

BACK STABLE K -THEORY SCHUBERT CALCULUS

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ABSTRACT. We study the back stable K -theory Schubert calculus of the infinite flag variety. We define back stable (double) Grothendieck polynomials and double K -Stanley functions and establish coproduct expansion formulae. Applying work of Weigandt, we extend our previous results on bumpless pipedreams from cohomology to K -theory. We study finiteness and positivity properties of the ring of back stable Grothendieck polynomials, and divided difference operators in K -homology.

CONTENTS

1. Introduction	1
2. Thick infinite flag scheme	5
3. Grothendieck polynomials	12
4. Back stable Grothendieck polynomials	14
5. Back stable double Grothendieck polynomials	20
6. K -Bumpless pipedreams	28
7. K -homology and Hopf structure	34
8. The ring of back stable Grothendieck polynomials	38
9. Determinantal formulae for $G_\lambda(x a)$	49
10. Degeneracy loci	54
11. Further directions	55
Appendix A. Grothendieck Inversion	56
Appendix B. Some computations	57
References	60

1. INTRODUCTION

In [LLS21] we initiated the study of back stable Schubert calculus. In the present work, we generalize from cohomology to K -theory, and study back stable K -theory Schubert calculus. We assume the reader has some familiarity with our earlier work, and in the introduction we will emphasize some of the differences.

1.1. Back stable Grothendieck polynomials. Let w be a permutation of \mathbb{Z} moving finitely many integers and let $p \leq q$ be integers such that the interval of integers $[p, q]$ contains all nonfixed points of w . Lascoux and Schützenberger [LS] defined the Grothendieck polynomial $\mathfrak{G}_w(x_p, x_{p+1}, \dots, x_q) \in \mathbb{Z}[x_p, x_{p+1}, \dots, x_q]$, for the purpose of giving explicit polynomial representatives of classes of Schubert

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structure sheaves in the K -theory of the flag variety. A priori the definition of \mathfrak{G}_w depends on the interval $[p, q]$.

Whereas the back stable Schubert polynomials of [LLS21] are limits of Schubert polynomials, in this paper we will be concerned with the back stable Grothendieck polynomials, which by definition are the limits

$$\overleftarrow{\mathfrak{G}}_w(x) := \lim_{p \rightarrow -\infty, q \rightarrow \infty} \mathfrak{G}_w(x_p, x_{p+1}, \dots, x_q).$$

It turns out that the q -limit is unnecessary, either by a well-known forward stability property of the Grothendieck polynomials or by Proposition 4.4. The p -limit is important: the back-stable Grothendieck polynomials are in fact formal series involving all of the variables x_i for $i \leq d$ where $d \in \mathbb{Z}$ is the maximum index such that $w(d) > w(d+1)$.¹

The same definition of back stable Grothendieck polynomials has appeared in the work of Marberg and Pawłowski [MP], who studied the principal specializations ($x_i \mapsto q^{i-1}$) of these power series. We remark that back stable Schubert polynomials have been generalized to the involution setting by Pawłowski [Paw].

1.2. Infinite flag varieties. Infinite-dimensional flag varieties, such as flag varieties of Kac-Moody groups, come in a number of algebro-geometric variants. Of interest to us is a thin flag variety Fl that is an ind-finite variety and a thick flag variety $\overline{\mathrm{Fl}}$ that is an infinite-dimensional scheme. There are other versions such as semi-infinite flag varieties and geometric models based on loop groups which will not feature in this work. The K -groups in this work are certain Grothendieck groups of coherent sheaves, and the choice of scheme structure plays a more significant role than in our earlier work in cohomology. For a discussion of K -groups on thin and thick flag varieties in the Kac-Moody setting, we refer the reader to Kumar [Kum] and Baldwin and Kumar [BK].

Whereas in [LLS21] we only considered the thin infinite flag variety, in the present work we also consider the thick infinite flag variety. Thick infinite flag varieties were studied by Kashiwara [Kas] and we give a mostly elementary treatment in §2. A different and elegant approach to infinite flag varieties is also given by Anderson [And21]. We show in Theorem 5.25 and Corollary 5.26 that back stable double Grothendieck polynomials represent classes of Schubert structure sheaves in the equivariant K -group $K_T(\overline{\mathrm{Fl}})$ of the thick infinite flag variety $\overline{\mathrm{Fl}}$.

1.3. Coproduct formula. Like their cohomological counterparts, back stable Grothendieck polynomials (and their double versions) satisfy a coproduct formula (Theorem 4.15), decomposing them into a symmetric part and a finite part:

$$(1.1) \quad \overleftarrow{\mathfrak{G}}_w = \sum_{\substack{u * v = w \\ v \in S_{\neq 0}}} (-1)^{\ell(u) + \ell(v) - \ell(w)} G_u \mathfrak{G}_v.$$

Here, $u * v$ denotes the Demazure or Hecke product, G_u denotes the K -Stanley symmetric function [Buc02a, FK]², and \mathfrak{G}_v is the Grothendieck polynomial [LS]. We deduce (1.1) from the coproduct formula in K -theory *affine* Schubert calculus [LLS+].

¹We prefer to include the q -limit in the definition of $\overleftarrow{\mathfrak{G}}_w$; otherwise we would have to pick a particular q and write $\overleftarrow{\mathfrak{G}}_w(\dots, x_{q-1}, x_q)$.

²The symmetric function G_w is usually called a *stable Grothendieck polynomial*, but to avoid conflicts in terminology we use a different name.

1.4. Double K -Stanley symmetric functions. The back stable double Grothendieck polynomials $\overleftarrow{\mathfrak{G}}_w(x; a)$ are defined as limits of double Grothendieck polynomials $\mathfrak{G}_w(x; a)$ in a similar manner. We define the *double K -Stanley symmetric functions* as the image $G_w(x||a) := \eta_a(\overleftarrow{\mathfrak{G}}_w(x; a))$ under an algebra homomorphism $\eta_a : \overleftarrow{R}(x; a) \rightarrow \hat{\Lambda}(x||a)$ (see §5.9) from back stable power series to symmetric power series. The symmetric functions $G_w(x||a)$ appear to be novel. (Setting $a = 0$, we recover the K -Stanley symmetric function G_w .) We obtain as a subset the Grassmannian double K -Stanley symmetric functions $G_\lambda(x||a)$. These functions form a basis of a ring $\Gamma(x||a)$ (Theorem 8.22), a double analogue of Buch's Γ ring [Buc02a]. We explain determinantal formulae for $G_\lambda(x||a)$ in §9, connecting our construction to the literature [And17, HIMN].

We conjecture (Conjecture 8.23) that the coefficients of the expansion of $G_w(x||a)$ in the basis $\{G_\lambda(x||a)\}$ have alternating signs. The analogous expansion coefficients of double Stanley functions $F_w(x||a)$ into double Schur functions $\{s_\lambda(x||a)\}$ were shown to be positive in [LLS21, Theorem 4.22] using the quantum equals affine phenomenon.

1.5. K -bumpless pipedreams. In [LLS21] we introduced *bumpless pipe-dreams* to give explicit monomial expansions for back stable double Schubert polynomials. Weigandt [Wei] connected bumpless pipedreams to earlier *alternating sign matrix* formulae of Lascoux [Las], and thereby obtained formulae for double Grothendieck polynomials in terms of K -bumpless pipe-dreams. Weigandt's work immediately gives a formula for back stable double Grothendieck polynomials in terms of K -bumpless pipedreams, which we state in Theorem 6.2. We also use K -bumpless pipedreams to give formulae for the double K -Stanley symmetric functions (Theorem 6.5 and Corollary 6.6), and expansion formulae (Theorem 6.9 and Corollary 6.10) for $\overleftarrow{\mathfrak{G}}_w(x; a)$ and $G_w(x||a)$ in terms of double K -Stanley symmetric functions.

Recently, bumpless pipedreams have found applications in the study of diagonal Gröbner degenerations of matrix Schubert varieties [HPW, Kle, KW], in the study of products of Schubert polynomials [Hua00, Hua01], and in other applications [FGS, BS, Xio].

1.6. K -homology. In §7, we study a basis dual to the Grassmannian double K -Stanley symmetric functions $G_\lambda(x||a)$. We call these symmetric functions $g_\lambda(y||a)$ the *K -Molev functions*. These symmetric functions are K -theory analogues of Molev's dual Schur functions $\hat{s}_\lambda(y||a)$ [Mol]. At $a = 0$, the symmetric functions $g_\lambda(y||a)$ reduce to the dual stable Grothendieck polynomials $g_\lambda(y)$ studied by Lam and Pylyavskyy [LP]; see also [Len00]. Geometrically, the functions $g_\lambda(y||a)$ form a basis of the equivariant K -group $K^T(\text{Gr})$ of the thin infinite Grassmannian. We show in Theorem 7.6 that the $g_\lambda(y||a)$ can be obtained recursively by applying K -homology divided difference operators. In Theorem 7.8, we sketch the relation between our $g_\lambda(y||a)$ and the deformation of symmetric functions studied by Knutson and Lederer [KL].

1.7. The algebra of back stable Grothendieck polynomials. We define the *algebra of back stable Grothendieck polynomials*

$$B := \bigoplus_w \mathbb{Z} \overleftarrow{\mathfrak{G}}_w$$

to be the span of all back stable Grothendieck polynomials. We show in Theorem 8.11 that B is a ring, or equivalently, the product of back stable Grothendieck polynomials is finite. This finiteness is quite nontrivial; for example, it fails to hold for the K -theory of the affine flag variety of SL_2 . The ring B is an infinite flag variety version of Buch's Γ ring spanned by K -Stanley functions [Buc02a], an analogous ring for the infinite Grassmannian.

We show in Theorem 8.14 that after adjoining an element $\Omega = \Omega[x_-]$, the ring B can be decomposed as a tensor product $\Gamma[\Omega] \otimes R^+$, where R^+ is spanned by finite Grothendieck polynomials.

We conjecture that the similar finiteness also holds for back stable double Grothendieck polynomials. Curiously, we show in that Proposition 8.28 that this would hold if the positivity conjecture (Conjecture 8.23) were true.

1.8. Flagged Grothendieck polynomials. In the course of studying the relationship between Grothendieck polynomials and their symmetrizations given by the K -Stanley functions, we found it natural to study a family of polynomials which interpolate between them. These are the flagged Grothendieck polynomials, which we introduce in §8.2 for an arbitrary permutation using divided differences. We prove in Proposition 8.5 a monomial expansion for flagged Grothendieck polynomials that generalizes the Fomin-Kirillov formula for Grothendieck polynomials. For the special cases of vexillary and 321-avoiding permutations, the flagged Grothendieck polynomials were defined combinatorially in [KMY08] [KMY09] [Mat].

1.9. K -classes of degeneracy loci. Fulton [Ful] realized the double Schubert polynomial as a universal formula for the cohomology class of a degeneracy locus Ω_w defined by rank conditions on composite maps in a sequence of maps of vector bundles living on a common base. Recently, Anderson and Fulton [AF] studied what might be called the “back stable limit” of this degeneracy locus construction, obtaining *enriched Schubert polynomials*, which specialize (and are nearly equivalent) to the back stable Schubert polynomials of [LLS21].

Buch [Buc02b] observed that the K -class of the structure sheaf of Ω_w in a flag bundle, has a *universal* formula given by the double Grothendieck polynomial $\mathfrak{G}_w(x; a)$, based on the work of Fulton and Lascoux [FL], who showed that after a certain change of variable, the double Grothendieck polynomials map to classes of structure sheaves of opposite Schubert varieties in the flag variety. Very recently, Buch [Buc+] gave us a detailed explanation of a limit of the degeneracy locus construction of double Grothendieck polynomials and suggested the result should be the back stable double Grothendieck polynomials. He also suggested to apply the degeneracy loci formulae in [BKTY05] and [BKSTY]. The result coincides with one of our formulae for back stable Grothendieck polynomials, and is explained in §10.

1.10. Further directions/relations. We were unable to pursue many obvious avenues of investigation, for example, the study of the ideal sheaf basis, and the relation to K -theory affine Schubert calculus [LSS, Mor]. We briefly discuss these ideas in §11.

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2. THICK INFINITE FLAG SCHEME

2.1. Dynkin type $A_{\mathbb{Z}}$. The Dynkin diagram of type $A_{\mathbb{Z}}$ has Dynkin node set $I = \mathbb{Z}$ and simple bonds $(i, i+1)$ for all $i \in \mathbb{Z}$. It has weight lattice $X' = \bigoplus_{i \in I} \mathbb{Z} \Lambda_i$ with basis of fundamental weights Λ_i . For $i \in \mathbb{Z}$, the simple coroot $\alpha_i^\vee \in \text{Hom}_{\mathbb{Z}}(X', \mathbb{Z})$ is defined by $\langle \alpha_i^\vee, \Lambda_j \rangle = \delta_{ij}$ for $i, j \in \mathbb{Z}$. For $i \in \mathbb{Z}$, let $\epsilon_i := \Lambda_i - \Lambda_{i-1}$. For $i \in \mathbb{Z}$, let $\alpha_i := \epsilon_i - \epsilon_{i+1}$ be the simple root. Let $Q := \bigoplus_{i \in I} \mathbb{Z} \alpha_i$ and $X := \bigoplus_{i \in \mathbb{Z}} \mathbb{Z} \epsilon_i$. We have

$$(2.1) \quad Q \subset X \subset X'.$$

Let the Weyl group $S_{\mathbb{Z}}$ be the subgroup of $\text{Aut}(X')$ generated by the following reflections s_i :

$$(2.2) \quad s_i \lambda = \lambda - \langle \alpha_i^\vee, \lambda \rangle \alpha_i \quad \text{for } i \in I \text{ and } \lambda \in X'.$$

We have

$$s_i(\epsilon_j) = \begin{cases} \epsilon_{i+1} & \text{if } j = i \\ \epsilon_i & \text{if } j = i+1 \\ \epsilon_j & \text{if } j \notin \{i, i+1\}. \end{cases}$$

Thus the restriction of the action of $S_{\mathbb{Z}}$ to the basis $\{\epsilon_i \mid i \in I\}$ of the sublattice $X \subset X'$ is the permutation representation on the set \mathbb{Z} : s_i exchanges i and $i+1$ and fixes other integers.

The set Φ of roots are the elements of the form $w\alpha_i$ for $i \in I$ and $w \in S_{\mathbb{Z}}$; they have the form $\alpha_{ij} = \epsilon_i - \epsilon_j$ for $i, j \in \mathbb{Z}$ with $i \neq j$. Let $s_{ij} \in S_{\mathbb{Z}}$ be the associated reflection; acting on \mathbb{Z} it exchanges i and j and fixes other integers. Let Φ^+ be the set of positive roots, the α_{ij} with $i < j$. Let $\Phi^- = -\Phi^+$.

2.2. Dynkin diagram automorphisms. Let $\text{Aut}(A_{\mathbb{Z}})$ denote the group of automorphisms of the diagram $A_{\mathbb{Z}}$, the permutations of the node set $I = \mathbb{Z}$ which preserve adjacency. $\text{Aut}(A_{\mathbb{Z}})$ is generated by the Dynkin shift $\gamma(i) = i+1$ for all $i \in I$ and the Dynkin reversal $\omega(i) = -i$ for all $i \in I$. This is an infinite dihedral group: $\omega^2 = \text{id}$ and

$$(2.3) \quad \omega \gamma \omega = \gamma^{-1}.$$

The group $\text{Aut}(A_{\mathbb{Z}})$ acts by automorphisms on X' : we have $\zeta(\Lambda_i) = \Lambda_{\zeta(i)}$ for all $\zeta \in \text{Aut}(A_{\mathbb{Z}})$ and $i \in I$. In particular, we have

$$(2.4) \quad \gamma(\epsilon_i) = \epsilon_{i+1}$$

$$(2.5) \quad \omega(\epsilon_i) = -\epsilon_{1-i} \quad \text{for all } i \in I.$$

There is an induced action of $\text{Aut}(A_{\mathbb{Z}})$ by automorphisms on $S_{\mathbb{Z}}$: $\zeta(s_i) = s_{\zeta(i)}$ for all $\zeta \in \text{Aut}(A_{\mathbb{Z}})$ and $i \in I$. We have

$$(2.6) \quad \gamma(s_i) = s_{i+1}$$

$$(2.7) \quad \omega(s_i) = s_{-i} \quad \text{for all } i \in I.$$

2.3. Thin infinite flag variety. Let $\mathbb{C}((t))$ and $\mathbb{C}((t^{-1}))$ denote the Laurent polynomial rings. Denote

$$E_a := \left\{ \sum_{i=a}^{\infty} c_i t^i \right\} \subset \mathbb{C}((t)) \quad \text{and} \quad F_a := \left\{ \sum_{i=a}^{-\infty} c_i t^i \right\} \subset \mathbb{C}((t^{-1}))$$

so that we have $\cdots \subset E_1 \subset E_0 \subset E_{-1} \subset \cdots$ and $\cdots \subset F_{-1} \subset F_0 \subset F_1 \subset \cdots$.

A subspace $\Lambda \subset \mathbb{C}((t))$ (resp. $\Xi \subset \mathbb{C}((t^{-1}))$) is called *admissible* if we have $E_N \subset \Lambda \subset E_{-N}$ (resp. $F_{-N} \subset \Xi \subset F_N$) for some N . The *Sato Grassmannian* Gr^\bullet (resp. Gr_-^\bullet) consists of the set of all admissible subspaces in $\mathbb{C}((t))$ (resp. $\mathbb{C}((t^{-1}))$) and has the structure of an ind-variety over \mathbb{C} .

The virtual dimensions $\text{vdim}(\Lambda)$ and $\text{vdim}(\Xi)$ are given by

$$\begin{aligned}\text{vdim}(\Lambda) &:= \dim(\Lambda/(\Lambda \cap E_0)) - \dim(E_0/(\Lambda \cap E_0)) \\ \text{vdim}(\Xi) &:= \dim(\Xi/(\Xi \cap F_0)) - \dim(F_0/(\Xi \cap F_0)).\end{aligned}$$

The virtual dimension measures the size of the subspace Λ (resp. Ξ) relative to the standard subspace E_0 (resp. F_0). Let $\text{Gr}^{(d)} \subset \text{Gr}^\bullet$ (resp. $\text{Gr}_-^{(d)} \subset \text{Gr}_-^\bullet$) consist of the admissible subspaces of virtual dimension d . Thus, Gr^\bullet is the disconnected union of the $\text{Gr}^{(d)}$ for $d \in \mathbb{Z}$.

An *admissible flag* in $\mathbb{C}((t))$ of virtual dimension 0 is a sequence

$$\Lambda_\bullet = \{\cdots \subset \Lambda_1 \subset \Lambda_0 \subset \Lambda_{-1} \subset \cdots\}$$

of admissible subspaces satisfying the conditions: (1) $\text{vdim}(\Lambda_i) = -i$, and (2) for some N , we have $\Lambda_i = E_i$ for all i with $|i| \geq N$. Similarly, we define admissible flags in $\mathbb{C}((t^{-1}))$. The *thin infinite flag variety* Fl (resp. Fl_-) consists of the set of all admissible flags in $\mathbb{C}((t))$ (resp. $\mathbb{C}((t^{-1}))$) of virtual dimension 0, and has the structure of an ind-variety over \mathbb{C} .

2.4. Thick infinite flag scheme. We describe the thick infinite flag scheme in an elementary fashion; see [Kas] for further details. We say that a subspace $V \subset \mathbb{C}((t^{-1}))$ is *opposed* to a subspace $\Xi \in \text{Gr}_-^\bullet$ if the composite map $V \subset \mathbb{C}((t^{-1})) \rightarrow \mathbb{C}((t^{-1}))/\Xi$ is an isomorphism, and that V is *thick* if it is opposed to some $\Xi \in \text{Gr}_-^\bullet$. Denote by $\overline{\text{Gr}}_\Xi$ the set of all $V \subset \mathbb{C}((t^{-1}))$ opposed to Ξ . The thick infinite Grassmannian $\overline{\text{Gr}}$ is the space of all thick subspaces, that is,

$$\overline{\text{Gr}} := \bigcup_{\Xi \in \text{Gr}_-^\bullet} \overline{\text{Gr}}_\Xi.$$

Each $\overline{\text{Gr}}_\Xi$ is an affine space of infinite dimension. For example, for $\Xi = F_0$, the vectors t, t^2, \dots , form a basis of $\mathbb{C}((t^{-1}))/F_0$. Thus any thick subspace V opposed to F_0 has a basis of the form $t + v_1, t^2 + v_2, \dots$ where $v_1, v_2, \dots \in F_0$, and the v_i are arbitrary. Similarly, for any $\Xi \in \text{Gr}_-^\bullet$, $\overline{\text{Gr}}_\Xi$ can be identified with $\Xi^\infty \cong \prod_1^\infty \Xi$. Identifying Ξ^∞ with $\mathbb{C}^\infty \cong \prod_{\mathbb{Z}} \mathbb{C} \cong \text{Spec}(k[x_i \mid i \in \mathbb{Z}])$, $\overline{\text{Gr}}_\Xi$ is endowed with the structure of an infinite-dimensional affine space.

