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RESEARCH ARTICLE

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Exceptional loci in Lefschetz theory

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Abstract

Let $\phi : X \to \mathbb{P}^n$ be a morphism of varieties. Given a hyperplane H in $\mathbb{P}^n$, there is a Gysin map from the compactly supported cohomology of $\phi^{-1}(H)$ to that of X. We give conditions on the degree of the cohomology under which this map is an isomorphism for all but a low-dimensional set of hyperplanes, generalizing results due to Skorobogatov, Benoist, and Poonen-Slavov. Our argument is based on Beilinson's theory of singular supports for étale sheaves.

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1 | INTRODUCTION AND STATEMENT OF RESULTS

In this note, we prove a generalization of the following theorem due to Benoist and written here in a form due to [9].

Theorem 1.1 ([2], Théorème 1.4). *Let $X \subset \mathbb{P}^N$ be a geometrically irreducible quasiprojective variety over a field k. Define $M_{\text{bad}} \subseteq \check{\mathbb{P}}^N$ as the locus of hyperplanes H such that $X_H := X \cap H$ is not geometrically irreducible. Then* $\operatorname{codim} M_{\text{bad}} \geqslant \dim X - 1$

This result is geometric, but has a cohomological reformulation: the top degree compactly supported cohomology groups of X and X_H are isomorphic. In this note, we prove a similar result for all cohomology of sufficiently high degree on X.

Theorem 1.2. *Let X be a separated scheme of finite type over a separably closed field k, let $\phi : X \to \mathbb{P}^n$ be a morphism, and let $\Lambda = \mathbb{Z}/\ell\mathbb{Z}$ for ℓ a prime power not divisible by the characteristic of k. Set* $r = \dim X \times_{\mathbb{P}^n} X$.

Then for each $c \geqslant 1$, there is a closed subscheme $Z_c \subset \check{\mathbb{P}}^n$ of dimension at most $n - c$ such that for $H \in \check{\mathbb{P}}^n \setminus Z_c$ and $q > c + r$ (respectively, $q = c + r$), the Gysin map:

$$H_c^{q-2}(\phi^{-1}(H), \Lambda(-1)) \to H_c^q(X, \Lambda)$$

is an isomorphism (respectively, surjective).

In the setting of Theorem 1.1, taking $q = 2\dim X$, we see that $M_{\text{bad}} \subset Z_{\dim X - 1}$, which has dimension $\leqslant n - \dim X + 1$, recovering the assertion of *loc. cit.*

We remind the construction of Gysin maps in Section 2.

Remark. If X, ϕ, and H are defined over an arbitrary field k' with separable closure k, then our construction shows that Z_c is naturally defined over k'. Moreover, a Gysin map $H_c^{q-2}(\phi^{-1}(H) \times_{\text{Spec(k}')} \text{Spec}(k), \Lambda(-1)) \to H_c^q(X \times_{\text{Spec(k}')} \text{Spec(k)}, \Lambda)$ is compatible with the action of $\text{Gal(k/k}')$, so if $H \notin Z_c$ is defined over k' and q is as in Theorem 1.2, the Gysin map is an isomorphism or surjection of Galois representations.

The main new tool in the proof of this result is Beilinson and Saito's works [1, 10] on the singular support of constructible sheaves in arbitrary characteristic. In Section 2, we will use their work to prove a result, Theorem 2.2, that is at the core of the argument; we will also collect a couple lemmas we will need. In Section 3, we will use these tools to prove Theorem 1.2.

1.1 | Past results

Theorem 1.2 generalizes a number of results beyond Theorem 1.1. Poonen–Slavov [9] establish the $q = 2\dim X$ case of Theorem 1.2 under the additional assumption that ϕ has equidimensional fibers. If ϕ is the immersion of a normal projective complex variety and $c = 1$, Theorem 1.2 is Corollary 7.4.1 of [5], and is proven as a special case of the paper's Lefschetz hyperplane theorem for intersection homology.

And if ϕ is the closed immersion of a smooth projective variety, this result is known; by [11, Theorem 2.1], there exists an isomorphism $H_c^{q-2}(\phi^{-1}(H), \Lambda(-1)) \to H_c^q(X, \Lambda)$ so long as $q \geqslant n + s + 3$, where s is the dimension of the singular locus of $\phi^{-1}(H)$. So the locus where there exists no isomorphism $H_c^{q-2}(\phi^{-1}(H), \Lambda(-1)) \to H_c^q(X, \Lambda)$ is contained in the locus of hyperplanes such that $\phi^{-1}(H)$ is singular in dimension at least $q - n - 2$. This locus in turn has codimension at least $q - n - 1$.

