

Pacific Journal of Mathematics

**EXCEPTIONAL COMPLETE INTERSECTION MAPS
OF LOCAL RINGS**

SRIKANTH B. IYENGAR, JANINA C. LETZ, JIAN LIU AND JOSH POLLITZ

EXCEPTIONAL COMPLETE INTERSECTION MAPS OF LOCAL RINGS

SRIKANTH B. IYENGAR, JANINA C. LETZ, JIAN LIU AND JOSH POLLITZ

This work concerns surjective maps $\varphi : R \rightarrow S$ of commutative noetherian local rings with kernel generated by a regular sequence that is part of a minimal generating set for the maximal ideal of R . The main result provides criteria for detecting such exceptional complete intersection maps in terms of the lattices of thick subcategories of the derived category of complexes of finite length homology. A key input is a characterization of such maps in terms of the truncated Atiyah class of φ .

Introduction

This work is part of an ongoing project to understand properties of commutative noetherian rings, and of maps between them, in terms of the triangulated structure of their derived categories. Building on the work of Dwyer, Greenlees and Iyengar [20], Pollitz [35] established a homotopical description of locally complete intersection rings as those commutative noetherian rings such that any nonzero thick subcategory of their bounded derived category contains a nonzero perfect complex. Similar homotopical characterizations of the complete intersection property have since appeared in the works of Briggs, Grifo and the authors [14; 15; 30]; the most general result is contained in [15].

A surjective map $\varphi : R \rightarrow S$ of local rings is a *complete intersection* map if its kernel is generated by an R -regular sequence. The focus of the present manuscript is on a special class of complete intersection maps where, in addition, the regular sequence is part of a minimal generating set for the maximal ideal of R . Such maps arise naturally in various contexts; for example, the diagonal of a smooth map has this property locally. They also arose in recent work of Iyengar with Brochard and

This work was partly supported by National Science Foundation DMS grants 2001368 (S. B. Iyengar) and 2002173 (J. Pollitz) and by the Alexander von Humboldt Foundation in the framework of an Alexander von Humboldt Professorship endowed by the German Federal Ministry of Education and Research (J. C. Letz).

MSC2020: primary 13B10; secondary 13D03, 13D09.

Keywords: exceptional complete intersection, Hochschild cohomology, truncated Atiyah class, thick subcategory.

Khare [16], where they are called *exceptional complete intersection maps*; we adopt this terminology.

Our motivation for the present undertaking comes from [15], which raises the prospect of defining locally complete intersection maps in the broader setting of tensor triangulated categories. However, to realize this goal requires a homotopical description of the diagonal of a smooth map, and more generally, of a weakly regular map. Our main result provides such a description for general exceptional complete intersection maps; see also the discussion in §5.8. Let $D_b^{\text{fl}}(R)$ be the subcategory of the derived category of R consisting of complexes with finite length homology. The statement involves the functor $\varphi_* : D_b^{\text{fl}}(S) \rightarrow D_b^{\text{fl}}(R)$ induced by restriction along φ .

Theorem A. *Let $\varphi : R \rightarrow S$ be a surjective map of commutative noetherian local rings. Then φ is exceptional complete intersection if and only if $\text{proj dim}_R S$ is finite and φ_* induces an isomorphism between the lattice of thick subcategories of $D_b^{\text{fl}}(S)$ and that of $D_b^{\text{fl}}(R)$.*

See Corollary 5.10 for a more precise statement. The exceptional complete intersection property does not localize and so, unsurprisingly, our characterization is in terms of the complexes supported only at the maximal ideals of R and S . One cannot, in general, relax this hypothesis to allow thick subcategories with bigger support; see Example 3.7 and Lemma 5.3. It is also worth pointing out that the characterizations of the complete intersection property in [14; 15; 35] use objects from $D_b^{\text{fl}}(R)$ in an essential way.

Questions about the thick subcategories of $D_b^f(R)$, the bounded derived category of R , can often be reduced to those about the thick subcategories of $D_b^f(R_p)$ as p ranges over the prime ideals in R ; this is the upshot of local-to-global principles from [12]. When the ring R is Cohen–Macaulay, Theorem A allows one to further reduce to the case of artinian rings. This idea is illustrated in Corollary 5.13.

For each S -complex M there is a naturally defined morphism in $D(S)$ called the *truncated Atiyah class of φ at M* , denoted $\text{at}^\varphi(M)$, which was introduced in [27]; see §2.1. The following characterization of exceptional complete intersection maps in terms of the truncated Atiyah class of k is an input in our proof of Theorem A.

Theorem B. *Let $\varphi : R \rightarrow S$ be a surjective map of commutative noetherian local rings, and let k be their common residue field. Then φ is exceptional complete intersection if and only if $\text{proj dim}_R S$ is finite and $\text{at}^\varphi(k) = 0$.*

This result is part of Theorem 3.4. It is essentially contained in [3, Proposition 2.8] and [15, §3.11], though couched in terms of actions of Hochschild cohomology. Our contribution is to clarify the connection with the Atiyah class; see, especially, Lemma 2.8.

Theorem B also sheds light on the faithfulness (or lack thereof) of the restriction functor $\varphi_* : D_b^f(S) \rightarrow D_b^f(R)$, at least when φ is complete intersection. In this case, φ_* is faithful if and only if $\text{at}^\varphi(M)$ is zero for each M in $D(S)$; see §3.8. This holds, for example, when φ is an algebra retract, but not always, even for exceptional complete intersection maps. This is connected to the “lifting problem”; see Example 3.9. This connection also makes it clear that in **Theorem B** the vanishing of $\text{at}^\varphi(k)$ cannot be replaced with the stronger conclusion that $\text{at}^\varphi(M)$ is zero for each M in $D(S)$; see Examples 3.7 and 3.9. One can thus view the nonvanishing of the truncated Atiyah class as a homological obstruction to φ admitting an algebra retract.

1. Hochschild cohomology

The Atiyah class of a map is closely connected to the Hochschild cohomology of the algebras involved. In this section we record the necessary facts about Hochschild cohomology and its characteristic action on derived categories. Some constructions require dg (differential graded) algebras and the corresponding derived category of dg modules; everything we need is explained in [4; 10].

1.1. We only consider graded-commutative dg algebras, so let A be such a dg algebra. The derived category of dg A -modules is denoted $D(A)$, and its suspension functor is denoted Σ . One has the usual bifunctors $\text{RHom}_A(-, -)$ and $- \otimes_A^L -$ on $D(A)$; for dg A -modules M and N , we set

$$\text{Ext}_A^*(M, N) := H^*(\text{RHom}_A(M, N)) \quad \text{and} \quad \text{Tor}_*^A(M, N) := H_*(M \otimes_A^L N).$$

An element α in $\text{Ext}_A^n(M, N)$ corresponds to a morphism $M \rightarrow \Sigma^n N$ in $D(A)$; by a slight abuse of notation we also write α for this morphism. Given $\alpha : M \rightarrow \Sigma^n M$ we write α^i for the composition

$$M \xrightarrow{\alpha} \Sigma^n M \xrightarrow{\Sigma^n \alpha} \dots \xrightarrow{\Sigma^{n(i-1)} \alpha} \Sigma^{ni} M$$

in $D(A)$, and say α is *nilpotent* if $\alpha^i = 0$ in $D(A)$ for some $i \geq 0$.

1.2. Let $\varphi : R \rightarrow S$ be a map of commutative rings. Set $S_R^e := S \otimes_R^L S$, the (derived) enveloping algebra of S over R , and let $\mu_R^S : S_R^e \rightarrow S$ be the multiplication map. We can model S_R^e as $S \otimes_R A$ where φ factors as maps of dg algebras

$$R \rightarrow A \xrightarrow{\epsilon} S$$

with A a dg R -algebra such that each R -module A_i is flat, and ϵ is a quasi-isomorphism. The multiplication map μ_R^S is modeled by $1 \otimes \epsilon : S \otimes_R A \rightarrow S$. This is independent of the choice of factorization of φ in the sense that if $R \rightarrow A' \rightarrow S$ is another such factorization of φ , then $A' \otimes_R S \simeq A \otimes_R S$ as dg algebras augmented to S . In particular, S is fixed under the equivalence $D(S \otimes_R A) \equiv D(S \otimes_R A')$ induced by the aforementioned quasi-isomorphism of dg algebras.