As shown in [Kas], $\overline{\text{Gr}}$ has the structure of a separated scheme over \mathbb{C} , and $\overline{\text{Gr}}_\Xi \subset \overline{\text{Gr}}$ are affine open subschemes. We also define the subscheme $\overline{\text{Gr}}^i \subset \overline{\text{Gr}}$ by

$$\overline{\text{Gr}}^i := \bigcup_{\text{vdim}(\Xi)=i} \overline{\text{Gr}}_\Xi$$

and we note that $\overline{\text{Gr}}^i \cap \overline{\text{Gr}}^j = \emptyset$ if $i \neq j$.

There is an injection $\text{Gr}^\bullet \hookrightarrow \overline{\text{Gr}}$, described as follows. Let $\Lambda \in \text{Gr}^\bullet$. Pick N so that $E_{-N} \supset \Lambda \supset E_N$. Let $\mathcal{B} \subset \Lambda \cap \bigoplus_{i=-N}^{N-1} \mathbb{C}t^i$ project to a basis of Λ/E_N . Define

$$\overline{\Lambda} := \text{span}(\mathcal{B} \cup \{t^N, t^{N+1}, \dots\}) \subset \mathbb{C}((t)).$$

Then $\overline{\Lambda}$ is a thick subspace (opposed to $E_{-N-1} \oplus W$ for any complement W of $\text{span}(\mathcal{B})$ in $\bigoplus_{i=-N}^{N-1} \mathbb{C}t^i$), and the map $\Lambda \mapsto \overline{\Lambda}$ induces an injection $\text{Gr}^\bullet \hookrightarrow \overline{\text{Gr}}$.

Let $\Xi_\bullet \in \text{Fl}_-$ be an admissible flag with $\text{vdim}(\Xi_i) = i$. A sequence V_\bullet of subspaces of $\mathbb{C}((t^{-1}))$

$$\cdots \subset V_1 \subset V_0 \subset V_{-1} \subset \cdots$$

where V_i is opposed to Ξ_i is called a *thick flag opposed to Ξ_\bullet* . Let $\overline{\text{Fl}}_{\Xi_\bullet}$ denote the set of thick flags opposed to Ξ_\bullet . Let $\dots, w_{-1}, w_0, w_1, \dots \in \mathbb{C}((t^{-1}))$ be such that $\mathbb{C}((t^{-1}))/\Xi_i$ is spanned by w_i, w_{i+1}, \dots . If V_\bullet is opposed to Ξ_\bullet , then there exist vectors

$$v_k = w_k + \sum_{j=k-1}^{-\infty} a_{kj} w_j \in \mathbb{C}((t^{-1}))$$

such that V_i is the span of v_i, v_{i+1}, \dots . The coefficients $a_{kj} \in \mathbb{C}$ are arbitrary and uniquely determine V_\bullet , endowing $\overline{\text{Fl}}_{\Xi_\bullet}$ with the structure of an infinite-dimensional affine space. The *thick infinite flag scheme*

$$(2.8) \quad \overline{\text{Fl}} = \bigcup_{\Xi_\bullet \in \text{Fl}_-} \overline{\text{Fl}}_{\Xi_\bullet}$$

is the space of all thick flags, and has the structure of a separated scheme. The construction $\Lambda \mapsto \overline{\Lambda}$ induces an injection $\text{Fl} \hookrightarrow \overline{\text{Fl}}$. The thin standard flag $E_\bullet \in \text{Fl}$ gives a thick standard flag $\overline{E}_\bullet \in \overline{\text{Fl}}$.

We use the flag $F_\bullet \in \text{Fl}_-$ as the basepoint of Fl_- . For $w \in S_{\mathbb{Z}}$, we have a point $wF_\bullet \in \text{Fl}_-$ given by

$$(wF)_i = \prod_{j \in w((-\infty, i])} \mathbb{C}t^j.$$

Similarly, we have $wE_\bullet \in \text{Fl}$ and $w\overline{E}_\bullet \in \overline{\text{Fl}}$.

Lemma 2.1. *We have $\overline{\text{Fl}} = \bigcup_{w \in S_{\mathbb{Z}}} \overline{\text{Fl}}_{wF_\bullet}$.*

Proof. Let $V_\bullet \in \overline{\text{Fl}}$. Then there exists N such that for $|i| \geq N$, we have that V_i is opposed to F_i . The statement then reduces to the corresponding statement for the flag variety of the finite-dimensional vector space F_N/F_{-N} . Namely, any flag G_\bullet in \mathbb{C}^{2N} is opposed to (at least) one of the $(2N)!$ flags wH_\bullet , where $w \in S_{2N}$ is a permutation and H_\bullet is some choice of basepoint flag. \square

Example 2.2. Define

$$V_i = \begin{cases} \text{span}(t^{i+1}, t^{i+2}, t^{i+3}, t^{i+4}, \dots) & \text{if } i \text{ is even} \\ \text{span}(t^i, t^{i+2}, t^{i+3}, t^{i+4}, \dots) & \text{if } i \text{ is odd} \end{cases},$$

so $V_0 = \text{span}(t, t^2, t^3, t^4, \dots)$, $V_1 = \text{span}(t, t^3, t^4, \dots)$, $V_2 = \text{span}(t^3, t^4, \dots)$. It is clear that $\cdots \subset V_2 \subset V_1 \subset V_0 \subset \cdots$ and each V_i is opposed to some admissible subspace. So it belongs to the set that Kashiwara has in [Kas, p. 190]. However, it does not belong to $\overline{\text{Fl}}$ from (2.8) since it is not true that V_m is opposed to F_m for large m , so it cannot be opposed to any admissible flag.

We also define the thick partial flag schemes $\overline{\text{Fl}}_k$ for $k \in \mathbb{Z}$, consisting of partial flags V_\bullet in $\mathbb{C}((t^{-1}))$ indexed by $i \in \mathbb{Z} \setminus \{k\}$, with V_i opposed to Ξ_i for some $\Xi_\bullet \in \text{Fl}_-$. There is a natural morphism $\overline{\text{Fl}} \rightarrow \overline{\text{Fl}}_k$ defined by forgetting V_k .

2.5. Infinite Schubert varieties. Let $B \subset \text{End}(\mathbb{C}((t^{-1})))$ consist of the invertible linear transformations $\varphi : \mathbb{C}((t^{-1})) \rightarrow \mathbb{C}((t^{-1}))$ satisfying

$$\varphi(t^i) \in c_i t^i + \bigoplus_{j > i} \mathbb{C} t^j \quad c_i \neq 0.$$

Let $B_- \subset \text{End}_{\mathbb{C}}(\mathbb{C}((t^{-1})))$ consist of the invertible linear transformations φ satisfying

$$\varphi(t^i) \in c_i t^i + \prod_{j < i} \mathbb{C} t^j \quad c_i \neq 0.$$

Let $T = T_{\mathbb{Z}} = B \cap B_-$. The groups B, B_- act on $\overline{\text{Fl}}$. The group B acts on Fl , but B_- does not. We have that $B \cdot E_{\bullet} = E_{\bullet}$ and $B \cdot \overline{E}_{\bullet} = \overline{E}_{\bullet}$. The (thin) Schubert cells and (thin) Schubert varieties are defined to be

$$\Omega_w := B \cdot w E_{\bullet} \subset \text{Fl} \quad \text{and} \quad X_w := \overline{B \cdot w E_{\bullet}} \subset \text{Fl}.$$

We have the decomposition $\text{Fl} = \bigsqcup_{w \in S_{\mathbb{Z}}} \Omega_w$. Furthermore, $\Omega_w \cong \mathbb{C}^{\ell(w)}$ and X_w is an irreducible projective variety of dimension $\ell(w)$.

An order ideal in $S_{\mathbb{Z}}$ is a subset $S \subset S_{\mathbb{Z}}$ such that if $x < y \in S_{\mathbb{Z}}$ with $y \in S$ then $x \in S$. Let $K_T(\text{Fl})$ denote the K -group of T -equivariant coherent sheaves on Fl . Since Fl is an ind-scheme, the K -group $K_T(\text{Fl})$ is the inductive limit

$$\varinjlim_S K_T(\bigcup_{w \in S} X_w) = \bigoplus_{w \in S_{\mathbb{Z}}} K_T(\text{pt})[\mathcal{O}_{X_w}].$$

The (thick) opposite Schubert cells and (thick) opposite Schubert varieties are defined to be

$$\Omega^w := B_- \cdot w \overline{E}_{\bullet} \subset \overline{\text{Fl}} \quad \text{and} \quad X^w := \overline{B_- \cdot w \overline{E}_{\bullet}} \subset \overline{\text{Fl}}.$$

We have the decomposition $\overline{\text{Fl}} = \bigsqcup_{w \in S_{\mathbb{Z}}} \Omega^w$. Furthermore, Ω^w is an affine space of infinite dimension and of codimension $\ell(w)$ in $\overline{\text{Fl}}$.

For any finite order ideal $S \subset S_{\mathbb{Z}}$, let $\Omega^S = \bigcup_{w \in S} w \Omega^e = \bigsqcup_{w \in S} \Omega^w$. Let $K_T(\Omega^S)$ be the Grothendieck group of coherent T -equivariant \mathcal{O}_{Ω^S} -modules. One may show that $K_T(\Omega^S) \cong \bigoplus_{w \in S} K_T(\text{pt})[\mathcal{O}_{X^w}]$ [KS, Lemma 2.3]. Define

$$K_T(\overline{\text{Fl}}) = \varprojlim_S K_T(\Omega^S) = \prod_{w \in S_{\mathbb{Z}}} K_T(\text{pt})[\mathcal{O}_{X^w}].$$

2.6. NilHecke ring, localization and the GKM ring. We recall some results from [KS]. These results are stated in the Kac-Moody setting, but the proofs are valid for our thick infinite flag scheme $\overline{\text{Fl}}$.

We have $K_T^*(\text{pt}) \cong R(T) \cong \mathbb{Z}[X]$ (see §2.1 for the definition of X) with the image of $\lambda \in X$ written e^{λ} . The infinite symmetric group $S_{\mathbb{Z}}$ acts on T and therefore on $R(T) = \mathbb{Z}[X]$. Let $R(T)_{\text{loc}} = R(T)[(1 - e^{\alpha})^{-1} \mid \alpha \in \Phi]$ and let $\mathbb{K}_{\text{loc}} = R(T)_{\text{loc}}[S_{\mathbb{Z}}]$ be the twisted group ring with multiplication $(fv)(gw) = fv(g)vw$ for $f, g \in R(T)_{\text{loc}}$ and $v, w \in S_{\mathbb{Z}}$. Then \mathbb{K}_{loc} acts on $R(T)_{\text{loc}}$. Let $D_i \in \mathbb{K}_{\text{loc}}$ be the element $(1 - e^{-\alpha_i})^{-1}(1 - e^{-\alpha_i} s_i)$ for $i \in \mathbb{Z}$. Let \mathbb{K} be the K -theoretic nilHecke algebra, the subring of \mathbb{K}_{loc} generated by $R(T)$ and the D_i . The algebra \mathbb{K} acts naturally on $R(T)$. In this context D_i is known as the *Demazure operator* [Dem].

Let $\text{Fun}(S_{\mathbb{Z}}, R(T))$ be the $R(T)$ -algebra of functions $f : S_{\mathbb{Z}} \rightarrow R(T)$ under pointwise product, and similarly define $\text{Fun}(S_{\mathbb{Z}}, R(T)_{\text{loc}})$. The algebra \mathbb{K} acts on

$\text{Fun}(S_{\mathbb{Z}}, R(T)_{\text{loc}})$ by

$$(2.9) \quad D_i(f)(w) = \frac{1}{1 - e^{-w(\alpha_i)}} f(w) - \frac{e^{-w(\alpha_i)}}{1 - e^{-w(\alpha_i)}} f(ws_i).$$

The T -fixed points on $\overline{\text{Fl}}$ are the flags $w\overline{E}_{\bullet}$, and we identify $\overline{\text{Fl}}^T$ with $S_{\mathbb{Z}}$. For each $w \in S_{\mathbb{Z}}$, localization at w gives a $R(T)$ -module homomorphism $i_w^* : K_T(\overline{\text{Fl}}) \rightarrow R(T)$ [KS, §2]. It satisfies

$$(2.10) \quad i_w^*([\mathcal{O}_{X^v}]) = \begin{cases} \prod_{\alpha \in \Phi^+ \cap w(\Phi^-)} (1 - e^\alpha) & \text{if } v = w \\ 0 & \text{unless } v \leq w. \end{cases}$$

The map $S_{\mathbb{Z}} \cong \overline{\text{Fl}}^T \hookrightarrow \overline{\text{Fl}}$ induces the $R(T)$ -algebra homomorphism $\text{res} : K_T(\overline{\text{Fl}}) \rightarrow \text{Fun}(S_{\mathbb{Z}}, R(T))$ given by $\text{res}(f)(w) = i_w^*(f)$. Define

$$(2.11) \quad \psi^v = \text{res}([\mathcal{O}_{X^v}]) \quad \text{for } v \in S_{\mathbb{Z}}.$$

Recall that for $i \in \mathbb{Z}$ we have the projection $p_i : \overline{\text{Fl}} \rightarrow \overline{\text{Fl}}_i$ to the i -th minimal thick partial flag scheme. We have [KS, Cor. 3.3]

$$(2.12) \quad p_i^* p_{i*}(\mathcal{O}_{X^v}) = \begin{cases} [\mathcal{O}_{X^{vs_i}}] & \text{if } vs_i < v \\ [\mathcal{O}_{X^v}] & \text{if } vs_i > v. \end{cases}$$

The following result is due to Kashiwara; see [LSS, Prop. 3.3].

Proposition 2.3. *The following diagram commutes:*

$$\begin{array}{ccc} K_T(\overline{\text{Fl}}) & \xrightarrow{\text{res}} & \text{Fun}(S_{\mathbb{Z}}, R(T)) \hookrightarrow \text{Fun}(S_{\mathbb{Z}}, R(T)_{\text{loc}}) \\ p_i^* p_{i*} \downarrow & & \downarrow D_i \\ K_T(\overline{\text{Fl}}) & \xrightarrow{\text{res}} & \text{Fun}(S_{\mathbb{Z}}, R(T)) \hookrightarrow \text{Fun}(S_{\mathbb{Z}}, R(T)_{\text{loc}}) \end{array}$$

Proposition 2.4. *There are functions $\{\psi^v \mid v \in S_{\mathbb{Z}}\} \subset \text{Fun}(S_{\mathbb{Z}}, R(T))$ which are uniquely determined by the following conditions:*

- (1) $\psi^v(\text{id}) = \delta_{\text{id},v}$ for $v \in S_{\mathbb{Z}}$.
- (2) If $ws_i < w$ then

$$(2.13) \quad \psi^v(w) = \begin{cases} \psi^v(ws_i) & \text{if } vs_i > v \\ e^{-w(\alpha_i)} \psi^v(ws_i) + (1 - e^{-w(\alpha_i)}) \psi^{vs_i}(w) & \text{if } vs_i < v. \end{cases}$$

- (3) We have

$$(2.14) \quad \psi^v(w) \in \mathbb{Z}[Q] \quad \text{for all } v, w \in S_{\mathbb{Z}}.$$

Proof. Part (1) follows from the support condition (2.10). Part (2) follows from

$$(2.15) \quad D_i(\psi^v) = \begin{cases} \psi^{vs_i} & \text{if } vs_i < v \\ \psi^v & \text{otherwise,} \end{cases}$$

which holds by equation (2.12), Proposition 2.3, and the definition of ψ^v . Equation (2.14) holds by induction. \square

Let Ψ be the set of $\psi \in \text{Fun}(S_{\mathbb{Z}}, R(T))$ such that

$$(2.16) \quad 1 - e^\alpha \mid \psi(s_\alpha w) - \psi(w) \quad \text{for all } \alpha \in \Phi \text{ and } w \in S_{\mathbb{Z}}.$$

Proposition 2.5. *The space Ψ is an $R(T)$ -subalgebra of $\text{Fun}(S_{\mathbb{Z}}, R(T))$. Moreover*

$$(2.17) \quad \Psi = \prod_{v \in S_{\mathbb{Z}}} R(T)\psi^v.$$

The ring Ψ is called the GKM ring [GKM]. In the case of Kac-Moody flag varieties the analogous condition is due to Kostant and Kumar [KK]. The action of \mathbb{K} on $\text{Fun}(S_{\mathbb{Z}}, R(T))$ preserves Ψ .

Proposition 2.6. *The map res induces an isomorphism $K_T(\overline{\text{Fl}}) \cong \Psi$ of $R(T)$ -algebras and \mathbb{K} -algebras where D_i acts by $p_i^* p_{i*}$.*

If there is an action of a group G on an algebra R and an R -module M , then we say that M is a G -equivariant R -module if $g(a) \cdot g(m) = g(a \cdot m)$ for all $g \in G$, $a \in R$, and $m \in M$.

Proposition 2.7.

- (a) *The group $\text{Aut}(A_{\mathbb{Z}})$ acts on \mathbb{K} by conjugation and acts on $R(T)$, making $R(T)$ into an $\text{Aut}(A_{\mathbb{Z}})$ -equivariant \mathbb{K} -module.*
- (b) *$\text{Aut}(A_{\mathbb{Z}})$ acts on $\text{Fun}(S_{\mathbb{Z}}, R(T))$ by conjugation, stabilizing the subring Ψ , making $\text{Fun}(S_{\mathbb{Z}}, R(T))$ and Ψ into $\text{Aut}(A_{\mathbb{Z}})$ -equivariant \mathbb{K} -modules.*
- (c) *For every $\zeta \in \text{Aut}(S_{\mathbb{Z}})$ and $v \in S_{\mathbb{Z}}$, we have $\zeta(\psi^v) = \psi^{\zeta(v)}$.*

2.7. Multiplicative formal group law. Let A be a ring. Define the binary operation \oplus on A by

$$(2.18) \quad a \oplus b := a + b - ab.$$

For $b \in A$ such that $1 - b$ is a unit in A , define

$$(2.19) \quad \ominus b := \frac{-b}{1 - b} \quad \text{for } b \neq 1.$$

$$(2.20) \quad a \ominus b := a \oplus (\ominus b) = \frac{a - b}{1 - b}.$$

The operation \oplus is commutative and associative with neutral element 0 and $b \ominus b = 0$ for $b \neq 1$.

2.8. From exponentials to polynomials. Define the Laurent polynomial ring R and its subrings R^+ and R^- , as follows:

$$(2.21) \quad R := \mathbb{Z}[(1 - x_i)^{\pm 1} \mid i \in \mathbb{Z}]$$

$$(2.22) \quad R^+ := \mathbb{Z}[(1 - x_i) \mid i \in \mathbb{Z}_{>0}][(1 - x_i)^{-1} \mid i \in \mathbb{Z}_{\leq 0}]$$

$$(2.23) \quad R^- := \mathbb{Z}[(1 - x_i) \mid i \in \mathbb{Z}_{\leq 0}][(1 - x_i)^{-1} \mid i \in \mathbb{Z}_{>0}].$$

Note that $\mathbb{Z}[\ominus(x_i)] = \mathbb{Z}[(1 - x_i)^{-1}]$.

Let $\Theta : R(T) \rightarrow R$ be the ring isomorphism given by $\Theta(e^{\epsilon_i}) = (1 - x_i)^{-1}$ for $i \in \mathbb{Z}$. This is merely a renaming of the generators of a Laurent polynomial ring, but is convenient for combinatorial applications. We have

$$(2.24) \quad \Theta(1 - e^{\alpha_{ij}}) = x_j \ominus x_i \quad \text{for all } i, j \in \mathbb{Z} \text{ with } i \neq j.$$

The isomorphism Θ is $S_{\mathbb{Z}}$ -equivariant.

