Iyengar et al. *Res Math Sci* _####################_
https://doi.org/10.1007/s40687-022-00321-7

Research in the Mathematical Sciences

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Cohomological supports of tensor products of modules over commutative rings

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To Jürgen Herzog on his 80th birthday.

Abstract

This works concerns cohomological support varieties of modules over commutative local rings. The main result is that the support of a derived tensor product of a pair of differential graded modules over a Koszul complex is the join of the supports of the modules. This generalizes, and gives another proof of, a result of Dao and the third author dealing with Tor-independent modules over complete intersection rings. The result for Koszul complexes has a broader applicability, including to exterior algebras over local rings.

Keywords: Koszul complex, Dg modules, Cohomological support, Tensor products, Join, BGG correspondence

Mathematics Subject Classification: 13D02, 16E45 (primary); 13D07, 13H10 (secondary)

Introduction

Throughout, we fix a Koszul complex E over a (commutative noetherian) local ring $(R, \mathfrak{m}, k)$ on a list of elements $\boldsymbol{f} = f_1, \ldots, f_c$ in $\mathfrak{m}$. As explained in [25] studying the homological properties of differential graded (abbreviated to dg), E-modules allows one to unify and extend the results about quotients $R \to R/(\boldsymbol{f})$ when $\boldsymbol{f}$ is an R-regular sequence as well as those about exterior algebras over R. The dg E-modules perfect when regarded as R-complexes—in the sense that they are quasi-isomorphic to a bounded complex of finite rank free R-modules—are the ones that exhibit especially structured homological phenomena; see, for example, [1,4,6,9,11,16,20,26]. The homological properties of such a dg E-module M are often encoded in its cohomological support, denoted $\mathrm{V}_E(M)$, which is a naturally associated Zariski closed subset of $\mathbb{P}_k^{c-1}$; cf. 5.2.

The main result of this article is the following.

Theorem *For dg E-modules M, N that are perfect over R, there is an equality*

$$\mathrm{V}_E(M \otimes_E^{\mathrm{L}} N) = \mathrm{Join}(\mathrm{V}_E(M), \mathrm{V}_E(N)) .$$

The join of closed subsets U, V of $\mathbb{P}_k^{c-1}$, denoted $\mathrm{Join}(U, V)$, is the closure of the union of lines connecting a point from U to a point from V; see 1.1 for details. Specializing the

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Journal: 40687 Article No.: 0321 TYPESET DISK LE CP Disp.:2022/4/12 Pages: 15 Layout: BMC-OneCol

theorem above to the case where R is a regular ring, $\boldsymbol{f}$ is an R-regular sequence, and M, N are finitely generated $R/(\boldsymbol{f})$-modules satisfying $\operatorname{Tor}_i^E(M, N) = 0$ for all $i \geq 1$ recovers [15, Theorem 3.1]. The proof in *loc. cit.* involves a series reductions and ad hoc geometric arguments. Besides generalizing this result, a main point of this article is to offer a simpler proof by a passage to an exterior algebra, as briefly described below.

The theorem above is proved in Sect. 6. As a corollary, we deduce that when R is Gorenstein and $\operatorname{RHom}_E(M, N)$ is perfect as an R-complex, there is an equality

$$\mathrm{V}_E(\operatorname{RHom}_E(M, N)) = \operatorname{Join}(\mathrm{V}_E(M), \mathrm{V}_E(N)).$$

This is Corollary 6.4 and it generalizes [15, Theorem 4.7]. Theorem 6.6 relates the support of the dg module $M \otimes^{\mathsf{L}}_E N$ to that of its homology modules, namely, $\operatorname{Tor}_i^E(M, N)$, thereby providing a positive answer to [15, Question 2].

The key ingredient in our work is a functor, denoted t, from the derived category of dg E-modules $\mathsf{D}(E)$ to the derived category of dg Λ-modules $\mathsf{D}(\Lambda)$ where Λ is an exterior algebra on Σk^c; see Sect. 5. The relevance of this functor arises from Lemma 5.3 which identifies $\mathrm{V}_E(M)$ with $\mathrm{V}_\Lambda(\mathsf{t}M)$, and that as dg Λ-modules

$$\mathsf{t}(M \otimes^{\mathsf{L}}_E N) \simeq \mathsf{t}M \otimes^{\mathsf{L}}_\Lambda \mathsf{t}N\,.$$

The expression for $\mathrm{V}_E(M \otimes^{\mathsf{L}}_E N)$ in the theorem above is a consequence of Proposition 4.4 that asserts if X, Y are dg Λ-modules with finite-dimensional homology, then

$$\mathrm{V}_\Lambda(X \otimes^{\mathsf{L}}_\Lambda Y) = \operatorname{Join}(\mathrm{V}_\Lambda(X), \mathrm{V}_\Lambda(Y)). \tag{†}$$

This equality is in turn deduced using a contravariant version, from [2], of the Bernstein-Gelfand-Gelfand correspondence functor:

$$\mathsf{d}\colon \mathsf{D}(\Lambda) \to \mathsf{D}(\mathcal{S}),$$

where $\mathcal{S}$ is the symmetric algebra on $\Sigma^{-2}k^c$. The main calculation in the proof of (†) is the interaction between tensor products and the functor d, namely: Given dg Λ-modules X, Y with homology finite dimensional over k, there is an isomorphism of dg $\mathcal{S}$-modules

$$\mathsf{d}(X \otimes^{\mathsf{L}}_\Lambda Y) \simeq \mathsf{d}X \otimes_k \mathsf{d}Y\,,$$

where the right-hand side is regarded as a dg $\mathcal{S}$-module through a natural map of k-algebras $\Delta\colon \mathcal{S} \to \mathcal{S} \otimes_k \mathcal{S}$, which makes $\mathcal{S}$ into a Hopf algebra; see (1.1.1). Given this result, (†) follows by a standard argument concerning supports of modules over polynomial rings, discussed in Sect. 1; see especially Lemma 1.4. The isomorphism above, which is folklore, is contained in Proposition 4.4.

1 Joins and supports

Let k be a field. In what follows, we encounter graded k-vector spaces W whose natural grading is lower and also those whose natural grading is upper. It is convenient to adopt the convention that W has both an upper and a lower grading, with $W^i = W_{-i}$ for each integer i. We indicate the primary grading when necessary.

Fix a finite-dimensional graded k-space $W{:=}\{W^i\}_{i\in\mathbb{Z}}$ concentrated in positive even degrees. Let $\mathcal{S}{:=}\operatorname{Sym}_k W$ be the symmetric algebra (over k) on W and $\operatorname{Proj}\mathcal{S}$ the set of homogeneous prime ideals of $\mathcal{S}$ not containing the irrelevant maximal ideal $\mathcal{S}^{>0}$ of $\mathcal{S}$, equipped with the Zariski topology. In this section, we recall some basics on joins of closed subsets of $\operatorname{Proj}\mathcal{S}$ and of supports of graded $\mathcal{S}$-modules. Our standard references are [18, Sect. 1.3], for joins, and [19], for supports.

1.1 The map $W \to W \oplus W$ given by $w \mapsto (w, 0) + (0, w)$ induces a map

$$\Delta\colon \mathcal{S} \to \mathcal{S} \otimes_k \mathcal{S} \tag{1.1.1}$$

of graded k-algebras and makes $\mathcal{S}$ into a graded Hopf algebra over k. It also defines a rational map

$$\delta\colon \operatorname{Proj}(\mathcal{S} \otimes_k \mathcal{S}) \dashrightarrow \operatorname{Proj}\mathcal{S}$$

that is defined (and regular) off of the anti-diagonal D in $\operatorname{Proj}(\mathcal{S} \otimes_k \mathcal{S})$; here, D is the image of the embedding $\operatorname{Proj}\mathcal{S} \hookrightarrow \operatorname{Proj}(\mathcal{S} \otimes_k \mathcal{S})$ determined by the map $W \oplus W \to W$ given by $(w_1, w_2) \mapsto w_1 + w_2$.