2 | THE GYSIN MAP FOR BOUNDED COMPLEXES

2.1 | Notation and the basic setup

Let k be a separably closed field, let ℓ be a prime power not divisible by the characteristic of k, and set $\Lambda = \mathbb{Z}/\ell\mathbb{Z}$. A *variety* is a separated scheme of finite type over k. For the remainder of this paper, sheaves will be constructible étale sheaves of Λ-modules, and for any variety V we will use $D(V)$ to denote the bounded derived category of constructible sheaves of Λ-modules on V. Given a k-point $H \in \check{\mathbb{P}}^n$, let $i : H \to \mathbb{P}^n$ denote the corresponding inclusion, and let $j : \mathbb{P}^n \setminus H \to \mathbb{P}^n$ be the inclusion of the complement. For the remainder of the paper, all functors will be derived; for instance, we will use j_* to denote the derived pushforward associated to j. Finally, we will use the notation $\mathbb{H}^q(\mathcal{F})$ to denote the hypercohomology of $\mathcal{F} \in D(V)$ in degree q.

On the level of sheaves, the map in Theorem 1.2 is induced by applying $R\Gamma$ to a composition of two arrows in $D(\mathbb{P}^n)$. Given a bounded complex of sheaves $\mathcal{F}$ on $\mathbb{P}^n$, the two maps we will use are:

- the counit map $i_! i^! \mathcal{F} \to \mathcal{F}$, which appears in the localization triangle:

$$i_! i^! \mathcal{F} \to \mathcal{F} \to j_* j^* \mathcal{F} \xrightarrow{+1},$$

- the Gysin map $i^* \mathcal{F}(-1)[-2] \to i^! \mathcal{F}$, as discussed in [4].

Applying $R\Gamma$ to these two maps produces maps:

$$\mathbb{H}^q(H, i^! \mathcal{F}) \to \mathbb{H}^q(\mathcal{F})$$

and:

$$\mathbb{H}^{q-2}(H, i^* \mathcal{F}(-1)) \to \mathbb{H}^q(H, i^! \mathcal{F}),$$

respectively. Our main effort will be proving that these maps are isomorphisms or surjections assuming certain hypotheses on $\mathcal{F}$ and H.

2.2 | The counit map

Of the two maps above, the counit map is the easier one to understand, as its cone is $j_* j^* \mathcal{F}$.

Lemma 2.1. *Let $\mathcal{F} \in D(\mathbb{P}^n)$. Fix an integer r. Suppose that for all p, the sheaf $\mathcal{H}^p(\mathcal{F})$ has support of dimension at most $r - p$. Then the sheaf $\mathcal{H}^p(j_* j^* \mathcal{F})$ has support of dimension at most $r - p$ and we have $\mathbb{H}^q(\mathbb{P}^n, j_* j^* \mathcal{F}) = 0$ for $q > r$.*

Proof. Let $\mathcal{F}$ and r satisfy the hypotheses of the lemma. For each p, the support of $\mathcal{H}^p(j^* \mathcal{F}) \cong j^* \mathcal{H}^p(\mathcal{F})$ has dimension at most $r - p$. Since $\mathbb{P}^n \setminus H$ is affine, we then have that affine vanishing [8, Theorem VI.7.3] implies that $R^s j_*(\mathcal{H}^p(j^* \mathcal{F}))$ is supported in dimension at most $r - p - s$ and $H^s(\mathcal{H}^p(j^* \mathcal{F})) = 0$ if $s + p > r$. The results follow by applying a spectral sequence to the filtered complex $\tau^{\geqslant p} \mathcal{F}$. ☐

Remark. The hypothesis of the lemma is equivalent to asking that $\mathcal{F}$ sit in perverse degrees $\leqslant r$. From this perspective, the claim follows from right exactness of affine pushforwards with respect to the perverse t-structure.

2.3 | The Gysin map

The following result is the most nonstandard ingredient of the proof of Theorem 1.2.

Theorem 2.2. *Let $\mathcal{F}$ be an object of $D(\mathbb{P}^n)$, let $\Phi \subset \mathbb{P}^n \times \check{\mathbb{P}}^n$ be the universal hyperplane, and let π_1, π_2 denote the projections. Then there is a closed subscheme $Z \subset \Phi$ of dimension $\leqslant n-1$ such that for $H \in \check{\mathbb{P}}^n$, the cone of the Gysin map:*

$$i^*\mathcal{F}(-1)[-2] \to i^!\mathcal{F},$$

associated to $i : H \to \mathbb{P}^n$ is supported on $\pi_1(Z \cap \pi_2^{-1}(H))$.