Let $\tilde{\mathbb{K}}$ be the algebra generated by R and elements $\tilde{\pi}_i$ for $i \in I$, equipped with an isomorphism $\Theta : \mathbb{K} \rightarrow \tilde{\mathbb{K}}$ which restricts to the isomorphism $\Theta : R(T) \rightarrow R$ and satisfies $\Theta(D_i) = \tilde{\pi}_i$. We have

$$\begin{aligned}\tilde{\pi}_i &= \Theta(D_i) \\ &= \Theta((1 - e^{-\alpha_i})^{-1}(1 - e^{-\alpha_i}s_i)) \\ &= (x_i \ominus x_{i+1})^{-1}(1 - (1 - x_{i+1})^{-1}(1 - x_i)s_i) \\ &= (x_i - x_{i+1})^{-1}(1 - x_{i+1} - (1 - x_i)s_i) \\ &= (x_i - x_{i+1})^{-1}(1 - s_i)(1 - x_{i+1}) \\ &= A_i \circ (1 - x_{i+1}),\end{aligned}$$

where

$$(2.25) \quad A_i = (x_i - x_{i+1})^{-1}(1 - s_i)$$

is the divided difference operator [LLS21]. Consider now the $\text{Aut}(A_{\mathbb{Z}})$ -equivariant \mathbb{K} -module structure on $R(T)$. By (2.5), we have

$$(2.26) \quad \omega(e^{\epsilon_i}) = e^{-\epsilon_{1-i}} \quad \text{for } i \in I,$$

$$(2.27) \quad \gamma(e^{\epsilon_i}) = e^{\epsilon_{i+1}} \quad \text{for } i \in I.$$

Via the isomorphism Θ , we obtain an $\text{Aut}(A_{\mathbb{Z}})$ -equivariant $\tilde{\mathbb{K}}$ -module structure on R . The elements ω and γ of $\text{Aut}(S_{\mathbb{Z}})$ induce automorphisms of R and $\tilde{\mathbb{K}}$ that we denote by $\tilde{\omega}$ and γ respectively. By (2.26), we have $\tilde{\omega}((1 - x_i)^{-1}) = 1 - x_{1-i}$. Thus for all $i \in \mathbb{Z}$, we have

$$(2.28) \quad \tilde{\omega}(x_i) = \ominus(x_{1-i})$$

$$(2.29) \quad \tilde{\omega}(\tilde{\pi}_i) = \tilde{\omega}(\tilde{\pi}_{-i})$$

$$(2.30) \quad \gamma(x_i) = x_{i+1}$$

$$(2.31) \quad \gamma(\tilde{\pi}_i) = \tilde{\pi}_{i+1}.$$

Furthermore, we have

$$(2.32) \quad \tilde{\omega}\gamma\tilde{\omega} = \gamma^{-1}$$

on $\tilde{\mathbb{K}}$ and on R , by (2.3).

Let $R(a)$ be the ring isomorphic to R , using the variables a_i instead of x_i . We write $\Theta_a : R(T) \rightarrow R(a)$ for the isomorphism involving the a_i variables.

Let $(\Theta_a)_* : \text{Fun}(S_{\mathbb{Z}}, R(T)) \cong \text{Fun}(S_{\mathbb{Z}}, R(a))$ be defined by $(\Theta_a)_*(f) = \Theta_a \circ f$. Let $\tilde{\Psi} \subset \text{Fun}(S_{\mathbb{Z}}, R(a))$ be the image of Ψ under $(\Theta_a)_*$. Applying Θ_a to (2.16), using (2.24), and observing that $a_j \ominus a_i$ and $a_j - a_i$ generate the same ideal of $R(a)$, we see that $\tilde{\Psi}$ is the subring of functions $f : S_{\mathbb{Z}} \rightarrow R(a)$ such that

$$(2.33) \quad a_j - a_i \mid f(w) - f(s_{ij}w) \quad \text{for all } w \in S_{\mathbb{Z}} \text{ and } i, j \in \mathbb{Z} \text{ with } i \neq j.$$

This induces an isomorphism $(\Theta_a)_* : \Psi \rightarrow \tilde{\Psi}$. Finally, for $v \in S_{\mathbb{Z}}$, let $\tilde{\psi}_v \in \tilde{\Psi}$ be defined by $\tilde{\psi}^v = (\Theta_a)_*(\psi^v) = \Theta_a \circ \psi^v$.

3. GROTHENDIECK POLYNOMIALS

3.1. **Demazure operators.** For $i \in \mathbb{Z}$ and $w \in S_{\mathbb{Z}}$, define

$$s_i * w := \begin{cases} s_i w & \text{if } s_i w > w \\ w & \text{otherwise.} \end{cases}$$

This defines a monoid $(S_{\mathbb{Z}}, *)$ called the 0-Hecke monoid. The operation $*$ is called the *Hecke product* or *Demazure product*. Recall the operators $\tilde{\pi}_i \in \tilde{\mathbb{K}}$ defined in §2.8.

Lemma 3.1.

- (1) The $\tilde{\pi}_i$ satisfy the braid relations $\tilde{\pi}_i \tilde{\pi}_{i+1} \tilde{\pi}_i = \tilde{\pi}_{i+1} \tilde{\pi}_i \tilde{\pi}_{i+1}$, and $\tilde{\pi}_i^2 = \tilde{\pi}_i$.
- (2) The $\tilde{\pi}_i$ generate a monoid isomorphic to the 0-Hecke monoid.

3.2. **Subgroups of permutations.** Our notation for symmetric groups and partitions follows [LLS21]. Define the following subgroups of $S_{\mathbb{Z}}$:

$$(3.1) \quad S_+ = \langle s_i \mid i \in \mathbb{Z}_{>0} \rangle$$

$$(3.2) \quad S_- = \langle s_i \mid i \in \mathbb{Z}_{<0} \rangle$$

$$(3.3) \quad S_{\neq 0} = S_- \times S_+ = \langle s_i \mid i \in \mathbb{Z} \setminus \{0\} \rangle.$$

3.3. **Grothendieck polynomials.** For $w \in S_n$, the *Grothendieck polynomial* $\mathfrak{G}_w \in \mathbb{Z}[x_1, \dots, x_n]$ is defined by [LS] [FK, §4]

$$(3.4) \quad \mathfrak{G}_{w_0} := x_1^{n-1} x_2^{n-2} \cdots x_{n-1}$$

$$(3.5) \quad \mathfrak{G}_w := \tilde{\pi}_i \mathfrak{G}_{ws_i} \quad \text{for } ws_i > w.$$

Example 3.2. The Grothendieck polynomials \mathfrak{G}_w for $w \in S_3$ are given below with $\mathfrak{G}_{s_1 s_2 s_1}$ at the top.

$$\begin{array}{ccc} & x_1^2 x_2 & \\ \swarrow \tilde{\pi}_1 & & \searrow \tilde{\pi}_2 \\ x_1 x_2 & & x_1^2 \\ \downarrow \tilde{\pi}_2 & & \downarrow \tilde{\pi}_1 \\ x_1 & & x_1 \oplus x_2 \\ \searrow \tilde{\pi}_1 & & \swarrow \tilde{\pi}_2 \\ 1 & & \end{array}$$

Remark 3.3. By Lemma 3.1, the polynomials $\mathfrak{G}_w \in \mathbb{Z}[x_1, \dots, x_n]$ for $w \in S_n$ are well-defined. Moreover for $w \in S_+ = \bigcup_{n \geq 1} S_n$, \mathfrak{G}_w is independent of n in the sense that for any n such that $w \in S_n$, $\mathfrak{G}_{\iota(w)} = \mathfrak{G}_w$ under the standard embedding $\iota : S_n \rightarrow S_{n+1}$.

Fomin and Kirillov give the following monomial expansion of \mathfrak{G}_w .

Proposition 3.4. [FK, Prop. 3.3] *For $w \in S_+$, we have*

$$(3.6) \quad \mathfrak{G}_w = \sum_{\substack{s_{a_1} * s_{a_2} * \cdots * s_{a_L} = w \\ 1 \leq i_1 \leq i_2 \leq \cdots \leq i_L \\ a_k \leq a_{k+1} \Rightarrow i_k < i_{k+1} \\ i_k \leq a_k}} (-1)^{L-\ell(w)} x_{i_1} x_{i_2} \cdots x_{i_L},$$

where L is arbitrary.

Proposition 3.5. *The set $\{\mathfrak{G}_w \mid w \in S_+\}$ is a \mathbb{Z} -basis of $\mathbb{Z}[x_+] = \mathbb{Z}[x_i \mid i \in \mathbb{Z}_{\geq 0}]$.*

Proof. By Proposition 3.4, the lowest homogeneous part of \mathfrak{G}_w coincides with the Billey-Jockusch-Stanley formula for the Schubert polynomial [BJS]. The Schubert polynomials form a basis [LLS21, Thm. 2.7]. This shows that the \mathfrak{G}_w are linearly independent. The Monk Rule of [Len03] gives a finite Grothendieck polynomial expansion of any product $x_k \mathfrak{G}_w$ for $k \geq 1$ and $w \in S_+$. In particular, iterating the Monk rule one may expand any monomial (times $\mathfrak{G}_{\text{id}} = 1$) as a finite linear combination of Grothendieck polynomials. This proves that $\{\mathfrak{G}_w \mid w \in S_+\}$ span $\mathbb{Z}[x_+]$. \square

3.4. Negative Grothendieck polynomials. Recall the automorphism ω of $S_{\mathbb{Z}}$ from §2.2 and $\tilde{\omega}$ of R from §2.8. The automorphism $\omega \in \text{Aut}(S_{\mathbb{Z}})$ restricts to an isomorphism $S_- \cong S_+$. For $w \in S_-$, define the *negative Grothendieck polynomial* $\mathfrak{G}_w \in R^+$ by

$$(3.7) \quad \mathfrak{G}_w := \tilde{\omega}(\mathfrak{G}_{\omega(w)}) \quad \text{for } w \in S_-.$$

Example 3.6. The polynomials \mathfrak{G}_w for $w \in \langle s_{-1}, s_{-2} \rangle$ are given below. The top polynomial is $\mathfrak{G}_{s_{-1}s_{-2}s_{-1}}$.

$$\begin{array}{ccc}
 & (\ominus x_0)^2 (\ominus x_{-1}) & \\
 \swarrow \tilde{\pi}_{-1} & & \searrow \tilde{\pi}_{-2} \\
 (\ominus x_0) (\ominus x_{-1}) & & (\ominus x_0)^2 \\
 \downarrow \tilde{\pi}_{-2} & & \downarrow \tilde{\pi}_{-1} \\
 \ominus x_0 & & (\ominus x_0) \oplus (\ominus x_{-1}) \\
 \searrow \tilde{\pi}_{-1} & & \swarrow \tilde{\pi}_{-2} \\
 & 1 &
 \end{array}$$

Proposition 3.7. *The set $\{\mathfrak{G}_w \mid w \in S_-\}$ is a \mathbb{Z} -basis of $\mathbb{Z}[(1 - x_i)^{-1} \mid i \in \mathbb{Z}_{\leq 0}]$.*

Proof. The map $\tilde{\omega}$ restricts to an isomorphism

$$\mathbb{Z}[x_i \mid i \in \mathbb{Z}_{>0}] \rightarrow \mathbb{Z}[(1 - x_i)^{-1} \mid i \in \mathbb{Z}_{\leq 0}]$$

sending the basis $\{\mathfrak{G}_w \mid w \in S_+\}$ to the basis $\{\mathfrak{G}_w \mid w \in S_-\}$. \square

For $w \in S_{\neq 0} = S_- \times S_+$, there is a unique factorization $w = uv$ with $u \in S_-$ and $v \in S_+$. Define

$$(3.8) \quad \mathfrak{G}_w := \mathfrak{G}_u \mathfrak{G}_v.$$

Lemma 3.8. *For $w \in S_{\mathbb{Z}}$, we have*

$$(3.9) \quad \tilde{\omega}(\mathfrak{G}_w) = \mathfrak{G}_{\omega(w)} \quad \text{for } w \in S_{\neq 0}.$$

Proposition 3.9. *The set $\{\mathfrak{G}_w \mid w \in S_{\neq 0}\}$ is a \mathbb{Z} -basis of $R^+ = \mathbb{Z}[(1 - x_i) \mid i \in \mathbb{Z}_{>0}] \oplus \mathbb{Z}[(1 - x_i)^{-1} \mid i \in \mathbb{Z}_{\leq 0}]$.*

Proof. Follows by tensoring the bases in Propositions 3.5 and 3.7. \square

4. BACK STABLE GROTHENDIECK POLYNOMIALS

4.1. **Back stable rings.** Let

$$x_+ := (x_1, x_2, x_3, \dots), \quad x_- = x_{\leq 0} := (x_0, x_{-1}, x_{-2}, \dots).$$

Let $\Lambda = \Lambda(x_-)$ denote the ring of symmetric functions in the variables x_- with coefficients in \mathbb{Z} and let $\hat{\Lambda}$ denote the graded completion of Λ . That is, an element $f \in \hat{\Lambda}$ is a formal linear combination $f = f_1 + f_2 + \dots$ where $f_i \in \Lambda$ is homogeneous of degree i . Similarly, we define $\hat{\Lambda}(x_+)$ using positive variables. When no decorations are present, we assume negative variables are used: $\hat{\Lambda}$ means $\hat{\Lambda}(x_-)$.

Let $p_k, e_k, h_k \in \hat{\Lambda}$ denote the power sum, elementary, and homogeneous symmetric functions. We have isomorphisms $\hat{\Lambda} \cong \mathbb{Z}[[e_1, e_2, \dots]]$ and $\hat{\Lambda} \cong \mathbb{Z}[[h_1, h_2, \dots]]$. Despite the fact that $\hat{\Lambda}(x_-) \supsetneq \mathbb{Z}[[p_1, p_2, \dots]]$, since $\hat{\Lambda}(x_-) \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}[[p_1, p_2, \dots]]$ we define maps from $\hat{\Lambda}$ by describing the images of p_k . That is, we define \mathbb{Q} -algebra homomorphisms using the topological generators p_k and in all cases they restrict to ring homomorphisms over the integers. Define the back stable rings

$$\begin{aligned} \overleftarrow{R} &:= \hat{\Lambda}(x_-) \otimes R = R[[e_1, e_2, \dots]] \\ \overleftarrow{R}^+ &:= \hat{\Lambda}(x_-) \otimes R^+ = R^+[[e_1, e_2, \dots]]. \end{aligned}$$

The Dynkin shift automorphism $\gamma \in \text{Aut}(A_{\mathbb{Z}})$ from §2.1 induces the ring automorphism of \overleftarrow{R} defined by $\gamma(x_i) = x_{i+1}$ for $i \in \mathbb{Z}$ and $\gamma(p_k(x_-)) = p_k(x_-) + x_1^k$ for all $k \geq 1$.

We identify $\hat{\Lambda}(x_+)$ with $\hat{\Lambda}(x_-)$ by setting

$$(4.1) \quad p_k(x_+) = -p_k(x_-) \quad \text{for } k \geq 1.$$

This is consistent with the $S_{\mathbb{Z}}$ -action: the element $\sum_{i \in \mathbb{Z}} x_i^k$ is set to zero, and is $S_{\mathbb{Z}}$ -symmetric. Under (4.1) we have $e_n(x_+) = (-1)^n h_n(x_-)$ and $h_n(x_+) = (-1)^n e_n(x_-)$, thus preserving symmetric functions with integral coefficients.

4.2. Conjugation automorphism. Recall $\tilde{\omega} \in \text{Aut}(R)$ from §2.8. Since the p_k are algebraically independent topological generators of $\hat{\Lambda}$, we may extend $\tilde{\omega}$ to an automorphism of \overleftarrow{R} by

$$(4.2) \quad \tilde{\omega}(p_k) := (-1)^{k+1} \sum_{r \geq 0} \binom{r+k-1}{k-1} p_{k+r} \quad \text{for } k \geq 1.$$

Heuristically, we have $\tilde{\omega}(f(x)) = f(\tilde{\omega}(x)) = f(\ominus(x_{1-i}))$ for $f \in \hat{\Lambda}$. Using (4.1), this induces the map (4.2).

We call $\tilde{\omega}$ the (K -theoretic) *conjugation automorphism*. Note that $\tilde{\omega}$ differs from the automorphism ω defined in [LLS21], and restricts to an automorphism of $\hat{\Lambda}$ that differs from the usual conjugation automorphism ω of symmetric functions. The automorphisms $\tilde{\omega}$ and γ define an action of $\text{Aut}(A_{\mathbb{Z}})$ on \overleftarrow{R} .

Proposition 4.1. *The maps γ and $\tilde{\omega}$ define an action of $\text{Aut}(A_{\mathbb{Z}})$ on \overleftarrow{R} . That is,*

$$(4.3) \quad \tilde{\omega} \circ \gamma \circ \tilde{\omega} = \gamma^{-1}.$$

Proof. As this relation already holds on R by (2.32), it is enough to check the identity applied to p_k for $k \geq 1$. We have

$$\begin{aligned}
(\gamma \circ \tilde{\omega})(p_k) &= \gamma((-1)^{k+1} \sum_{r \geq 0} \binom{r+k-1}{k-1} p_{k+r}) \\
&= (-1)^{k+1} \left(\sum_{r \geq 0} \binom{r+k-1}{k-1} p_{k+r} + \sum_{r \geq 0} \binom{r+k-1}{k-1} x_1^{k+r} \right) \\
&= \tilde{\omega}(p_k) - \left(\frac{-x_1}{1-x_1} \right)^k \\
&= \tilde{\omega}(p_k - x_0^k) \\
&= \tilde{\omega} \circ \gamma^{-1}(p_k).
\end{aligned}$$

□

The conjugation automorphism $\tilde{\omega}$ restricts to an automorphism of \overleftarrow{R}^+ .

4.3. Group law negation automorphism. Denote by $\ominus : R \rightarrow R$ the involutive ring automorphism of R defined by $x_i \mapsto \ominus(x_i)$ for all i . We will write $f(\ominus(x))$ for $\ominus(f(x))$ for $f(x) \in R$. The map \ominus restricts to an isomorphism $R^+ \cong R^-$. It satisfies the operator identities

$$(4.4) \quad \ominus \circ w \circ \ominus = w$$

$$(4.5) \quad \ominus \circ \tilde{\pi}_i \circ \ominus = s_i \tilde{\pi}_i s_i =: \tilde{\pi}_i^\ominus = A_i(x_i - 1).$$

Lemma 4.2.

$$(4.6) \quad \tilde{\pi}_i^\ominus(\mathfrak{G}_w(\ominus x)) = \begin{cases} \mathfrak{G}_{ws_i}(\ominus x) & \text{if } ws_i < w \\ \mathfrak{G}_w(\ominus x) & \text{otherwise.} \end{cases}$$

We extend \ominus to an automorphism of \overleftarrow{R} , called the *group law negation automorphism*, by setting

$$\ominus(p_k) := -\tilde{\omega}(p_k) = (-1)^k \sum_{r \geq 0} \binom{r+k-1}{k-1} p_{k+r} \quad \text{for } k \geq 1.$$

4.4. Antipode automorphism. The *antipode* S is the involutive \mathbb{Q} -algebra automorphism on \overleftarrow{R} defined by $S(p_k) = -p_k$ for all $k \geq 1$ and $S(x_i) = x_{1-i}$ for $i \in \mathbb{Z}$. We have

$$(4.7) \quad \ominus = S \circ \tilde{\omega} = \tilde{\omega} \circ S.$$

Proposition 4.3. *We have $\ominus \circ \gamma = \gamma \circ \ominus$ and $S \circ \gamma^{-1} = \gamma \circ S$.*

4.5. Back stable Grothendieck polynomials. For $w \in S_{\mathbb{Z}}$ and an interval $[p, q] \subset \mathbb{Z}$ that contains all integers moved by w , let $\mathfrak{G}_w^{[p, q]}$ be the usual Grothendieck polynomial except computed using variables x_p, x_{p+1}, \dots, x_q instead of x_1, x_2, \dots . That is,

$$(4.8) \quad \mathfrak{G}_w^{[p, q]} := \gamma^{p-1} \mathfrak{G}_{\gamma^{1-p}(w)}$$

where γ denotes both $\gamma \in \text{Aut}(\overleftarrow{R})$ from §4.1 and $\gamma \in \text{Aut}(S_{\mathbb{Z}})$ from §2.2.

The *back stable Grothendieck polynomial* $\overleftarrow{\mathfrak{G}}_w \in \overleftarrow{R}$ is defined by

$$(4.9) \quad \overleftarrow{\mathfrak{G}}_w := \lim_{\substack{p \rightarrow -\infty \\ q \rightarrow \infty}} \mathfrak{G}_w^{[p,q]}.$$

It is immediate that we have

$$(4.10) \quad \tilde{\pi}_i \mathfrak{G}_w = \begin{cases} \mathfrak{G}_{ws_i} & \text{if } ws_i < w \\ \mathfrak{G}_w & \text{otherwise.} \end{cases}$$

Taking the limit of Proposition 3.4 gives the following formula in which the indices i_k are integers.