Given closed subsets $U:=\mathcal{V}(\mathcal{I})$ and $V:=\mathcal{V}(\mathcal{J})$ of $\operatorname{Proj}\mathcal{S}$, consider

$$J(U, V):= \operatorname{Proj}(\mathcal{S}/\mathcal{I} \otimes_k \mathcal{S}/\mathcal{J})$$

viewed as a closed subset of $\operatorname{Proj}(\mathcal{S} \otimes_k \mathcal{S})$. The *join* of U and V, denoted $\operatorname{Join}(U, V)$, is the closure in $\operatorname{Proj}\mathcal{S}$ of the set

$$\delta\,(J(U, V) \setminus D)\,.$$

When k is algebraically closed, the Nullstellensatz identifies $\operatorname{Proj}\mathcal{S}$ with projective space $\mathbb{P}_k^{d-1}$ where $d = \dim_k W$. Under this identification, the join of U and V is the closure of the union of lines in $\operatorname{Proj}\mathcal{S}$ containing a point u in U and a point v in V.

Remark 1.1 The join can also be defined as follows: Consider the rational map

$$\delta'\colon \operatorname{Proj}(\mathcal{S} \otimes_k \mathcal{S}) \dashrightarrow \operatorname{Proj}\mathcal{S}\,,$$

that is regular of the diagonal in $\operatorname{Proj}(\mathcal{S} \otimes_k \mathcal{S})$, induced by the k-algebra map $\mathcal{S} \to \mathcal{S} \otimes_k \mathcal{S}$ determined by $w \mapsto w \otimes 1 - 1 \otimes w$. The linear automorphism α of $\operatorname{Proj}(\mathcal{S} \otimes_k \mathcal{S})$ determined by

$$w \otimes 1 \mapsto w \otimes 1 \quad \text{and} \quad 1 \otimes w \mapsto -1 \otimes w$$

fixes $J(U, V)$ for any pair of closed subsets U, V of $\operatorname{Proj}\mathcal{S}$, maps D bijectively to D' and $\delta = \delta'\alpha$. Hence,

$$\delta(J(U, V) \setminus D) = \delta'(J(U, V) \setminus D'),$$

where the right-hand side is the definition of the join used in [18, Sect. 1.3]. That is the definitions of joins from *loc. cit.* and 1.1 coincide. We opt for the latter as the isomorphism in Proposition 4.4 respects Δ.

1.2 Let X be a graded $\mathcal{S}$-module. The *support of* X over $\mathcal{S}$ is the subset

$$\operatorname{Supp}_{\mathcal{S}} X:=\{\mathfrak{p} \in \operatorname{Proj}\mathcal{S} \mid X_{\mathfrak{p}} \neq 0\}\,,$$

where $X_{\mathfrak{p}}$ denotes the homogeneous localization of X at $\mathfrak{p}$. Following Foxby [19], the *small support of* X is

$$\operatorname{supp}_{\mathcal{S}} X:=\{\mathfrak{p} \in \operatorname{Proj}\mathcal{S} \mid X \otimes^{\mathrm{L}}_{\mathcal{S}} \kappa(\mathfrak{p}) \neq 0\}\,,$$

where $\kappa(\mathfrak{p})$ is the graded field $\mathcal{S}_{\mathfrak{p}}/\mathfrak{p}\mathcal{S}_{\mathfrak{p}}$. Consider the closed subset

$$\mathcal{V}(\operatorname{ann}_{\mathcal{S}} X):=\{\mathfrak{p} \in \operatorname{Proj}\mathcal{S} \mid \mathfrak{p} \supseteq \operatorname{ann}_{\mathcal{S}} X\}$$

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of $\operatorname{Proj}\mathcal{S}$. In general, there are inclusions

$$\operatorname{supp}_{\mathcal{S}} X \subseteq \operatorname{Supp}_{\mathcal{S}} X \subseteq \mathcal{V}(\operatorname{ann}_{\mathcal{S}} X). \tag{1.3.1}$$

Moreover, $\operatorname{Supp}_{\mathcal{S}} X$ is the specialization closure of $\operatorname{supp}_{\mathcal{S}} X$; see [10, Lemma 2.2]. Equalities hold when the $\mathcal{S}$-module X is finitely generated.

Lemma 1.4 *Let X, Y be finitely generated graded $\mathcal{S}$-modules. There is an equality*

$$\operatorname{Supp}_{\mathcal{S}}(X \otimes_k Y) = \operatorname{Join}(\operatorname{Supp}_{\mathcal{S}} X, \operatorname{Supp}_{\mathcal{S}} Y),$$

where $X \otimes_k Y$ is regarded as a graded $\mathcal{S}$-module via (1.1.1).

Proof As a matter of notation, we write $\mathcal{S}^{\mathrm{e}}$ for $\mathcal{S} \otimes_k \mathcal{S}$ and use $\overline{(-)}$ for closure in the Zariski topology. For any finite generated $\mathcal{S}^{\mathrm{e}}$-module N and $\mathfrak{p} \in \operatorname{Proj}\mathcal{S}$, one has

$$N \otimes^{\mathrm{L}}_{\mathcal{S}} \kappa(\mathfrak{p}) \simeq N \otimes^{\mathrm{L}}_{\mathcal{S}^{\mathrm{e}}} (\mathcal{S}^{\mathrm{e}} \otimes^{\mathrm{L}}_{\mathcal{S}} \kappa(\mathfrak{p})).$$

This leads to the following equivalences:

$$\begin{aligned} \mathfrak{p} \in \operatorname{supp}_{\mathcal{S}} N &\iff \operatorname{supp}_{\mathcal{S}^{\mathrm{e}}}(N) \cap \operatorname{supp}_{\mathcal{S}^{\mathrm{e}}}(\mathcal{S}^{\mathrm{e}} \otimes^{\mathrm{L}}_{\mathcal{S}} \kappa(\mathfrak{p})) \neq \emptyset \\ &\iff \operatorname{supp}_{\mathcal{S}^{\mathrm{e}}}(N) \cap (\delta^{-1}(\mathfrak{p}) \smallsetminus D) \neq \emptyset \\ &\iff \mathfrak{p} \in \delta(\operatorname{supp}_{\mathcal{S}^{\mathrm{e}}}(N) \smallsetminus D). \end{aligned}$$

Applying this observation to $N := X \otimes_k Y$ justifies the last equality below:

$$\begin{aligned} \operatorname{Join}(\operatorname{Supp}_{\mathcal{S}} X, \operatorname{Supp}_{\mathcal{S}} Y) &= \overline{\delta(\operatorname{Supp}_{\mathcal{S}^{\mathrm{e}}}(X \otimes_k Y) \smallsetminus D)} \\ &= \overline{\delta(\operatorname{supp}_{\mathcal{S}^{\mathrm{e}}}(X \otimes_k Y) \smallsetminus D)} \\ &= \overline{\operatorname{supp}_{\mathcal{S}}(X \otimes_k Y)}. \end{aligned}$$

The first equality holds as X, Y are finitely generated over $\mathcal{S}$, while the second equality holds because $X \otimes_k Y$ is finitely generated over $\mathcal{S}^{\mathrm{e}}$. Thus, for the desired statement, it suffices to verify that

$$\overline{\operatorname{supp}_{\mathcal{S}}(X \otimes_k Y)} = \operatorname{Supp}_{\mathcal{S}}(X \otimes_k Y).$$

To that end, given 1.2, it suffices to verify that $\operatorname{Supp}_{\mathcal{S}}(X \otimes_k Y)$ is closed in $\operatorname{Proj}\mathcal{S}$. As an $\mathcal{S}$-module $X \otimes_k Y$ need not be finitely generated, but it is finitely generated over $\mathcal{S}^{\mathrm{e}}$, and that suffices.

