a field. Thus one has a map

$$(2.2.1) \quad \mathrm{Tor}^F(\ell, \ell) \otimes_\ell \mathrm{Tor}^F(\ell, \ell) \longrightarrow \mathrm{Tor}^F(\ell, \ell)$$

that makes $\mathrm{Tor}^F(\ell, \ell)$ is a graded-commutative ℓ -algebra. This is where the commutativity comes in play.

The coproduct on $\mathrm{Tor}^F(\ell, \ell)$ is induced by the natural map

$$\ell \otimes_F^L \ell \longrightarrow \ell \otimes_F^L \ell \otimes_F^L \ell \simeq (\ell \otimes_F^L \ell) \otimes_\ell^L (\ell \otimes_F^L \ell)$$

induced by the assignment $x \otimes y \mapsto x \otimes 1 \otimes y$. Using the fact that the Künneth map is an isomorphism, this induces the map

$$(2.2.2) \quad \mathrm{Tor}^F(\ell, \ell) \longrightarrow \mathrm{Tor}^F(\ell, \ell) \otimes_\ell \mathrm{Tor}^F(\ell, \ell).$$

It is easy to verify that this map is a morphism of ℓ -algebras, and that it defines a coproduct on $\mathrm{Tor}^F(\ell, \ell)$.

The antipode on $\mathrm{Tor}^F(\ell, \ell)$ is induced by the twisting map

$$\ell \otimes_F^L \ell \longrightarrow \ell \otimes_F^L \ell, \quad \text{where } a \otimes b \mapsto (-1)^{|a||b|} b \otimes a.$$

The computations that show that with the structures defined above, $\mathrm{Tor}^F(\ell, \ell)$ is a Hopf algebra over ℓ are straightforward; see [21] that deals with the case where F is a ring. The same arguments carry over to our context as well.

2.3. In this paragraph we assume that the dg algebra is F *degreewise noetherian* by which we mean that the ring F_0 is noetherian, and the F_0 -modules F_i are finitely generated for $i \geq 1$; recall that $F_i = 0$ for $i < 0$. These conditions imply that $\mathrm{rank}_\ell \mathrm{Tor}_i^F(\ell, \ell)$ is finite for each i , and zero for $i < 0$. The adjunction isomorphism

$$\mathrm{Hom}_\ell(\ell \otimes_F^L \ell, \ell) \cong \mathrm{RHom}_F(\ell, \ell)$$

induces an isomorphism of ℓ -vector spaces

$$\mathrm{Ext}_F(\ell, \ell) \cong \mathrm{Hom}_\ell(\mathrm{Tor}^F(\ell, \ell), \ell).$$

Thus, as per the discussion in 1.2, the Hopf algebra structure on $\mathrm{Tor}^F(\ell, \ell)$ induces such a structure on $\mathrm{Ext}_F(\ell, \ell)$, and since the former is commutative, and latter is cocommutative. The multiplication thus defined on $\mathrm{Ext}_F(\ell, \ell)$ is the usual one given by composition.

3. HOCHSCHILD COHOMOLOGY

In this section we introduce a Hopf algebra structure on the Hochschild cohomology of commutative algebras with field coefficients, and connect this with the Hopf algebra $\text{Ext}_F(\ell, \ell)$ defined in the previous section, for a suitable F . Both descriptions will be crucial: we need the Hochschild approach in Sections 4 and 5, and the Yoneda approach in Section 6.

Throughout $\varphi: R \rightarrow S$ will be a morphism of dg algebras, as in 2.1. The main case of interest is when R and S are rings, but it is convenient to work in the greater generality, for it allows one to replace rings with semi-free dg models when needed.

3.1. We write S_R^e for the derived enveloping algebra, $S \otimes_R^L S$, of S over R . It is convenient to replace the R -algebra S by a suitable model and assume $S_R^e = S \otimes_R S$. We consider the morphisms of dg algebras:

$$\mu: S_R^e \rightarrow S \quad \text{and} \quad \tau: S_R^e \rightarrow S_R^e,$$

where μ is the multiplication map and τ is the twisting map, defined on pure tensors by $s \otimes t \mapsto (-1)^{|s||t|} t \otimes s$.

An S_R^e -module is the same as an S -bimodule on which the R -action is symmetric, and it will be useful to keep both perspectives in mind. Given S_R^e -modules M and N , the tensor product $M \otimes_S N$ is again an S_R^e -module, with left and right actions inherited from the left and right factors, respectively:

$$s \cdot (m \otimes n) := sm \otimes n \quad \text{and} \quad (m \otimes n) \cdot s := m \otimes ns$$

for s in S and $m \otimes n$ in $M \otimes_S N$. Viewing S viewed as a bimodule via μ , the natural maps $M \otimes_S S \rightarrow M$ and $S \otimes_S N \rightarrow N$ are isomorphisms of bimodules. We repeatedly exploit the fact that this induces a structure of a tensor-triangulated category on $\mathbf{D}(S_R^e)$, with product $- \otimes_S^L -$ and unit S , so that

$$(3.1.1) \quad S \otimes_S^L M \simeq M \quad \text{and} \quad M \otimes_S^L S \simeq M$$

for each S_R^e -module M . This tensor product is not symmetric, and this leads to interesting structures on cohomology, as will become clear soon.

3.2. Given an S_R^e -module M , let $\text{HH}^*(S/R; M)$ be the *Hochschild cohomology* of the R -algebra S , with coefficients in M ; thus

$$\text{HH}^*(S/R; M) := \text{Ext}_{S_R^e}^*(S, M).$$

This is also called the Shukla cohomology, the derived Hochschild cohomology, or the Hochschild-Quillen cohomology [27].

3.3. Let $\varphi: R \rightarrow S$ be as above. In the remainder of the section we fix a surjective map $S \rightarrow \ell$, where ℓ is a field. In addition, we ask that R and S be degreewise noetherian, in the sense of 2.3, and that the map $\varphi_0: R_0 \rightarrow S_0$ be *essentially of finite type*; that is to say, S_0 is a localisation of a finitely generated R_0 -algebra. This implies that the dg algebra S_R^e can be chosen degreewise noetherian as well. In this context we construct operations making

$$\text{HH}^*(S/R; \ell)$$

into a graded, cocommutative Hopf ℓ -algebra.

We fix a semi-free dg algebra model $\varepsilon: \tilde{S} \xrightarrow{\sim} S$ over S_R^e and compute Hochschild cohomology of an S -bimodule M using the complex

$$\text{Hom}_{S_R^e}(\tilde{S}, M).$$

By our hypotheses of R and S , we can and will choose $\tilde{S}$ to be degreewise noetherian.