Proposition 4.4. *For $w \in S_{\mathbb{Z}}$, we have*

$$(4.11) \quad \overleftarrow{\mathfrak{G}}_w = \sum_{\substack{s_{a_1} * s_{a_2} * \dots * s_{a_L} = w \\ i_1 \leq i_2 \leq \dots \leq i_L \\ a_k \leq a_{k+1} \Rightarrow i_k < i_{k+1} \\ i_k \leq a_k}} (-1)^{L-\ell(w)} x_{i_1} x_{i_2} \cdots x_{i_L},$$

where L is arbitrary.

Example 4.5. Consider $\overleftarrow{\mathfrak{G}}_{s_0}$. For every $L \geq 1$, s_0 may be obtained as the Hecke product of L copies of s_0 . For this Hecke factorization the associated sequence (i_1, \dots, i_L) can be any sequence of integers with $i_1 < i_2 < \dots < i_L \leq 0$. Therefore, we have $\overleftarrow{\mathfrak{G}}_{s_0} = \sum_{L \geq 1} (-1)^{L-1} e_L$.

See Appendix B for more examples of back stable Grothendieck polynomials.

4.6. $\text{Aut}(A_{\mathbb{Z}})$ -action on back stable Grothendieck polynomials.

Proposition 4.6. *For $w \in S_{\mathbb{Z}}$, we have*

$$(4.12) \quad \overleftarrow{\mathfrak{G}}_{\gamma(w)} = \gamma(\overleftarrow{\mathfrak{G}}_w).$$

Proof. Follows from Proposition 4.4. \square

Proposition 4.7. *For $w \in S_{\mathbb{Z}}$, we have*

$$(4.13) \quad \overleftarrow{\mathfrak{G}}_{\omega(w)} = \tilde{\omega}(\overleftarrow{\mathfrak{G}}_w).$$

Proof. This holds by Proposition 5.23 and Proposition 5.21, using the fact that $\tilde{\omega}$ (defined in §5.6) and $\tilde{\omega}$ commute. \square

4.7. K -Stanley functions. There is a ring homomorphism $\eta_0 : \overleftarrow{R} \rightarrow \hat{\Lambda}$ given by $x_i \mapsto 0$ for all $i \in \mathbb{Z}$, and $p_k \mapsto p_k$ for all $k \geq 1$. This “sets all x_i to zero except those in $\hat{\Lambda}(x_-)$ ”.

Define the K -Stanley function $G_w \in \hat{\Lambda}(x_-)$ for $w \in S_{\mathbb{Z}}$ by

$$(4.14) \quad G_w := \eta_0(\overleftarrow{\mathfrak{G}}_w).$$

Remark 4.8. We call G_w the K -Stanley function because it is the K -theoretic analogue of a Stanley function in cohomology [LLS21]. For $w \in S_+$, the (forward-)stable Grothendieck polynomial of [FK] is the element $G_w(x_+) \in \hat{\Lambda}(x_+)$ defined by

$$G_w(x_+) := \lim_{n \rightarrow \infty} \mathfrak{G}_{\gamma^n(w)},$$

which by (3.6) is

$$(4.15) \quad G_w(x_+) = \sum_{\substack{s_{a_1} * s_{a_2} * \cdots * s_{a_p} = w \\ 1 \leq i_1 \leq i_2 \leq \cdots \leq i_p \\ a_k \leq a_{k+1} \Rightarrow i_k < i_{k+1}}} (-1)^{p-\ell(w)} x_{i_1} x_{i_2} \cdots x_{i_p}.$$

Note that this gives the same symmetric series as the definition in (4.14), only with variables x_+ rather than x_- .

In this work we use the name “ K -Stanley” in lieu of “stable Grothendieck” because the latter produces a conflict in the equivariant setting as there are three different versions of limiting double Grothendieck polynomial: (1) a back stable limit (the back stable double Grothendieck polynomial) which is not symmetric in the x variables, (2) a forward stable limit (the super K -Stanley function), which is supersymmetric, and (3) an equivariant analogue of (4.14) (the double K -Stanley functions).

Proposition 4.9. *We have*

$$(4.16) \quad \eta_0 \circ \gamma = \eta_0$$

$$(4.17) \quad G_{\gamma(w)} = G_w \quad \text{for all } w \in S_{\mathbb{Z}}.$$

Proof. As both sides are algebra maps one may check (4.16) on the topological algebra generators, which is straightforward. Equation (4.17) follows from equation (4.16). \square

Proposition 4.10. *For $w \in S_{\mathbb{Z}}$, we have*

$$(4.18) \quad G_w = \sum_{\substack{s_{a_1} * s_{a_2} * \cdots * s_{a_p} = w \\ i_1 \leq i_2 \leq \cdots \leq i_p \leq 0 \\ a_k \leq a_{k+1} \Rightarrow i_k < i_{k+1}}} (-1)^{p-\ell(w)} x_{i_1} x_{i_2} \cdots x_{i_p}.$$

Proof. By (4.17) we may assume that $w \in S_+$. Since $ws_i > w$ for $i \in \mathbb{Z}_{<0}$, $\overleftarrow{\mathfrak{G}}_w$ is S_- -symmetric. Hence may write $\overleftarrow{\mathfrak{G}}_w = \sum_{\alpha} f_{\alpha} x_+^{\alpha}$ where $f_{\alpha} \in \hat{\Lambda}(x_-)$ with only finitely many $f_{\alpha} \neq 0$. Applying η_0 to (4.11) we have $G_w = f_0$. But f_0 equals the right hand side of (4.18), the sum of monomials in (4.11) having only x_i with $i \leq 0$. \square

Proposition 4.11. *For $w \in S_{\mathbb{Z}}$, we have*

$$(4.19) \quad G_{\omega(w^{-1})} = G_w.$$

Proof. By (4.17) one may assume $w \in S_+$ so that $w \in S_n$ for some n . By [FK, Cor. 5.10] and the S_n -invariance of

$$G_w(x_1, \dots, x_n) := G_w(\dots, 0, 0, x_1, x_2, \dots, x_n, 0, 0, \dots),$$

we have

$$(4.20) \quad G_w(x_1, \dots, x_n) = \sum_{u*v=w} (-1)^{\ell(u)+\ell(v)-\ell(w)} \mathfrak{G}_u w_0(\mathfrak{G}_{w_0 v^{-1} w_0})$$

$$(4.21) \quad = \sum_{u*v=w} (-1)^{\ell(u)+\ell(v)-\ell(w)} w_0(\mathfrak{G}_u) \mathfrak{G}_{w_0 v^{-1} w_0}.$$

Since conjugation by w_0 is a length-preserving automorphism of S_n and taking inverses is a length-preserving anti-automorphism of S_n , $u*v = w$ if and only if

$(w_0 v^{-1} w_0) * (w_0 u^{-1} w_0) = w_0 w^{-1} w_0$. Applying (4.20) for $w_0 w^{-1} w_0$ and (4.21) for w we have

$$\begin{aligned} G_{w_0 w^{-1} w_0}(x_1, \dots, x_n) &= \sum_{u*v=w} (-1)^{\ell(u)+\ell(v)-\ell(w)} \mathfrak{G}_{w_0 v^{-1} w_0} w_0(\mathfrak{G}_u) \\ &= G_w(x_1, \dots, x_n). \end{aligned}$$

Letting $n \rightarrow \infty$ we deduce that $G_w = G_{w_0 w^{-1} w_0}$ in $\hat{\Lambda}$.

The Dynkin reversal on S_n given by conjugation by w_0 , sends s_i to s_{n-i} , whereas ω sends s_i to s_{-i} . Both are group homomorphisms, so $w_0 w^{-1} w_0 = \gamma^n \omega(w)$. The result follows by (4.17). \square

Proposition 4.12. *For all $w \in S_{\mathbb{Z}}$, we have*

$$(4.22) \quad \tilde{\omega}(G_w) = G_{\omega(w)}$$

$$(4.23) \quad SG_w(x) = G_{w^{-1}}(\ominus x).$$

Proof. Equation (4.22) follows from Proposition 4.7 and applying the specialization $x_i \mapsto 0$, which commutes with $\tilde{\omega}$. For (4.23), we calculate

$$SG_{w^{-1}}(\ominus x) = (S \circ \ominus) G_{w^{-1}} = \tilde{\omega} G_{w^{-1}} = G_{\omega(w^{-1})} = G_w$$

by (4.7), (4.22), and (4.19). Applying S yields (4.23). \square

4.8. Coproduct on symmetric functions. Let $\Delta : \hat{\Lambda} \rightarrow \hat{\Lambda} \otimes \hat{\Lambda}$ be the coproduct: $\Delta(p_k) = p_k \otimes 1 + 1 \otimes p_k$. We identify $\hat{\Lambda} \otimes \hat{\Lambda}$ with symmetric series in two sets of variables, one set for each tensor factor. If we use x_- for the first factor and a_- for the second then $\Delta(p_k) = p_k(x_-) + p_k(a_-)$.

Proposition 4.13. *For $w \in S_{\mathbb{Z}}$, we have*

$$(4.24) \quad \Delta(G_w) = \sum_{u*v=w} (-1)^{\ell(u)+\ell(v)-\ell(w)} G_u \otimes G_v.$$

Proof. Plug two sets of variables into (4.18). \square

4.9. Superization. For $f(x) \in \hat{\Lambda}$ let $f(x/a)$ denote the image of f in $\hat{\Lambda} \otimes \hat{\Lambda} = \hat{\Lambda}(x) \otimes \hat{\Lambda}(a)$ under superization, which is the coproduct Δ followed by $\text{id} \otimes S$ where the antipode S acts on the “a” variables. The map $f \mapsto f(x/a)$ is the unique \mathbb{Z} -algebra homomorphism sending $p_k(x_-)$ to $p_k(x/a) = p_k(x_-) - p_k(a_-)$, which for historical reasons we denote by $p_k(x||a)$. We call $G_w(x/a)$ the *super K-Stanley function*.

Proposition 4.14. *For $w \in S_{\mathbb{Z}}$, we have*

$$(4.25) \quad G_w(x/a) = \sum_{u*v=w} (-1)^{\ell(u)+\ell(v)-\ell(w)} G_{u^{-1}}(\ominus(a)) G_v(x)$$

$$(4.26) \quad = \sum_{u*v=w} (-1)^{\ell(u)+\ell(v)-\ell(w)} G_{v^{-1}}(\ominus(a)) G_u(x).$$

Proof. This holds by Propositions 4.12 and 4.13. \square

4.10. Coproduct formula. The coproduct Δ on $\hat{\Lambda}$ can be extended to a map $\Delta : R \rightarrow \hat{R}$ giving R the structure of a $\hat{\Lambda}$ -comodule, by defining $\Delta(x_i) = 1 \otimes x_i$ for $i \in \mathbb{Z}$. It restricts to a map $R^+ \rightarrow \hat{R}^+$ that makes R^+ into a $\hat{\Lambda}$ -comodule.

Theorem 4.15. *For $w \in S_{\mathbb{Z}}$, we have*

$$(4.27) \quad \Delta(\overleftarrow{\mathfrak{G}}_w) = \sum_{u*v=w} (-1)^{\ell(u)+\ell(v)-\ell(w)} G_u \otimes \overleftarrow{\mathfrak{G}}_v,$$

$$(4.28) \quad \overleftarrow{\mathfrak{G}}_w = \sum_{\substack{u*v=w \\ v \in S_{\neq 0}}} (-1)^{\ell(u)+\ell(v)-\ell(w)} G_u \mathfrak{G}_v.$$

Proof. We first derive (4.27) from (4.28). By Proposition 4.13, we have

$$\begin{aligned} \Delta(\overleftarrow{\mathfrak{G}}_w) &= \Delta \left(\sum_{\substack{u*v=w \\ v \in S_{\neq 0}}} (-1)^{\ell(u)+\ell(v)-\ell(w)} G_u \mathfrak{G}_v \right) \\ &= \sum_{\substack{u_1*u_2*v=w \\ v \in S_{\neq 0}}} (-1)^{\ell(u_1)+\ell(u_2)+\ell(v)-\ell(w)} G_{u_1} \otimes G_{u_2} \mathfrak{G}_v \\ &= \sum_{u_1*z=w} (-1)^{\ell(u_1)+\ell(z)-\ell(w)} G_{u_1} \otimes \overleftarrow{\mathfrak{G}}_z. \end{aligned}$$

The equality (4.28) can be deduced from [LLS+, Theorem 4.7] by taking a limit; see [LLS+, Section 6.2] for an explanation of this limit in the very similar cohomology setting. Presumably, (4.28) could also be deduced from a direct combinatorial argument similar to the proof of the coproduct formula in [LLS21]. \square

Example 4.16. We compute G_{s_1} and $G_{s_{-1}}$ using Theorem 4.15. By (4.17) we have $G_{s_1} = G_{s_{-1}} = G_{s_0}$.

$$\begin{aligned} \overleftarrow{\mathfrak{G}}_{s_1} &= G_{s_1} + \mathfrak{G}_{s_1} - G_{s_1} \mathfrak{G}_{s_1} = G_{s_0} \oplus x_1 \\ \overleftarrow{\mathfrak{G}}_{s_{-1}} &= G_{s_0} \ominus x_0. \end{aligned}$$

4.11. Grassmannian K -Stanley functions. We denote by $S_{\mathbb{Z}}^0$ the subset of Grassmannian elements, the set of $w \in S_{\mathbb{Z}}$ such that $ws_i > w$ for all $i \in \mathbb{Z} \setminus \{0\}$. Let \mathbb{Y} be Young's lattice of partitions. There is a bijection $\mathbb{Y} \rightarrow S_{\mathbb{Z}}^0$ denoted $\lambda \mapsto w_{\lambda}$ [LLS21]. Consider the tableau of shape λ in which the box (i, j) in the i -th row and j -th column is filled with the simple reflection s_{j-i} . Then $w_{\lambda} \in S_{\mathbb{Z}}^0$ is the element with reduced word given by reading the rows of this tableau from right to left, starting with the bottom row.

Example 4.17. For $\lambda = (3, 2)$ we have $w_{\lambda} = (s_0 s_{-1})(s_2 s_1 s_0)$ where the parentheses separate the reflections according to their row in the tableau.

s_0	s_1	s_2
s_{-1}	s_0	

Define the *Grassmannian K -Stanley function*

$$(4.29) \quad G_{\lambda} := G_{w_{\lambda}} \quad \text{for } \lambda \in \mathbb{Y}.$$

Some examples of G_{λ} are given in Appendix B.

Lemma 4.18. *For $\lambda \in \mathbb{Y}$, we have*

$$(4.30) \quad G_\lambda = \overleftarrow{\mathfrak{G}}_{w_\lambda}.$$

Proof. Since $w_\lambda \in S_{\mathbb{Z}}^0$, it follows that $\overleftarrow{\mathfrak{G}}_{w_\lambda}$ is $S_{\neq 0}$ -symmetric, that is, $\overleftarrow{\mathfrak{G}}_{w_\lambda} \in \hat{\Lambda}$. Thus $G_\lambda = G_{w_\lambda} = \eta_0(\overleftarrow{\mathfrak{G}}_{w_\lambda}) = \overleftarrow{\mathfrak{G}}_{w_\lambda}$, as required. \square

Proposition 4.19. *For $\lambda \in \mathbb{Y}$, we have*

$$(4.31) \quad \tilde{\omega}(G_\lambda) = G_{\lambda'}$$

Proof. We have $\omega(w_\lambda) = w_{\lambda'}$. This is a special case of Proposition 4.7 for w_λ . \square

The following is easily deduced from e.g. the Schur expansion of G_λ in [Len00, Thm. 2.2].

Proposition 4.20. *For all $r \geq 1$, we have*

$$(4.32) \quad G_r = \sum_{i \geq 0} (-1)^r s_{r,1^i}$$

$$(4.33) \quad G_{1^r} = \sum_{i \geq 0} (-1)^i \binom{i+r-1}{r-1} s_{1^{r+i}}.$$

5. BACK STABLE DOUBLE GROTHENDIECK POLYNOMIALS

Recall the Laurent polynomial rings $R(a)$ and $R = R(x)$ from §2.8. Define the $R(a)$ -algebras

$$(5.1) \quad R(x; a) = R(x) \otimes R(a)$$

$$(5.2) \quad R(x; a)^+ = R(x)^+ \otimes R(a).$$

For $w \in S_{\mathbb{Z}}$, let w^x (resp. w^a) denote the action of w on the x (resp. a) variables in $R(x; a)$. We use similar superscript notation for other operators.

5.1. Double Grothendieck polynomials. For $w \in S_n$, the *double Grothendieck polynomial* $\mathfrak{G}_w \in R(x; a)$ is defined by

$$(5.3) \quad \mathfrak{G}_{w_0}(x; a) = \prod_{i+j \leq n} (x_i \ominus a_j)$$

$$(5.4) \quad \mathfrak{G}_w(x; a) = \tilde{\pi}_i^x(\mathfrak{G}_{ws_i}(x; a)) \quad \text{if } ws_i > w.$$

Example 5.1. The double Grothendieck polynomials $\mathfrak{G}_w(x; a)$ for $w \in S_3$ are given below.

$$\begin{array}{ccc}
 & (x_1 \ominus a_1)(x_1 \ominus a_2)(x_2 \ominus a_1) & \\
 & \swarrow \tilde{\pi}_1 \quad \searrow \tilde{\pi}_2 & \\
 (x_1 \ominus a_1)(x_2 \ominus a_1) & & (x_1 \ominus a_1)(x_1 \ominus a_2) \\
 \downarrow \tilde{\pi}_2 & & \downarrow \tilde{\pi}_1 \\
 x_1 \ominus a_1 & & (x_1 \ominus a_1) \oplus (x_2 \ominus a_2) \\
 & \searrow \tilde{\pi}_1 \quad \swarrow \tilde{\pi}_2 & \\
 & 1 &
 \end{array}$$

Remark 5.2. The literature defines $\mathfrak{G}_w(x; a)$ using \oplus instead of \ominus . This has the convenient feature of avoiding denominators. However our convention is the most natural with respect to localization.

One may show that $\mathfrak{G}_w(x; a)$ is well-defined for $w \in S_+$.

Lemma 5.3. *For $w \in S_+$, we have*

$$(5.5) \quad \mathfrak{G}_w(x; 0) = \mathfrak{G}_w(x).$$

Proposition 5.4. *The set $\{\mathfrak{G}_w(x; a) \mid w \in S_+\}$ is a $R(a)$ -basis of $R(a) \otimes \mathbb{Z}[x_i \mid i \in \mathbb{Z}_{>0}]$.*

Proof. Follows by Lemma 5.3 and Proposition 3.9. \square

5.2. Negative double Grothendieck polynomials. The automorphism $\tilde{\omega}$ of $R(x)$ defined by (2.28) can be extended to a ring automorphism of $R(x; a)$ by letting $\tilde{\omega}(a_i) = \ominus(a_{1-i})$ for all $i \in \mathbb{Z}$.