Indeed, let G be a finite generating set for $X \otimes_k Y$ over $\mathcal{S}^{\mathrm{e}}$ and T the $\mathcal{S}$-submodule of $X \otimes_k Y$ generated by G; here, $\mathcal{S}$ is acts via the diagonal map (1.1.1). Since T is finitely generated over $\mathcal{S}$, one gets the first equality below:

$$\begin{aligned} \mathcal{V}(\operatorname{ann}_{\mathcal{S}} T) &= \operatorname{Supp}_{\mathcal{S}} T \\ &\subseteq \operatorname{Supp}_{\mathcal{S}}(X \otimes_k Y) \\ &\subseteq \mathcal{V}(\operatorname{ann}_{\mathcal{S}}(X \otimes_k Y)) \\ &= \mathcal{V}(\operatorname{ann}_{\mathcal{S}} T). \end{aligned}$$

The containments are from 1.2; the last equality holds as $\operatorname{ann}_{\mathcal{S}}(X \otimes_k Y) = \operatorname{ann}_{\mathcal{S}} T$. Thus, the inclusions above are equalities, as desired. □

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2 Dg modules over graded algebras

Let $A = \{A_i\}_{i\in\mathbb{Z}}$ be a strictly graded-commutative dg algebra. Its homology algebra, $\mathrm{H}(A)$, is thus also strictly graded-commutative.

2.1 A dg A-module F is *semifree* provided it admits an exhaustive filtration

$$0 = F(-1) \subseteq F(0) \subseteq F(1) \subseteq \ldots \subseteq F,$$

where each subquotient $F(i)/F(i-1)$ is a coproduct of suspensions of A. A *semifree resolution* of a dg A-module M is a surjective quasi-isomorphism of dg A-modules $F \xrightarrow{\simeq} M$ where F is a semifree dg A-module. Such resolutions of M exist and any two are unique up to homotopy equivalence; see, for example, [17, 6.6].

2.2 Let M be a dg A-module and fix $F \xrightarrow{\simeq} M$ a semifree resolution over A. By [17,], the functors $F \otimes_A -$ and $\mathrm{Hom}_A(F, -)$ preserve (surjective) quasi-isomorphisms. Hence by replacing objects with their semifree resolutions, we obtain bi-functors $- \otimes^{\mathsf{L}}_A -$ and $\mathrm{RHom}_A(-, -)$ on $\mathsf{D}(A)$; that is to say,

$$M \otimes^{\mathsf{L}}_A - := F \otimes_A - \quad \text{and} \quad \mathrm{RHom}_A(M, -) := \mathrm{Hom}_A(F, -)\,.$$

As usual, we set

$$\mathrm{Tor}^A_*(M, N) := \mathrm{H}_*(M \otimes^{\mathsf{L}}_A N) \quad \text{and} \quad \mathrm{Ext}^*_A(M, N) := \mathrm{H}^*(\mathrm{RHom}_A(M, N))\,.$$

As A is graded-commutative, these are graded $\mathrm{H}(A)$-modules.

2.3 The derived category of dg A-modules is denoted $\mathsf{D}(A)$, and it is regarded as a triangulated category in the standard way; see, for example, [5, Sect. 2]. The suspension functor associates with each dg A-module M the dg module ΣM with

$$(\Sigma M)_i = M_{i-1}, \quad \partial^{\Sigma M} = -\partial^M \quad \text{and} \quad a \cdot \Sigma m = (-1)^{|a|} am\,,$$

where $|a|$ denotes the degree a. A *thick* subcategory of a triangulated category is a triangulated subcategory that is closed under retracts.

2.4 Let A be a dg algebra over a field k. We define several thick subcategories of $\mathsf{D}(A)$ that will be of interest in what follows.

Let $\mathsf{D}^{\mathsf{f}}_{+}(A)$ denote the full subcategory of $\mathsf{D}(A)$ consisting of dg A-modules M with each $\mathrm{H}_i(M)$ finite dimensional and $\mathrm{H}_i(M) = 0$ for all $i \ll 0$; define $\mathsf{D}^{\mathsf{f}}_{-}(A)$ analogously where the second condition is replaced with $\mathrm{H}_i(M) = 0$ for all $i \gg 0$. We let $\mathsf{D}^{\mathsf{f}}_{\mathsf{b}}(A)$ denote $\mathsf{D}^{\mathsf{f}}_{+}(A) \cap \mathsf{D}^{\mathsf{f}}_{-}(A)$. That is, $\mathsf{D}^{\mathsf{f}}_{\mathsf{b}}(A)$ consists exactly of those dg A-modules whose homology is finite-dimensional over k. We write $\mathsf{Perf}(A)$ for the thick subcategory of $\mathsf{D}(A)$ generated by A; see [5, Theorem 4.2] for an alternative characterization.

3 Exterior algebras

In this section, $V := \{V_i\}_{i\in\mathbb{Z}}$ is a finite graded k-space concentrated in positive odd degrees. Set $(-)^\vee := \mathrm{Hom}_k(-, k)$, the graded dual, and $W := \Sigma^{-1}(V^\vee)$. Let

$$\Lambda := \bigwedge_k V \quad \text{and} \quad \mathcal{S} := \mathrm{Sym}_k\, W\,;$$

the former being the exterior algebra, over k, on V. Set $\Gamma := \mathcal{S}^\vee$ with the standard $\mathcal{S}$-module structure: For $\alpha \in \Gamma$ and $\chi \in \mathcal{S}$, one has

$$\chi \cdot \alpha := \alpha(\chi \cdot -)\,.$$

We view Λ as a graded Hopf algebra, with coproduct $\Lambda \to \Lambda \otimes_k \Lambda$ induced by the map of k-spaces $v \mapsto (v, 1) + (1, v)$, for $v \in V$. Hence for any left dg Λ-module, the antipode defines a dg Λ-module structure on $M^\vee$. Also, for a pair of dg Λ-modules M, N, their tensor product $M \otimes_k N$ is regarded as a dg Λ-module through the coproduct. See, for example, [7, Remark 5.2]. We also view $\mathcal{S}$ as a graded Hopf algebra over k, with coproduct defined in 1.1.

Notation 3.1 Fix a basis $e_1, \ldots, e_c$ for V, and let $\chi_1, \ldots, \chi_c$ be the dual basis for W; thus χ_i has lower degree $-|e_i| - 1$. These determine isomorphisms

$$\Lambda \cong \bigwedge (ke_1 \oplus \ldots \oplus ke_c) \quad \text{and} \quad \mathcal{S} \cong k[\chi_1, \ldots, \chi_c].$$

3.2 For a dg Λ-module M, its *universal resolution* $\mathsf{u}M$ is the dg $(\Lambda \otimes_k \mathcal{S})$-module with underlying graded $(\Lambda \otimes_k \mathcal{S})$-module $\Lambda \otimes_k \Gamma \otimes_k M$, with $\Lambda \otimes_k \mathcal{S}$ acting by left multiplication on the two left factors, and differential

$$1 \otimes 1 \otimes \partial^M + \sum_{i=1}^{c} (1 \otimes \chi_i \otimes e_i - e_i \otimes \chi_i \otimes 1).$$

The canonical projection $\mathsf{u}M \to M$ is a semifree resolution of M over Λ; see [3, Proposition 2.6] or [5, Sect. 7]. Moreover, since $\mathsf{u}M$ is a dg module over $\Lambda \otimes_k \mathcal{S}$, the graded k-space $\operatorname{Hom}_\Lambda(\mathsf{u}M, -)$ retains a dg $\mathcal{S}$-module structure and so

$$\operatorname{Ext}^*_\Lambda(M, -) = \operatorname{H}^*(\operatorname{Hom}_\Lambda(\mathsf{u}M, -))$$

is a graded $\mathcal{S}$-module.