We view S as a dg S_R^c -module through the multiplication map μ_R^S .

1.3. The Hochschild cohomology of S over R is the graded-commutative S -algebra

$$\mathrm{HH}^*(S/R) := \mathrm{Ext}_{S_R^c}^*(S, S).$$

For each M in $\mathrm{D}(S)$ there is a morphism of graded S -algebras

$$\chi_M : \mathrm{HH}^*(S/R) \rightarrow \mathrm{Ext}_S^*(M, M)$$

induced by the functor $- \otimes_S^L M : \mathrm{D}(S^c) \rightarrow \mathrm{D}(S)$. We call χ_M the *characteristic map of M* . It defines a central action of $\mathrm{HH}^*(S/R)$ on $\mathrm{D}(S)$: for N in $\mathrm{D}(S)$ and maps α in $\mathrm{HH}^*(S/R)$ and β in $\mathrm{Ext}_S(M, N)$, one has

$$\chi_N(\alpha)\beta = (-1)^{|\alpha||\beta|}\beta\chi_M(\alpha).$$

1.4. A subset U of $\mathrm{HH}^*(S/R)$ acts *trivially*, respectively, *nilpotently*, on M if for each $\alpha \in U$, the element $\chi_M(\alpha)$ in $\mathrm{Ext}_S(M, M)$ is zero, respectively, nilpotent. For example, $\mathrm{HH}^{\geq 1}(S/R)$ acts trivially on the S -complex $S \otimes_R^L L$ for any R -complex L . The subcategory of $\mathrm{D}(S)$ consisting of S -complexes on which U acts nilpotently is thick; that is to say, it is a triangulated subcategory of $\mathrm{D}(S)$ closed under direct summands. These observation will be used often in what follows.

2. The truncated Atiyah class

In this section we recall the construction of the truncated Atiyah class, introduced by Huybrechts and Thomas [27], and discuss its connection to Hochschild cohomology. While the truncated Atiyah class, like the Atiyah class itself, can be defined for any map of rings, we only need it for surjective maps.

Throughout, $\varphi : R \rightarrow S$ will be a surjective map of commutative noetherian rings with kernel I . The discussion below is extracted from [15].

2.1. The multiplication map μ_R^S from § 1.2 extends in an exact triangle

$$S_R^c \xrightarrow{\mu} S \rightarrow \Sigma J \rightarrow$$

in $\mathrm{D}(S_R^c)$. From the triangle above, it is clear that

$$H_i(J) = \begin{cases} \mathrm{Tor}_{\geq 1}^R(S, S) & \text{if } i \geq 1, \\ 0 & \text{if } i \leq 0. \end{cases}$$

Thus the canonical truncation yields a morphism $J \rightarrow \Sigma I/I^2$ in $\mathrm{D}(S_R^c)$. The *truncated Atiyah class* of φ , denoted at^φ , is the composition of morphisms

$$S \rightarrow \Sigma J \rightarrow \Sigma^2 I/I^2$$

in $\mathrm{D}(S_R^c)$. Consider the map sending α in $\mathrm{Hom}_S(I/I^2, S)$ to the composition

$$S \xrightarrow{\mathrm{at}^\varphi} \Sigma^2 I/I^2 \xrightarrow{\Sigma^2 \alpha} \Sigma^2 S$$

in $D(S_R^c)$. By [15, Lemma 2.8], this is an isomorphism

$$\delta^\varphi : \text{Hom}_S(I/I^2, S) \xrightarrow{\cong} \text{HH}^2(S/R).$$

For each M in $D(S)$ and α in $\text{Hom}_S(I/I^2, S)$, applying $-\otimes_S^L M$ yields the following commutative diagram in $D(S)$:

$$(2.1.1) \quad \begin{array}{ccc} M & \xrightarrow{\text{at}^\varphi \otimes_S^L M} & \Sigma^2 I/I^2 \otimes_S^L M \\ & \searrow \chi_M(\delta^\varphi(\alpha)) & \downarrow \Sigma^2 \alpha \otimes 1 \\ & & \Sigma^2 M \end{array}$$

The horizontal map in this diagram is the *truncated Atiyah class* of φ at M , denoted $\text{at}^\varphi(M)$. Thus we can interpret $\text{at}^\varphi(M)$ as a class in $\text{Ext}_S^2(M, I/I^2 \otimes_S^L M)$.

2.2. As discussed in [27], the truncated Atiyah class $\text{at}^\varphi(M)$ can be recovered from the universal Atiyah class

$$\text{At}^\varphi : S \rightarrow \Sigma L(\varphi)$$

where $L(\varphi)$ is the cotangent complex of φ ; see [28; 36]. Namely, $\text{at}^\varphi(M)$ factors as

$$M \xrightarrow{\text{At}^\varphi \otimes_S^L M} \Sigma L(\varphi) \otimes_S^L M \rightarrow \Sigma^2 I/I^2 \otimes_S^L M$$

where the second map is induced by the soft truncation using $H_0(L(\varphi)) = 0$ and $H_1(L(\varphi)) = I/I^2$. The cotangent complex and universal Atiyah class of φ are defined via simplicial resolutions, and since this machinery is not needed in this article, we direct the interested reader to the references in this paragraph for details.

2.3. Suppose now that the surjective map $\varphi : R \rightarrow S$ is complete intersection of codimension c ; that is to say, that the ideal I is generated by an R -regular sequence of length c . Then the S -module I/I^2 is a free S -module of rank c , and the map δ^φ induces the isomorphism of S -algebras

$$\text{Sym}(\delta^\varphi) : \text{Sym}_S(\text{Hom}_S(I/I^2, S)) \xrightarrow{\cong} \text{HH}^*(S/R)$$

where $\text{Hom}_S(I/I^2, S)$ sits in cohomological degree 2. See [6, Section 3] for a proof. Choosing a basis for the S -module I/I^2 gives rise to a description of $\text{HH}^*(S/R)$ as a polynomial ring over S , in c variables of cohomological degree 2. These are the operators of Gulliksen [24] and Eisenbud [21]; see also [8].

Lemma 2.4. *Let $\varphi : R \rightarrow S$ be a surjective complete intersection map, and M an S -complex. One has $\text{at}^\varphi(M) = 0$ if and only if $\text{HH}^2(S/R)$ acts trivially on M .*

Proof. Set $I := \text{Ker } \varphi$ and consider the composition in $D(S)$

$$M \xrightarrow{\text{at}^\varphi(M)} \Sigma^2 I/I^2 \otimes_S^L M \simeq \Sigma^2 I/I^2 \otimes_S M.$$

The desired result follows from this, (2.1.1), and the freeness of I/I^2 over S . $\square$

2.5. Assume φ is complete intersection of codimension one. By §2.3, $\mathrm{HH}^2(S/R)$ is isomorphic to S , and so choosing a basis element η for the free S -module $\mathrm{HH}^2(S/R)$ identifies $\mathrm{at}^\varphi(M)$ with $\chi_M(\eta)$ in $\mathrm{Ext}_S^2(M, M)$. Finally, from the triangle defining the Atiyah class, one obtains the exact triangle in $\mathrm{D}(S)$

$$M \xrightarrow{\chi_M(\eta)} \Sigma^2 M \rightarrow \Sigma S \otimes_R^L \varphi_*(M) \rightarrow$$

where $\varphi_* : \mathrm{D}(S) \rightarrow \mathrm{D}(R)$ is the restriction functor; see, for example, [3].

The arguments in the next section use some basic results on André–Quillen cohomology and homotopy Lie algebras of local rings. We recall the relevant points below; for further details see [29; 32] and [4, Section 10].