Recall that ω acts on $S_{\mathbb{Z}}$ by (2.7). For $w \in S_-$, define the *negative double Grothendieck polynomials* by

$$\mathfrak{G}_w(x; a) := \tilde{\omega}(\mathfrak{G}_{\omega(w)}(x; a)).$$

Example 5.5. We have $\mathfrak{G}_{s_1}(x; a) = x_1 \ominus a_1$. Thus

$$\mathfrak{G}_{s_{-1}}(x; a) = \tilde{\omega}(\mathfrak{G}_{s_1}) = \tilde{\omega}(x_1 \ominus a_1) = (\ominus x_0) \ominus (\ominus a_0) = a_0 \ominus x_0.$$

Proposition 5.6. *The set $\{\mathfrak{G}_w(x; a) \mid w \in S_-\}$ is a $R(a)$ -basis of $R(a)[(1 - x_i)^{-1} \mid i \in \mathbb{Z}_{\leq 0}]$.*

Proof. This follows from the fact that $\tilde{\omega}$ restricts to a $R(a)$ -algebra isomorphism

$$R(a)[x_i \mid i \in \mathbb{Z}_{>0}] \rightarrow R(a)[(1 - x_i)^{-1} \mid i \in \mathbb{Z}_{\leq 0}]$$

$$\mathfrak{G}_w(x; a) \mapsto \mathfrak{G}_{\omega(w)}(x; a) \quad \text{for } w \in S_+.$$

\square

For $w \in S_{\neq 0}$, write $w = uv$ with $u \in S_+$ and $v \in S_-$. Define

$$\mathfrak{G}_w(x; a) := \mathfrak{G}_u(x; a)\mathfrak{G}_v(x; a) \quad \text{for } w \in S_{\neq 0}.$$

Proposition 5.7. *The set $\{\mathfrak{G}_w(x; a) \mid w \in S_{\neq 0}\}$ is a $R(a)$ -basis of $R(x; a)^+$.*

Proof. Follows immediately from Propositions 5.4 and 5.6. \square

Proposition 5.8. *For $w \in S_{\neq 0}$, we have*

$$(5.6) \quad \mathfrak{G}_w(x; a) = \sum_{u*v=w} (-1)^{\ell(u)+\ell(v)-\ell(w)} \mathfrak{G}_{u^{-1}}(\ominus a) \mathfrak{G}_v(x),$$

$$(5.7) \quad \mathfrak{G}_w(x) = \sum_{u*v=w} (-1)^{\ell(u)+\ell(v)-\ell(w)} \mathfrak{G}_u(a) \mathfrak{G}_v(x; a).$$

Proof. For (5.6), by factoring and applying $\tilde{\omega}$ one may reduce to the case that $w \in S_+$. Then $w \in S_n$ for some n , in which case (5.6) holds by [FK, Prop. 3.2, Lemma 5.5]. For (5.7) apply Proposition A.1. \square

Example 5.9. We have $\mathfrak{G}_{s_1}(x; a) = x_1 \ominus a_1$. Using (5.6) with the Hecke factorizations $u*v = s_1$ given by (u, v) equal to (s_1, id) , (id, s_1) , and (s_1, s_1) we have

$$\begin{aligned} \mathfrak{G}_{s_1}(x; a) &= \mathfrak{G}_{s_1}(\ominus a) \mathfrak{G}_{\text{id}}(x) + \mathfrak{G}_{\text{id}}(\ominus(a)) \mathfrak{G}_{s_1}(x) - \mathfrak{G}_{s_1}(\ominus a) \mathfrak{G}_{s_1}(x) \\ &= \mathfrak{G}_{s_1}(\ominus a) \oplus \mathfrak{G}_{s_1}(x) = \ominus(a_1) \oplus x_1 = x_1 \ominus a_1. \end{aligned}$$

Proposition 5.10. *For $w \in S_{\neq 0}$, we have*

$$(5.8) \quad \mathfrak{G}_{w^{-1}}(x; a) = \mathfrak{G}_w(\ominus a; \ominus x).$$

Proof. This follows from (5.6) using that $u*v = w$ if and only if $v^{-1} * u^{-1} = w^{-1}$. \square

Proposition 5.11. *We have*

$$(5.9) \quad \tilde{\pi}_i^{a, \ominus} \mathfrak{G}_w(x; a) = \begin{cases} \mathfrak{G}_{s_i w}(x; a) & \text{if } s_i w < w \\ \mathfrak{G}_w(x; a) & \text{otherwise.} \end{cases}$$

Proof. Follows by Proposition 5.10 and Lemma 4.2. \square

5.3. Double symmetric function ring. Let $\mathbb{S} = \mathbb{Z}[[a_i \mid i \in \mathbb{Z}]]$ be the ring of formal power series in variables a_i for $i \in \mathbb{Z}$ with coefficients in \mathbb{Z} . Let $\hat{\Lambda}(x||a) := \hat{\Lambda}(x/a) \otimes_{\mathbb{Z}} \mathbb{S}$ be the \mathbb{S} -algebra of formal power series in $e_k(x/a)$ with coefficients in \mathbb{S} . The ring $\hat{\Lambda}(x||a)$ is a Hopf algebra over \mathbb{S} with coproduct such that the elements $p_k(x||a)$ are primitive, counit $\epsilon_a : \hat{\Lambda}(x||a) \rightarrow \mathbb{S}$ given by $\epsilon_a(p_k(x||a)) = 0$ for $k \geq 1$, and antipode $S(p_k(x||a)) = -p_k(x||a)$ for all $k \geq 1$.

5.4. Double back stable rings. Define the \mathbb{S} -algebras, called *double back stable rings*,

$$(5.10) \quad \overleftarrow{R}(x; a) = \hat{\Lambda}(x||a) \otimes_{R(a)} R(x; a)$$

$$(5.11) \quad \overleftarrow{R}(x; a)^+ = \hat{\Lambda}(x||a) \otimes_{R(a)} R(x; a)^+.$$

The infinite symmetric group $S_{\mathbb{Z}}$ has two commuting actions on $\overleftarrow{R}(x; a)$, one acting on x variables and the other on a variables, including “the variables in $\hat{\Lambda}(x||a)$ ”, where $p_k(x||a)$ is as in §4.8:

$$\begin{aligned} s_i^x(p_k(x||a)) &= p_k(x||a) + \delta_{i0}(x_1^k - x_0^k) \\ s_i^a(p_k(x||a)) &= p_k(x||a) + \delta_{i0}(a_0^k - a_1^k). \end{aligned}$$

The algebra $\hat{\Lambda}(x||a)$ is the \mathbb{S} -subalgebra of $S_{\neq 0}^x$ -invariants in $\overleftarrow{R}(x; a)$.

Remark 5.12. Since formal series in $p_k(x||a)$ are allowed, in order to admit an $S_{\mathbb{Z}}$ -action, series in the a_i must also be allowed. This is why \mathbb{S} is used for the coefficient ring rather than $R(a)$.

5.5. $\text{Aut}(A_{\mathbb{Z}})$ -action on double back stable rings. The group $\text{Aut}(A_{\mathbb{Z}})$ acts on $\overleftarrow{R}(x; a)$ by ring automorphisms. Define $\gamma : \overleftarrow{R}(x; a) \rightarrow \overleftarrow{R}(x; a)$ by

$$\gamma(x_i) = x_{i+1}, \quad \gamma(a_i) = a_{i+1}, \quad \gamma(p_k(x||a)) = p_k(x||a) + x_1^k - a_1^k.$$

Let $\tilde{\omega}$ be the ring automorphism of $\overleftarrow{R}(x; a)$ extending the automorphism $\tilde{\omega}$ of $R(x; a)$ in §5.2 by

$$(5.12) \quad \tilde{\omega}(p_k(x||a)) = (-1)^{k+1} \sum_{r \geq 0} \binom{r+k-1}{k-1} p_{k+r}(x||a) \quad \text{for all } k \geq 1.$$

This is consistent with the definition of $\tilde{\omega}(p_k(x_-))$ and the parallel definition of $\tilde{\omega}(p_k(a_-))$ in §4.1 (using the convention $p_k(a_+) = -p_k(a_-)$).

Proposition 5.13. *The maps γ and $\tilde{\omega}$ define an action of $\text{Aut}(A_{\mathbb{Z}})$ on $\overleftarrow{R}(x; a)$. That is,*

$$(5.13) \quad \tilde{\omega} \circ \gamma \circ \tilde{\omega} = \gamma^{-1}.$$

Proof. This can be readily verified on the generators. \square

Proposition 5.14. *For all $w \in S_{\mathbb{Z}}$, we have*

$$(5.14) \quad \tilde{\omega}(G_w(x/a)) = G_{\omega(w)}(x/a).$$

Proof. Follows from Proposition 5.23 and Lemma 5.20. \square

5.6. Back stable double Grothendieck polynomials. Given $w \in S_{\mathbb{Z}}$, let $[p, q] \subset \mathbb{Z}$ be an interval that contains all elements of \mathbb{Z} moved by w . Define $\mathfrak{G}_w^{[p,q]}(x; a) \in R(x; a)$ by

$$\mathfrak{G}_w^{[p,q]}(x; a) = \gamma^{p-1}(\mathfrak{G}_{\gamma^{1-p}(w)}(x; a)).$$

Define the *back stable double Grothendieck polynomial* $\overleftarrow{\mathfrak{G}}_w(x; a)$ by

$$\overleftarrow{\mathfrak{G}}_w(x; a) = \lim_{\substack{p \rightarrow -\infty \\ q \rightarrow \infty}} \mathfrak{G}_w^{[p,q]}(x; a).$$

It is immediate that we have

$$(5.15) \quad \tilde{\pi}_i^x \mathfrak{G}_w(x; a) = \begin{cases} \mathfrak{G}_{ws_i}(x; a) & \text{if } ws_i < w \\ \mathfrak{G}_w(x; a) & \text{otherwise.} \end{cases}$$

Proposition 5.15. *We have*

$$(5.16) \quad \tilde{\pi}_i^{a, \ominus}(\overleftarrow{\mathfrak{G}}_w(x; a)) = \begin{cases} \overleftarrow{\mathfrak{G}}_{s_i w}(x; a) & \text{if } s_i w < w \\ \mathfrak{G}_w(x; a) & \text{otherwise.} \end{cases}$$

Proof. Follows from Proposition 5.11. \square

Proposition 5.16. *For $w \in S_{\mathbb{Z}}$, we have*

$$(5.17) \quad \overleftarrow{\mathfrak{G}}_w(x; a) = \sum_{u*v=w} (-1)^{\ell(u)+\ell(v)-\ell(w)} \overleftarrow{\mathfrak{G}}_{u^{-1}}(\ominus(a)) \overleftarrow{\mathfrak{G}}_v(x).$$

Proof. This holds by Proposition 5.8 and the definitions. \square

Proposition 5.17. *For $w \in S_{\mathbb{Z}}$, we have*

$$(5.18) \quad \overleftarrow{\mathfrak{G}}_w(x; a) = \sum_{\substack{u*v*z=w \\ u, z \in S_{\neq 0}}} (-1)^{\ell(u)+\ell(v)+\ell(z)-\ell(w)} \mathfrak{G}_{u^{-1}}(\ominus a) G_v(x/a) \mathfrak{G}_z(x).$$

Proof. Let $w = u * v$, $u^{-1} = u_1 * v_1$ and $v = u_2 * v_2$ with $v_1, v_2 \in S_{\neq 0}$, so that $w = v_1^{-1} * u_1^{-1} * u_2 * v_2$. By Propositions 5.16, 4.15, and 4.14 we have

$$\begin{aligned} \overleftarrow{\mathfrak{G}}_w(x; a) &= \sum_{u * v = w} (-1)^{\ell(u) + \ell(v) - \ell(w)} \overleftarrow{\mathfrak{G}}_{u^{-1}}(\ominus(a)) \overleftarrow{\mathfrak{G}}_v(x) \\ &= \sum_{\substack{v_1^{-1} * u_1^{-1} * u_2 * v_2 = w \\ v_1, v_2 \in S_{\neq 0}}} (-1)^{\ell(u_1) + \ell(v_1) + \ell(u_2) + \ell(v_2) - \ell(w)} \\ &\quad \mathfrak{G}_{v_1}(\ominus(a)) G_{u_1}(\ominus(a)) G_{u_2}(x) \mathfrak{G}_{v_2}(x) \\ &= \sum_{\substack{v_1^{-1} * y * v_2 = w \\ v_1, v_2 \in S_{\neq 0}}} (-1)^{\ell(v_1) + \ell(y) + \ell(v_2) - \ell(w)} \mathfrak{G}_{v_1}(\ominus(a)) G_y(x/a) \mathfrak{G}_{v_2}(x). \quad \square \end{aligned}$$

Example 5.18. We have

$$\begin{aligned} \overleftarrow{\mathfrak{G}}_{s_0}(x; a) &= G_{s_0}(x/a) = G_1(x/a) \\ \overleftarrow{\mathfrak{G}}_{s_1}(x; a) &= \mathfrak{G}_{s_1}(\ominus a) + G_{s_1}(x/a) + \mathfrak{G}_{s_1}(x) \\ &\quad - \mathfrak{G}_{s_1}(\ominus a) G_{s_1}(x/a) - \mathfrak{G}_{s_1}(\ominus a) \mathfrak{G}_{s_1}(x) - G_{s_1}(x/a) \mathfrak{G}_{s_1}(x) \\ &\quad + \mathfrak{G}_{s_1}(\ominus a) G_{s_1}(x/a) \mathfrak{G}_{s_1}(x) \\ &= \mathfrak{G}_{s_1}(\ominus a) \oplus G_1(x/a) \oplus \mathfrak{G}_{s_1}(x) \\ &= \ominus(a_1) \oplus G_1(x/a) \oplus x_1 = G_1(x/a) \oplus \mathfrak{G}_{s_1}(x; a). \end{aligned}$$

Proposition 5.19. For $w \in S_{\mathbb{Z}}$, we have

$$(5.19) \quad \overleftarrow{\mathfrak{G}}_{w^{-1}}(x; a) = \overleftarrow{\mathfrak{G}}_w(\ominus a; \ominus x).$$

Proof. This holds by Proposition 5.16 and the fact that $u * v = w$ if and only if $v^{-1} * u^{-1} = w^{-1}$. \square

Lemma 5.20. For all $w \in S_{\mathbb{Z}}$, we have

$$(5.20) \quad G_w(x/a) = \overleftarrow{\mathfrak{G}}_w(x; a)|_{x_i \mapsto 0, a_i \mapsto 0}.$$

Proof. This follows by Proposition 5.17 and the fact that $\mathfrak{G}_z(x)|_{x_i \mapsto 0} = 0$ for $z \neq \text{id}$. \square

Define the forgetful ring homomorphism $\text{For} : \overleftarrow{R}(x; a) \rightarrow \overleftarrow{R}$ by

$$\begin{aligned} \text{For}(p_k(x|a)) &= p_k(x_-) \\ \text{For}(x_i) &= x_i \\ \text{For}(a_i) &= 0. \end{aligned}$$

Proposition 5.21. For all $w \in S_{\mathbb{Z}}$, $\text{For}(\overleftarrow{\mathfrak{G}}_w(x; a)) = \overleftarrow{\mathfrak{G}}_w(x)$.

Proof. Follows from Propositions 5.17 and 4.15. \square

5.7. $\text{Aut}(A_{\mathbb{Z}})$ -action on back stable double Grothendieck polynomials. The group $\text{Aut}(A_{\mathbb{Z}})$ of automorphisms of the Dynkin diagram $A_{\mathbb{Z}}$, permutes the back stable double Grothendieck polynomials. By Theorem 5.25 and Proposition 2.7 the following hold.

Proposition 5.22. For $w \in S_{\mathbb{Z}}$, we have

$$(5.21) \quad \overleftarrow{\mathfrak{G}}_{\gamma(w)}(x; a) = \gamma(\overleftarrow{\mathfrak{G}}_w(x; a)).$$

Proposition 5.23. *For $w \in S_{\mathbb{Z}}$, we have*

$$(5.22) \quad \overleftarrow{\mathfrak{G}}_{\omega(w)}(x; a) = \tilde{\omega}(\overleftarrow{\mathfrak{G}}_w(x; a)).$$

5.8. Back stable double Grothendieck polynomials are the equivariant Schubert basis. For $f \in \overleftarrow{R}(x; a)$ and $w \in S_{\mathbb{Z}}$ define

$$f|_w = f(wa; a) = \epsilon_a(w^x(f(x; a))).$$

The map $f \mapsto f|_w$ is an \mathbb{S} -algebra homomorphism $\overleftarrow{R}(x; a) \rightarrow \mathbb{S}$.

Proposition 5.24.

- (1) *For $w, y \in S_+$, $\overleftarrow{\mathfrak{G}}_w(x; a)|_y = \mathfrak{G}_w(x; a)|_y$.*
- (2) *For $w, y \in S_{\mathbb{Z}}$ let $M \in \mathbb{Z}_{\geq 0}$ be such that $\gamma^M(w), \gamma^M(y) \in S_+$. Then*

$$\overleftarrow{\mathfrak{G}}_w(x; a)|_y = \gamma^{-M} \mathfrak{G}_{\gamma^M(w)}(x; a)|_{\gamma^M(y)}.$$

Proof. For (1) consider (5.18). Since $y \in S_+$, $G_v(x/a)$ is invariant under y^a and vanishes at $x = ya$ unless $v = \text{id}$. Therefore by (5.6) we have

$$\begin{aligned} \overleftarrow{\mathfrak{G}}_w(x; a)|_y &= \sum_{\substack{u \ast z = y \\ u, z \in S_{\neq 0}}} (-1)^{\ell(u) + \ell(z) - \ell(w)} \mathfrak{G}_{u^{-1}}(\ominus a) \mathfrak{G}_z(ya) \\ &= \mathfrak{G}_w(x; a)|_y. \end{aligned}$$

Part (2) follows from part (1) and (5.21). \square

Theorem 5.25. *There is a \mathbb{S} -algebra and $S_{\mathbb{Z}}$ -equivariant embedding $\text{res} : \overleftarrow{R}(x; a) \rightarrow \mathbb{S} \otimes_{R(a)} \tilde{\Psi}$ defined by $\text{res}(f)(w) = f|_w$ for all $w \in S_{\mathbb{Z}}$. Moreover, $\text{res}(\overleftarrow{\mathfrak{G}}_v) = \tilde{\psi}^v$ for all $v \in S_{\mathbb{Z}}$.*

Proof. We observe that the values of $f|_w$ are in \mathbb{S} . Let $f \in \overleftarrow{R}(x; a)$. $w^x(f) - (s_{ij}w)^x(f)$ is s_{ij}^x -antisymmetric and therefore divisible by $x_i - x_j$. It follows that $f|_w - f|_{s_{ij}w}$ is a multiple of $a_i - a_j$. Thus $\text{res}(f) \in \mathbb{S} \otimes_{R(a)} \tilde{\Psi}$ by (2.33).

Next we show that $\text{res}(\overleftarrow{\mathfrak{G}}_v(x; a)) = \tilde{\psi}^v$ for all $v \in S_{\mathbb{Z}}$. By Proposition 2.4, this is equivalent to showing that

$$(5.23) \quad \overleftarrow{\mathfrak{G}}_v(a; a) = \delta_{\text{id}, v} \quad \text{for } v \in S_{\mathbb{Z}}$$

and if $ws_i < w$ then

$$(5.24) \quad \begin{aligned} \overleftarrow{\mathfrak{G}}_v(wa; a) &= \\ &\begin{cases} \overleftarrow{\mathfrak{G}}_v(ws_i a; a) & \text{if } vs_i > v \\ \frac{1 - a_{w(i)}}{1 - a_{w(i+1)}} \overleftarrow{\mathfrak{G}}_v(ws_i a; a) + \frac{a_{w(i)} - a_{w(i+1)}}{1 - a_{w(i+1)}} \overleftarrow{\mathfrak{G}}_{vs_i}(wa; a) & \text{if } vs_i < v. \end{cases} \end{aligned}$$

Equation (5.23) follows by Proposition 5.24 and (A.11).

Suppose $w \neq \text{id}$. Let $i \in I$ be such that $ws_i < w$. Suppose first that $vs_i > v$. Then $\overleftarrow{\mathfrak{G}}_v(x; a) = \tilde{\pi}_i^x \overleftarrow{\mathfrak{G}}_v(x; a)$. Since the image of $\tilde{\pi}_i^x$ is s_i^x -invariant, we deduce that $\overleftarrow{\mathfrak{G}}_v(wa; a) = \overleftarrow{\mathfrak{G}}_v(ws_i a; a)$ as required.