3.3 Let M be dg Λ-module with M_i degreewise finite dimensional over k for each i and 0 for $i \ll 0$; up to quasi-isomorphism any object in $\mathsf{D}^{\mathsf{f}}_{+}(\Lambda)$ has this form. There is an isomorphism of dg $\mathcal{S}$-modules

$$\operatorname{Hom}_\Lambda(\mathsf{u}M, k) \cong \mathcal{S} \otimes_k M^\vee, \tag{3.3.1}$$

where the right-hand term has differential $1 \otimes \partial^{M^\vee} + \sum_{i=1}^{c} \chi_i \otimes e_i$; we denote the dg $\mathcal{S}$-module on the right by $\mathcal{S}_M$. From this isomorphism and [25, Proposition 1.2.8], $\operatorname{Hom}_\Lambda(\mathsf{u}M, k)$ is a semifree dg $\mathcal{S}$-module.

The contravariant functor $\operatorname{Hom}_\Lambda(\mathsf{u}(-), k)$ induces the exact functor

$$\mathsf{d}\colon \mathsf{D}(\Lambda)^{\mathsf{op}} \to \mathsf{D}(\mathcal{S}).$$

By [2], this restricts to an exact equivalence

$$\mathsf{d}\colon \mathsf{D}^{\mathsf{f}}_{+}(\Lambda)^{\mathsf{op}} \xrightarrow{\equiv} \mathsf{D}^{\mathsf{f}}_{-}(\mathcal{S}),$$

that further restricts to equivalences

$$\mathsf{D}^{\mathsf{f}}_{\mathsf{b}}(\Lambda)^{\mathsf{op}} \xrightarrow{\equiv} \mathsf{Perf}(\mathcal{S}) \quad \text{and} \quad \mathsf{Perf}(\Lambda)^{\mathsf{op}} \xrightarrow{\equiv} \mathsf{D}^{\mathsf{f}}_{\mathsf{b}}(\mathcal{S}).$$

One has also the functor $\operatorname{Hom}_\Lambda(\mathsf{u}k, -)$ that induces an exact functor

$$\mathsf{b}\colon \mathsf{D}(\Lambda) \to \mathsf{D}(\mathcal{S})$$

which restricts to equivalences

$$\mathsf{D}^{\mathsf{f}}_{\mathsf{b}}(\Lambda) \xrightarrow{\equiv} \mathsf{Perf}(\mathcal{S}) \quad \text{and} \quad \mathsf{Perf}(\Lambda) \xrightarrow{\equiv} \mathsf{D}^{\mathsf{f}}_{\mathsf{b}}(\mathcal{S}).$$

cf. [5]. There is the following commutative diagram

$$\begin{tikzcd} \mathsf{D}^{\mathsf{f}}_{+}(\Lambda)^{\mathsf{op}} \arrow[r, "\mathsf{d}"] \arrow[d, "(-)^{\vee}"'] & \mathsf{D}^{\mathsf{f}}_{-}(\mathcal{S})\,. \\ \mathsf{D}^{\mathsf{f}}_{-}(\Lambda) \arrow[ur, "\mathsf{b}"'] & \end{tikzcd}$$

3.4 The functors b, d defined above determine two notions of cohomological support for dg Λ-modules. Namely, for a dg Λ-module M, consider subsets of $\operatorname{Proj} \mathcal{S}$

$$\begin{aligned} \mathrm{V}^{\mathsf{b}}_{\Lambda}(M) &:= \operatorname{Supp}_{\mathcal{S}} \mathrm{H}(\mathsf{b}M) = \operatorname{Supp}_{\mathcal{S}} \operatorname{Ext}_{\Lambda}(k, M), \\ \mathrm{V}^{\mathsf{d}}_{\Lambda}(M) &:= \operatorname{Supp}_{\mathcal{S}} \mathrm{H}(\mathsf{d}M) = \operatorname{Supp}_{\mathcal{S}} \operatorname{Ext}_{\Lambda}(M, k)\,. \end{aligned}$$

In [14], the supports $\mathrm{V}^{\mathsf{b}}_{\Lambda}(-)$ are used to classify the thick subcategories of $\mathsf{D}^{\mathsf{f}}_{\mathsf{b}}(\Lambda)$. Our focus will be on $\mathrm{V}^{\mathsf{d}}_{\Lambda}(-)$ but it is worth recording their relationship.

Proposition 3.1 *Let M be in $\mathsf{D}^{\mathsf{f}}_{+}(\Lambda)$. There is an equality $\mathrm{V}^{d}_{\Lambda}(M) = \mathrm{V}^{b}_{\Lambda}(M^{\vee})$. Moreover, if M is in $\mathsf{D}^{\mathsf{f}}_{\mathsf{b}}(\Lambda)$, then $\mathrm{V}^{d}_{\Lambda}(M) = \mathrm{V}^{b}_{\Lambda}(M)$.*

Proof The first equality is immediate from $\mathsf{b}((-)^{\vee}) = \mathsf{d}$; see 3.3. The second equality follows from the first. Indeed, it is easy to check that if N is in the thick subcategory generated by N', then

$$\mathrm{V}^{\mathsf{b}}_{\Lambda}(N) \subseteq \mathrm{V}^{\mathsf{b}}_{\Lambda}(N') \quad \text{and} \quad \mathrm{V}^{\mathsf{d}}_{\Lambda}(N) \subseteq \mathrm{V}^{\mathsf{d}}_{\Lambda}(N')\,.$$

When M is in $\mathsf{D}^{\mathsf{f}}_{\mathsf{b}}(\Lambda)$, the dg Λ-modules M and $M^{\vee}$ generate the same thick subcategory—see [23, Sect. 4]—so the second equality follows from the first. □

4 Support for tensor products, I

As in the previous section, $V := \{V_i\}_{i>0}$ is a finite graded k-space concentrated in positive even degrees, and

$$\Lambda := \bigwedge V \quad \text{and} \quad \mathcal{S} := \operatorname{Sym}_k W\,,$$

where $W := \Sigma^{-1}(V^{\vee})$. In this section, we analyze the interaction between the functor $\mathsf{d}\colon \mathsf{D}^{\mathsf{f}}_{+}(\Lambda)^{\mathsf{op}} \to \mathsf{D}^{\mathsf{f}}_{-}(\mathcal{S})$, from 3.3, and the tensor products $\otimes_k$ and $\otimes^{\mathrm{L}}_{\Lambda}$. The main results, Proposition 4.2 and Proposition 4.4, are folklore but we could not find adequate references, so we give complete proofs; see also Remark 4.3.