3.4. The product on $\mathrm{HH}^*(S/R; \ell)$ uses only the S -bimodule structure of $\tilde{S}$. The tensor product $\tilde{S} \otimes_S \tilde{S}$ is viewed as an S -bimodule in the usual way; see 3.1. It is semi-free over S_R^e since $\tilde{S}$ is assumed so. There are two natural quasi-isomorphisms $\tilde{S} \otimes_S \tilde{S} \rightarrow \tilde{S}$, namely $1 \otimes_S \varepsilon$ and $\varepsilon \otimes_S 1$, and both represent the same morphism in $\mathrm{D}(S_R^e)$ since they are coequalised by the quasi-isomorphism ε . This yields the quasi-isomorphism below:

$$\begin{aligned} \mathrm{Hom}_{S_R^e}(\tilde{S}, \ell) \otimes_\ell \mathrm{Hom}_{S_R^e}(\tilde{S}, \ell) &\longrightarrow \mathrm{Hom}_{S_R^e}(\tilde{S} \otimes_S \tilde{S}, \ell) \xleftarrow{\sim} \mathrm{Hom}_{S_R^e}(\tilde{S}, \ell) \\ f_1 \otimes f_2 &\mapsto (\tilde{S} \otimes_S \tilde{S} \xrightarrow{f_1 \otimes_S f_2} \ell \otimes_S \ell = \ell). \end{aligned}$$

In cohomology, the composite map, which entails inverting the one on the right, defines the cup product $f_1 \smile f_2$; see also the discussion in 4.2, and also [16, §4]. The unit is the augmentation $S \rightarrow \ell$ composed with ε .

3.5. The coproduct on $\mathrm{HH}^*(S/R; \ell)$ uses the multiplication map $\tilde{\mu}: \tilde{S} \otimes_{S_R^e} \tilde{S} \rightarrow \tilde{S}$. In contrast to the bimodule tensor product used to define the cup product, the tensor product here treats both factors $\tilde{S}$ as S_R^e -modules in the *same* way, exploiting commutativity. This induces the map on the left below:

$$\tilde{\Delta}: \mathrm{Hom}_{S_R^e}(\tilde{S}, \ell) \xrightarrow{\mathrm{Hom}(\tilde{\mu}, \ell)} \mathrm{Hom}_{S_R^e}(\tilde{S} \otimes_{S_R^e} \tilde{S}, \ell) \xleftarrow{\cong} \mathrm{Hom}_{S_R^e}(\tilde{S}, \ell) \otimes_\ell \mathrm{Hom}_{S_R^e}(\tilde{S}, \ell).$$

The isomorphism is standard; it exists because $\tilde{S}$ is degreewise noetherian. Passing to cohomology, and using the Künneth isomorphism, yields the coproduct on $\mathrm{HH}^*(S/R; \ell)$. In fact $\tilde{\Delta}$ makes $\mathrm{Hom}_{S_R^e}(\tilde{S}, \ell)$ into a dg coalgebra over ℓ , with counit the dual of the structure map $S_R^e \rightarrow \tilde{S}$.

3.6. The antipode on $\mathrm{HH}^*(S/R; \ell)$ is defined using the twisting map τ on S_R^e . Since $\tilde{S}$ is semi-free over S_R^e and ε is a quasi-isomorphism of dg algebras, one can construct a commutative diagram

$$\begin{array}{ccc} \tilde{S} & \xrightarrow{\tilde{\tau}} & \tau_*(\tilde{S}) \\ \varepsilon \downarrow & & \downarrow \varepsilon \\ S & \xlongequal{\quad} & S \end{array}$$

of dg S_R^e -modules, where τ_* is the restriction of scalars along the twisting map $\tau: S_R^e \rightarrow S_R^e$. This defines the antipode $\sigma := \mathrm{Ext}_\tau(\tilde{\tau}, \ell)$. Thus, given an S_R^e -linear chain map $f: \tilde{S} \rightarrow \ell$ its antipode $\sigma(f)$ is the composition

$$\tilde{S} \xrightarrow{\tilde{\tau}} \tau_*(\tilde{S}) \xrightarrow{\tau_*(f)} \tau_*(\ell) = \ell.$$

Here is the result announced at the beginning of the section.

Theorem 3.7. *With $R \rightarrow S \rightarrow \ell$ as in 3.3, the operations described above endow $\mathrm{HH}^*(S/R; \ell)$ with the structure of a graded, cocommutative, Hopf ℓ -algebra.*

Proof. The associativity and unitality of the cup product—as well as the coassociativity and counitality of the coproduct—can be verified in a straightforward way. The graded cocommutativity of $\mathrm{HH}^*(S/R; \ell)$ follows from the graded commutativity of $\tilde{S}$. For the bialgebra identity, take chain maps f and g in $\mathrm{Hom}_{S_R^e}(\tilde{S}, \ell)$, and

consider the following commutative diagram

$$\begin{array}{ccc} (\tilde{S} \otimes_S \tilde{S}) \otimes_{S_R^e} (\tilde{S} \otimes_S \tilde{S}) & \longrightarrow & \tilde{S} \otimes_S \tilde{S} \xrightarrow{f \otimes_S g} \ell \\ (\varepsilon \otimes 1) \otimes (\varepsilon \otimes 1) \downarrow \sim & & \varepsilon \otimes 1 \downarrow \sim \\ \tilde{S} \otimes_{S_R^e} \tilde{S} & \longrightarrow & \tilde{S}, \end{array}$$

where the top row is given by $(x \otimes y) \otimes (u \otimes v) \mapsto (-1)^{|y||x|} xu \otimes yv$. The upper path represents $(\smile \otimes \smile)(1 \otimes \tau \otimes 1)(\Delta(f) \otimes \Delta(g))$, while the lower path (inverting the quasi-isomorphism) represents $\Delta(f \smile g)$, so the diagram is witness to the bialgebra identity.

It remains to show that σ is an antipode; the argument for this anticipates the proof of Theorem 4.7; confer also the discussion in 1.4. Let f be a chain map in $\text{Hom}_{S_R^e}(\tilde{S}, \ell)$ with $\Delta(f) = \sum f_1 \otimes f_2$. We must show that

$$\sum f_1 \smile \sigma(f_2) = \eta \varepsilon(f) = \sum \sigma(f_1) \smile f_2.$$

We verify the equality on the left; the argument for the one on the right is similar. It is sufficient to verify that $\sum f_1 \smile \sigma(f_2) = 0$ when f is in the kernel of the counit ε . Denote by $\iota: S \rightarrow S_R^e$ the inclusion into the left tensor factor, and consider the commutative diagram

$$\begin{array}{ccc} \tilde{S} \otimes_S \tilde{S} & \xrightarrow{\sum f_1 \smile \sigma(f_2)} & \ell \\ \kappa \downarrow & & \parallel \\ \mu_* \iota_*(\tilde{S}) & \xrightarrow{\mu_* \iota_*(f)} & \mu_* \iota_*(\ell). \end{array}$$

Here $\kappa(x \otimes y) = x\tilde{\tau}(y)$; one checks directly that κ is well-defined and S_R^e -linear. Finally, if $\varepsilon(f) = 0$ then $\mu_* \iota_*(f) = 0$, since already the class of $\iota_*(f)$ vanishes in

$$\text{Hom}_S(\iota_*(\tilde{S}), \ell) \simeq \text{Hom}_S(S, \ell) = \ell. \quad \square$$

Next, we identify the Hochschild Hopf algebra just constructed with a Yoneda Hopf algebra of the previous section.

3.8. We remain in the context of 3.3, and set $F := \ell \otimes_R^L S$, viewed as a dg ℓ -algebra. The dg algebra S_R^e can be chosen degreewise noetherian, so the same property is inherited by F and its homology algebra.