Suppose $vs_i < v$. Then $\overleftarrow{\mathfrak{G}}_{vs_i}(x; a) = \tilde{\pi}_i^x \overleftarrow{\mathfrak{G}}_v(x; a)$. This yields

$$\overleftarrow{\mathfrak{G}}_{vs_i}(wa; a) = \frac{(1 - a_{w(i+1)}) \overleftarrow{\mathfrak{G}}_v(wa; a) - (1 - a_{w(i)}) \overleftarrow{\mathfrak{G}}_v(ws_i a; a)}{a_{w(i)} - a_{w(i+1)}}$$

which rearranges to (5.24), as required. \square

Corollary 5.26. *The back stable Grothendieck polynomials $\overleftarrow{\mathfrak{G}}_w$ represent the Schubert classes in the K -theory $K(\overline{\mathrm{Fl}})$. The back stable double Grothendieck polynomials $\overleftarrow{\mathfrak{G}}_w(x; a)$ represent the Schubert classes in the torus-equivariant K -theory $K_T(\overline{\mathrm{Fl}})$.*

5.9. Double K -Stanley functions. Let $\eta_a : \overleftarrow{R}(x; a) \rightarrow \hat{\Lambda}(x||a)$ be the \mathbb{S} -algebra homomorphism sending $x_i \mapsto a_i$ and $p_r(x||a) \mapsto p_r(x||a)$. Define the *double K -Stanley function* $G_w(x||a) \in \hat{\Lambda}(x||a)$ by

$$(5.25) \quad G_w(x||a) := \eta_a(\overleftarrow{\mathfrak{G}}_w(x; a)).$$

Proposition 5.27. *For $w \in S_{\mathbb{Z}}$, we have*

$$(5.26) \quad G_w(x||a) = \sum_{\substack{u*v=z=w \\ u, z \in S_{\neq 0}}} (-1)^{\ell(u)+\ell(v)+\ell(z)-\ell(w)} \mathfrak{G}_{u^{-1}}(\ominus(a)) G_v(x/a) \mathfrak{G}_z(a).$$

Proof. This follows immediately from Proposition 5.17. \square

Example 5.28. Since $G_{s_0 s_1} = G_{s_{-1} s_0} = G_{11}$ and $G_{s_0} = G_1$ we have

$$\begin{aligned} G_{s_0 s_1}(x||a) &= G_{s_0 s_1}(x/a) + G_{s_0}(x/a) \mathfrak{G}_{s_1}(a) - G_{s_0 s_1}(x/a) \mathfrak{G}_{s_1}(a) \\ &= G_{s_0 s_1}(x/a) + G_{s_0}(x/a) a_1 - G_{s_0 s_1}(x/a) a_1 \\ &= (1 - a_1) G_{11}(x/a) + a_1 G_1(x/a). \end{aligned}$$

Since $\mathfrak{G}_{s_{-1}}(\ominus a) = a_0$ we have

$$\begin{aligned} G_{s_{-1} s_0}(x||a) &= \mathfrak{G}_{s_{-1}}(\ominus a) G_{s_0}(x/a) + G_{s_{-1} s_0}(x/a) - \mathfrak{G}_{s_{-1}}(\ominus a) G_{s_{-1} s_0}(x/a) \\ &= (1 - a_0) G_{11}(x/a) + a_0 G_1(x/a). \end{aligned}$$

Proposition 5.29. *For $w \in S_{\mathbb{Z}}$, we have*

$$\tilde{\omega}(G_w(x||a)) = G_{\omega(w)}(x||a).$$

Proof. It is straightforward to check that $\eta_a \circ \tilde{\omega} = \tilde{\omega}$. Thus (5.27) follows from Proposition 5.22. \square

Proposition 5.30. *For all $w \in S_{\mathbb{Z}}$, we have*

$$(5.27) \quad G_{w^{-1}}(x||a) = G_w(x||a)|_{x \mapsto \ominus a, a \mapsto \ominus x}.$$

Proof. The transformation $x \mapsto \ominus a, a \mapsto \ominus x$ commutes with η_a . The result follows from (5.19). \square

5.10. Coproduct formula. The Hopf algebra structure on $\hat{\Lambda}(x||a)$ is defined by letting $p_k(x||a)$ be primitive for all $k \geq 1$. We give $\overleftarrow{R}(x; a)$ the structure of a $\hat{\Lambda}(x||a)$ -comodule by letting Δ act on the tensor factor $\hat{\Lambda}(x||a)$ in $\overleftarrow{R}(x; a)$.

Theorem 5.31. *For $w \in S_{\mathbb{Z}}$, we have*

$$(5.28) \quad \Delta(\overleftarrow{\mathfrak{G}}_w(x; a)) = \sum_{u*v=w} (-1)^{\ell(u)+\ell(v)-\ell(w)} G_u(x||a) \otimes \overleftarrow{\mathfrak{G}}_v(x; a)$$

$$(5.29) \quad \overleftarrow{\mathfrak{G}}_w(x; a) = \sum_{\substack{u*v=w \\ v \in S_{\neq 0}}} (-1)^{\ell(u)+\ell(v)-\ell(w)} G_u(x||a) \mathfrak{G}_v(x; a).$$

Proof. Equation (5.29) follows from Propositions 5.17, 5.27, and (5.7).

For (5.28), using Propositions 5.17, 5.27, and A.2 we have

$$\begin{aligned}
& \sum_{u*v=w} \pm G_u(x||a) \otimes \overleftarrow{\mathfrak{G}}_v(x; a) \\
&= \sum_{\substack{u_1*v_1*z_1*u_2*v_2*z_2=w \\ u_i, z_i \in S_{\neq 0}}} \pm \mathfrak{G}_{u_1^{-1}}(\ominus(a)) G_{v_1}(x/a) G_{z_1}(a) \\
&\quad \otimes \mathfrak{G}_{u_2}^{-1}(\ominus(a)) G_{v_2}(x/a) G_{z_2}(x) \\
&= \sum_{\substack{u_1*v_1*v_2*z_2=w \\ u_1, z_2 \in S_{\neq 0}}} \pm \mathfrak{G}_{u_1^{-1}}(\ominus(a)) G_{v_1}(x/a) \\
&\quad \otimes G_{v_2}(x/a) G_{z_2}(x) \\
&= \sum_{\substack{u_1*v*z_2=w \\ u_1, z_2 \in S_{\neq 0}}} \pm \mathfrak{G}_{u_1^{-1}}(\ominus(a)) \Delta(G_v(x/a)) \mathfrak{G}_{z_2}(x) \\
&= \Delta(\overleftarrow{\mathfrak{G}}_w(x; a)). \quad \square
\end{aligned}$$

5.11. Grassmannian double K -Stanley functions. Recalling w_λ from §4.11, define

$$(5.30) \quad G_\lambda(x||a) := G_{w_\lambda}(x; a) \quad \text{for } \lambda \in \mathbb{Y}.$$

Lemma 5.32. *For $\lambda \in \mathbb{Y}$, we have*

$$(5.31) \quad G_\lambda(x||a) = \overleftarrow{\mathfrak{G}}_{w_\lambda}(x; a).$$

Proof. For any Hecke factorization $w_\lambda = u * v * z$ with $u, z \in S_{\neq 0}$, we have $z = \text{id}$. The result then follows from the definition (5.25) by comparing (5.18) and (5.26). \square

Corollary 5.33. *The Grassmannian K -Stanley functions $\{G_\lambda \mid \lambda \in \mathbb{Y}\}$ represent the basis of structure sheaves of opposite Schubert varieties in the K -theory $K(\overline{\text{Gr}}^0)$. The Grassmannian double K -Stanley functions $\{G_\lambda(x||a) \mid \lambda \in \mathbb{Y}\}$ represent the structure sheaves of opposite Schubert varieties in the torus-equivariant K -theory $K_T(\overline{\text{Gr}}^0)$.*

Proof. Since $K_T(\overline{\text{Gr}}^0) = (K_T(\overline{\text{Fl}}))^{S_{\neq 0}^x}$, the statement about $G_\lambda(x||a)$ follows from Corollary 5.26, Lemma 5.32, and the fact that the $G_\lambda(x||a)$ are precisely the $S_{\neq 0}^x$ -invariant Schubert basis elements for $K_T(\overline{\text{Fl}})$. Applying the forgetful homomorphism $K_T(\overline{\text{Gr}}) \rightarrow K(\overline{\text{Gr}})$, the equivariant Schubert basis is sent to the Schubert basis. The proof is completed by applying Proposition 5.21. \square

Proposition 5.34. *For $\lambda \in \mathbb{Y}$, we have*

$$\tilde{\omega}(G_\lambda(x||a)) = G_{\lambda'}(x||a).$$

Proof. Follows from Proposition 5.29 and the fact that $\omega(w_\lambda) = w_{\lambda'}$. \square

A rook strip is a skew shape ν/μ which has at most one box in each row and in each column. Write $\nu/\mu \in \mathcal{RS}$ if ν/μ is a rook strip. The Durfee square of λ is largest square partition contained in λ ; denote its side length by $d(\lambda)$.

If $\mu \subset \lambda$, we define

$$(5.32) \quad w_{\lambda/\mu} := w_\lambda w_\mu^{-1}.$$

Lemma 5.35. *Let $\mu \subset \lambda \in \mathbb{Y}$ and $u \in S_{\neq 0}$. Then $u * w_\mu = w_\lambda$ if and only if $u = w_{\lambda/\nu}$ where $\nu \subset \mu$, $d(\nu) = d(\lambda)$, and $\mu/\nu \in \mathcal{RS}$.*

The coproduct formula gives the transition matrix between the Grassmannian double K -Stanley functions and the super K -Stanley symmetric functions.

Proposition 5.36. *For $\lambda \in \mathbb{Y}$, we have*

$$(5.33) \quad G_\lambda(x||a) = \sum_{\substack{\nu \subset \mu \subset \lambda \\ d(\nu) = d(\lambda) \\ \mu/\nu \in \mathcal{RS}}} (-1)^{|\mu|-|\nu|} \mathfrak{G}_{w_{\lambda/\nu}^{-1}}(\ominus a) G_\mu(x/a)$$

$$(5.34) \quad G_\lambda(x/a) = \sum_{\substack{\nu \subset \mu \subset \lambda \\ d(\nu) = d(\lambda) \\ \mu/\nu \in \mathcal{RS}}} (-1)^{|\mu|-|\nu|} \mathfrak{G}_{w_{\lambda/\nu}}(a) G_\mu(x||a).$$

Proof. Equation (5.34) follows from (5.33) using Proposition A.1. Equation (5.33) is an instance of Proposition 5.17. In this application $z = \text{id}$ since $w_\lambda \in S_{\mathbb{Z}}^0$. The Proposition follows from Lemma 5.35. \square

In §9, we give determinantal formulae for $G_\lambda(x||a)$.

Example 5.37. By Proposition 5.36 and using that $\mathfrak{G}_{s_1}(a) = a_1$, $\mathfrak{G}_{s_{-1}}(a) = \ominus(a_0)$, and $\mathfrak{G}_{s_1 s_{-1}}(a) = a_1(\ominus(a_0))$, we have

$$\begin{aligned} G_2(x/a) &= (1 - \mathfrak{G}_{s_1}(a)) G_2(x||a) + \mathfrak{G}_{s_1}(a) G_1(x||a) \\ &= (1 - a_1) G_2(x||a) + a_1 G_1(x||a) \\ G_{11}(x/a) &= (1 - \mathfrak{G}_{s_{-1}}(a)) G_{11}(x||a) + \mathfrak{G}_{s_{-1}}(a) G_1(x||a) \\ &= (1 - \ominus(a_0)) G_{11}(x||a) + \ominus(a_0) G_1(x||a) \\ G_{21}(x/a) &= (1 - \mathfrak{G}_{s_1}(a) - \mathfrak{G}_{s_{-1}}(a) + \mathfrak{G}_{s_1 s_{-1}}(a)) G_{21}(x||a) \\ &\quad + (\mathfrak{G}_{s_{-1}}(a) - \mathfrak{G}_{s_1 s_{-1}}(a)) G_2(x||a) + (\mathfrak{G}_{s_1}(a) - \mathfrak{G}_{s_1 s_{-1}}(a)) G_{11}(x||a) \\ &\quad + \mathfrak{G}_{s_1 s_{-1}}(a) G_1(x||a) \\ &= (1 - (a_1 \ominus a_0)) G_{21}(x||a) + \ominus(a_0)(1 - a_1) G_2(x||a) \\ &\quad + a_1(1 - \ominus(a_0)) G_{11}(x||a) + a_1(\ominus(a_0)) G_1(x||a). \end{aligned}$$

6. K -BUMPLESS PIPEDREAMS

In [LLS21], we introduced bumpless pipedreams and showed that back stable (double) Schubert polynomials can be obtained as sums over bumpless pipedreams. Weigandt [Wei] connected bumpless pipedreams to alternating sign matrices and a formula of Lascoux [Las], and thereby obtained a bumpless pipedream formula for (double) Grothendieck polynomials.

6.1. Back stable double Grothendieck polynomials. Recall that a bumpless pipedream is a tiling of the plane by the tiles: empty, NW elbow, SE elbow, horizontal line, crossing, and vertical line.



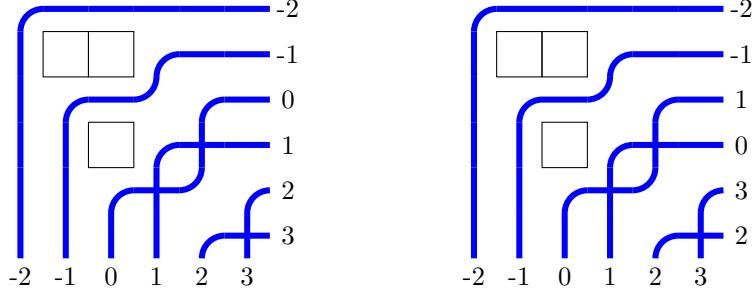


FIGURE 1. A K -bumpless pipedream for $w = s_0 s_2 = (\dots, -2, -1, 1|0, 3, 2, \dots)$ with weight $\text{wt} = -(x_{-1} \ominus a_{-1})(x_{-1} \ominus a_0)(x_1 \ominus a_0)(1 - (x_0 \ominus a_1)(1 - (x_2 \ominus a_2)))$. In the left hand diagram the labels are Cartesian. In the right hand diagram, the row labels give the permutation $w(D)$, whose computation only considers the first time two pipes cross and ignores later crossings.

We shall use **matrix coordinates** for unit squares in the plane. Thus row coordinates increase from top to bottom, column coordinates increase from left to right, and (i, j) indicates the square in row i and column j . A K -bumpless pipedream is a bumpless pipedream D covering the whole plane, such that for all $N \gg 0$ and all $N \ll 0$, there is a pipe traveling north from (∞, N) to the square (N, N) where it turns east and travels towards (N, ∞) . The permutation $w(D) \in S_{\mathbb{Z}}$ of a K -bumpless pipedream is obtained as follows. For each $i \in \mathbb{Z}$, there is a pipe that heads north from (∞, i) . We follow this pipe until it heads east towards (j, ∞) , ignoring all crossings between pairs of pipes that have already crossed (reading pipes from SW to NE). Then $w(D)$ is determined by $w(j) = i$, as $i \in \mathbb{Z}$ varies.

The weight $\text{wt}(D)$ of a K -bumpless pipedream D is given by

$$(6.1) \quad \text{wt}(D) := \prod_{\text{empty tiles } (i,j)} (-(x_i \ominus a_j)) \prod_{\text{NW-elbows } (i,j)} (1 - (x_i \ominus a_j)),$$

where the first product is over empty tiles (i, j) and the second product is over elbow tiles (i, j) that connect the north and west sides.

Example 6.1. The one-line notation for $w \in S_{\mathbb{Z}}$ is the list

$$(\dots, w(-2), w(-1), w(0)|w(1), w(2), \dots)$$

with the vertical divider separating the images of nonpositive and positive integers. In one line notation, $w = s_0 s_2 = (\dots, -2, -1, 1|0, 3, 2, \dots)$ where \dots denotes fixed points. Figure 1 shows a K -bumpless pipedream D for w , where we have only drawn the region $\{(i, j) \mid i, j \in [-2, 3]\}$ (the rest of the pipes head north, turn once and head east). In the left picture, the empty tiles have been indicated, as have the row and column numbers. In the right picture, we have indicated the calculation of $w(D)$, labeling each pipe by the column where it enters the picture. The pipes labeled 0 and 1 intersect twice, and the second intersection is ignored when computing $w(D)$.

The following result follows from [Wei, Theorem 1.1], reproduced as Theorem 6.13 below.

Theorem 6.2. *Let $w \in S_{\mathbb{Z}}$. Then*

$$\overleftarrow{\mathfrak{G}}_w(x; a) = (-1)^{\ell(w)} \sum_D \text{wt}(D)$$

where the summation runs over all K -bumpless pipedreams D with permutation $w(D) = w$.

Example 6.3. Let $w = s_0$. Then for each $j \geq 0$, there is one K -bumpless pipedream D_j for w with one empty tile $(-j, -j)$ and j NW-elbow tiles $(-k, -k)$ for $0 \leq k < j$ and

$$\text{wt}(D_j) = -(x_{-j} \ominus a_{-j}) \prod_{0 \leq k < j} (1 - (x_{-k} \ominus a_{-k})).$$

By Proposition 5.36, we have $\overleftarrow{\mathfrak{G}}_{s_0}(x; a) = G_1(x/a)$ and by Proposition 4.14, we have $G_1(x/a) = G_1(x) + G_1(\ominus a) - G_1(x)G_1(\ominus a) = 1 - \prod_{j \geq 0} (1 - x_{-j})/(1 - a_{-j})$. Using this, one checks that indeed $\overleftarrow{\mathfrak{G}}_{s_0}(x; a) = \sum_{j \geq 0} (x_{-j} \ominus a_{-j}) \prod_{0 \leq k < j} (1 - (x_{-k} \ominus a_{-k}))$.

6.2. Pipedreams for Grassmannian double K -Stanley functions. Let $\lambda \in \mathbb{Y}$. Recall from [LLS21] that a λ -halfplane pipedream is a bumpless pipedream in the upper halfplane $\mathbb{Z}_{\leq 0} \times \mathbb{Z}$ such that the crossing tile is not used, and

- (1) there are (unlabeled) pipes entering from the southern boundary in the columns indexed by $I \subset \mathbb{Z}$;
- (2) setting $(I_+, I_-) = (I \cap \mathbb{Z}_{>0}, \mathbb{Z}_{\leq 0} \setminus I)$, we have $I_{\pm} = I_{w_{\lambda}, \pm}$ (see (7.1), (7.2));
- (3) the i -th eastmost pipe entering from the south heads off to the east in row $1 - i$. (Equivalently, for every row $i \in \mathbb{Z}_{\leq 0}$, there is some pipe heading towards (i, ∞) .)

Since crossing tiles are not used, there is no distinction between a halfplane pipedream and a K -halfplane pipedream.

The weight of a halfplane pipedream D is given by (6.1) (this is different from the weight used in [LLS21]).

Example 6.4. Let $\lambda = (5, 3, 2, 2)$. In Figure 2 the Rothe pipedream (see [LLS21, Section 5.2]) and another λ -halfplane bumpless pipedream are depicted.

Theorem 6.5. *Let $\lambda \in \mathbb{Y}$. Then*

$$G_{\lambda}(x||a) = (-1)^{|\lambda|} \sum_D \text{wt}(D)$$

where the summation runs over all λ -halfplane pipedreams.

Proof of Theorem 6.5. Let $w_{\lambda} \in S_{\mathbb{Z}}$ be a Grassmannian permutation, and set $I = w_{\lambda}^{-1}(\mathbb{Z}_{\leq 0}) = I_{w_{\lambda}, +} \sqcup (\mathbb{Z}_{\leq 0} \setminus I_{w_{\lambda}, -})$ and $I' = \mathbb{Z} \setminus I$. Let D be a K -bumpless pipedream for w_{λ} . Then the pipes labeled by I head off to the east in the rows labeled by $\mathbb{Z}_{\leq 0}$, while the pipes labeled by I' head off to the east in the rows labeled by $\mathbb{Z}_{>0}$. Furthermore, pipes of each type do not cross pipes of the same type. It follows immediately that the part $D_{\leq 0}$ of D that lies in rows indexed by $\mathbb{Z}_{\leq 0}$ contains no crossing tiles and is a halfplane pipedream.

On the other hand, we claim that the bottom half $D_{>0}$ of D that lies in rows indexed by $\mathbb{Z}_{>0}$ depends only on w_{λ} and furthermore contains no empty tiles, and no NW-elbows. Indeed, $D_{>0}$ is given as follows: any pipe labeled by $i \in I_+$ travels northward until row 0, and any pipe labeled by $i \in I_-$ travels northward until row $w^{-1}(i)$, turns and travels eastward. This description can be proven, for example,

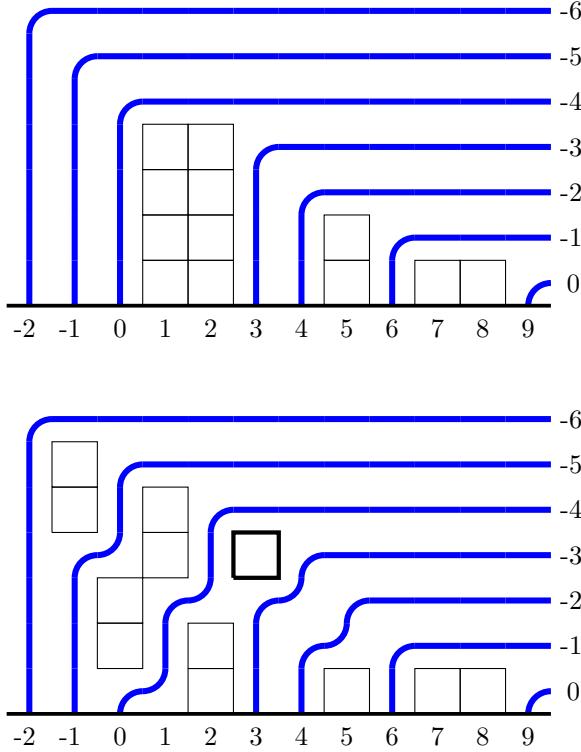


FIGURE 2. The Rothe and another $(5, 3, 2, 2)$ -halfplane bumpless pipedream.

by descending induction on the label $i \in \mathbb{Z}$. (Note that it follows that D is actually a bumpless pipedream – there are no pipes that cross twice.) We also deduce that the top half $D_{\leq 0}$ is thus a λ -halfplane pipedream.