Lemma 4.1 *For $\mathcal{S}$-modules X, Y with finitely generated homology,*

$$\operatorname{Supp}_{\mathcal{S}} \mathrm{H}(X \otimes^{\mathrm{L}}_{\mathcal{S}} Y) = \operatorname{Supp}_{\mathcal{S}} \mathrm{H}(X) \cap \operatorname{Supp}_{\mathcal{S}} \mathrm{H}(Y)\,.$$

Proof Since X, Y have finitely generated homology, there are equalities

$$\begin{aligned} \operatorname{Supp}_{\mathcal{S}} \mathrm{H}(X) &= \{\mathfrak{p} \in \operatorname{Proj} \mathcal{S} \mid X \otimes^{\mathrm{L}}_{\mathcal{S}} \kappa(\mathfrak{p}) \not\simeq 0\}, \\ \operatorname{Supp}_{\mathcal{S}} \mathrm{H}(Y) &= \{\mathfrak{p} \in \operatorname{Proj} \mathcal{S} \mid Y \otimes^{\mathrm{L}}_{\mathcal{S}} \kappa(\mathfrak{p}) \not\simeq 0\}\,. \end{aligned}$$

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See, for instance, [14, Theorem 2.4]. Since $\mathcal{S}$ has finite global dimension, the $\mathcal{S}$-module $\mathrm{H}(X \otimes^{\mathrm{L}}_{\mathcal{S}} Y)$ is also finitely generated and so

$$\operatorname{Supp}_{\mathcal{S}} \mathrm{H}(X \otimes^{\mathrm{L}}_{\mathcal{S}} Y) = \{\mathfrak{p} \in \operatorname{Proj} \mathcal{S} \mid X \otimes^{\mathrm{L}}_{\mathcal{S}} Y \otimes^{\mathrm{L}}_{\mathcal{S}} \kappa(\mathfrak{p}) \not\simeq 0\}.$$

The desired equality follows from the ones above and the isomorphism

$$X \otimes^{\mathrm{L}}_{\mathcal{S}} Y \otimes^{\mathrm{L}}_{\mathcal{S}} \kappa(\mathfrak{p}) \simeq (X \otimes^{\mathrm{L}}_{\mathcal{S}} \kappa(\mathfrak{p})) \otimes_{\kappa(\mathfrak{p})} (Y \otimes^{\mathrm{L}}_{\mathcal{S}} \kappa(\mathfrak{p})).$$ □

The result below records the relationship between d and tensor products.

Proposition 4.2 *For M, N in $\mathsf{D}^{\mathsf{f}}_{+}(\Lambda)$, there is an isomorphism of dg $\mathcal{S}$-modules*

$$\mathsf{d}(M \otimes_k N) \simeq \mathsf{d}M \otimes^{\mathrm{L}}_{\mathcal{S}} \mathsf{d}N.$$

Furthermore if M, N are in $\mathsf{D}^{\mathsf{f}}_{\mathsf{b}}(\Lambda)$, then

$$\mathrm{V}^{d}_{\Lambda}(M \otimes_k N) = \mathrm{V}^{d}_{\Lambda}(M) \cap \mathrm{V}^{d}_{\Lambda}(N).$$

Proof Replacing M and N with semifree resolutions over Λ, we may assume both M and N are bounded below and degreewise finite dimensional over k, as in 3.3. Let Φ denote the composition of the isomorphisms of dg $\mathcal{S}$-modules

$$(\mathcal{S} \otimes_k M^{\vee}) \otimes_{\mathcal{S}} (\mathcal{S} \otimes_k N^{\vee}) \longrightarrow (\mathcal{S} \otimes_{\mathcal{S}} \mathcal{S}) \otimes_k M^{\vee} \otimes_k N^{\vee} \longrightarrow \mathcal{S} \otimes_k M^{\vee} \otimes_k N^{\vee},$$

where the first one is the twist isomorphism given by

$$(s \otimes \alpha) \otimes (s' \otimes \beta) \mapsto (s \otimes s') \otimes (\alpha \otimes \beta)$$

and the second map is the multiplication isomorphism. It is straightforward to see

$$\Phi \circ \sum_{i=1}^{c} (\chi_i \otimes e_i) \otimes 1 + 1 \otimes (\chi_i \otimes e_i) = \sum_{i=1}^{c} \chi_i \otimes (e_i \otimes 1 + 1 \otimes e_i) \circ \Phi.$$

As M, N are degreewise finite dimensional and bounded below, there is a natural isomorphism of dg Λ-modules

$$(M \otimes_k N)^{\vee} \cong M^{\vee} \otimes_k N^{\vee}.$$

Hence, Φ yields an isomorphism

$$\mathcal{S}_M \otimes_{\mathcal{S}} \mathcal{S}_N \xrightarrow{\cong} \mathcal{S}_{M \otimes_k N}.$$

As a consequence, (3.3.1) establishes the isomorphisms in $\mathsf{D}(\mathcal{S})$:

$$\mathsf{d}(M \otimes_k N) \simeq \mathsf{d}M \otimes_{\mathcal{S}} \mathsf{d}N \simeq \mathsf{d}M \otimes^{\mathrm{L}}_{\mathcal{S}} \mathsf{d}N. \tag{4.2.1}$$

As for the statement regarding supports, consider the following equalities:

$$\begin{aligned}
\mathrm{V}^{\mathsf{d}}_{\Lambda}(M \otimes_k N) &= \operatorname{Supp}_{\mathcal{S}} \mathrm{H}(\mathsf{d}(M \otimes_k N)) \\
&= \operatorname{Supp}_{\mathcal{S}} \mathrm{H}(\mathsf{d}M \otimes^{\mathrm{L}}_{\mathcal{S}} \mathsf{d}N) \\
&= \operatorname{Supp}_{\mathcal{S}} \mathrm{H}(\mathsf{d}M) \cap \operatorname{Supp}_{\mathcal{S}} \mathrm{H}^*(\mathsf{d}N) \\
&= \mathrm{V}^{\mathsf{d}}_{\Lambda}(M) \cap \mathrm{V}^{\mathsf{d}}_{\Lambda}(N).
\end{aligned}$$

The second equality is from (4.2.1), while the third is Lemma 4.1. □

Remark 4.3 Buchweitz proved that if M, N are graded Λ-modules that are bounded below and are degreewise finite rank over k, then

$$\mathsf{b}(M \otimes_k N) \simeq \mathsf{b}M \otimes^{\mathsf{L}}_{\mathcal{S}} \mathsf{b}N\,; \tag{3.3.2}$$

see [13, (9.4.10)]. It is easy to see that this isomorphism holds for all pairs of objects in $\mathsf{D}^{\mathsf{f}}_{+}(\Lambda)$. From the equality $\mathsf{b}((-)^{\vee}) = \mathsf{d}$, the isomorphisms in (3.3.2) can also be deduced from (and imply) the ones in Proposition 4.2.

Proposition 4.4 *For M, N in $\mathsf{D}^{\mathsf{f}}_{+}(\Lambda)$, there is an isomorphism of dg $\mathcal{S}$-modules*

$$d(M \otimes^{\mathsf{L}}_{\Lambda} N) \simeq dM \otimes_k dN,$$

where the right-hand side is a dg $\mathcal{S}$-module through the diagonal $\Delta\colon \mathcal{S} \to \mathcal{S} \otimes_k \mathcal{S}$ described in (1.1.1). Furthermore, if M, N are in $\mathsf{D}^{\mathsf{f}}_{\mathsf{b}}(\Lambda)$, then

$$\mathrm{V}^{d}_{\Lambda}(M \otimes^{\mathsf{L}}_{\Lambda} N) = \mathrm{Join}(\mathrm{V}^{d}_{\Lambda}(M), \mathrm{V}^{d}_{\Lambda}(N))\,.$$

Proof Replacing M and N with suitable resolutions, we can assume M and N are bounded above, degreewise finite dimensional over k, and semifree. As in 3.2, we consider $\Gamma = \mathcal{S}^{\vee}$, regarded as a graded $\mathcal{S}$-module. Forgetting differentials, one has a commutative diagram

$$\begin{tikzcd}
\mathsf{u}M \otimes_{\Lambda} \mathsf{u}N \arrow[r, dotted, "\Phi"] \arrow[d, "\cong"'] & \mathsf{u}(M \otimes_{\Lambda} N) \\
\Lambda \otimes_k \Gamma^{\otimes 2} \otimes_k M \otimes_k N \arrow[r, "1\otimes\mu\otimes\pi"] & \Lambda \otimes_k \Gamma \otimes_k (M \otimes_{\Lambda} N) \arrow[u, "\cong"']
\end{tikzcd}$$

of graded $\mathcal{S}$-modules, where the map on the bottom is defined using the multiplication $\mu\colon \Gamma \otimes_k \Gamma \to \Gamma$ map which is dual to the diagonal $\Delta\colon \mathcal{S} \to \mathcal{S} \otimes_k \mathcal{S}$ in (1.1.1), and $\pi : M \otimes_k N \to M \otimes_{\Lambda} N$ is the canonical projection. It is straightforward to check Φ is a Λ-linear morphism of complexes that is compatible with the canonical augmentations to $M \otimes_{\Lambda} N$. Thus, Φ is a comparison map between semifree resolutions of $M \otimes^{\mathsf{L}}_{\Lambda} N$ over Λ, and so it is a homotopy equivalence.