2.6. Let $\varphi : R \rightarrow S$ be a surjective map of local rings, and let k be their common residue field. We write $\mathrm{D}^i(S/R; k)$ for the i -th André–Quillen cohomology module of φ with coefficients in k . We need elementary properties whose proofs are contained in the references above.

First, there is a natural identification

$$\mathrm{D}^1(S/R; k) \cong \mathrm{Hom}_k(I/\mathfrak{m}_R I, k)$$

where $\mathfrak{m}_R$ is the maximal ideal of R and I is the kernel of φ . Second, there is a canonically induced exact sequence of k -spaces

$$0 \rightarrow \mathrm{D}^1(k/S; k) \rightarrow \mathrm{D}^1(k/R; k) \rightarrow \mathrm{D}^1(S/R; k) \xrightarrow{\partial} \mathrm{D}^2(k/S; k) \rightarrow \cdots$$

called the Jacobi–Zariski sequence corresponding to the composition $R \xrightarrow{\varphi} S \rightarrow k$.

2.7. Fix a local ring R with residue field k . Let $\pi(R)$ denote the homotopy Lie algebra of R . It is a graded Lie algebra over k whose universal enveloping algebra is $\mathrm{Ext}_R(k, k)$. By the Poincaré–Birkhoff–Witt theorem, a k -basis for $\mathrm{Ext}_R(k, k)$ consists of all compositions of the form

$$\zeta_1^{i_1} \zeta_2^{i_2} \cdots \zeta_n^{i_n}$$

where ζ_j is in $\pi(R)$ with $i_j \geq 0$ when $|\zeta_j|$ is even and i_j is either 0 or 1 when $|\zeta_j|$ is odd; see, for example, [4, Theorem 10.2.1]. In particular, a nonzero element in $\pi(R)$ of even degree is not nilpotent when regarded as an element of $\mathrm{Ext}_R(k, k)$.

Let $\varphi : R \rightarrow S$ be a surjective map of local rings. There is a naturally defined map of graded Lie algebras

$$\pi(\varphi) : \pi(S) \rightarrow \pi(R).$$

An important point is that for $i = 1, 2$, the map φ determines a commutative diagram

$$(2.7.1) \quad \begin{array}{ccccc} D^i(k/S; k) & \xrightarrow{\cong} & \pi^i(S) & \hookrightarrow & \text{Ext}_S^i(k, k) \\ D^i(k/\varphi; k) \downarrow & & \downarrow \pi^i(\varphi) & & \downarrow \text{Ext}_\varphi^i(k, k) \\ D^i(k/R; k) & \xrightarrow{\cong} & \pi^i(R) & \hookrightarrow & \text{Ext}_R^i(k, k) \end{array}$$

Consider the diagram of S -modules

$$(2.7.2) \quad \begin{array}{ccccc} \text{HH}^2(S/R) & \xleftarrow[\cong]{\delta\varphi} & \text{Hom}_S(I/I^2, S) & \longrightarrow & \text{Hom}_S(I/I^2, k) \\ \downarrow \chi_k & & \searrow \psi & & \downarrow \cong \\ & & & & D^1(S/R; k) \\ & & & & \downarrow \bar{\partial} \\ \text{Ext}_S^2(k, k) & \xleftarrow[\iota]{} & \pi^2(S) & \xleftarrow[\cong]{} & D^2(k/S; k) \end{array}$$

where ι is the canonical inclusion and the unlabeled horizontal arrow is induced by $S \rightarrow k$; the map ψ is obtained by applying $\text{Hom}_{D(S)}(-, \Sigma^2 k)$ to the composition

$$k \xrightarrow{\text{at}^\varphi(k)} \Sigma^2 I/I^2 \otimes_S^L k \rightarrow \Sigma^2 I/I^2 \otimes_S k$$

where the second map is the augmentation.

Lemma 2.8. *Diagram (2.7.2) is commutative, and $\text{Im}(\psi) \supseteq \chi_k(\text{HH}^2(S/R))$ with equality when φ is complete intersection.*

Proof. By [15, §3.11], the image of ψ is $\text{Ker } \pi^2(\varphi)$. Now the commutativity of (2.7.2), and hence the desired containment, follows from the observations in §2.6 and (2.7.1). When φ is complete intersection, the containment is an equality since I/I^2 is a free S -module. $\square$

3. Exceptional complete intersections

In this section we prove Theorem B from the introduction, and present examples that witness some, perhaps, surprising behavior.

3.1. Let $\varphi : R \rightarrow S$ be a surjective map of noetherian local rings; set $I := \text{Ker } \varphi$. Following [16] we say φ is *exceptional complete intersection* provided I is generated by a regular sequence that can be extended to a minimal generating set for $\mathfrak{m}_R$, the maximal ideal of R . The latter condition means that the natural map

$$I/\mathfrak{m}_R I \rightarrow \mathfrak{m}_R/\pi_R^2$$

is one-to-one. We refer to [16], where this notion is introduced for not necessarily surjective maps of local rings, for various characterizations of exceptional complete

intersection maps in terms of numerical invariants associated to R and S . Here is one more, extracted from the proof of [16, Theorem 3.1].

Lemma 3.2. *Let $\varphi : R \rightarrow S$ be a surjective map of noetherian local rings. The map φ is exceptional complete intersection if and only if $\text{proj dim}_R S$ is finite and the natural map $I/\mathfrak{m}_R I \rightarrow \mathfrak{m}_R/\mathfrak{m}_R^2$ is one-to-one.*

Proof. Since a complete intersection map has finite projective dimension, the “only if” direction is clear. As to the converse: We can assume $I \neq 0$. Then the hypothesis that $\text{proj dim}_R S$ is finite means that I contains a nonzero divisor, say x , by a result of Auslander and Buchsbaum [17, Corollary 1.4.7]. Since $I \not\subseteq \mathfrak{m}_R^2$, we can ensure that x is not in $\mathfrak{m}_R^2$ by a variation of the prime avoidance argument [17, Lemma 1.2.2]. Setting $R' := R/xR$, the map φ factors as

$$R \rightarrow R' \rightarrow S.$$

By the choice of x , the map $R \rightarrow R'$ is exceptional complete intersection. Thus since $\text{proj dim}_R S$ is finite, so is $\text{proj dim}_{R'} S$, by a result of Nagata to be stated shortly. We can verify easily that the hypothesis on I is inherited by the kernel of the map $R' \rightarrow S$. We can thus proceed by induction on the number of generators of I to deduce that φ is exceptional complete intersection. $\square$

We now state the result of Nagata [33, Corollary 27.5] used above; this will be generalized later in Theorem 4.1.

3.3. *When $\varphi : R \rightarrow S$ is exceptional complete intersection and M is a finitely generated S -module, $\text{proj dim}_R M$ is finite if and only if $\text{proj dim}_S M$ is finite.*

The next result contains Theorem B (page 276). Most of the groundwork for the proof has been laid in Section 2.

Theorem 3.4. *Let $\varphi : R \rightarrow S$ be a surjective map of noetherian local rings with $\text{proj dim}_R S$ finite. The following conditions are equivalent:*

- (1) φ is exceptional complete intersection.
- (2) φ is complete intersection and $\text{HH}^2(S/R)$ acts trivially on k .
- (3) φ is complete intersection and $\text{HH}^2(S/R)$ acts nilpotently on k .
- (4) $\text{at}^\varphi(k) = 0$, where k is the residue field of S .

Proof. (1) $\Rightarrow$ (2) From the assumption and [3, Proposition 2.8] it follows that $\text{HH}^2(S/R)$ acts trivially on k .

(2) $\Leftrightarrow$ (3) This follows from Lemma 2.8, since nonzero elements of $\pi^2(S)$ cannot be nilpotent when regarded as maps in $\text{Ext}_S^2(k, k)$; see §2.7.

(2) $\Rightarrow$ (4) This is contained in Lemma 2.4.