The map $S \rightarrow \ell$ induces the morphism

$$S_R^e = S \otimes_R^L S \rightarrow \ell \otimes_R^L S = F$$

of dg algebras. Moreover the composition $S_R^e \xrightarrow{\mu} S \rightarrow \ell$ factors through this map and induces quasi-isomorphisms

$$F \otimes_{S_R^e}^L S = (\ell \otimes_R^L S) \otimes_{S_R^e}^L S \simeq \ell \otimes_S^L S \simeq \ell,$$

where the first one is the standard diagonal isomorphism. Apply $\text{RHom}_F(-, \ell)$ and using adjunction yields an isomorphism

$$\text{RHom}_{S_R^e}(S, \ell) \simeq \text{RHom}_F(\ell, \ell).$$

In summary, there is an isomorphism of ℓ -vector spaces

$$(3.8.1) \quad \text{HH}^*(S/R; \ell) := \text{Ext}_{S_R^e}^*(S, \ell) \cong \text{Ext}_F^*(\ell, \ell).$$

Here is an explicit description in terms of morphisms: Given $\zeta: S \rightarrow \Sigma^n \ell$ in $D(S_R^e)$ the corresponding morphism in $D(F)$ is the composition

$$\ell \cong \ell \otimes_S^L S \xrightarrow{\ell \otimes_S^L \Sigma^n \zeta} \Sigma^n \ell \otimes_S^L \ell \longrightarrow \Sigma^n \ell,$$

where the map on the right is induced by the multiplication on ℓ . On the other hand, given $\xi: \ell \rightarrow \Sigma^n \ell$ in $D(F)$ composing with the map $S \rightarrow \ell$ gives the morphism

$$S \longrightarrow \ell \xrightarrow{\xi} \Sigma^n \ell$$

in $D(S_R^e)$. A straightforward computation shows that these assignments are inverse isomorphisms yielding (3.8.1).

3.9. Since F is degreewise noetherian, from the discussion in 2.3 there is a natural structure of a Hopf ℓ -algebra on $\text{Ext}_F^*(\ell, \ell)$. On the other hand, $\text{HH}^*(S/R; \ell)$ is also a Hopf ℓ -algebra, by Theorem 3.7. The result below should come as no surprise.

Proposition 3.10. *The isomorphism $\text{HH}^*(S/R; \ell) \cong \text{Ext}_F^*(\ell, \ell)$ in (3.8.1) is compatible with the Hopf algebra structures.*

Proof. We only need to check compatibility with product and coproduct, for the antipode is determined by those structures.

We can assume that S is semi-free over R and so identify F with the dg ℓ -algebra $\ell \otimes_R S$. Let $\varepsilon: \tilde{S} \rightarrow S$ be a semi-free dg model of S over S_R^e . Since $\tilde{S}$ is also semi-free over S , applying $\ell \otimes_S (-)$ to ε yields a quasi-isomorphism

$$X := \ell \otimes_S \tilde{S} \xrightarrow{\varepsilon'} \ell.$$

of dg algebras over F . Note that X is semi-free over F .

Applying $\ell \otimes_S (-)$ to the multiplication map on $\tilde{S}$ yields the multiplication map on X . The coproducts on $\text{HH}^*(S/R; \ell)$ and $\text{Ext}_F^*(\ell, \ell)$ are induced by these maps—see the discussion in 2.2, 2.3, and 3.5—so we deduce that they coincide.

As to the product structures, the one on $\text{HH}^*(S/R; \ell)$ is induced by the quasi-isomorphism of dg S_R^e -algebras

$$\tilde{S} \xleftarrow{1 \otimes \varepsilon} \tilde{S} \otimes_S \tilde{S},$$

as explained in 3.4. This induces the top row of the following diagram

$$\begin{array}{ccc} X \otimes_F X \cong (X \otimes_\ell \ell) \otimes_F X & \xleftarrow[\simeq]{1 \otimes \varepsilon' \otimes 1} & (X \otimes_\ell X) \otimes_F X \\ & \searrow & \downarrow \\ & & X \otimes_F X \otimes_F X \end{array}$$

In the top row, the action of F on $X \otimes_\ell \ell$ is through X . The diagonal map is defined by the assignment $x \otimes y \mapsto x \otimes 1 \otimes y$. The diagram commutes in the derived category of dg ℓ -modules, for the two maps in question are coequalized by the quasi-isomorphism $1 \otimes \varepsilon' \otimes 1$. It remains to observe that the diagonal map induces the product on $\text{Ext}_F^*(\ell, \ell)$; see 2.2 and 2.3. $\square$

4. ACTIONS OF HOCHSCHILD COHOMOLOGY

This section concerns the action of $\text{HH}^*(S/R; \ell)$ on certain cohomology modules arising from bimodules. The main new result is Theorem 4.7 and that is at the heart of all that follows. We prepare the ground to state and prove it.

4.1. Throughout $\varphi: R \rightarrow S$ and $S \rightarrow \ell$ are morphisms of dg algebras such that:

- (1) R_0 is a noetherian ring and the R_0 -module R_i is finitely generated for $i \geq 1$;
- (2) S_0 is essentially of finite type as an R_0 -algebra, and the S_0 -module S_i is finitely generated for $i \geq 1$;
- (3) $S \rightarrow \ell$ is a surjective map with ℓ a field.

These are hypothesis from 3.3 onward, and we freely draw upon the discussion in the preceding section.

4.2. Given S_R^e -modules M and N one gets an external product

$$(4.2.1) \quad \mathrm{Ext}_{S_R^e}(M, \ell) \otimes_{\ell} \mathrm{Ext}_{S_R^e}(N, \ell) \xrightarrow{\cap} \mathrm{Ext}_{S_R^e}(M \otimes_S^L N, \ell),$$

where $M \otimes_S^L N$ is viewed as an S_R^e -module as explained in 3.1. Associativity of tensor products implies that this product is associative in the obvious sense. Setting $M = S = N$ one gets that $\mathrm{Ext}_{S_R^e}(S, \ell)$, that is to say, $\mathrm{HH}^*(S/R; \ell)$ is a graded ℓ -algebra. This algebra structure is the same as the one introduced in 3.4. Moreover, given (3.1.1), specialising M or N to S yields the following:

Proposition 4.3. *For each M in $\mathrm{D}(S_R^e)$, the product defined via (4.2.1) endows $\mathrm{Ext}_{S_R^e}(M, \ell)$ with a natural structure of a $\mathrm{HH}^*(S/R; \ell)$ -bimodule.* $\square$

One can describe this explicitly on models; this will be useful later on.

4.4. We work with models as in 3.3. Let M be a semi-free S_R^e -module. The map

$$\tilde{S} \otimes_S M \otimes_S \tilde{S} \longrightarrow M, \quad \text{where } s \otimes m \otimes s' \mapsto sms'$$

is evidently S_R^e -linear, where the bimodule structure on the left is the natural one, via the outer factors of the tensor product. It is also a quasi-isomorphism, by (3.1.1). Given chain maps α, β in $\mathrm{Hom}_{S_R^e}(\tilde{S}, \ell)$ and f in $\mathrm{Hom}_{S_R^e}(M, \ell)$, the class of $\alpha \cdot f \cdot \beta$ is represented by the commutative diagram

$$\begin{array}{ccc} M & \xleftarrow{\sim} & \tilde{S} \otimes_S M \otimes_S \tilde{S} \\ \alpha \cdot f \cdot \beta \downarrow & & \downarrow \alpha \otimes f \otimes \beta \\ \ell & \xleftarrow{\quad} & \ell \otimes_S \ell \otimes_S \ell \end{array}$$

in $\mathrm{D}(S_R^e)$, where the arrow in the bottom is given by the multiplication on ℓ .