Applying $\mathrm{Hom}_{\Lambda}(-, k)$ to Φ yields the top map in the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\Lambda}(\mathsf{u}(M \otimes_{\Lambda} N), k) \arrow[r, "\Phi^{\vee}"] \arrow[d, "\cong"'] & \mathrm{Hom}_{\Lambda}(\mathsf{u}M \otimes_{\Lambda} \mathsf{u}N, k) \arrow[d, "\cong"] \\
\mathcal{S}_{M \otimes_{\Lambda} N} \arrow[r, "\Delta\otimes\pi^*"] & \mathcal{S}_M \otimes_k \mathcal{S}_N
\end{tikzcd}$$

of dg $\mathcal{S}$-modules, where $\mathcal{S}_M \otimes_k \mathcal{S}_N$ is viewed as a dg $\mathcal{S}$-module through Δ. The vertical parallel maps are isomorphisms by (3.3.1); the one on the right also uses the standard isomorphisms

$$\begin{aligned}
\mathrm{Hom}_{\Lambda}(\mathsf{u}M \otimes_{\Lambda} \mathsf{u}N, k) &\cong \mathrm{Hom}_{\Lambda}(\mathsf{u}M, \mathrm{Hom}_{\Lambda}(\mathsf{u}N, k)) \\
&\cong \mathrm{Hom}_{\Lambda}(\mathsf{u}M, k) \otimes_k \mathrm{Hom}_{\Lambda}(\mathsf{u}N, k)\,.
\end{aligned}$$

This is where the assumption that both M and N are degreewise finite rank and bounded below is needed. As Φ is a homotopy equivalence, from the commutativity of the diagram above it follows that $\Delta \otimes \pi^*$ is a homotopy equivalence of dg $\mathcal{S}$-modules justifying the first assertion; cf. 3.3.

With this in hand, we have

$$\mathrm{H}(\mathsf{d}(M \otimes^{\mathsf{L}}_{\Lambda} N)) \cong \mathrm{H}(\mathsf{d}M \otimes_k \mathsf{d}N) \cong \mathrm{H}(\mathsf{d}M) \otimes_k \mathrm{H}(\mathsf{d}N),$$

where the second map is the Künneth isomorphism. This gives the second of the following equalities:

$$\begin{aligned}\mathrm{V}^{\mathsf{d}}_{\Lambda}(M \otimes^{\mathrm{L}}_{\Lambda} N) &= \operatorname{Supp}_{\mathcal{S}} \mathrm{H}(\mathsf{d}(M \otimes^{\mathrm{L}}_{\Lambda} N)) \\ &= \operatorname{Supp}_{\mathcal{S}} (\mathrm{H}(\mathsf{d}M) \otimes_k \mathrm{H}(\mathsf{d}N)) \\ &= \operatorname{Join}(\operatorname{Supp}_{\mathcal{S}} \mathrm{H}(\mathsf{d}M), \operatorname{Supp}_{\mathcal{S}} \mathrm{H}(\mathsf{d}N)) \\ &= \operatorname{Join}(\mathrm{V}^{\mathsf{d}}_{\Lambda}(M), \mathrm{V}^{\mathsf{d}}_{\Lambda}(N)).\end{aligned}$$

The third equality is Lemma 1.4. □

5 Passage to the exterior algebra

Throughout this section and the next $(R, \mathfrak{m}, k)$ is a commutative noetherian local ring. Fix a list of elements $\boldsymbol{f} = f_1, \dots, f_c$ in $\mathfrak{m}$ and set

$$E := R\langle e_1, \dots, e_c \mid \partial e_i = f_i \rangle ,$$

the Koszul complex on $\boldsymbol{f}$ over R, regarded as a local dg R-algebra in the standard way. One could take R to be a local dg algebra where $\boldsymbol{f}$ is a list of cycles in even degrees, contained in the maximal ideal of R; we stick to the situation above for ease of exposition. Two special cases are worth mention.

Remark 5.1 When $\boldsymbol{f}$ forms an R-regular sequence, the augmentation $E \xrightarrow{\simeq} R/(\boldsymbol{f})$ is a quasi-isomorphism of dg algebras and the map $R \to R/(\boldsymbol{f})$ is complete intersection. When $\boldsymbol{f}$ is the zero sequence, E is the exterior algebra over R on c generators of degree one.

Set $\Lambda := k \otimes_R E$ and $V := \Lambda_1$. We identify $e_1, \dots, e_c$ with their images in Λ; they are a basis for the k-space V. Set $W := \Sigma^{-1}(V^{\vee})$, and

$$\mathcal{S} := \operatorname{Sym}_k W .$$

Let $\chi_1, \dots, \chi_c$ be the basis of W dual to $e_1, \dots, e_c$.

5.2 Let M be a dg E-module whose homology is finitely generated over R. Let F be a dg E-module that is semifree as a dg R-module, and $F \xrightarrow{\simeq} M$ an E-linear quasi-isomorphism. By [25, Proposition 4.2.8], $\operatorname{RHom}_E(M, k)$ can be equipped with a dg $\mathcal{S}$-module structure through the isomorphism

$$\operatorname{RHom}_E(M, k) \simeq \mathcal{S} \otimes_k \operatorname{Hom}_R(F, k),$$

where the differential of the complex on the right is

$$1 \otimes \partial^{\operatorname{Hom}_E(F,k)} + \sum_{i=1}^{c} \chi_i \otimes \operatorname{Hom}(e_i, k) ;$$

we let $\mathcal{C}_F$ denote this dg $\mathcal{S}$-module. Following [24, Definition 3.3.1], the *cohomological support of M over E* is

$$\mathrm{V}_E(M) = \operatorname{Supp}_{\mathcal{S}} \operatorname{Ext}^*_E(M, k) = \operatorname{Supp}_{\mathcal{S}} \mathrm{H}^*(\mathcal{C}_F) .$$

A bridge to exterior algebras has been used effectively to acquire cohomological information on these support varieties when R is regular and $\boldsymbol{f}$ is an R-regular sequence; see, for

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instance, [7,14,23]. This path is still sensible at this generality and can be used to establish results over E, as we do now.

Consider the functor $\mathsf{t}\colon \mathsf{D}(E) \to \mathsf{D}(\Lambda)$ given by $k \otimes^{\mathsf{L}}_R -$. In the statement below, the construction of the dg $\mathcal{S}$-module $\mathcal{S}_{\mathsf{t}F}$ is given in 3.3.

Lemma 5.3 *Let M be a dg E-module with finitely generated homology over R and fix $F \xrightarrow{\simeq} M$ a quasi-isomorphism of dg E-modules where F is semifree when regarded as dg R-module. There is the following isomorphism of dg $\mathcal{S}$-modules*

$$\mathcal{C}_F \cong \mathcal{S}_{\mathsf{t}F}\,.$$

In particular, $\mathrm{V}_E(M) = \mathrm{V}^{\mathsf{d}}_\Lambda(\mathsf{t}M)$.