(4) $\Rightarrow$ (1) Given that $\text{proj dim}_R S$ is finite, it suffices to prove that the natural map $I/\mathfrak{m}_R I \rightarrow \mathfrak{m}_R/\mathfrak{m}_R^2$ is one-to-one, or equivalently, that its k -vector space dual is onto, and to apply [Lemma 3.2](#). By assumption the commutative diagram (2.7.2) yields $\tilde{\partial} = 0$ in the Jacobi–Zariski sequence corresponding to $R \rightarrow S \rightarrow k$ from §2.6. Thus the latter reduces to an exact sequence

$$0 \rightarrow (\mathfrak{m}_S/\mathfrak{m}_S^2)^\vee \rightarrow (\mathfrak{m}_R/\mathfrak{m}_R^2)^\vee \rightarrow (I/\mathfrak{m}_R I)^\vee \rightarrow 0$$

where $(-)^\vee$ denotes the vector space dual and the maps are the canonically induced ones. This is the desired conclusion. $\square$

3.5. Recall from §2.2 that in $D(S)$ there is a commutative diagram

$$\begin{array}{ccc} k & \xrightarrow{\text{At}^\varphi \otimes_S k} & \Sigma L(\varphi) \otimes_S k \\ \parallel & & \downarrow \\ k & \xrightarrow{\text{at}^\varphi(k)} & \Sigma^2 I/I^2 \otimes_S^L k \end{array}$$

where the map on the right is induced by the soft truncation $\tau : L(\varphi) \rightarrow \Sigma I/I^2$; it is an isomorphism when φ is complete intersection. It follows from this observation and [Theorem 3.4](#) that φ is exceptional complete intersection if and only if $\text{proj dim}_R S$ is finite and the map $\text{At}^\varphi \otimes_S k$ is zero in $D(S)$.

3.6. Let $D_b^{\text{fl}}(S)$ denote the full subcategory of $D(S)$ consisting of S -complexes M such that the S -module $H(M)$ has finite length. Since $D_b^{\text{fl}}(S)$ is the smallest thick subcategory of $D(S)$ containing k — see, for example, [20, Example 3.5] — it follows from [Theorem 3.4](#) that when φ is exceptional complete intersection, the action of $\text{HH}^2(S/R)$ on $D_b^{\text{fl}}(S)$ is nilpotent; however there may be no bound on the nilpotence degree. Moreover $\text{HH}^2(S/R)$ need not act nilpotently on objects not in $D_b^{\text{fl}}(S)$. The example below illustrates these points.

Example 3.7. Let k be a field, and consider the hypersurface ring

$$R := k[[x, y, z]]/(x^2 - yz)$$

and the exceptional complete intersection map

$$\varphi : R \rightarrow S := R/(z) \cong k[[x, y]]/(x^2).$$

Let η denote the class dual to the class of z in $(z)/(z^2)$, viewed as an element of $\text{HH}^2(S/R)$ via the isomorphism δ^φ .

Set $M := S/(x)$, regarded as an S -module via the surjection $S \rightarrow S/(x)$. The minimal resolution of M over S is $\cdots \xrightarrow{x} S \xrightarrow{x} S \rightarrow 0$. From there it is clear that $\text{Ext}_S^i(M, M) = M$ for all $i \geq 0$. Using the construction of cohomology operators from [21, Section 1], one sees that multiplication by $\chi_M(\eta)$ on $\text{Ext}_S(M, M)$ is

represented by

$$M \xrightarrow{y^*} M.$$

In particular $\mathrm{HH}^2(S/R)$ does not act nilpotently on M .

In the same vein, for $N := S/(x, y^n)$, where n is any positive integer, one gets

$$\mathrm{Ext}_S^i(N, N) = \begin{cases} N & \text{if } i = 0, \\ N^{\oplus 2} & \text{if } i > 0, \end{cases}$$

and the action of $\chi_N(\eta)$ is represented by

$$N^{\oplus 2} \xrightarrow{\begin{pmatrix} y & 0 \\ 0 & y \end{pmatrix}} N^{\oplus 2}.$$

Thus the action of $\mathrm{HH}^i(S/R)$ on N is nontrivial for $0 \leq i < 2n$ and trivial for $i \geq 2n$.

3.8. Let $\varphi : R \rightarrow S$ be complete intersection. Consider the restriction functor

$$\varphi_* : \mathrm{D}(S) \rightarrow \mathrm{D}(R).$$

It is straightforward from [Lemma 2.4](#) and [§ 2.5](#) to see the following are equivalent:

- (1) φ_* is faithful on morphisms.
- (2) $\mathrm{HH}^2(S/R)$ acts trivially on each object of $\mathrm{D}(S)$.
- (3) $\mathrm{at}^\varphi(M)$ is zero for each object M in $\mathrm{D}(S)$.

When φ admits an algebra section — that is to say, a map of rings $\sigma : S \rightarrow R$ such that $\varphi\sigma = \mathrm{id}^S$ — the restriction functor φ_* is faithful and so $\mathrm{at}^\varphi(M)$ vanishes for each M in $\mathrm{D}(S)$.

Contrast this with the exceptional complete intersection map in [Example 3.7](#) where a class of modules present a nonzero, even nonnilpotent, truncated Atiyah class. Hence, when φ is exceptional complete intersection, the nonvanishing of truncated Atiyah classes serve as homological obstructions to the regular sequence generating $\mathrm{Ker} \varphi$ being, loosely speaking, independent power series variables in R . These can also be viewed as obstructions to the notion of weak liftability discussed in [\[2; 37\]](#).

Example 3.9. Assume $\varphi : R \rightarrow S$ is exceptional complete intersection with R , and hence S , a complete regular local ring. Whether φ_* is faithful depends on the characteristic of R . When R is equicharacteristic, φ admits an algebra section; hence by [§ 3.8](#) we conclude that φ_* is faithful.

In contrast, an example of Hochster [\[25, Example 1\]](#) settled Grothendieck's lifting problem in the negative. By combining a calculation of Dao [\[19, Example 3.5\]](#) and a result of Yoshino [\[37, Theorem 3.11\]](#), the module considered by Hochster has nonvanishing truncated Atiyah class over an exceptional complete intersection map whose base ring is a mixed characteristic regular local ring.

4. Homological invariants

In this section we track the change of certain homological invariants along exceptional complete intersection maps. The results obtained suggest that, notwithstanding the remarks in §3.8 and Example 3.9, homological properties of the source and target of an exceptional complete intersection map are tightly linked.

As before, given a local ring S , we write $D_b^f(S)$ for the full subcategory of $D(S)$ consisting of objects with finitely generated homology.

Theorem 4.1. *Let $\varphi : R \rightarrow S$ be exceptional complete intersection and M, N in $D_b^f(S)$. Then the following hold:*

- (1) $\text{Ext}_R^i(M, N) = 0$ for $i \gg 0$ if and only if $\text{Ext}_S^i(M, N) = 0$ for $i \gg 0$;
- (2) $\text{Tor}_i^R(M, N) = 0$ for $i \gg 0$ if and only if $\text{Tor}_i^S(M, N) = 0$ for $i \gg 0$.

Proof. We first reduce to the case when $H(M)$ has finite length. Let K be the Koszul complex on a system of parameters for S and consider the S -complex $K \otimes_S M$. Then the length of the S -module $H(K \otimes_S M)$ is finite. Moreover, since $K \otimes_S M$ can be realized as an iterated mapping cone of maps defined by multiplication by an element of S , a straightforward argument yields that

$$\begin{aligned} \text{Ext}_R^i(M, N) = 0 \text{ for } i \gg 0 &\iff \text{Ext}_R^i(K \otimes_S M, N) = 0 \text{ for } i \gg 0, \\ \text{Ext}_S^i(M, N) = 0 \text{ for } i \gg 0 &\iff \text{Ext}_S^i(K \otimes_S M, N) = 0 \text{ for } i \gg 0; \end{aligned}$$

see, for instance, [20, §2.9]. Thus replacing M by $K \otimes_S M$ we can assume the length of $H(M)$ is finite.