Recall from Theorem 3.7 that $\mathrm{HH}^*(S/R; \ell)$ is a Hopf algebra. Its two-sided action on $\mathrm{Ext}_{S_R^e}(M, \ell)$ can thus be combined into a single adjoint action, as explained in 1.4. Here is a chain-level description.

4.5. Let α in $\mathrm{Hom}_{S_R^e}(\tilde{S}, \ell)$ be a chain map and using 3.5 write

$$\tilde{\Delta}(\alpha) = \sum \alpha_1 \otimes \alpha_2.$$

A caveat: The maps α_1 and α_2 need not be chain maps, though they can be chosen to be so, at the expense of replacing equality above by an equality of cohomology classes. Observe that, by definition, one has

$$(4.5.1) \quad \alpha(xy) = \sum (-1)^{|\alpha_2||y|} \alpha_1(x) \alpha_2(y) \quad \text{for all } x, y \in \tilde{S}.$$

This is just the chain-level version of (1.2.1).

Let $f: M \rightarrow \ell$ be a chain map. It follows from the description of the coproduct 3.5 and the antipode 3.6 that the adjoint action of the class of α on the class of f is represented by the following commutative diagram

$$(4.5.2) \quad \begin{array}{ccc} M & \xleftarrow{\sim} & \tilde{S} \otimes_S M \otimes_S \tilde{S} \\ \text{ad}(\alpha) \cdot f \downarrow & & \downarrow \sum (-1)^{|\alpha_2| |f|} \alpha_1 \otimes f \otimes (\alpha_2 \tilde{\tau}) \\ \ell & \xleftarrow{\quad} & \ell \otimes_S \ell \otimes_S \ell \end{array}$$

in $D(S_R^e)$. It can be checked directly that the map on the right is a chain map.

We need one more piece of structure: For any S -module M , the Hochschild cohomology algebra $\text{HH}^*(S/R; \ell)$ acts on $\text{Ext}_S(M, \ell)$ through Yoneda composition. A chain level description of this *characteristic action* is given below.

4.6. Let M be a semi-free S -module and consider the S -module $\tilde{S} \otimes_S M$, where S acts through the left-hand factor. The map $\tilde{S} \otimes_S M \rightarrow M$, where $s \otimes m \mapsto sm$, is evidently S -linear; it is also a quasi-isomorphism, by (3.1.1). Given chain maps α in $\text{Hom}_{S_R^e}(\tilde{S}, \ell)$ and f in $\text{Hom}_S(M, \ell)$, the class of $\alpha \cdot f$ is represented by

$$\begin{array}{ccc} M & \xleftarrow{\sim} & \tilde{S} \otimes_S M \\ \alpha \cdot f \downarrow & & \downarrow \alpha \otimes f \\ \ell & \xleftarrow{\quad} & \ell \otimes_S \ell, \end{array}$$

where the arrow in the bottom row is given by multiplication on ℓ .

The result below is the one we have been working towards. It is about the restriction functor $\mu_*: D(S) \rightarrow D(S_R^e)$ associated to the multiplication map $\mu: S_R^e \rightarrow S$.

Theorem 4.7. *Let $R \rightarrow S \rightarrow \ell$ be as in 4.1. For each S -module M the map*

$$\mu_*: \text{Ext}_S(M, \ell) \longrightarrow \text{Ext}_{S_R^e}(\mu_*(M), \ell)$$

is linear with respect to the characteristic action of $\text{HH}^(S/R; \ell)$ on the left and its adjoint action on the right.*

Proof. The proof becomes a direct computation, once we recall the actions involved in a convenient form. We can assume we are in the context of 3.3, and also that the S -module M is semi-free. Thus elements in $\text{HH}^*(S/R; \ell)$ and $\text{Ext}_S(M, \ell)$ are represented by chain maps in $\text{Hom}_{S_R^e}(\tilde{S}, \ell)$ and $\text{Hom}_S(M, \ell)$, respectively.

Since M is semi-free over S , the modules

$$\tilde{S} \otimes_S \mu_*(M) \otimes_S \tilde{S} \quad \text{and} \quad \tilde{S} \otimes_S M$$

are semi-free over S_R^e and S , respectively. As usual, the S -bimodule structure on the module on the left is via the outer factors of the tensor product. Both these modules are quasi-isomorphic to M , as explained in 4.4 and 4.6. It can be checked directly that the map

$$\begin{aligned} \kappa: \tilde{S} \otimes_S \mu_*(M) \otimes_S \tilde{S} &\longrightarrow \mu_*(\tilde{S} \otimes_S M) \\ x \otimes m \otimes y &\mapsto (-1)^{|m||y|} x \tilde{\tau}(y) \otimes m \end{aligned}$$

is well-defined, S_R^e -linear, and a lifting of the identity on M across μ . Thus the map μ_* in the statement of the theorem is induced by the map

$$\text{Hom}_S(\tilde{S} \otimes_S M, \ell) \xrightarrow{\text{Hom}_\mu(\kappa, \ell)} \text{Hom}_{S_R^e}(\tilde{S} \otimes_S \mu_*(M) \otimes_S \tilde{S}, \ell).$$

The action of $\mathrm{HH}^*(S/R; \ell)$ on the source and target of this map have been described in 4.6 and 4.4, respectively.

Let α in $\mathrm{Hom}_{S_R^e}(\tilde{S}, \ell)$ be a chain map, and write $\tilde{\Delta}(\alpha) = \sum \alpha_1 \otimes \alpha_2$ as in 3.5. Let f in $\mathrm{Hom}_S(M, \ell)$ be a chain map, set

$$g := \sum (-1)^{|\alpha_2||f|} \alpha_1 \otimes \mu_*(f) \otimes (\alpha_2 \tilde{\tau}) : \tilde{S} \otimes_S \mu_*(M) \otimes_S \tilde{S} \longrightarrow \ell \otimes_S \ell \otimes_S \ell,$$

and consider the following diagram of S_R^e -modules:

$$\begin{array}{ccccc} \mu_*(M) & \xleftarrow{\sim} & \tilde{S} \otimes_S \mu_*(M) \otimes_S \tilde{S} & \xrightarrow{\kappa} & \mu_*(\tilde{S} \otimes_S M) \xrightarrow{\sim} M \\ \mathrm{ad}(\alpha) \cdot \mu_*(f) \downarrow & & \downarrow g & & \downarrow \alpha \otimes f \quad \downarrow \alpha \cdot f \\ \ell & \longleftarrow & \ell \otimes_S \ell \otimes_S \ell & \longrightarrow & \ell \longleftarrow \ell \otimes_S \ell \longrightarrow \ell, \end{array}$$

where the squares on the left and on the right commute, by 4.5 and 4.6 respectively. We claim that the diagram in the middle is also commutative: going down from the top of the rectangle along the left edge is the map

$$x \otimes m \otimes y \mapsto (-1)^{|f||x|} \alpha(x \tilde{\tau}(y)) f(m).$$

This computation uses (4.5.1), and the fact that $f(m) = 0$ unless $|f| = -|m|$. We leave it to the reader's pleasure to verify that going the other way gives the same map, with the correct sign. This justifies the equivariance of μ_* . $\square$

5. THE ATIYAH CHARACTER

In this section $\varphi: R \rightarrow S$ is a map of commutative noetherian rings and $L(\varphi)$ is its cotangent complex. We sometimes write $\mathrm{Ext}_S^*(L(\varphi), N)$ for André-Quillen cohomology $D^*(R/S; N)$ as it makes certain constructions transparent. We begin by sketching the construction of the cotangent complex; for details see [2, 23, 27].