Proof For the isomorphism, since $\mathfrak{m}\operatorname{Hom}_R(F,k) = 0$ the E-action on $\operatorname{Hom}_R(F,k)$ factors through Λ. It is immediate to check the adjunction isomorphism

$$\alpha : \operatorname{Hom}_R(F,k) \xrightarrow{\cong} \operatorname{Hom}_k(\mathsf{t}F,k)$$

is one of Λ-modules. Therefore from the definitions of $\mathcal{C}_F$ and $\mathcal{S}_{\mathsf{t}F}$ in 5.2 and 3.3, respectively, the map

$$1 \otimes \alpha : \mathcal{C}_F \to \mathcal{S}_{\mathsf{t}F}$$

is an isomorphism of dg $\mathcal{S}$-modules. The equality of supports follows:

$$\begin{aligned}
\mathrm{V}_E(M) &= \operatorname{Supp}_{\mathcal{S}} \mathrm{H}(\mathcal{C}_F)\\
&= \operatorname{Supp}_{\mathcal{S}} \mathrm{H}(\mathcal{S}_{\mathsf{t}F})\\
&= \operatorname{Supp}_{\mathcal{S}} \mathrm{H}(\mathcal{S}_{\mathsf{t}M})\\
&= \operatorname{Supp}_{\mathcal{S}} \mathrm{H}(\mathsf{dt}M)\\
&= \mathrm{V}^{\mathsf{d}}_\Lambda(\mathsf{t}M)\,;
\end{aligned}$$

the second equality holds by the established isomorphism above and the others are clear from the various definitions. □

Remark 5.4 Suppose $\boldsymbol{f}$ is an R-regular sequence and M a finitely generated R-module such that $\boldsymbol{f}M = 0$. The cohomological support of M over E agrees with support variety of M introduced by Avramov in [1], and further developed in the work of Avramov and Buchweitz [4].

More generally, without the assumption $\boldsymbol{f}$ is regular, $\mathrm{V}_E(M)$ specializes to the support sets of Jorgensen [22] and Avramov and Iyengar [8]; cf. [25, Sect. 6.2]. When M has finite projective dimension over R, the cohomological support $\mathrm{V}_E(M)$ agrees with those above; hence Lemma 5.3 reveals how, in this setting, all of these supports are cohomological supports over an exterior algebra.

5.5 Let $\mathsf{D_b}(E/R)$ denote the full subcategory of $\mathsf{D}(E)$ consisting of objects M such that M is perfect when regarded as an object of $\mathsf{D}(R)$ via restriction of scalars. That is, if $\eta\colon R \to E$ is the structure map and $\eta_*\colon \mathsf{D}(E) \to \mathsf{D}(R)$ denotes the restriction of scalars functor along η, then M is in $\mathsf{D_b}(E/R)$ if and only if $\eta_*(M)$ is isomorphic in $\mathsf{D}(R)$ to a bounded complex of finite rank free R-modules. In particular, M has bounded and finitely generated homology over R. When R is regular $\mathsf{D_b}(E/R)$ is just the bounded derived category of dg E-modules.

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The result below is a particular case of a theorem of Gulliksen [21] and Avramov, Gashasrov, and Peeva [6].

Proposition 5.6 *For a dg E-module M, the following conditions are equivalent:*

(1) $\operatorname{Tor}^R(k, M)$ *is finitely generated over* k;
(2) $\operatorname{Ext}_\Lambda(\mathsf{t}M, k)$ *is finitely generated over* $\mathcal{S}$;
(3) $\operatorname{Ext}_\Lambda(k, \mathsf{t}M)$ *is finitely generated over* $\mathcal{S}$.

Moreover, when $\mathrm{H}(M)$ *is finite over* R*, the conditions above are equivalent to:*

(4) M *is in* $D_b(E/R)$.

Proof The equivalence of (1), (2), and (3) is from a special case of [25, Theorem 4.3.2], and the fact (1) and (4) are equivalent when $\mathrm{H}(M)$ is finite over R, is classical; see, for example, [12, Corollary 1.3.2]. □

6 Support for tensor products, II

The notation in this section is as in the previous one. The result below is the theorem announced in the introduction.

Theorem 6.1 *Suppose E is a Koszul complex over a local ring $(R, \mathfrak{m}, k)$ on a finite list of elements in $\mathfrak{m}$. For M, N in $D_b(E/R)$,*

$$\mathrm{V}_E(M \otimes^{\mathrm{L}}_E N) = \operatorname{Join}(\mathrm{V}_E(M), \mathrm{V}_E(N))\,.$$

Proof We need only pass to the exterior algebra:

$$\begin{aligned}
\mathrm{V}_E(M \otimes^{\mathrm{L}}_E N) &= \mathrm{V}^{\mathsf{d}}_\Lambda(\mathsf{t}(M \otimes^{\mathrm{L}}_E N))\\
&= \mathrm{V}^{\mathsf{d}}_\Lambda(\mathsf{t}M \otimes^{\mathrm{L}}_\Lambda \mathsf{t}N)\\
&= \operatorname{Join}(\mathrm{V}^{\mathsf{d}}_\Lambda(\mathsf{t}M), \mathrm{V}^{\mathsf{d}}_\Lambda(\mathsf{t}N))\\
&= \operatorname{Join}(\mathrm{V}_E(M), \mathrm{V}_E(N))\,.
\end{aligned}$$

The first and fourth equalities hold by Lemma 5.3; the second one follows from the isomorphism

$$\mathsf{t}(M \otimes^{\mathrm{L}}_E N) \simeq \mathsf{t}M \otimes^{\mathrm{L}}_\Lambda \mathsf{t}N\,.$$

By Proposition 5.6, the dg Λ-modules $\mathsf{t}M, \mathsf{t}N$ are in $\mathrm{D}^{\mathsf{f}}_{\mathsf{b}}(\Lambda)$ and so the third equality holds by Proposition 4.4. □

Remark 6.2 There is an alternative proof of Theorem 6.1, using the Hopf algebra structure on $\operatorname{Ext}^*_E(k, k)$. The key point is that for any dg E-modules the maps

$$\operatorname{Ext}^*_E(M, k) \otimes_k \operatorname{Ext}^*_E(N, k) \to \operatorname{Ext}^*_E(M \otimes^{\mathrm{L}}_E N, k \otimes^{\mathrm{L}}_E k) \to \operatorname{Ext}^*_E(M \otimes^{\mathrm{L}}_E N, k) \qquad (6.2.1)$$

are $\operatorname{Ext}^*_E(k, k)$-linear. The second map is induced by multiplication, $k \otimes^{\mathrm{L}}_E k \to k$. In (6.2.1), the graded Ext-module on the left is given an $\operatorname{Ext}^*_E(k, k)$-module structure through the diagonal

$$\operatorname{Ext}^*_E(k, k) \to \operatorname{Ext}^*_E(k, k) \otimes_k \operatorname{Ext}^*_E(k, k)\,.$$

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This is a straightforward calculation. However, this approach requires a bit of background on dg algebras with divided powers and suitably adapting classical material to this more general setting; cf. [21]. The main point is that $\operatorname{Ext}^*_E(k,k)$ is generated, as a k-algebra, by primitives induced by derivations that respect divided powers on the minimal semifree resolution of k over E.

One can identify $\mathcal{S}$ as a Hopf subalgebra of $\operatorname{Ext}^*_E(k,k)$ so the maps in (6.2.1) are also $\mathcal{S}$-linear. When M, N are in $\mathsf{D_b}(E/R)$, the $\mathcal{S}$-modules $\operatorname{Ext}^*_E(M,k)$, $\operatorname{Ext}^*_E(N,k)$ are finite over $\mathcal{S}$, see Proposition 5.6, so the assertion of Theorem 6.1 follows directly from Lemma 1.4 once noting the composition in (6.2.1) is an isomorphism.