In the proof of Theorem 1.2, we will use the following corollary of this theorem.

Corollary 2.3. *With $\mathcal{F}$ as in Theorem 2.2, for any positive integer c there is a closed subscheme $Z_c \subset \check{\mathbb{P}}^n$ of dimension at most $n-c$ such that for any $H \in \check{\mathbb{P}}^n \setminus Z_c$ the cone of the Gysin map associated to $i : H \to \mathbb{P}^n$ is supported on a subscheme of dimension at most $c-2$.*

Proof. Take Z as in Theorem 2.2. Let $Z_c \subset \check{\mathbb{P}}^n$ be the closed subscheme where the fibers of the projection $Z \to \check{\mathbb{P}}^n$ have dimension $\geqslant c-1$. As $\dim Z = n-1$, we have $\dim Z_c \leqslant n-c$, and it satisfies the conclusion by definition of Z. □

The theory of *singular support* for étale sheaves was developed by Beilinson [1] and Saito [10]. We use it in the following form. Note that in what follows, given a map $f : U \to Y$, we use df to denote the map on cotangent sheaves $df : f^*(T^*Y) \to T^*U$.

Theorem 2.4 (Theorem 1.3, [1]). *Let X be a smooth variety of dimension n and let $\mathcal{F}$ be a bounded constructible complex on it.*

Then there exists a closed, conical subscheme $SS(\mathcal{F}) \subset T^*X$ of dimension n such that for every pair $h : U \to X$ smooth and $f : U \to Y$ a morphism to a smooth variety Y such that $df^{-1}(dh(SS(\mathcal{F})))$ is contained in the zero section $Y \subset T^*Y$, the map f is locally acyclic with respect to $h^*(\mathcal{F})$ (in the sense of [3], 2.12).

Remark. The most serious part of the theorem is the calculation of the dimension of $SS(\mathcal{F})$.

Local acyclicity in turn guarantees that the Gysin map is an isomorphism:

Lemma 2.5. *Suppose X is smooth and let $i : D \to X$ be the embedding of a smooth divisor. Suppose $\mathcal{F} \in D(X)$ has the property that the intersection:*

$$SS(\mathcal{F})|_D \cap N^*_{X/D} \subset T^*X \underset{X}{\times} D$$

is contained in the zero section.

Then the Gysin map:

$$i^*\mathcal{F}(-1)[-2] \to i^!\mathcal{F}$$

is an isomorphism.

Proof. This result is local on X, so we may assume there is a smooth map $f : X \to \mathbb{A}^1$ with $D = f^{-1}(0)$. By Theorem 2.4, f is locally acyclic with respect to $\mathcal{F}$; indeed, by Lemma 4.7 of [7], the pair $(f, \mathcal{F})$ is *strongly locally acylic*. Then, by Theorem 6.8 of [7], the Gysin map $i^*\mathcal{F}(-1)[-2] \to i^!\mathcal{F}$ is an isomorphism. □

Proof of Theorem 2.2. Recall that Φ is canonically isomorphic to $\mathbb{P}T^*\mathbb{P}^n$. Let $Z = \mathbb{P}SS(\mathcal{F})$ as in Theorem 2.4. By the theorem, it has dimension $n - 1$.

Fix $H \in \check{\mathbb{P}}^n$, and let $B \subset H$ be the subscheme $\pi_1(Z \cap \pi_2^{-1}(H))$. By construction, the closed embedding $H \setminus B \to \mathbb{P}^n \setminus B$ satisfies the hypotheses of Lemma 2.5, so the cone of the Gysin map is supported on B as desired. □

3 | PROOF OF THEOREM 1.2

In this section, we continue to use notation from the previous section.

Let $\phi : X \to \mathbb{P}^n$ be a morphism of separated schemes of finite type over k. In this section, we establish Theorem 1.2 for the map ϕ. Let Λ_X denote the constant sheaf on X with fiber Λ, and set $r = \dim X \times_{\mathbb{P}^n} X$.