The stated formula for $G_\lambda(x||a) = \overleftarrow{\mathfrak{G}}_{w_\lambda}(x||a)$ now follows from Theorem 6.2. \square

We restate Theorem 6.5 using semistandard tableaux. Given $k \in \mathbb{Z}$ and a box $s = (i, j) \in \lambda$ define

$$(6.2) \quad \text{wt}(k, s, \lambda) = (x_k \ominus a_{k+\ell(\lambda)+j-i}).$$

Corollary 6.6. *Let $\lambda \in \mathbb{Y}$. Then*

$$(6.3) \quad G_\lambda(x||a) = (-1)^{|\lambda|} \sum_T \prod_{s \in \lambda} \left(\text{wt}(T(s), s, \lambda) \prod_k (1 - \text{wt}(k, s, \lambda)) \right)$$

where T runs over the semistandard tableaux of shape λ with entries in $\mathbb{Z}_{\leq 0}$, s runs over the boxes in λ and $k \in \mathbb{Z}_{\leq 0}$ runs over values such that $T(s) < k$ and replacing the s -th entry of T by k results in a semistandard tableau.

Proof. Every λ -halfplane bumpless pipedream can be obtained from the Rothe bumpless pipedream, the unique one that has no NW elbow tiles. This corresponds to the unique semistandard tableau of shape λ having λ_i copies of the value $1 - i$. Moreover each droop moves an empty tile one row north and one row west. Since the tiles move diagonally, reading along diagonals from northwest to southeast starting

from the southwestmost empty tiles, the row indices of the empty tiles define the values in a corresponding diagonal in the semistandard tableau. This yields a bijection between the λ -halfplane bumpless pipedreams and the semistandard tableaux. The NW elbow tiles correspond to entries k that can be increased while preserving semistandardness. \square

Example 6.7. The semistandard tableaux for the pipedreams of in Figure 2 are given by

-3	-3	-1	0	0	
-2	-2	0			
-1	-1				
0	0				

-5	-4	-3	0	0	
-4	-3	0			
-2	-1				
-1	0				

The outlined empty tile in the second pipedream in Figure 2 corresponds to the outlined -3 tableau entry. The fact that this -3 can be replaced by the larger elements -2 and -1 corresponds to the presence of the two NW elbow tiles that are directly to the southeast of the outlined empty tile. The contribution of the box of the outlined -3 is given by

$$(x_{-3} \ominus a_3)(1 - (x_{-2} \ominus a_4))(1 - (x_{-1} \ominus a_5)).$$

Remark 6.8.

- (1) In the nonequivariant setting the original formula for G_λ uses set-valued tableaux [Buc02a]. These are in bijection with marked bumpless pipedreams [Wei, §7].
- (2) We biject from λ -halfplane pipedreams to semistandard tableaux with non-positive entries. Upon replacing entries i by $1 - i$ one obtains reverse semistandard tableaux with entries $\{1, 2, \dots\}$. In this context our formula is an equivariant upgrade of the symmetric Grothendieck special case of [SY, Theorem 1.3]. This result is also implicit in [BSW].
- (3) In the more general case of vexillary permutations, a more complicated version of (6.3) was given in part 3 of the second corollary in §1.2 of [KMY08]. The article [SY] makes the observation that using reverse tableaux often leads to simpler formulas.

6.3. Expansion formulae. Let $w \in S_n$. A w -rectangular K -bumpless pipedream is a K -bumpless pipedream in the $n \times 2n$ rectangular region $R_n := \{(i, j) \mid i \in [1, n] \text{ and } j \in [1 - n, n]\}$. The pipes are labeled $1 - n, 2 - n, \dots, 0, 1, \dots, n$ entering the south boundary from left to right. The positively labeled pipes exit the east boundary, and determine $w \in S_n$ using the same prescription as for K -bumpless pipedreams. The nonpositively labeled pipes exit the north boundary, and these pipes cannot intersect any other pipe. The weight of a w -rectangular K -bumpless pipedream is again given by (6.1). We also associate a partition $\lambda(D)$ to an S_n -rectangular K -bumpless pipedream: it is obtained by reading the north boundary edges from right to left, to then obtain the boundary of a partition inside a $n \times n$ box, where empty edges correspond to steps to the left, and edges with a pipe exiting correspond to downward steps.

Theorem 6.9. *Let $w \in S_n$. Then*

$$\overleftarrow{\mathfrak{G}}_w(x; a) = (-1)^{\ell(w)} \sum_D (-1)^{|\lambda(D)|} \text{wt}(D) G_{\lambda(D)}(x || a)$$

where the summation is over w -rectangular pipedreams.

Proof. Let D be a K -bumpless pipedream for $w \in S_n$. The pipes labeled by $n+1, n+2, \dots$ travel northward and turn east in the row corresponding to their labels. The pipes labeled by $1, 2, \dots, n$ travel northward until row n , perform one or more turns inside the square $[1, n] \times [1, n]$ and then travel eastward once they exit the square in one of the rows $1, 2, \dots, n$. The pipes labeled by $0, -1, -2, \dots$, cannot cross other pipes. In particular, the pipe labeled $i \in \mathbb{Z}_{\leq 0}$ travels northward until row $n+i-1$ before it makes its first turn. Thus the pipes labeled $0, -1, -2, \dots, 1-n$ travel northward until row n , possibly perform some turns, and then exit the north boundary of the rectangle R_n . The pipes labeled $-n, -n-1, \dots$ travel vertically at least until row 0.

To summarize: (1) the top half $D_{\leq 0}$ of D is a λ -halfplane pipedream for some $\lambda \in \mathbb{Y}$; (2) the interesting part of the bottom half $D_{>0}$ of D is contained in the rectangular region R_n , which in particular contains all the empty tiles and NW-elbows of $D_{>0}$. Since $\text{wt}(D) = \text{wt}(D_{\leq 0})\text{wt}(D_{>0})$, we obtain the stated formula by combining Theorem 6.2 with Theorem 6.5. \square

The following result follows immediately from Theorem 6.9 and the definition of $G_w(x||a)$.

Corollary 6.10. *Let $w \in S_n$. Then*

$$G_w(x||a) = (-1)^{\ell(w)} \sum_D (-1)^{|\lambda(D)|} \eta_a(\text{wt}(D)) G_{\lambda(D)}(x||a)$$

where the summation is over w -rectangular pipedreams.

Example 6.11. Let $w = s_1 \in S_2$. Theorem 6.9 gives

$$\overleftarrow{\mathfrak{G}}_{s_1}(x||a) = (x_1 \ominus a_1) + (1 - (x_1 \ominus a_1))G_1(x||a),$$

and noting that $G_1(x||a) = G_1(x/a)$ this agrees with Example 5.18. Corollary 6.10 gives

$$G_{s_1}(x||a) = G_1(x||a).$$

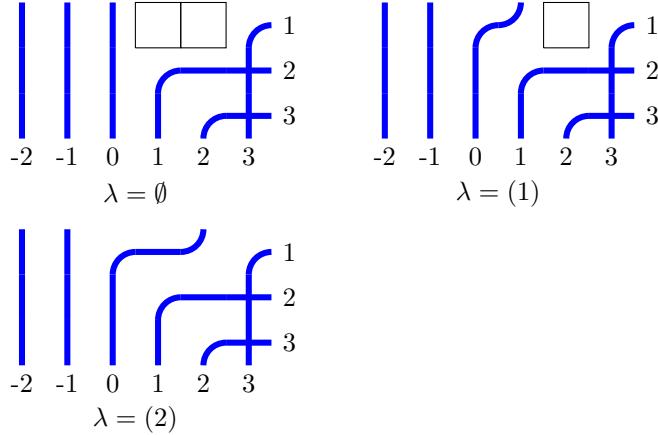
Example 6.12. Let $w = s_2 s_1 \in S_3$. In one line notation, $w(1, 2, 3) = (3, 1, 2)$. The w -rectangular K -bumpless pipedreams are shown in Figure 3. For first halfplane bumpless pipedream D the sequence of top boundary edges translates to left, left, down, down, down. This is the edge sequence of the partition $\lambda(D) = \emptyset$. By Theorem 6.9, we have

$$\begin{aligned} \overleftarrow{\mathfrak{G}}_w(x; a) &= (x_1 \ominus a_1)(x_1 \ominus a_2) + (x_1 \ominus a_2)(1 - (x_1 \ominus a_1))G_1(x||a) \\ &\quad + (1 - (x_1 \ominus a_2))G_2(x||a), \end{aligned}$$

and by Corollary 6.10, we have

$$G_w(x||a) = (a_1 \ominus a_2)G_1(x||a) + (1 - (a_1 \ominus a_2))G_2(x||a).$$

6.4. Weigandt's formula for double Grothendieck polynomials. Let $w \in S_n$. A w -square K -bumpless pipedream is a bumpless pipedream in the $n \times n$ square region $[n] \times [n]$. The pipes are labeled $1, \dots, n$ entering the south boundary from left to right, and all pipes exit the east boundary. The permutation $w \in S_n$ is determined as for K -bumpless pipedreams. The weight $\text{wt}(D)$ of a w -square K -bumpless pipedream is defined by (6.1) as before. Weigandt's formula [Wei, Theorem 1.1] for double Grothendieck polynomials is the following.

FIGURE 3. w -rectangular K -bumpless pipedreams for $w = s_2s_1$.

Theorem 6.13. *Let $w \in S_n$. Then*

$$\mathfrak{G}_w(x; a) = (-1)^{\ell(w)} \sum_D \text{wt}(D)$$

where the summation is over all w -square K -bumpless pipedreams D .

6.5. Hecke bumpless pipedreams. Let D be a w -square K -bumpless pipedream. Following Weigandt [Wei], we call D a *Hecke bumpless pipedream* if all the empty tiles are in the northeast corner, where they form a partition shape $\lambda = \lambda(D)$, called the shape of D . The following result was proven by Weigandt [Wei].

Theorem 6.14. *The coefficient k_w^λ of G_λ in G_w (see (8.1)) is equal to $(-1)^{\ell(w)-|\lambda|}$ times the number of w -Hecke bumpless pipedream with shape λ .*

Proof. Substituting $a = 0$ into Corollary 6.10, only rectangular pipedreams with no empty tiles contribute. Erasing the nonpositively labeled pipes from such a rectangular pipedream gives a w -Hecke bumpless pipedream. \square

Theorem 6.14 is a K -theoretic analogue of [LLS21, Theorem 5.14], a direct bijective proof of which was given by Fan, Guo and Sun [FGS]. Weigandt [Wei] gave a more general bijection between Hecke bumpless pipedreams and decreasing tableaux [BKSTY].

7. K -HOMOLOGY AND HOPF STRUCTURE

7.1. Hopf structure on GKM ring. Let $\Psi_{\text{Gr}} \subset \Psi$ denote the subspace of functions ψ satisfying $\psi(v) = \psi(w)$ if $vS_{\neq 0} = wS_{\neq 0}$, and similarly define $\tilde{\Psi}_{\text{Gr}} \subset \tilde{\Psi}$. Then Ψ_{Gr} (resp. $\tilde{\Psi}_{\text{Gr}}$) has basis (allowing infinite sums) $\{\psi^{w_\lambda} \mid \lambda \in \mathbb{Y}\}$ (resp. $\{\tilde{\psi}^{w_\lambda} \mid \lambda \in \mathbb{Y}\}$). We have $K_T(\overline{\text{Gr}}^0) \simeq \Psi_{\text{Gr}}$.

The map $\text{res} : \overleftarrow{R}(x; a) \rightarrow \mathbb{S} \otimes_{R(a)} \tilde{\Psi}$ of Theorem 5.25 restricts to a map $\text{res} : \Gamma(x||a) \rightarrow \tilde{\Psi}$ with image given by $\bigoplus_\lambda R(a)\tilde{\psi}^{w_\lambda}$. Following [LLS21], we now describe a Hopf structure on Ψ_{Gr} that is compatible with the bialgebra structure on $\Gamma(x||a)$.

For any $w \in S_{\mathbb{Z}}$, let

$$(7.1) \quad I_{w,+} := \mathbb{Z}_{>0} \cap w(\mathbb{Z}_{\leq 0})$$

$$(7.2) \quad I_{w,-} := \mathbb{Z}_{\leq 0} \cap w(\mathbb{Z}_{>0}).$$

The map $w \mapsto (I_{w,+}, I_{w,-})$ is a bijection from $S_{\mathbb{Z}}^0$ to pairs of finite sets (I_+, I_-) such that $I_+ \subset \mathbb{Z}_{>0}$, $I_- \subset \mathbb{Z}_{\leq 0}$, and $|I_+| = |I_-|$. There is a partial multiplication map $S_{\mathbb{Z}}^0 \times S_{\mathbb{Z}}^0 \rightarrow S_{\mathbb{Z}}^0$. The product of $x \in S_{\mathbb{Z}}^0$ and $y \in S_{\mathbb{Z}}^0$ is equal to $z \in S_{\mathbb{Z}}^0$ if (1) $I_{x,+} \cap I_{y,+} = \emptyset = I_{x,-} \cap I_{y,-}$ and (2) $I_{x,\pm} \cup I_{y,\pm} = I_{z,\pm}$.

The following result is proved in the same manner as [LLS21, Proposition 7.11].

Proposition 7.1. *There is a coproduct $\Delta : \tilde{\Psi}_{\text{Gr}} \rightarrow \tilde{\Psi}_{\text{Gr}} \hat{\otimes}_{R(a)} \tilde{\Psi}_{\text{Gr}}$ such that the map $\text{res} : \Gamma(x||a) \rightarrow \tilde{\Psi}_{\text{Gr}}$ is a $R(a)$ -bialgebra morphism. The coproduct satisfies*

$$(7.3) \quad \Delta(\psi)|_{x \otimes y} = \psi|_{xy}$$

whenever $x, y \in S_{\mathbb{Z}}^0$ and $xy \in S_{\mathbb{Z}}^0$ is defined.

7.2. K -homology basis. Let $\Lambda(y)$ denote the \mathbb{Z} -algebra of symmetric functions in $y = y_{\leq 0} = (y_0, y_{-1}, y_{-2}, \dots)$ and $\tilde{\Lambda}(y||a) = \prod_{\lambda \in \mathbb{Y}} R(a)s_{\lambda}(y)$ the completion of $R(a) \otimes \Lambda(y)$ whose elements are formal (possibly infinite) $R(a)$ -linear combinations $\sum_{\lambda \in \mathbb{Y}} a_{\lambda}s_{\lambda}(y)$ of Schur functions, with $a_{\lambda} \in R(a)$. The ring $\tilde{\Lambda}(y||a)$ is a $R(a)$ -Hopf algebra with coproduct $\Delta(p_k(y)) = 1 \otimes p_k(y) + p_k(y) \otimes 1$.

Define the Cauchy kernel

$$\Omega[(x_- - a_-)y] = \prod_{i,j \leq 0} \frac{1 - a_i y_j}{1 - x_i y_j} = \exp \left(\sum_{k \geq 0} \frac{1}{k} p_k(x||a) p_k(y) \right).$$

This induces the structure of dual $R(a)$ -Hopf algebras on $\tilde{\Lambda}(y||a)$ and (a completion of) $\Gamma(x||a)$. Write $\langle \cdot, \cdot \rangle$ for the corresponding pairing $\Gamma(x||a) \otimes_{R(a)} \tilde{\Lambda}(y||a) \rightarrow R(a)$. Then by definition

$$(7.4) \quad \langle s_{\lambda}(x/a), s_{\mu}(y) \rangle = \delta_{\lambda, \mu}.$$

Let $g_{\lambda}(y)$ be the *dual stable Grothendieck polynomials* of [LP]. They are defined by

$$(7.5) \quad \langle G_{\lambda}(x/a), g_{\mu}(y) \rangle = \delta_{\lambda, \mu}.$$

Define the *K -Molev functions* $g_{\lambda}(y||a) \in \tilde{\Lambda}(y||a)$ by duality with the Grassmannian double K -Stanley functions $G_{\lambda}(x||a)$:

$$(7.6) \quad \langle G_{\lambda}(x||a), g_{\mu}(y||a) \rangle = \delta_{\lambda, \mu}.$$

The ring $\tilde{\Lambda}(y||a)$ consists of formal $R(a)$ -linear combinations of the $g_{\lambda}(y||a)$. At $a = 0$, the polynomials $g_{\lambda}(y||a)$ reduce to the dual stable Grothendieck polynomials $g_{\lambda}(y)$. The functions $g_{\lambda}(y||a)$ are K -theoretic analogues of Molev's dual Schur functions.

Remark 7.2. Since the Grassmannian double K -Stanley functions $G_{\lambda}(x||a)$ represent the structure sheaves of Schubert varieties in the equivariant K -theory $K_T(\overline{\text{Gr}}^0)$ of the thick infinite Grassmannian (Corollary 5.33), the functions $g_{\lambda}(y||a)$ represent the dual basis of ideal sheaves of boundaries of Schubert varieties in the K -group $K^T(\overline{\text{Gr}}^0)$ of finitely supported equivariant coherent sheaves on the thin infinite

Grassmannian. Note that the Schubert varieties of $\overline{\mathrm{Gr}}^0$ and of $\mathrm{Gr}^{(0)}$ are “opposite”. See [Kum, BK] for details on the duality between thin and thick K -groups, and [LSS] for a discussion of the Hopf structure.

Recall the element $w_{\lambda/\mu} \in S_{\mathbb{Z}}$ from (5.32). Proposition 5.36 implies the following.

Proposition 7.3. *For $\mu \in \mathbb{Y}$, we have*

$$(7.7) \quad g_{\mu}(y) = \sum_{\substack{\nu \subset \mu \subset \lambda \\ d(\nu) = d(\lambda) \\ \mu/\nu \in \mathcal{RS}}} (-1)^{|\mu| - |\nu|} \mathfrak{G}_{w_{\lambda/\nu}^{-1}}(\ominus(a)) g_{\lambda}(y||a)$$

$$(7.8) \quad g_{\mu}(y||a) = \sum_{\substack{\nu \subset \mu \subset \lambda \\ d(\nu) = d(\lambda) \\ \mu/\nu \in \mathcal{RS}}} (-1)^{|\mu| - |\nu|} \mathfrak{G}_{w_{\lambda/\nu}}(a) g_{\lambda}(y).$$

7.3. K -Homology divided difference operators. Recall that by convention we have $x := x_{\leq 0} = (\dots, x_{-1}, x_0)$ and similarly, $y := y_{\leq 0}$ and $a := a_{\leq 0}$.

For $i \in \mathbb{Z}$, define the operators (see (8.15))

$$(7.9) \quad \tilde{s}_i^a = \Omega[(x - a)y] s_i^a \Omega[(a - x)y]$$

$$(7.10) \quad \delta_i^{a,\ominus} = \Omega[(x - a)y] \tilde{\pi}_i^{a,\ominus} \Omega[(a - x)y].$$

It is clear that these operators, being conjugate to the operators s_i^a and $\tilde{\pi}_i^{a,\ominus}$ respectively, satisfy the type A braid relations. In operator expressions, a symmetric function or polynomial f denotes left multiplication by f . We have

$$(7.11) \quad \tilde{s}_i^a \delta_i^{a,\ominus} = \delta_i^{a,\ominus}$$

$$(7.12) \quad \delta_i^{a,\ominus} f = \delta_i^{a,\ominus}(f) + \tilde{s}_i^a(f) \tilde{\pi}_i^{a,\ominus}.$$

Let $\alpha_i = a_i - a_{i+1}$ for $i \in \mathbb{Z}$. Since $\Omega[(a - x)y]$ is s_i^a invariant for $i \neq 0$ we have

$$(7.13) \quad \tilde{s}_i^a = s_i^a \quad \text{for } i \neq 0$$

$$(7.14) \quad \delta_i^{a,\ominus} = \tilde{\pi}_i^{a,\ominus} \quad \text{for } i \neq 0.$$

$$(7.15) \quad \tilde{s}_0^a = \Omega[-\alpha_0 y] s_0^a$$

$$(7.16) \quad \delta_0^{a,\ominus} = \alpha_0^{-1} (1 - \tilde{s}_0)(a_0 - 1).$$

The diagonal index of a box in row i and column j is by definition $j - i$. For $\lambda \in \mathbb{Y}$ and $d \in \mathbb{Z}$, let $\lambda + d$ denote the partition obtained by adding a corner to λ in the d -th diagonal if such a corner exists, and $\lambda + d := \lambda$ if such a partition does not exist. Define $\lambda - d$ similarly for removal of the corner in diagonal d .