5.1. As usual S_R^e is the derived enveloping algebra of S over R . Following [27, §6], let $R \rightarrow P \xrightarrow{\sim} S$ be a simplicial resolution of S , so that $S_R^e = S \otimes_R P$ is a simplicial model for the derived enveloping algebra. Set

$$J := \mathrm{Ker}(S \otimes_R P \rightarrow S)$$

so J/J^2 is the cotangent complex of φ , by [27, §6]. Thus there is an exact sequence

$$(5.1.1) \quad 0 \longrightarrow L(\varphi) \longrightarrow (S \otimes_R P)/J^2 \longrightarrow S \longrightarrow 0$$

of complexes over $S \otimes_R P = S_R^e$. The corresponding connecting morphism

$$\mathrm{At}^\varphi: S \longrightarrow \Sigma L(\varphi)$$

in $D(S_R^e)$ is the *universal Atiyah class* of φ . This map was defined by Illusie [23, IV.2.3.6.2], whilst the terminology is borrowed from [15].

5.2. Fixing an S -module N , one can dualise the universal Atiyah class, with respect to N , along the multiplication map $\mu: S_R^e \rightarrow S$ to obtain

$$\mathrm{At}_\varphi(N): \mathrm{Ext}_S^*(L(\varphi), N) \longrightarrow \mathrm{HH}^{*+1}(S/R; N).$$

This is a map of degree one from the André-Quillen cohomology to the Hochschild cohomology of S over R , both with coefficients in N . We call this map the *Atiyah*

character of φ with coefficients in N , mainly because of Theorem 5.3. By definition $\text{At}_\varphi^n(N)$, the component in degree n , factors as

$$(5.2.1) \quad \text{Ext}_S^n(\text{L}(\varphi), N) \xrightarrow{\mu_*} \text{Ext}_{S_R^e}^n(\text{L}(\varphi), N) \xrightarrow{(\text{At}^\varphi)^\vee} \text{Ext}_{S_R^e}^{n+1}(S, N),$$

where $(\text{At}^\varphi)^\vee$ is used as a shorthand for $\text{Ext}_{S_R^e}(\text{At}^\varphi, N)$.

Specialising N to ℓ and applying Proposition 4.3 and Theorem 4.7 yields:

Theorem 5.3. *Let R be a commutative noetherian ring, $\varphi: R \rightarrow S$ a map essentially of finite type, and $S \rightarrow \ell$ a surjection onto a field. The Atiyah character*

$$\text{At}_\varphi(\ell): D^*(S/R; \ell) \longrightarrow \text{HH}^{*+1}(S/R; \ell)$$

with coefficients in ℓ , is equivariant with respect to the characteristic action of $\text{HH}^(S/R; \ell)$ on the left and the adjoint action on the right.* $\square$

5.4. Recall that $H_0(\text{L}(\varphi))$ is the module of Kähler differentials of S over R . Thus one can identify $\text{Ext}_S^0(\text{L}(\varphi), N)$ with $\text{Der}_R(S, N)$, the module of R -linear derivations from S into N . The Atiyah character in degree zero is a familiar isomorphism

$$\text{At}_\varphi^0(N): \text{Der}_R(S, N) \xrightarrow{\cong} \text{HH}^1(S/R; N).$$

When φ is surjective, with kernel I , one has

$$(5.4.1) \quad H_0(\text{L}(\varphi)) = 0 \quad \text{and} \quad H_1(\text{L}(\varphi)) \cong I/I^2.$$

Thus one gets an isomorphism

$$\text{Ext}_S^1(\text{L}(\varphi), N) \cong \text{Hom}_S(H_1(\text{L}(\varphi)), N) \cong \text{Hom}_S(I/I^2, N).$$

The degree one part of the Atiyah character will play a crucial role in the sequel.

Lemma 5.5. *When φ is surjective, the Atiyah character is bijective in degree one:*

$$\text{At}_\varphi^1(N): \text{Ext}_S^1(\text{L}(\varphi), N) \xrightarrow{\cong} \text{HH}^2(S/R; N).$$

Proof. Consider the construction of the cotangent complex from 5.1. It is shown in [27, §6] that since φ is surjective, $H_i(J^2) = 0$ for $i \leq 1$, and it follows from this that $\text{Ext}_{S_R^e}^i((S \otimes_R P)/J^2, N) = 0$ for $i \leq 2$. Hence the exact sequence arising from the triangle (5.1.1) degenerates into an isomorphism

$$0 \longrightarrow \text{Ext}_{S_R^e}^1(\text{L}(\varphi), N) \xrightarrow{(\text{At}^\varphi)^\vee} \text{Ext}_{S_R^e}^2(S, N) \longrightarrow 0.$$

The last ingredient is to observe that in (5.2.1) the map μ_1 is also an isomorphism:

$$\text{Ext}_S^1(\text{L}(\varphi), N) = \text{Hom}_S(H_1(\text{L}(\varphi)), N) = \text{Hom}_{S_R^e}(H_1(\text{L}(\varphi)), N) = \text{Ext}_{S_R^e}^1(\text{L}(\varphi), N).$$

Since $\text{At}_\varphi(N) = (\text{At}^\varphi)^\vee \circ \mu_*$ this completes the proof. $\square$

6. LOCAL COMPLETE INTERSECTION MAPS

This section is entirely about surjective maps of local rings. It contains a proof of Theorem B from the introduction. The argument uses Theorem 5.3 and a characterisation of the complete intersection property for a local map in terms of nilpotent elements in its homotopy Lie algebra.

6.1. Let R be a noetherian local ring, $\varphi: R \rightarrow S$ a surjective map, ℓ the residue field of S , and $F := \ell \otimes_R^L S$. Avramov [5, §3] attaches to φ a graded Lie algebra, called the *homotopy Lie algebra* of φ , and denoted $\pi(\varphi)$; see also [9]. It is an ℓ -vector subspace of $\text{Ext}_F(\ell, \ell)$ consisting of primitive elements, closed under the commutator, which endows it with the structure of a Lie algebra. We view $\pi(\varphi)$ as a subspace of the primitives of $\text{HH}^*(S/R; \ell)$, using (3.8.1).