(1) Since the length of $H(M)$ is finite, the $\text{HH}^*(S/R)$ -action on $\text{Ext}_S(M, M)$, and hence also on $\text{Ext}_S(M, N)$, is nilpotent; see §3.6. The desired equivalence is now immediate from [24, Theorem 3.1] and [9, Theorem 4.2], which state that the R -module $\text{Ext}_R(M, N)$ is finitely generated if and only if the $\text{HH}^*(S/R)$ -module $\text{Ext}_S(M, N)$ is finitely generated.

(2) With $(-)^{\vee}$ denoting Matlis duality of S -modules, there are isomorphisms

$$\text{Tor}_i^R(M, N)^{\vee} \cong \text{Ext}_R^i(M^{\vee}, N) \quad \text{and} \quad \text{Tor}_i^S(M, N)^{\vee} \cong \text{Ext}_S^i(M^{\vee}, N).$$

Since $H(M)$ has finite length, so does $H(M^{\vee})$; the desired equivalence now follows from (1) and the faithfulness of Matlis duality. $\square$

Setting $N := k$ in either part of Theorem 4.1 recovers Nagata's theorem (see §3.3), whilst setting $M := k$ in Theorem 4.1(1) yields an analogous result for injective dimension. Here is a more precise statement; see [7] for the definitions of the projective dimension and injective dimension of a complex.

Corollary 4.2. *If $\varphi : R \rightarrow S$ is exceptional complete intersection, then for each M in $D_b^f(S)$ there are equalities*

$$\begin{aligned}\mathrm{proj\,dim}_R M &= \mathrm{proj\,dim}_S M + \dim R - \dim S, \\ \mathrm{inj\,dim}_R M &= \mathrm{inj\,dim}_S M + \dim R - \dim S.\end{aligned}$$

Proof. As noted above, from [Theorem 4.1](#) we know $\mathrm{proj\,dim}_R M$ is finite if and only if $\mathrm{proj\,dim}_S M$ is; so assume both values are finite. Now we can apply a generalization of the Auslander–Buchsbaum formula to complexes in [\[23, Theorem II\]](#) to obtain the first and third equalities in this computation:

$$\begin{aligned}\mathrm{proj\,dim}_R M &= \mathrm{depth} R - \mathrm{depth}_R M \\ &= \mathrm{depth} R - \mathrm{depth} S + \mathrm{depth} S - \mathrm{depth}_S M \\ &= \mathrm{depth} R - \mathrm{depth} S + \mathrm{proj\,dim}_S M \\ &= \dim R - \dim S + \mathrm{proj\,dim}_S M.\end{aligned}$$

The last equality holds because φ is complete intersection.

For the equality involving injective dimensions, from [Theorem 4.1](#), we can assume $\mathrm{inj\,dim}_R M$ and $\mathrm{inj\,dim}_S M$ are both finite. Now observe that

$$\begin{aligned}\mathrm{inj\,dim}_R M &= \mathrm{depth} R - \inf\{i \in \mathbb{Z} : H_i(M) \neq 0\} \\ &= \mathrm{depth} R - \mathrm{depth} S + \mathrm{depth} S - \inf\{i \in \mathbb{Z} : H_i(M) \neq 0\} \\ &= \mathrm{depth} R - \mathrm{depth} S + \mathrm{inj\,dim}_S M \\ &= \dim R - \dim S + \mathrm{inj\,dim}_S M\end{aligned}$$

where the first and third equalities hold by [\[11, Lemma 1.4\]](#). □

5. Homotopical characterization

In this section, $\varphi : R \rightarrow S$ is a surjective map of commutative noetherian local rings with (common) residue field k . Set $I := \mathrm{Ker} \varphi$. The main result in this section is [Theorem 5.6](#). An essential point is that when φ is complete intersection, information regarding finite building, in the sense recalled below, is related to nilpotence of the action of $\mathrm{HH}^2(S/R)$.

Given an S -complex M we write $\mathrm{thick}_S(M)$ for the smallest thick subcategory of $D(S)$ containing M ; see [\[13, Section 2\]](#) or [\[10, Section 2\]](#). Any S -complex N in $\mathrm{thick}_S(M)$ is said to be *finitely built* from M ; we write $M \models_S N$ to indicate that this is so.

Hopkins [\[26\]](#) and Neeman [\[34\]](#) proved that if M and N in $D_b^f(S)$ are S -complexes of finite projective dimension, and $\mathrm{supp}_S H(M) \supseteq \mathrm{supp}_S H(N)$, then $M \models_S N$. Here $\mathrm{supp}_S(-)$ denotes support. This result has the following consequence.

5.1. Let M in $D_b^f(S)$ be an S -complex of finite projective dimension and $H(M) \neq 0$. If K is the Koszul complex on a finite set of elements that generate an $\mathfrak{m}_S$ -primary ideal, then $M \models_S K$.

Lemma 5.2. *If $\text{proj dim}_R S$ is finite, then $\text{thick}_R(M) = \text{thick}_R(S \otimes_R^L M)$ for any M in $D(S)$.*

Proof. The desired result translates to $M \models_R S \otimes_R^L M$ and $S \otimes_R^L M \models_R M$.

As $\text{proj dim}_R S$ is finite, $R \models_R S$, and so applying $-\otimes_R^L M$ yields $M \models_R S \otimes_R^L M$.

Let K be the Koszul complex on a generating set for the ideal I . As $\text{proj dim}_R S$ is finite, it follows that $S \models_R K$; see §5.1. Applying $-\otimes_R^L M$, one sees that

$$S \otimes_R^L M \models_R K \otimes_R M.$$

It remains to observe that since R acts on M through S , in $D(R)$ the complex $K \otimes_R M$ is isomorphic to a finite direct sum of suspensions of M . $\square$

Our interest is in the statement corresponding to Lemma 5.2 in $D(S)$. Let M be an object of $D(S)$. In what follows, $S \otimes_R^L M$ is always regarded as an S -complex via the left S -action. That is, $S \otimes_R^L M$ denotes $S \otimes_R^L \varphi_*(M)$ as an object of $D(S)$.

By [20, Theorem 8.3], we have

$$(5.2.1) \quad M \models_S S \otimes_R^L M$$

when $\varphi : R \rightarrow S$ is a surjective complete intersection homomorphism, but the reverse building is not guaranteed. The obstruction is the nonnilpotence of the action of $\text{HH}^2(S/R)$ on M ; this is the content of the result below.

Lemma 5.3. *Let $\varphi : R \rightarrow S$ be a surjective complete intersection map. For an S -complex N the following conditions are equivalent:*

- (1) $\text{HH}^2(S/R)$ acts nilpotently on N .
- (2) $S \otimes_R^L N$ finitely builds N in $D(S)$.
- (3) For any S -complex M , we have $M \models_R N \iff M \models_S N$.

Proof. (1) $\Rightarrow$ (2) As $\text{HH}^2(S/R)$ acts nilpotently on N , any factorization of φ into a composition of complete intersection maps

$$R \rightarrow R' \rightarrow S$$

has the property that $\text{HH}^2(R'/R)$ and $\text{HH}^2(S/R')$ act nilpotently on N in $D(R')$ and $D(S)$, respectively; this follows from [8, §3.1] or [3, Proposition 2.3]. Furthermore, there is an isomorphism

$$S \otimes_{R'}^L (R' \otimes_R^L N) \simeq S \otimes_R^L N$$

in $D(S)$. Hence, by induction on the codimension of φ , it is enough to show (2) holds when φ is complete intersection of codimension one.

In this case, there is an exact triangle

$$N \xrightarrow{\chi_N(\eta)} \Sigma^2 N \rightarrow \Sigma S \otimes_R^L N \rightarrow$$

where η is a basis element for the free S -module $\mathrm{HH}^2(S/R)$; see §2.5. Iteratively applying the octahedral axiom yields

$$(5.3.1) \quad S \otimes_R^L N \models_S \mathrm{cone}(\chi_N(\eta)^n)$$

for any $n \geq 1$. The assumption that $\mathrm{HH}^2(S/R)$ acts nilpotently on N means there is some $n \geq 1$ such that $\chi_N(\eta)^n = 0$ and so $\mathrm{cone}(\chi_N(\eta)^n) \simeq \Sigma N \oplus \Sigma^{2n} N$ in $\mathrm{D}(S)$. Combining this with (5.3.1) we obtain the desired result.