Consider the equivalence

$$(-)^{\dagger}\colon \mathsf{D_b}(E/R)^{\mathsf{op}} \longrightarrow \mathsf{D_b}(E/R),$$

where $M^{\dagger} := \operatorname{RHom}_E(M,E)$ for each M.

Lemma 6.3 *If M is in $\mathsf{D_b}(E/R)$, then $\operatorname{V}_E(M) = \operatorname{V}_E(M^{\dagger})$.*

Proof. As M is perfect over R there is an isomorphism of dg Λ-modules

$$\mathsf{t}(M^{\dagger}) \simeq \operatorname{RHom}_{\Lambda}(\mathsf{t}M, \Lambda)\,.$$

By [23, Theorem 4.1], $\mathsf{t}M$ and $\operatorname{RHom}_{\Lambda}(\mathsf{t}M,\Lambda)$ generate the same thick subcategory in $\mathsf{D}(\Lambda)$. Thus the second equality below holds:

$$\operatorname{V}_E(M) = \operatorname{V}^{\mathsf{d}}_{\Lambda}(\mathsf{t}M) = \operatorname{V}^{\mathsf{d}}_{\Lambda}(\operatorname{RHom}_{\Lambda}(\mathsf{t}M,\Lambda)) = \operatorname{V}^{\mathsf{d}}_{\Lambda}(\mathsf{t}(M^{\dagger})) = \operatorname{V}_E(M^{\dagger})\,. \quad \square$$

Corollary 6.4 *If R is Gorenstein and $\operatorname{RHom}_E(M,N)$ belongs to $\mathsf{D_b}(E/R)$, then*

$$\operatorname{V}_E(\operatorname{RHom}_E(M,N)) = \operatorname{Join}(\operatorname{V}_E(M), \operatorname{V}_E(N)).$$

Proof As R is Gorenstein and the R-modules $\operatorname{H}(M)$, $\operatorname{H}(N)$ are finitely generated, there is an isomorphism

$$\operatorname{RHom}_E(M,N)^{\dagger} \simeq M \otimes^{\mathsf{L}}_E N^{\dagger}.$$

Thus, the second equality below holds

$$\begin{aligned}
\operatorname{V}_E(\operatorname{RHom}_E(M,N)) &= \operatorname{V}_E(\operatorname{RHom}_E(M,N)^{\dagger}) \\
&= \operatorname{V}_E(M \otimes^{\mathsf{L}}_E N^{\dagger}) \\
&= \operatorname{Join}(\operatorname{V}_E(M), \operatorname{V}_E(N^{\dagger})) \\
&= \operatorname{Join}(\operatorname{V}_E(M), \operatorname{V}_E(N))\,;
\end{aligned}$$

the first and fourth equalities are by Lemma 6.3 and the third is by Theorem 6.1. $\square$

Remark 6.5 In light of Theorem 6.1, it would be interesting to determine whether Corollary 6.4 holds without the assumption that $\operatorname{RHom}_E(M,N)$ is in $\mathsf{D_b}(E/R)$.

The result below relates the cohomological support of $M \otimes^{\mathsf{L}}_E N$ to those of its homology modules. Specializing to the case R is regular and $\boldsymbol{f}$ is an R-regular sequence, yields a positive answer to [15, Question 2]. The containment in the statement of the theorem can be strict; see [15, Example 5.3].

Theorem 6.6 *Let M, N be in $D_b(E/R)$ and suppose the $\mathcal{S}$-modules $\operatorname{Ext}^*_E(M,k)$ and $\operatorname{Ext}^*_E(N,k)$ are generated in cohomological degrees at most s and t, respectively. There is a containment of closed subsets*

$$\mathrm{V}_E(M \otimes^{\mathrm{L}}_E N) \subseteq \bigcup_{i \le s+t} \mathrm{V}_E(\operatorname{Tor}^E_i(M,N)).$$

Proof By Proposition 4.4 and Lemma 5.3, one may identify $\operatorname{Ext}^*_E(M \otimes^{\mathrm{L}}_E N, k)$ with

$$\operatorname{Ext}^*_E(M,k) \otimes_k \operatorname{Ext}^*_E(N,k)$$

viewed as a graded $\mathcal{S}$-module via restriction along the diagonal map (1.1.1). Let T denote its graded $\mathcal{S}$-submodule of $\operatorname{Ext}^*_E(M \otimes^{\mathrm{L}}_E N, k)$ generated by

$$\bigoplus_{i+j \leqslant u} \operatorname{Ext}^i_E(M,k) \otimes_k \operatorname{Ext}^j_E(N,k),$$

where $u = s + t$. The $(\mathcal{S} \otimes_k \mathcal{S})$-module generated by T is $\operatorname{Ext}^*_E(M \otimes^{\mathrm{L}}_E N, k)$, so arguing as in the proof of Lemma 1.4, one gets an equality

$$\mathrm{V}_E(M \otimes^{\mathrm{L}}_E N) = \operatorname{Supp}_{\mathcal{S}} T. \tag{6.6.1}$$

Fix a semifree resolution $F \xrightarrow{\simeq} M \otimes^{\mathrm{L}}_E N$ over E, and let F' be the soft truncation of F in lower degrees at most u. Thus there is morphism of dg E-modules $\tau \colon F \to F'$ with the property that

$$\tau_{\leqslant u} \colon F_{\leqslant u} \to F'_{\leqslant u}$$

is the identity map. Hence,

$$\operatorname{Ext}(\tau, k)\colon \operatorname{Ext}^*_E(F', k) \to \operatorname{Ext}^*_E(M \otimes^{\mathrm{L}}_E N, k)$$

is an isomorphism in upper degrees at most u. In particular, under the identification discussed above

$$T \subseteq \operatorname{Im}\left(\operatorname{Ext}^*_E(F', k) \xrightarrow{\operatorname{Ext}(\tau,k)} \operatorname{Ext}^*_E(M \otimes^{\mathrm{L}}_E N, k)\right),$$

and hence, one has an inclusion

$$\operatorname{Supp}_{\mathcal{S}} T \subseteq \operatorname{Supp}_{\mathcal{S}} \operatorname{Ext}^*_E(F', k).$$

Since F' has bounded homology, it is in the thick subcategory of $\mathsf{D}(E)$ generated by $\mathrm{H}(F')$ regarded as dg E-module via the augmentation $E \to \mathrm{H}_0(E)$. Thus,

$$\operatorname{Supp}_{\mathcal{S}} T \subseteq \operatorname{Supp}_{\mathcal{S}} \operatorname{Ext}_E\left(\oplus_{i \leqslant u} \operatorname{Tor}^E_i(M,N), k\right) = \bigcup_{i \leqslant u} \mathrm{V}_E(\operatorname{Tor}^E_i(M,N)),$$

where for the first containment, we are also using the equality

$$\mathrm{H}(F') = \bigoplus_{i \leqslant u} \operatorname{Tor}^E_i(M,N).$$

Combining this with (6.6.1) finishes the proof. □

Remark 6.7 Theorem 6.6 implies that when R is regular and $M \otimes^{\mathrm{L}}_E N$ has finitely generated homology over R, the complexity of $M \otimes^{\mathrm{L}}_E N$, in the sense of [1, Sect. 3], is bounded above by the maximum of the complexities of $\operatorname{Tor}^E_i(M,N)$ for $i \le s + t$, where s and t are from Theorem 6.6.

Data availability

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

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Received: 4 January 2022 Accepted: 14 March 2022

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