6.2. Let φ be as above and suppose that $\text{proj dim}_R S$ is finite. The contrapositive of [10, Theorem C] due to Avramov and Halperin states: If for each α in $\pi^2(\varphi)$ and $\beta \in \pi(\varphi)$ there exists an integer $s \geq 1$ such that $(\text{ad } \alpha)^s(\beta) = 0$, then φ is complete intersection. That is, the kernel of φ is generated by a regular sequence.

This is one of the key ingredients in the proof of the result below. The statement involves the characteristic action of $\text{HH}^*(S/R; \ell)$ -action on $\text{Ext}_S(\text{L}(\varphi), \ell)$; see 4.6.

Theorem 6.3. *Let R be a noetherian local ring, $\varphi: R \rightarrow S$ a surjective map, and ℓ the residue field of S . If $\text{proj dim}_R S$ is finite, and there exists an integer n such that $\text{HH}^{\geq n}(S/R; \ell) \cdot \text{Ext}_S^1(\text{L}(\varphi), \ell) = 0$, then φ is complete intersection.*

Proof. By Theorem 5.3 the Atiyah character $\text{At}_\varphi(\ell)$ is equivariant with respect to the action of $\text{HH}^*(S/R; \ell)$. The hypotheses and Lemma 5.5 therefore yield

$$(6.3.1) \quad \text{ad}(x)(\alpha) = 0 \quad \text{for all } x \in \text{HH}^{\geq n}(S/R; \ell) \text{ and } \alpha \in \text{HH}^2(S/R; \ell).$$

We claim that this puts us in a context where the result recalled in 6.2 applies. To see this we exploit the fact that $\pi(\varphi)$ consists of primitive elements in $\text{HH}^*(S/R; \ell)$. Thus for any $\alpha \in \pi^2(\varphi)$ and $\beta \in \pi^{\geq n}(\varphi)$ one gets

$$\text{ad}(\alpha)(\beta) = [\alpha, \beta] = -[\beta, \alpha] = -\text{ad}(\beta)(\alpha) = 0,$$

where the first and last equality hold because α and β are primitive elements, see 1.4; the second one holds since α has degree 2 and the last one holds by (6.3.1). Then for any $s \geq \lceil n/2 \rceil$ and $\gamma \in \pi(\varphi)$ one gets that

$$\text{ad}(\alpha)^s(\gamma) = \text{ad}(\alpha)(\text{ad}(\alpha)^{s-1}(\gamma)) = 0$$

because the degree of $\text{ad}(\alpha)^{s-1}(\gamma)$ is $2(s-1) + |\gamma| \geq 2s \geq n$. Thus the result in 6.2 applies to yield that φ is complete intersection. $\square$

6.4. Let $\varphi: R \rightarrow S$ be a surjective local map with kernel I . Given (5.4.1), in $\text{D}(S)$ there is a natural morphism

$$\text{L}(\varphi) \longrightarrow \Sigma I/I^2.$$

When φ is complete intersection, $\text{proj dim}_R S$ is finite, the S -module I/I^2 is free, and the map above is a quasi-isomorphism. Theorem 6.5 gives a strong converse.

With $\mathfrak{m}$ denoting the maximal ideal of R , one has $I^2 \subseteq \mathfrak{m}I$. Composing the map above with the surjection $I/I^2 \rightarrow I/\mathfrak{m}I$ yields a morphism

$$(6.4.1) \quad \varepsilon: \text{L}(\varphi) \longrightarrow \Sigma I/\mathfrak{m}I$$

in $\text{D}(S)$. The result below is about this map.

Theorem 6.5. *Let $(R, \mathfrak{m})$ be a noetherian local ring and $\varphi: R \rightarrow S$ a surjective map with $\text{proj dim}_R S$ finite. If the morphism $\varepsilon: \text{L}(\varphi) \rightarrow \Sigma I/\mathfrak{m}I$ factors in $\text{D}(S)$ through a perfect complex, then φ is complete intersection.*

Proof. By hypothesis ε factors as a composition of morphisms $L(\varphi) \rightarrow P \rightarrow \Sigma I/\mathfrak{m}I$ in $D(S)$, where P is a perfect complex. With ℓ denoting the residue field of S , the map $\mathrm{Ext}_S(\varepsilon, \ell)$ factors as

$$\mathrm{Ext}_S(I/\mathfrak{m}I, \ell) \longrightarrow \mathrm{Ext}_S(P, \ell) \xrightarrow{\nu} \mathrm{Ext}_S(L(\varphi), \ell).$$

These maps are compatible with the characteristic action of $\mathrm{HH}^*(S/R; \ell)$. Using $H_i(L(\varphi)) = 0$ for $i \leq 0$ it is easy to verify that $\mathrm{Ext}_S^1(\varepsilon, \ell)$ is an isomorphism, and so ν^1 is surjective. As P is perfect, $\mathrm{Ext}_S^{\geq n+1}(P, \ell) = 0$ for some integer n , and so

$$\mathrm{HH}^{\geq n}(S/R; \ell) \cdot \mathrm{Ext}_S^1(P, \ell) = 0.$$

Hence the linearity of the maps with respect to the action of $\mathrm{HH}^*(S/R; \ell)$ and the surjectivity of ν^1 imply

$$\mathrm{HH}^{\geq n}(S/R; \ell) \cdot \mathrm{Ext}_S^1(L(\varphi), \ell) = 0.$$

Thus Theorem 6.5 applies and yields that φ is complete intersection. $\square$

6.6. Let $\varphi: R \rightarrow S$ be a surjective local map with $\mathrm{projdim}_R S$ finite, and set $I := \mathrm{Ker}(\varphi)$. If the second symbolic power $I^{(2)}$ satisfies $I^{(2)} \subseteq \mathfrak{m}I$, then Theorem 6.5 applies to say that if the S -module $I/I^{(2)}$ has finite projective dimension, then I is generated by a regular sequence, in which case $I/I^{(2)}$ is free; confer [20].

We end this section by sketching differential graded analogues of the above results. These remarks will not be needed in the sequel, but they may help to put the preceding discussion in context.

6.7. Let $\varphi: R \rightarrow S$ be a surjective local map, and ℓ the residue field of S . Using a minimal semi-free dg algebra resolution of φ in place of a simplicial resolution, one can define a dg version $L^{\mathrm{dg}}(\varphi)$ of the cotangent complex [11, 13]. The dg Atiyah class $\mathrm{dgAt}^\varphi: S \rightarrow \Sigma L^{\mathrm{dg}}(\varphi)$ can be defined analogously and from it the dg Atiyah character. Lemma 5.5 and Theorem 5.3 hold for $L^{\mathrm{dg}}(\varphi)$. There is a natural comparison map

$$\lambda: L^{\mathrm{dg}}(\varphi) \rightarrow L(\varphi).$$

When R contains a field of characteristic zero λ is a quasi-isomorphism; otherwise $L(\varphi)$ and $L^{\mathrm{dg}}(\varphi)$ are typically rather different; see [6, pp. 106].