(2) $\Rightarrow$ (3) Evidently, if $M \models_S N$ then $M \models_R N$. When $M \models_R N$, by applying the functor $S \otimes_R^L -$ we obtain

$$S \otimes_R^L M \models_S S \otimes_R^L N;$$

from this and the assumption in (2), we obtain that $S \otimes_R^L M \models_S N$. Finally

$$M \models_S S \otimes_R^L M$$

by (5.2.1), so combining this with the already established building $S \otimes_R^L M \models_S N$, we conclude that $M \models_S N$ as desired.

(3) $\Rightarrow$ (1) From Lemma 5.2 and the hypothesis in (3) we get $S \otimes_R^L N \models_S N$. As discussed in §1.3, the full subcategory of $\mathrm{D}(S)$ consisting of objects for which $\mathrm{HH}^2(S/R)$ is nilpotent is a thick subcategory of $\mathrm{D}(S)$. In particular, this subcategory is closed under finite building. Note that $\mathrm{HH}^2(S/R)$ has trivial action on $S \otimes_R^L N$ and so $\mathrm{HH}^2(S/R)$ acts nilpotently on N . $\square$

5.4. Let $\varphi_* : \mathrm{D}(S) \rightarrow \mathrm{D}(R)$ be the restriction functor and S a subcategory of $\mathrm{D}(S)$. We say that φ_* has *ascent of finite building on S* if whenever M and N are objects of S with $\varphi_*(M) \models_R \varphi_*(N)$, then $M \models_S N$. By a slight abuse of notation we will say φ has ascent of finite building on S .

As discussed in the introduction, much of this work is motivated by the homotopical characterizations of surjective complete intersection maps in [14; 15; 30; 35]. Before continuing to the main result of the article, we recall a method from [15] used to detect the complete intersection property among surjective maps of finite projective dimension.

5.5. Let K denote the Koszul complex on a minimal generating set for $\mathfrak{m}_S$ over S . If $\mathrm{proj\,dim}_R S$ is finite, then there exist $M_1, \dots, M_n$ in $\mathrm{D}_b^{\mathrm{fl}}(S)$ satisfying the following properties:

- (1) Each M_i finitely builds K in $\mathrm{D}(R)$.
- (2) If each M_i finitely builds K in $\mathrm{D}(S)$, then φ is complete intersection.

Explicitly, one can take the M_i as mapping cones in $D(S)$ on a generating set for $\text{Ker } \pi^2(\varphi)$; see §2.7. The first point is justified by [15, Lemmas 2.5 and 3.6] while the second condition holds by [15, Theorem 3.12].

Theorem 5.6. *A surjective map $\varphi : R \rightarrow S$ of commutative noetherian local rings is exceptional complete intersection if and only if $\text{proj dim}_R S$ is finite and φ has ascent of finite building on $D_b^{\text{fl}}(S)$.*

Proof. ($\Rightarrow$) As φ is exceptional complete intersection, §3.6 implies $\text{HH}^2(S/R)$ acts nilpotently on each object of $D_b^{\text{fl}}(S)$. Now the desired result follows from Lemma 5.3.

($\Leftarrow$) Since $\text{proj dim}_R S$ is finite, there exist $M_1, \dots, M_n$ in $D_b^{\text{fl}}(S)$ from §5.5. Using the assumption that φ has ascent of finite building, it follows from the two properties of the M_i from §5.5 that φ is complete intersection.

As $\text{proj dim}_R S$ is finite, $S \otimes_R^L k \models_R k$ in $D_b^{\text{fl}}(R)$. Since φ has ascent of finite building on $D_b^{\text{fl}}(S)$, we conclude that

$$S \otimes_R^L k \models_S k.$$

In particular $\text{HH}^2(S/R)$ acts nilpotently on k , for $\text{HH}^2(S/R)$ acts trivially on $S \otimes_R^L k$. Thus φ is exceptional complete intersection by Theorem 3.4. $\square$

Remark 5.7. Instead of invoking Theorem 3.4 in the proof of the necessary direction of Theorem 5.6, we can use the same argument that φ is complete intersection as before. Next we factor $R \rightarrow E \rightarrow S$ where E is the Koszul complex over R on a minimal generating set for $\text{Ker } \varphi$. The following commutative diagram is obtained by restricting scalars, and the vertical map is an equivalence since $E \xrightarrow{\sim} S$.

$$\begin{array}{ccc} D(R) & \longleftarrow & D(E) \\ & \nwarrow & \uparrow \cong \\ & & D(S) \end{array}$$

Now as φ has ascent of finite building on $D_b^{\text{fl}}(S)$, so does $R \rightarrow E$ on $D_b^{\text{fl}}(E)$.

The main point is that the Koszul complex over R on a minimal generating set for $\mathfrak{m}_R$, denoted K^R , is an object of $D_b^{\text{fl}}(E)$ with the property

$$E \otimes_R K^R \models_E K^R$$

if and only if φ is exceptional complete intersection; this can be verified by a direct calculation. Now as $R \rightarrow E$ has ascent of finite building on $D_b^{\text{fl}}(E)$ and one always has the finite building

$$E \otimes_R K^R \models_R K^R,$$

we obtain the desired conclusion from the previous remark.

Note that the argument sketched above justifies that when φ is complete intersection, φ is exceptional complete intersection if and only if $S \otimes_E^L K^R$ has finite projective dimension over S .

5.8. [Theorem 5.6](#) is a characterization of exceptional complete intersection maps purely in terms of the structure of $D(R)$ and $D(S)$ as triangulated categories. Namely, for a triangulated category T , consider its thick subcategory of compact objects, denoted T^c . Define T_b^{fl} to be the full subcategory of T consisting of objects X such that $\text{Hom}_T(C, X)$ is a finite length $\text{Hom}_T(C, C)$ -module for each C in T^c . In the context of $D(R)$, it is well known an R -complex belongs to $D(R)^c$ precisely when it has finite projective dimension over R , and $D(R)_b^{\text{fl}} = D_b^{\text{fl}}(R)$.

Let $\text{Thick } T$ be the lattice of thick subcategories of a triangulated category T .

5.9. Let $\varphi : R \rightarrow S$ be a surjective map of local rings, and $\varphi_* : D(S) \rightarrow D(R)$ the restriction functor. For a thick subcategory U of $D_b^{\text{fl}}(R)$, we write $F(U)$ to denote the smallest thick subcategory containing all objects M in $D_b^{\text{fl}}(S)$ such that $\varphi_*(M)$ is in U . If S is a thick subcategory of $D_b^{\text{fl}}(S)$, then $G(S)$ is the smallest thick subcategory containing all objects $\varphi_*(M)$ in $D_b^{\text{fl}}(R)$ such that M is in S . Thus there exist maps of lattices

$$\text{Thick } D_b^{\text{fl}}(R) \xrightleftharpoons[F]{F} \text{Thick } D_b^{\text{fl}}(S).$$

The following result is [Theorem A](#).

Corollary 5.10. *The map φ is exceptional complete intersection if and only if $\text{proj dim}_R S$ is finite and F and G are mutually inverse lattice isomorphisms.*

Proof. $(\Leftarrow)$ This follows immediately from [Theorem 5.6](#).

$(\Rightarrow)$ From [§3.6](#) and [Lemma 5.3](#), FG is the identity on $\text{Thick } D_b^{\text{fl}}(S)$.