We first note that $R^p\phi_!\Lambda_X$ is supported at points over which ϕ has fiber dimension at least $\frac{p}{2}$, because the stalk of $R^p\phi_!\Lambda_X$ at a point x is just $H_c^p(\phi^{-1}(X), \Lambda)$ by [12, Lemma 0F7L]; equivalently, $R^p\phi_!\Lambda_X$ is supported on a locus over which $X \times_{\mathbb{P}^n} X \to \mathbb{P}^n$ has fiber dimension p. As a consequence, $\mathcal{H}^p(\phi_!\Lambda_X)$ has support in dimension at most $r - p$. So by Lemma 2.1, we have that for any inclusion of a hyperplane $i : H \to \mathbb{P}^n$, the hypercohomology groups $\mathbb{H}^q(\mathbb{P}^n, j_*j^*\phi_!\Lambda_X)$ vanish for $q > r$. So we have that the counit map $\mathbb{H}^q(H, i^!\phi_!\Lambda_X) \to H_c^q(X, \Lambda)$ is an isomorphism if $q \geqslant r + 2$ and is a surjection if $q = r + 1$.

We now show that the Gysin map induces an isomorphism or surjection on cohomology. Fixing $c \geqslant 1$, let Z_c be the exceptional set in Corollary 2.3 relative to the complex of sheaves $\phi_!\Lambda_X$. Fix some k-point $H \in \check{\mathbb{P}}^n \setminus Z_c$, and let $Q \in D(H)$ denote the cone of the morphism:

$$i^*\phi_!\Lambda_X(-1)[-2] \to i^!\phi_!\Lambda_X.$$

By Corollary 2.3, Q is supported on a closed subscheme B of dimension at most $c - 2$. Moreover, $\mathcal{H}^p(j_*j^*\phi_!\Lambda_X)$ has support of dimension at most $\min(r - p, n)$ by Lemma 2.1. Since we also have that $R^p\phi_!\Lambda_X$ is supported in dimension $\min(r - p, n)$, the distinguished triangle $i_!i^!\phi_!\Lambda_X \to \phi_!\Lambda_X \to j_*j^*\phi_!\Lambda_X \to$ gives the bound on the dimension of the support of $\mathcal{H}^p(i_!i^!\phi_!\Lambda_X)$:

$$\dim \operatorname{supp}(\mathcal{H}^p(i_!i^!\phi_!\Lambda_X)) \leqslant \min(r - p + 1, n).$$

Likewise, from the distinguished triangle $j_!j^*\phi_!\Lambda_X \to \phi_!\Lambda_X \to i_*i^*\Lambda_X \to$, the sheaf $\mathcal{H}^p(i^*\phi_!\Lambda_X(-1)[-2])$ is supported on a set of dimension at most $\min(r - p + 2, n - 1)$. From these two observations and the defining triangle for Q, we see that $\mathcal{H}^p(Q)$ is supported on a subscheme of dimension at most $\min(r - p + 1, c - 2)$. Therefore, $\mathbb{H}^{q-p}(H, \mathcal{H}^p(Q)) = 0$ for $q - p > 2\min(r - p + 1, c - 2)$. Observe that if $q - p \leqslant 2\min(r - p + 1, c - 2)$, then $2(q - p) \leqslant 2(r - p + 1) + 2(c - 2)$, so $q \leqslant r + c - 1$. Therefore, whenever $q > r + c - 1$, $\mathbb{H}^{q-p}(H, \mathcal{H}^p(Q)) = 0$. Applying the Grothendieck spectral sequence, we find that $\mathbb{H}^q(H, Q) = 0$ for $q > r + c - 1$.

From this, we conclude that the Gysin map:

$$H_c^{q-2}(\phi^{-1}(H),\Lambda(-1)) \cong \mathbb{H}^{q-2}(H, i^*\phi_!\Lambda_X(-1)) \to \mathbb{H}^q(H, i^!\phi_!\Lambda_X)$$

is an isomorphism for $q > r + c$ and a surjection if $q = r + c$. Combining this with the counit map above, and noting $c \geqslant 1$, we have that the map on cohomology:

$$H_c^{q-2}(\phi^{-1}(H),\Lambda(-1)) \to H_c^q(X,\Lambda)$$

is an isomorphism for $q > r + c$ and a surjection if $q = r + c$, proving Theorem 1.2 for X.

Remark. In the above setting, note that $\phi_!(\Lambda_X) \in D(\mathbb{P}^n)$ is in perverse degrees $\leqslant r$. The argument shows that, suitably understood, the conclusion of Theorem 1.2 holds for any $\mathcal{F} \in D(\mathbb{P}^n)$ a bounded complex of constructible sheaves in perverse degrees $\leqslant r$.

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JOURNAL INFORMATION

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