In any case, the global Atiyah class At^φ factors through dgAt^φ via λ , so that the Atiyah character $\mathrm{At}_\varphi(\ell)$ factors as

$$\mathrm{Ext}_S(L(\varphi), \ell) \xrightarrow{\mathrm{Ext}_S(\lambda, \ell)} \mathrm{Ext}_S(L^{\mathrm{dg}}(\varphi), \ell) \longrightarrow \mathrm{HH}^*(S/R; \ell),$$

where the maps are compatible with the actions of $\mathrm{HH}^*(S/R; \ell)$. It turns out that the map on the right is bijective onto $\pi(\varphi)$. We plan to elaborate on this, and its connection to Lie algebras of local rings as introduced by André [1], elsewhere.

7. COTANGENT MODULES

In this section we prove Theorem A, and the other global results on locally complete intersection maps, announced in the Introduction.

7.1. In their full generality, locally complete intersection maps were defined by Avramov in [7]. We start with a local homomorphism $\varphi: R \rightarrow S$. By [8, 1.1], the

composition of φ with the completion map $S \rightarrow \widehat{S}$ at the maximal ideal of S admits a *regular factorisation*:

$$(7.1.1) \quad R \xrightarrow{\dot{\varphi}} R' \xrightarrow{\varphi'} \widehat{S},$$

where $\dot{\varphi}$ is a flat local map whose closed fibre is regular, and φ' is surjective. One says that φ is *complete intersection* if in some, equivalently, any, regular factorisation (7.1.1) the kernel of φ' is generated by a regular sequence; see [7, 3.3].

A map $\varphi: R \rightarrow S$ of noetherian rings is *locally complete intersection* if for each $\mathfrak{p}$ in $\text{Spec } S$ the localisation $\varphi_{\mathfrak{p}}: R_{\mathfrak{p} \cap R} \rightarrow S_{\mathfrak{p}}$ is complete intersection.

When φ is essentially of finite type the definition is already in [19]. In this context there is no need to complete S in (7.1.1), and one may take $\dot{\varphi}$ to be smooth.

7.2. Let S be a ring and M a complex S -modules such that $H_i(M) = 0$ for $i \ll 0$. Let $P \xrightarrow{\sim} M$ be a projective resolution of M ; thus P is a complex of projective S -modules with $P_i = 0$ for $i \ll 0$. If Q is another such resolution of M , then for each integer n , there is an quasi-isomorphism

$$P_{\geq n} \oplus \Sigma^n Q' \xrightarrow{\sim} Q_{\geq n} \oplus \Sigma^n P',$$

where P' and Q' are projective S -modules; see [12, 1.2]. In particular, one has an isomorphism of S -modules

$$H_n(P_{\geq n}) \oplus Q' \cong H_n(Q_{\geq n}) \oplus P'.$$

When M is a module, the S -module $H_n(P_{\geq n})$ is its n th syzygy module, and the discussion above says nothing more than that a syzygy module is well-defined, up to projective summands.

7.3. Let $\varphi: R \rightarrow S$ be a map of noetherian rings and $L(\varphi)$ its cotangent complex. Choose a projective resolution P of $L(\varphi)$ and for each integer n set

$$C_n(\varphi) := H_n(P_{\geq n}).$$

We call this S -module the *n th cotangent module* of φ ; by 7.2 it is independent of P , up to projective summands. In particular, the flat dimension, or the projective dimension, of $C_n(\varphi)$ is well-defined, and an invariant only of φ . Evidently $C_0(\varphi) \cong \Omega_{S/R}$, and in the case of a surjective homomorphism, when $S = R/I$ one has

$$C_0(\varphi) = 0 \quad \text{and} \quad C_1(\varphi) \cong I/I^2.$$

Therefore the cotangent modules can be considered as higher analogues of both the module of differentials and the conormal module. These objects were introduced by Avramov and Herzog [11]; in their context R is a field of characteristic zero and S is positively graded.

For $\mathfrak{p}$ in $\text{Spec } S$ there is a quasi-isomorphism $L(\varphi)_{\mathfrak{p}} \simeq L(\varphi_{\mathfrak{p}})$ by [2, IV.59 & V.27]. It follows that $C_n(\varphi)_{\mathfrak{p}} \cong C_n(\varphi_{\mathfrak{p}})$ up to projective $S_{\mathfrak{p}}$ -module summands.

Lemma 7.4. *Let $R \xrightarrow{\dot{\varphi}} R' \xrightarrow{\varphi'} S$ be maps of noetherian rings such that $\dot{\varphi}$ is smooth. Then for any integer $n \geq 1$ the flat dimension of $C_n(\varphi' \dot{\varphi})$ is equal to that of $C_n(\varphi')$.*

Proof. Setting $\varphi := \varphi' \dot{\varphi}$ yields the Jacobi-Zariski exact triangle

$$S \otimes_{R'}^L L(\dot{\varphi}) \longrightarrow L(\varphi) \longrightarrow L(\varphi') \longrightarrow$$

in $D(S)$; see [27, 5.1]. Let Q be a projective resolution of $L(\dot{\varphi})$, and P be a projective resolution of $L(\varphi)$. Then $S \otimes_{R'}^L L(\dot{\varphi}) \longrightarrow L(\varphi)$ may be represented by a map of

complexes $S \otimes_{R'} Q \rightarrow P$, and we may take $P' = \text{cone}(S \otimes_{R'} Q \rightarrow P)$ as a projective resolution of $L(\varphi')$. Having done this the truncated resolutions fit into a triangle

$$S \otimes_{R'} Q_{\geq n-1} \longrightarrow P_{\geq n} \longrightarrow P'_{\geq n} \longrightarrow .$$

Taking homology we obtain an exact sequence

$$H_n(S \otimes_{R'}^L L(\dot{\varphi})) \longrightarrow C_n(\varphi) \longrightarrow C_n(\varphi') \longrightarrow S \otimes_{R'} C_{n-1}(\dot{\varphi}) \longrightarrow 0.$$

Since $\dot{\varphi}$ is smooth, $L(\dot{\varphi})$ has flat dimension 0 by [2, XVI.17], and hence the same is true of the S -complex $S \otimes_{R'}^L L(\dot{\varphi})$. In particular $H_n(S \otimes_{R'}^L L(\dot{\varphi})) = 0$ and the module $S \otimes_{R'} C_{n-1}(\dot{\varphi})$ is flat, so the exact sequence is witness to the equality $\text{flat dim}_S C_n(\varphi) = \text{flat dim}_S C_n(\varphi')$. $\square$

When φ is locally complete intersection, the flat dimension of $L(\varphi)$ is at most one, and the S -modules $C_n(\varphi)$ are flat for $n \geq 1$. The result below provides a strong converse, at least for maps essentially of finite type. It generalises [11, Theorem 2.4] due to Avramov and Herzog, and answers a problem they pose in [11, §3].

Theorem 7.5. *Let $\varphi: R \rightarrow S$ be a map of noetherian rings, essentially of finite type and locally of finite flat dimension. If for some integer $n \geq 1$ the S -module $C_n(\varphi)$ has finite flat dimension, then φ is locally complete intersection.*

Proof. By 7.3 we may assume φ is local. Since φ is essentially of finite type it can be factored as $R \xrightarrow{\dot{\varphi}} R' \xrightarrow{\varphi'} S$, with $\dot{\varphi}$ smooth and with φ' surjective; see 7.1. Lemma 7.4 yields that $C_n(\varphi')$ has finite flat dimension. It follows from the fact that φ has finite flat dimension and $\dot{\varphi}$ is smooth that φ' has finite flat dimension as well; see, for instance, [8, Lemma 3.2]. Hence we may replace φ with φ' , and assume that it is surjective.