Fix a thick subcategory U in $\text{Thick } D_b^{\text{fl}}(R)$. To show $GF(U) = U$, it suffices to show that for each object L in U ,

$$(5.10.1) \quad \text{thick}_R L = \text{thick}_R \varphi_*(M) \quad \text{where} \quad M = S \otimes_R^L L.$$

Since $\text{proj dim}_R S$ is finite, $L \models_R \varphi_*(M)$, by [Lemma 5.2](#). Let K be the Koszul complex on a minimal generating set for $\mathfrak{m}_R$ and E the Koszul complex on a minimal generating set for $\text{Ker}(\varphi)$. By [§5.1](#), $E \models_R K$ and so

$$\varphi_*(M) \simeq E \otimes_R L \models_R K \otimes_R L.$$

Furthermore, as $H(L)$ has finite length over R , it follows that $K \otimes_R L \models_R L$ and so $\varphi_*(M) \models_R L$; thus, [\(5.10.1\)](#) holds. $\square$

We end this article with two applications. The first should be compared with the factorization theorems in [\[5, Lemma 5.7.1\]](#) and [\[15, Corollary 4.2\]](#)

Corollary 5.11. *Consider surjective maps $R \xrightarrow{\varphi} S \xrightarrow{\psi} T$ of commutative noetherian local rings. If φ and ψ are exceptional complete intersection, then $\psi\varphi$ is exceptional complete intersection. The converse holds when $\text{proj dim}_S T$ is finite.*

Proof. It is clear from the definition that the class of exceptional complete intersection maps is closed under composition. It also follows trivially from the characterization in [Theorem 5.6](#).

Assume that $\text{proj dim}_S T$ is finite and $\psi\varphi$ is exceptional complete intersection. It can be argued as in [\[15, Corollary 4.2\]](#) that $\text{proj dim}_R S$ is finite. Also, it is clear that ψ has ascent of finite building on $D_b^{\text{fl}}(S)$ since $\psi\varphi$ does. Hence, ψ is exceptional complete intersection. Now as ψ and $\psi\varphi$ have ascent of finite building on finite length objects, we can apply [Corollary 5.10](#) to deduce φ is exceptional complete intersection. $\square$

Another application of [Corollary 5.10](#) is provided when R is an equicharacteristic Cohen–Macaulay ring of *minimal multiplicity*. When R is Cohen–Macaulay, Abhyankar [\[1\]](#) proved the Hilbert–Samuel multiplicity of R is bounded below by $\text{codim } R + 1$; when equality holds R is said to be of minimal multiplicity.

5.12. For the rest of the section, assume R is an equicharacteristic Cohen–Macaulay ring of minimal multiplicity and infinite residue field k . There exists an exceptional complete intersection map $R \rightarrow S$ such that $\mathfrak{m}_S^2 = 0$; see [\[17, Exercise 4.6.14\]](#). As S is equicharacteristic, it has the form $k[x_1, \dots, x_n]/(x_1, \dots, x_n)^2$ for some $n \geq 0$.

The structure of the lattice $\mathcal{T} = \text{Thick } D_b^{\text{fl}}(R)$ is especially simple when R is also Gorenstein; this forces $n \leq 1$.

If $n = 0$, then R is regular and so $\mathcal{T}$ is simply $0 \subsetneq D_b^{\text{fl}}(R)$.

If $n = 1$, then the lattice $\mathcal{T}$ is

$$0 \subsetneq \text{thick}_R(K) \subsetneq D_b^{\text{fl}}(R),$$

where K is a Koszul complex on any system of parameters for R . This is a consequence of [Corollary 5.10](#) and [\[18; 31\]](#).

When R is not Gorenstein, $\text{Thick } D_b^{\text{fl}}(R)$ possesses a far wilder structure.

Corollary 5.13. *Let R be an equicharacteristic Cohen–Macaulay non-Gorenstein ring of minimal multiplicity with infinite residue field k . Then there exists an infinite descending binary tree of finitely generated thick subcategories in $\text{Thick } D_b^{\text{fl}}(R)$.*

Proof. The assumptions on R force S , from [§5.12](#), to be $k[x_1, \dots, x_n]/(x_1, \dots, x_n)^2$ with $n \geq 2$. Now the stated result is a consequence of [Corollary 5.10](#) and the work of Elagin and Lunts [\[22, Theorem A\]](#). $\square$

References

- [1] S. S. Abhyankar, “Local rings of high embedding dimension”, *Amer. J. Math.* **89** (1967), 1073–1077. [MR](#) [Zbl](#)
- [2] M. Auslander, S. Ding, and Ø. Solberg, “Liftings and weak liftings of modules”, *J. Algebra* **156**:2 (1993), 273–317. [MR](#) [Zbl](#)
- [3] L. L. Avramov, “Modules of finite virtual projective dimension”, *Invent. Math.* **96**:1 (1989), 71–101. [MR](#) [Zbl](#)
- [4] L. L. Avramov, “Infinite free resolutions”, pp. 1–118 in *Six lectures on commutative algebra*, edited by J. Elias et al., Progress in Mathematics **166**, Birkhäuser Verlag, Basel, 1998. [MR](#) [Zbl](#)
- [5] L. L. Avramov, “Locally complete intersection homomorphisms and a conjecture of Quillen on the vanishing of cotangent homology”, *Ann. of Math.* (2) **150**:2 (1999), 455–487. [MR](#) [Zbl](#)
- [6] L. L. Avramov and R.-O. Buchweitz, “Homological algebra modulo a regular sequence with special attention to codimension two”, *J. Algebra* **230**:1 (2000), 24–67. [MR](#) [Zbl](#)
- [7] L. L. Avramov and H.-B. Foxby, “Homological dimensions of unbounded complexes”, *J. Pure Appl. Algebra* **71**:2-3 (1991), 129–155. [MR](#) [Zbl](#)
- [8] L. L. Avramov and L.-C. Sun, “Cohomology operators defined by a deformation”, *J. Algebra* **204**:2 (1998), 684–710. [MR](#) [Zbl](#)
- [9] L. L. Avramov, V. N. Gasharov, and I. V. Peeva, “Complete intersection dimension”, *Inst. Hautes Études Sci. Publ. Math.* **86** (1997), 67–114. [MR](#) [Zbl](#)
- [10] L. L. Avramov, R.-O. Buchweitz, S. B. Iyengar, and C. Miller, “Homology of perfect complexes”, *Adv. Math.* **223**:5 (2010), 1731–1781. [MR](#) [Zbl](#)
- [11] L. L. Avramov, S. B. Iyengar, and J. Lipman, “Reflexivity and rigidity for complexes, I: Commutative rings”, *Algebra Number Theory* **4**:1 (2010), 47–86. [MR](#) [Zbl](#)
- [12] D. Benson, S. B. Iyengar, and H. Krause, “A local-global principle for small triangulated categories”, *Math. Proc. Cambridge Philos. Soc.* **158**:3 (2015), 451–476. [MR](#) [Zbl](#)
- [13] A. Bondal and M. van den Bergh, “Generators and representability of functors in commutative and noncommutative geometry”, *Mosc. Math. J.* **3**:1 (2003), 1–36. [MR](#) [Zbl](#)
- [14] B. Briggs, E. Grifo, and J. Pollitz, “Constructing nonproxy small test modules for the complete intersection property”, *Nagoya Math. J.* **246** (2022), 412–429. [MR](#) [Zbl](#)
- [15] B. Briggs, S. B. Iyengar, J. C. Letz, and J. Pollitz, “Locally complete intersection maps and the proxy small property”, *Int. Math. Res. Not.* **2022**:16 (2022), 12625–12652.
- [16] S. Brochard, S. B. Iyengar, and C. B. Khare, “A freeness criterion without patching for modules over local rings”, *J. Inst. Math. Jussieu* (2021), 1–13.
- [17] W. Bruns and J. Herzog, *Cohen–Macaulay rings*, revised ed., Cambridge Studies in Advanced Mathematics **39**, Cambridge University Press, 1998. [Zbl](#)
- [18] J. F. Carlson and S. B. Iyengar, “Thick subcategories of the bounded derived category of a finite group”, *Trans. Amer. Math. Soc.* **367**:4 (2015), 2703–2717. [MR](#) [Zbl](#)
- [19] H. Dao, “On liftable and weakly liftable modules”, *J. Algebra* **318**:2 (2007), 723–736. [MR](#) [Zbl](#)
- [20] W. Dwyer, J. P. C. Greenlees, and S. Iyengar, “Finiteness in derived categories of local rings”, *Comment. Math. Helv.* **81**:2 (2006), 383–432. [MR](#) [Zbl](#)
- [21] D. Eisenbud, “Homological algebra on a complete intersection, with an application to group representations”, *Trans. Amer. Math. Soc.* **260**:1 (1980), 35–64. [MR](#) [Zbl](#)
- [22] A. Elagin and V. A. Lunts, “Derived categories of coherent sheaves on some zero-dimensional schemes”, *J. Pure Appl. Algebra* **226**:6 (2022), art. id. 106939, 30 pp. [MR](#) [Zbl](#)