The homology modules $H_i(L(\varphi))$ are finitely generated and equal to 0 for $i \leq 0$. Thus there exists a resolution $P \xrightarrow{\sim} L(\varphi)$ consisting of finitely generated projective modules, with $P_i = 0$ for $i \leq 0$. In particular, we can assume $C_n(\varphi)$ is finitely generated, so the hypothesis gives that its projective dimension is finite. Moreover $H_1(P) = I/I^2$, where $I = \text{Ker}(\varphi)$, so there is a natural morphism of complexes $P \rightarrow \Sigma I/I^2$. Observe that this factors through the quotient complex

$$0 \longrightarrow C_n(\varphi) \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow 0.$$

Since the projective dimension of $C_n(\varphi)$ is finite, and each P_i is projective, the complex above is perfect. Thus in $D(S)$ the morphism $L(\varphi) \rightarrow \Sigma I/I^2$ factors through a perfect complex, and hence the same is true of the morphism

$$L(\varphi) \longrightarrow \Sigma(I/\mathfrak{m}I),$$

where $\mathfrak{m}$ is the maximal ideal of R ; see 6.4. We can now apply Theorem 6.5 to conclude that φ is complete intersection. $\square$

In Theorem 7.5, when $S = R/I$, the case $n = 1$ says that I is a locally complete intersection ideal as soon as the conormal module $C_1(\varphi) = I/I^2$ has finite projective dimension. This was a conjecture of Vasconcelos [29], resolved by the first author in [13]. Even in this special case, the proof given here is new. We will see soon that Theorem 7.5 also yields a new proof of a theorem of Avramov [7].

What happens in the case $n = 0$ is still unclear. Supposing that R is a field of characteristic zero, Vasconcelos conjectured that if $C_0(\varphi) = \Omega_{S/R}$ has finite

projective dimension over S , then S is a locally complete intersection ring [29]. In general this remains open, but see [11, 2.7] and [13, §3.4].

7.6. The flatness of the cotangent modules can be detected through the *André-Quillen homology functors*, namely

$$D_n(S/R; M) := H_n(L(\varphi) \otimes_S^L M).$$

Indeed, we can take a projective resolution P of $L(\varphi)$, and consider the natural map $C_n(\varphi) \rightarrow P_{n-1}$. For an S -module M , the hypothesis $D_n(S/R; M) = 0$ means exactly that the induced map

$$C_n(\varphi) \otimes_S M \longrightarrow P_{n-1} \otimes_S M$$

is injective. In particular, if $D_n(S/R; -) = 0$ on the category of S -modules, then $C_n(\varphi)$ is a *pure submodule* of P_{n-1} . Because of the exact sequence

$$0 \longrightarrow C_n(\varphi) \longrightarrow P_{n-1} \longrightarrow C_{n-1}(\varphi) \longrightarrow 0,$$

the flatness of P_{n-1} would then be inherited by both $C_n(\varphi)$ and $C_{n-1}(\varphi)$.

The next result implies immediately Quillen's conjecture [27, 5.6] that $L(\varphi)$ can only have finite flat dimension if φ is locally complete intersection, as proven originally by Avramov [7, Theorems 1.4, 1.5]. It also answers, in the affirmative, a question posed by him [3, Remark 2].

Corollary 7.7. *Let $\varphi: R \rightarrow S$ be a map of noetherian rings, locally of finite flat dimension. If $D_n(S/R; -) = 0$ for an $n \geq 1$, then φ is locally complete intersection.*

Proof. We may assume that φ is local by flat base change, and then we may take a regular factorisation of $R \rightarrow \widehat{S}$ as in (7.1.1). It follows from [7, 1.1] for $n = 1$, and from [7, 1.7] for $n \geq 2$, that if $D_n(S/R; -) = 0$ then as well $D_n(\widehat{S}/R'; -) = 0$. Hence we may assume φ is surjective.

By the observation in 7.6, the vanishing of $D_n(S/R; -)$ implies that both $C_n(\varphi)$ and $C_{n-1}(\varphi)$ are flat. So φ is complete intersection by Theorem 7.5. $\square$

We note that Theorem 7.5 is stronger than Corollary 7.7: The hypotheses of the later yield that *two* consecutive cotangent modules are flat, while the former theorem only requires a single cotangent module to have finite flat dimension.

7.8. The analogue of Corollary 7.7 for the homotopy Lie algebra says that for a surjective local homomorphism $\varphi: R \rightarrow S$ of finite flat dimension, if $\pi^n(\varphi) = 0$ for some $n \geq 2$ then φ is complete intersection. This was proven by Halperin in the case R is regular [22], and by Avramov in general [7]. The arguments of this paper can also be adapted to prove this result; indeed, the analogue of Theorem 7.5 works equally well using syzygies of the dg cotangent complex; see 6.7.

7.9. Corollary 7.7 is true as well for the André-Quillen cohomology functors, since $D^n(S/R; -) = 0$ implies $D_n(S/R; -) = 0$. In fact, the vanishing of $D^n(S/R; -)$ implies projectivity of the higher cotangent modules $C_n(\varphi)$ and $C_{n-1}(\varphi)$, which is a stronger condition than their flatness.

7.10. The results above can be restated geometrically. Let $f: X \rightarrow Y$ be a morphism of locally noetherian schemes. Illusie [23] associates to f a cotangent complex $L(f)$ in the derived category of quasi-coherent sheaves on X , from which we obtain the cotangent homology functors

$$D_n(X/Y; -) := H^{-n}(L(f) \otimes_X^L -).$$

By [23, 3.1.1] the vanishing of the cotangent homology functors can be detected on an open affine cover of X . Hence, it follows immediately from Corollary 7.7 that if f is locally of finite flat dimension and $D_n(X/Y; -) = 0$ for some $n \geq 1$ then f is locally complete intersection.

Now assume further that f is essentially of finite type, and that X has enough vector bundles (locally free sheaves); see [26, 1.2]. The cotangent complex $L(f)$ can then be represented as a bounded below complex P of vector bundles. We define the n th cotangent sheaf of f to be

$$C_n(f) := \operatorname{coker}(\partial: P_{n+1} \rightarrow P_n).$$

In this context we cannot assert that $C_n(f)$ is well-defined up to vector bundle summands. Instead we consider $C_n(f)$ as an object in the *singularity category* $D_{\text{sg}}(X)$ of Buchweitz [14] and Orlov [26]. Indeed, if $P \rightarrow P'$ is a quasi-isomorphism to another bounded below complex of vector bundles representing $L(f)$, then the induced map on the relevant cokernels has a perfect cone by 7.3, and is therefore an isomorphism in $D_{\text{sg}}(X)$. Finally, applying Theorem 7.5 locally yields the following statement:

Theorem 7.11. *If $f: X \rightarrow Y$ is a morphism of locally noetherian schemes, essentially of finite type and locally of finite flat dimension, and if $C_n(f) \simeq 0$ in $D_{\text{sg}}(X)$ for some $n \geq 1$ then f is locally complete intersection.* $\square$

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