- [23] H.-B. Foxby and S. Iyengar, “Depth and amplitude for unbounded complexes”, pp. 119–137 in *Commutative algebra* (Grenoble/Lyon, 2001), edited by L. L. Avramov et al., Contemp. Math. **331**, Amer. Math. Soc., Providence, RI, 2003. [MR](#) [Zbl](#)
- [24] T. H. Gulliksen, “A change of ring theorem with applications to Poincaré series and intersection multiplicity”, *Math. Scand.* **34** (1974), 167–183. [MR](#) [Zbl](#)
- [25] M. Hochster, “An obstruction to lifting cyclic modules”, *Pacific J. Math.* **61**:2 (1975), 457–463. [MR](#) [Zbl](#)
- [26] M. J. Hopkins, “Global methods in homotopy theory”, pp. 73–96 in *Homotopy theory* (Durham, 1985), edited by E. Rees and J. D. S. Jones, London Math. Soc. Lecture Note Ser. **117**, Cambridge Univ. Press, 1987. [MR](#) [Zbl](#)
- [27] D. Huybrechts and R. P. Thomas, “Deformation-obstruction theory for complexes via Atiyah and Kodaira–Spencer classes”, *Math. Ann.* **346**:3 (2010), 545–569. [MR](#) [Zbl](#)
- [28] L. Illusie, *Complexe cotangent et déformations, I*, Lecture Notes in Mathematics **239**, Springer, 1971. [MR](#) [Zbl](#)
- [29] S. Iyengar, “André–Quillen homology of commutative algebras”, pp. 203–234 in *Interactions between homotopy theory and algebra*, edited by L. L. Avramov et al., Contemp. Math. **436**, Amer. Math. Soc., Providence, RI, 2007. [MR](#) [Zbl](#)
- [30] J. C. Letz, “Local to global principles for generation time over commutative noetherian rings”, *Homology Homotopy Appl.* **23**:2 (2021), 165–182. [MR](#) [Zbl](#)
- [31] J. Liu and J. Pollitz, “Duality and symmetry of complexity over complete intersections via exterior homology”, *Proc. Amer. Math. Soc.* **149**:2 (2021), 619–631. [MR](#) [Zbl](#)
- [32] J. Majadas and A. G. Rodicio, *Smoothness, regularity and complete intersection*, London Mathematical Society Lecture Note Series **373**, Cambridge University Press, 2010. [MR](#) [Zbl](#)
- [33] M. Nagata, *Local rings*, Interscience Tracts in Pure and Applied Mathematics **13**, Wiley, New York, 1962. [MR](#) [Zbl](#)
- [34] A. Neeman, “The chromatic tower for $D(R)$ ”, *Topology* **31**:3 (1992), 519–532. [MR](#) [Zbl](#)
- [35] J. Pollitz, “The derived category of a locally complete intersection ring”, *Adv. Math.* **354** (2019), art. id. 106752, 18 pp. [MR](#) [Zbl](#)
- [36] D. Quillen, “Rational homotopy theory”, *Ann. of Math.* (2) **90** (1969), 205–295. [MR](#) [Zbl](#)
- [37] Y. Yoshino, “The theory of L -complexes and weak liftings of complexes”, *J. Algebra* **188**:1 (1997), 144–183. [MR](#) [Zbl](#)

Received July 15, 2021. Revised February 22, 2022.

SRIKANTH B. IYENGAR
 DEPARTMENT OF MATHEMATICS
 UNIVERSITY OF UTAH
 SALT LAKE CITY, UT
 UNITED STATES
iyengar@math.utah.edu

JIAN LIU
 SCHOOL OF MATHEMATICAL SCIENCES
 SHANGHAI JIAO TONG UNIVERSITY
 SHANGHAI
 CHINA
liuj231@sjtu.edu.cn

JANINA C. LETZ
 FACULTY OF MATHEMATICS
 BIELEFELD UNIVERSITY
 BIELEFELD
 GERMANY
jletz@math.uni-bielefeld.de

JOSH POLLITZ
 DEPARTMENT OF MATHEMATICS
 UNIVERSITY OF UTAH
 SALT LAKE CITY, UT
 UNITED STATES
pollitz@math.utah.edu

PACIFIC JOURNAL OF MATHEMATICS

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

msp.org/pjm

EDITORS

Don Blasius (Managing Editor)
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
blasius@math.ucla.edu

Matthias Aschenbrenner
Fakultät für Mathematik
Universität Wien
Vienna, Austria
matthias.aschenbrenner@univie.ac.at

Paul Balmer
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
balmer@math.ucla.edu

Vyjayanthi Chari
Department of Mathematics
University of California
Riverside, CA 92521-0135
chari@math.ucr.edu

Daryl Cooper
Department of Mathematics
University of California
Santa Barbara, CA 93106-3080
cooper@math.ucsb.edu

Wee Teck Gan
Mathematics Department
National University of Singapore
Singapore 119076
matgwt@nus.edu.sg

Robert Lipshitz
Department of Mathematics
University of Oregon
Eugene, OR 97403
lipshitz@uoregon.edu

Kefeng Liu
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
liu@math.ucla.edu

Jiang-Hua Lu
Department of Mathematics
The University of Hong Kong
Pokfulam Rd., Hong Kong
jhlu@maths.hku.hk

Sorin Popa
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
popa@math.ucla.edu

Paul Yang
Department of Mathematics
Princeton University
Princeton NJ 08544-1000
yang@math.princeton.edu

PRODUCTION

Silvio Levy, Scientific Editor, production@msp.org

See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2022 is US \$560/year for the electronic version, and \$760/year for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by *Mathematical Reviews*, *Zentralblatt MATH*, PASCAL CNRS Index, *Referativnyi Zhurnal*, *Current Mathematical Publications* and *Web of Knowledge* (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 1945-5844 electronic, 0030-8730 printed) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY

 **mathematical sciences publishers**
nonprofit scientific publishing

<http://msp.org/>

© 2022 Mathematical Sciences Publishers

PACIFIC JOURNAL OF MATHEMATICS

Volume 318 No. 2 June 2022

Divisors on surfaces isogenous to a product of mixed type with $p_g = 0$	233
DAVIDE FRAPPORTI and KYOUNG-SEOG LEE	
Elliptic surfaces of Kodaira dimension zero	249
KENTARO MITSUI	
Exceptional complete intersection maps of local rings	275
SRIKANTH B. IYENGAR, JANINA C. LETZ, JIAN LIU and JOSH POLLITZ	
A remark on the Gersten complex for Milnor K -theory	295
RAKESH PAWAR	
An algebraic construction of sum-integral interpolators	305
BENJAMIN FISCHER and JAMIE POMMERSHEIM	
Certain Fourier operators on GL_1 and local Langlands gamma functions	339
DIHUA JIANG and ZHILIN LUO	
The concordance crosscap number and rational Witt span of a knot	375
STANISLAV JABUKA	
On folded cluster patterns of affine type	401
BYUNG HEE AN and EUNJEONG LEE	
Semigroup rings as weakly Krull domains	433
GYU WHAN CHANG, VICTOR FADINGER and DANIEL WINDISCH	
Initial forms and a notion of basis for tropical differential equations	453
ALEX FINK and ZEINAB TOGHANI	