By Proposition 5.15, we have

$$(7.17) \quad \tilde{\pi}_i^{a,\ominus} G_{\lambda}(x||a) = G_{\lambda-i}(x||a) \quad \text{for all } \lambda \in \mathbb{Y} \text{ and } i \in \mathbb{Z}.$$

Let $\hat{\partial}_i^a = \tilde{\pi}_i^a - 1 = (1 - a_i) A_i^a$ and $\tau_i = \Omega[(x - a)y] \hat{\partial}_i^a \Omega[(a - x)y]$.

Lemma 7.4. *For $i \in \mathbb{Z}$, we have*

$$(7.18) \quad \hat{\partial}_i^a = s_i^a - \tilde{\pi}_i^{a,\ominus}$$

$$(7.19) \quad \tau_i = \tilde{s}_i^a - \delta_i^{a,\ominus}.$$

Proof. Equation 7.19 follows from (7.18) by conjugation by $\Omega[(x - a)y]$. We have

$$\begin{aligned} s_i^a - \tilde{\pi}_i^{a,\ominus} &= s_i^a - A_i^a(a_i - 1) \\ &= s_i^a - (1 + (a_{i+1} - 1)A_i^a) \\ &= (-\alpha_i - (a_{i+1} - 1)A_i^a) \\ &= (1 - a_i)A_i^a \\ \hat{\partial}_i^a &= \tilde{\pi}_i^a - 1 = -1 + A_i^a(1 - a_{i+1}) \\ &= (1 - a_i)A_i^a. \end{aligned}$$

□

Proposition 7.5. *For all $\mu \in \mathbb{Y}$ and $i \in \mathbb{Z}$, we have*

$$(7.20) \quad \tau_i(g_\mu(y||a)) = g_{\mu+i}(y||a).$$

Proof. Using (7.12), we have

$$\begin{aligned} \Omega[(x - a)y] &= \Omega[(x - a)y] \tilde{\pi}_i^{a,\ominus}(1) = \delta_i^{a,\ominus}(\Omega[(x - a)y]) \\ &= \delta_i^{a,\ominus} \sum_{\lambda} g_{\lambda}(y||a) G_{\lambda}(x||a) \\ &= \sum_{\lambda} (\delta_i^{a,\ominus}(g_{\lambda}(y||a)) G_{\lambda}(x||a) + \tilde{s}_i^a(g_{\lambda}(y||a)) \tilde{\pi}_i^{a,\ominus}(G_{\lambda}(x||a))) \\ &= \sum_{\lambda} (\delta_i^{a,\ominus}(g_{\lambda}(y||a)) G_{\lambda}(x||a) + \tilde{s}_i^a(g_{\lambda}(y||a)) G_{\lambda-i}(x||a)). \end{aligned}$$

Taking the coefficient of $G_{\mu}(x||a)$ we obtain

$$g_{\mu}(y||a) = \delta_i^{a,\ominus}(g_{\mu}(y||a)) + \tilde{s}_i^a(g_{\mu+i}(y||a)).$$

Acting by \tilde{s}_i^a , using (7.11), and rearranging we have

$$g_{\mu+i}(y||a) = (\tilde{s}_i^a - \delta_i^{a,\ominus})(g_{\mu}(y||a)) = \tau_i(g_{\mu}(y||a)).$$

□

The τ_i satisfy the type A braid relations and $\tau_i^2 = -\tau_i$. Thus $\tau_w = \tau_{i_1} \cdots \tau_{i_\ell}$ makes sense for any reduced decomposition $w = s_{i_1} \cdots s_{i_\ell} \in S_{\mathbb{Z}}$.

Theorem 7.6. *For any $\lambda \in \mathbb{Y}$, we have $g_{\lambda}(y||a) = \tau_{w_{\lambda}}(1)$.*

Example 7.7. We have

$$\begin{aligned} g_1(y||a) &= \tau_0(1) = \Omega[(x - a)y](1 - a_0)A_0^a\Omega[(a - x)y] \\ &= (1 - a_0)\alpha_0^{-1}\Omega[-a_0y](1 - s_0^a)\Omega[a_0y] \\ &= (1 - a_0)\alpha_0^{-1}(1 - \Omega[(a_1 - a_0)y]) \\ &= (1 - a_0) \sum_{i,j \geq 0} (-a_0)^i a_1^j s_{(j+1,1^i)}(y). \end{aligned}$$

In particular, setting all $a_i = 0$ we obtain $g_1(y) = s_1(y)$. Let

$$Z = \sum_{i,j \geq 0} (-a_0)^i a_1^j s_{(j+1,1^i)}(y).$$

We have

$$\begin{aligned}
g_{11}(y||a) &= \tau_{-1}g_1(y||a) = \hat{\partial}_i^a g_1(y||a) = (1 - a_{-1})A_{-1}^a(1 - a_0)Z \\
&= (1 - a_{-1})(1 + (1 - a_{-1})A_{-1}^a)Z \\
&= (1 - a_{-1})(1 + (1 - a_{-1})A_{-1}^a)(s_1(y) - a_0s_{11}(y) + a_1s_2(y) + \dots) \\
&= (1 - a_{-1})(s_1(y) - a_0s_{11}(y) + \dots) + (1 - a_{-1})^2(s_{11}(y) \dots).
\end{aligned}$$

We obtain $g_{11}(y) = s_{11} + s_1$. These computations of g_1 and g_{11} agree with [LP].

7.4. Connection to Knutson-Lederer. Knutson and Lederer [KL] define a deformation of the ring of symmetric functions, denoted R^{K^S} . The ring R^{K^S} has a basis $[X^\lambda] = [\mathcal{O}_{X^\lambda}]$ (representing structure sheaves of opposite Schubert varieties in the Grassmannian) as λ ranges over all partitions. The product structure of R^{K^S} is given by the direct sum operation on Grassmannians. We refer the reader to [KL] for the details and to [LLS21, Section 8.3] for a synopsis of the very similar situation in homology.

Let $h_\lambda(y||a) := \sum_{\mu \subset \lambda} g_\lambda(y||a)$. Whereas $g_\lambda(y||a)$ represents the ideal sheaf of a boundary of a Schubert variety in $K^T(\mathrm{Gr})$, the symmetric function $h_\lambda(y||a)$ represents the structure sheaf of the same Schubert variety. Further, let $h_\lambda(y||\delta)$ be obtained from $h_\lambda(y||a)$ by the specialization

$$a_i \longmapsto \begin{cases} \delta & \text{if } i > 0, \\ 0 & \text{if } i \leq 0. \end{cases}$$

See [LLS21, Section 8.3] for a more precise description. The following result is proved in the same manner as [LLS21, Theorem 8.12].

Theorem 7.8. *There is, up to a completion, an isomorphism of $\mathbb{Z}[\delta]$ -algebras*

$$R^{K^S} \rightarrow \Lambda(y)[\delta] \quad [X^\lambda] \mapsto h_\lambda(y||\delta)$$

where δ corresponds to $1 - \exp(-t)$ in [KL].

8. THE RING OF BACK STABLE GROTHENDIECK POLYNOMIALS

8.1. The subring of K -Stanley functions.

$$\Gamma = \Gamma(x_-) := \bigoplus_{\lambda \in \mathbb{Y}} \mathbb{Z}G_\lambda \subset \hat{\Lambda}$$

to be the span of Grassmannian K -Stanley functions. The structure of Γ was studied by Buch [Buc02a].

Proposition 8.1. [Buc02a] Γ is a commutative and cocommutative bialgebra containing G_w for all $w \in S_{\mathbb{Z}}$. In particular, the expansion

$$(8.1) \quad G_w = \sum_{\lambda} k_{\lambda}^w G_{\lambda}$$

exists and is finite, and furthermore we have $(-1)^{\ell(w)-|\lambda|} k_{\lambda}^w \in \mathbb{Z}_{\geq 0}$.

For $\lambda, \mu, \nu \in \mathbb{Y}$, define $c_{\mu\nu}^{\lambda} \in \mathbb{Z}$ by

$$G_{\mu}G_{\nu} = \sum_{\lambda} c_{\mu\nu}^{\lambda} G_{\lambda}.$$

By [Buc02a], we have the positivity property $c_{\mu\nu}^{\lambda} \in (-1)^{|\mu|+|\nu|-|\lambda|} \mathbb{Z}_{\geq 0}$.

Remark 8.2. Explicit combinatorial formulae for the product and coproduct structure constants of Γ with respect to the basis G_λ were given by Buch [Buc02a].

Remark 8.3. The following explicit tableau formula is given in [BKSTY, Theorem 1']: $(-1)^{\ell(w)-|\lambda|} k_\lambda^w$ is equal to the number of *decreasing* tableaux (rows strictly decrease from left to right and columns strictly decrease from top to bottom) T of shape λ whose column-reading word is a Hecke word for w , that is, if the column-reading word of T is $i_1 i_2 \cdots i_\ell$ then $s_{i_1} * s_{i_2} * \cdots * s_{i_\ell} = w$.

8.2. Flagged Grothendieck polynomials and K -Stanley functions via symmetrization. A polynomial truncation of the K -Stanley function G_w can be obtained from the Grothendieck polynomial \mathfrak{G}_w by symmetrization operators. The intermediate polynomials are what we shall call *flagged Grothendieck polynomials*.

Let $\theta_i := A_i \circ x_i(1 - x_{i+1})$ for $i \in \mathbb{Z}$ where A_i is defined in (2.25). These operators satisfy the braid relations and are idempotent: $\theta_i^2 = \theta_i$. Thus they generate a 0-Hecke algebra.

Say that a sequence $f = (f_1, f_2, \dots, f_n)$ of positive integers is *admissible* if $1 \leq f_1 \leq f_2 \leq \cdots \leq f_n \leq n$ and $f_i \geq i$ for all i . Then either $f = f_{\min} = (1, 2, \dots, n-1, n)$ or there is a minimum i such that $f_i > i$. Let f^- be f with f_i replaced by $f_i - 1$. Then define $\sigma_f \in S_n$ by

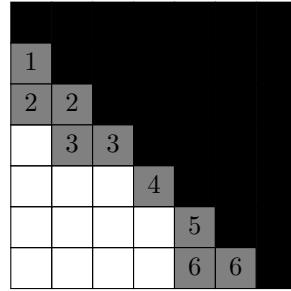
$$(8.2) \quad \sigma_f = \begin{cases} \text{id} & \text{if } f = f_{\min} \\ s_{f_i-1} \sigma_{f^-} & \text{if } i \text{ is minimum such that } f_i > i. \end{cases}$$

Also, define the sequence $f' = (f'_1, f'_2, \dots, f'_n)$ by $f'_i = \min\{j \mid f_j \geq i\}$.

Example 8.4. Let $n = 7$, $f = (3, 4, 4, 5, 7, 7, 7)$. Then $f' = (1, 1, 1, 2, 4, 5, 5)$ and

$$\sigma_f = (s_2 s_1)(s_3 s_2)(s_3)(s_4)(s_6 s_5)(s_6).$$

We illustrate the construction. In the following diagram the j -th column has size f_j and a box in row $i+1$ below the diagonal has corresponding simple reflection s_i . We have $f'_1 = 1$ and for $i \geq 2$, in the diagram f'_i is the leftmost column in the i -th row containing a gray or black square.



For $f = (f_1, f_2, \dots, f_n)$ admissible, define the *flagged Grothendieck polynomial* $\mathfrak{G}_{w,f}$ by

$$\mathfrak{G}_{w,f} := \theta_{\sigma_f}(\mathfrak{G}_w).$$

Proposition 8.5. *For $w \in S_+$, we have*

$$(8.3) \quad \mathfrak{G}_{w,f} = \sum_{\substack{s_{a_1} * s_{a_2} * \cdots * s_{a_p} = w \\ 1 \leq i_1 \leq i_2 \leq \cdots \leq i_p \\ a_k \leq a_{k+1} \Rightarrow i_k < i_{k+1} \\ i_k \leq f_{a_k}}} (-1)^{p-\ell(w)} x_{i_1} x_{i_2} \cdots x_{i_p},$$

where p is arbitrary.

Note that compared to (4.11), only the bound $i_k \leq a_k$ has been changed to $i_k \leq f_{a_k}$.

Corollary 8.6. *For $w \in S_n$, we have*

$$(8.4) \quad \theta_{w_0}(\mathfrak{G}_w) = G_w(x_1, \dots, x_n).$$

Proof. Apply Proposition 8.5 with $f = (n, n, \dots, n)$, which satisfies $\sigma_f = w_0$, and compare with Proposition 4.10. \square

To prove Proposition 8.5 we use 0-Hecke algebra generating functions as in [FK]. Consider the algebra \mathbb{A} over $P = \mathbb{Z}[x_1, \dots, x_n]$ generated by elements u_i for i in the Dynkin node set $I = \{1, 2, \dots, n-1\}$ of type A_{n-1} , which satisfy the type A_{n-1} braid relations and $u_i^2 = -u_i$ for all $i \in I$. For $w \in S_n$, the element $u_w = u_{i_1} u_{i_2} \cdots u_{i_\ell}$ is well-defined, where $w = s_{i_1} s_{i_2} \cdots s_{i_\ell}$ is a reduced expression. Then

$$(8.5) \quad \mathbb{A} = \bigoplus_{w \in S_n} P u_w.$$

We have $(1 + au_i)(1 + bu_i) = 1 + (a \oplus b)u_i$. In particular, $1 + x_i u_j$ is invertible in $\mathbb{A}_F := F \otimes_P \mathbb{A}$, where $F = \text{Frac}(P)$ denotes the fraction field of P . By [FK], we have the following identity:

$$(8.6) \quad \prod_{j=1}^{n-1} \prod_{i=n-1}^j (1 + x_j u_i) = \sum_{w \in S_n} \mathfrak{G}_w u_w$$

where in the inner product the index i goes from $n-1$ down to j going from left to right. Taking the coefficient of u_w one obtains the monomial expansion (3.6).

Proposition 8.7. *For admissible f , we have*

$$\mathfrak{G}_f(x, u) := \prod_{i=1}^n \prod_{j=n-1}^{f'_i} (1 + x_i u_j) = \sum_{w \in S_n} \mathfrak{G}_{w,f} u_w.$$

Proof. Follows by Lemmas 8.8 and 8.9. \square

Proof of Proposition 8.5. This follows by taking the coefficient of u_w in $\mathfrak{G}_f(x, u)$, noting that $j \geq f'_i$ if and only if $f_j \geq i$. \square

Lemma 8.8. *Suppose the value $0 < k < n$ occurs exactly once in f . Then*

$$(8.7) \quad \theta_k(\mathfrak{G}_{w,f}) = \mathfrak{G}_{w,g},$$

where g is obtained from f by replacing k by $k+1$ in f .

Proof. This is easily proved by induction. \square

Lemma 8.9. *With the same assumptions as in Lemma 8.8, we have*

$$\theta_k(\mathfrak{G}_f(u, x)) = \mathfrak{G}_g(u, x).$$

Here, the operator θ_k is acting on the coefficients of elements in \mathbb{A} .

Proof. Let j be such that $f_j = k$. We are assuming that $f_{j+1} > k$. Thus $f'_k = j$ and $f'_{k+1} = j+1$. The operator θ_k commutes with all operators $1 + x_i u_j$ except when $i \in \{k, k+1\}$. It therefore suffices to show that

$$\begin{aligned} \theta_k((1 + x_k u_{n-1}) \cdots (1 + x_k u_j)(1 + x_{k+1} u_{n-1}) \cdots (1 + x_{k+1} u_{j+1})) \\ = (1 + x_k u_{n-1}) \cdots (1 + x_k u_j)(1 + x_{k+1} u_{n-1}) \cdots (1 + x_{k+1} u_j). \end{aligned}$$

Without loss of generality we may assume $k = 1$ and $j = 1$. Let

$$h = \prod_{i=3}^{n-1} \prod_{j=n-1}^i (1 + x_i u_j).$$

This element is invertible in \mathbb{A}_F and commutes both with θ_1 and $1 + x_2 u_1$. It therefore suffices to show

$$\theta_1 \left(\prod_{i=1}^{n-1} \prod_{j=n-1}^i (1 + x_i u_j) \right) = \left(\prod_{i=1}^{n-1} \prod_{j=n-1}^i (1 + x_i u_j) \right) (1 + x_2 u_1)$$

as this is the required identity after multiplication by h . We have $\theta_1 = A_1 x_1 (1 - x_2) = (1 - x_2) + x_2 A_1 (1 - x_2) = (1 - x_2) + x_2 \hat{\pi}_1$. Using (8.6), we compute

$$\begin{aligned} \theta_1 \sum_{w \in S_n} \mathfrak{G}_w u_w &= (1 - x_2) \sum_{w \in S_n} \mathfrak{G}_w u_w + x_2 \sum_{\substack{w \in S_n \\ ws_1 < w}} \mathfrak{G}_{ws_1} u_w + x_2 \sum_{\substack{w \in S_n \\ ws_1 > w}} \mathfrak{G}_w u_w \\ &= \sum_{w \in S_n} \mathfrak{G}_w u_w + \sum_{\substack{w \in S_n \\ ws_1 < w}} (-x_2 \mathfrak{G}_w + x_2 \mathfrak{G}_{ws_1}) u_w, \end{aligned}$$

and

$$\begin{aligned} &\left(\sum_{w \in S_n} \mathfrak{G}_w u_w \right) (1 + x_2 u_1) \\ &= \sum_{w \in S_n} \mathfrak{G}_w u_w + x_2 \sum_{w \in S_n} \mathfrak{G}_w u_w u_1 \\ &= \sum_{w \in S_n} \mathfrak{G}_w u_w + x_2 \sum_{\substack{w \in S_n \\ ws_1 > w}} \mathfrak{G}_w u_{ws_1} - x_2 \sum_{\substack{w \in S_n \\ ws_1 < w}} \mathfrak{G}_w u_w \\ &= \sum_{w \in S_n} \mathfrak{G}_w u_w + x_2 \sum_{\substack{w \in S_n \\ ws_1 < w}} \mathfrak{G}_{ws_1} u_w - x_2 \sum_{\substack{w \in S_n \\ ws_1 < w}} \mathfrak{G}_w u_w, \end{aligned}$$

as required. \square

8.3. The subring of back stable Grothendieck polynomials. The back stable Grothendieck polynomials $\overleftarrow{\mathfrak{G}}_w$ are defined as elements of the ring \overleftarrow{R}^+ of §4.1. They are linearly independent because their lowest degree components are back stable Schubert polynomials, which are linearly independent. However, they do not span \overleftarrow{R}^+ , or even the “finite” subring $\Gamma \otimes R^+ \subset \overleftarrow{R}^+$. The following example shows that x_1 and s_1 are not finite linear combinations of $\overleftarrow{\mathfrak{G}}_w$.

Example 8.10. By (4.33), we have the Schur function

$$s_1 = \sum_{p \geq 1} G_{1^p} = \sum_{p \geq 1} \overleftarrow{\mathfrak{G}}_{s_{1-p} \cdots s_{-1} s_0}.$$

Applying γ , we obtain

$$s_1 + x_1 = \sum_{p \geq 1} \overleftarrow{\mathfrak{G}}_{s_{2-p} \cdots s_{-1} s_0 s_1}.$$

Subtracting we obtain x_1 as an infinite linear combination of back stable Grothendieck polynomials.

Let

$$(8.8) \quad B := \bigoplus_{w \in S_{\mathbb{Z}}} \mathbb{Z} \overleftarrow{\mathfrak{G}}_w$$

denote the subspace of \overleftarrow{R}^+ spanned by back stable Grothendieck polynomials. For $u, v, w \in S_+$ define $c_{uv}^w \in \mathbb{Z}$ by

$$\mathfrak{G}_u \mathfrak{G}_v = \sum_{w \in S_+} c_{uv}^w \mathfrak{G}_w.$$

It is shown in [Bri] that

$$(8.9) \quad (-1)^{\ell(u)+\ell(v)-\ell(w)} c_{uv}^w \in \mathbb{Z}_{\geq 0}.$$

Theorem 8.11. *For all $u, v \in S_{\mathbb{Z}}$, there are constants $\overleftarrow{c}_{uv}^w \in \mathbb{Z}$ such that*

$$(8.10) \quad \overleftarrow{\mathfrak{G}}_u \overleftarrow{\mathfrak{G}}_v = \sum_{w \in S_{\mathbb{Z}}} \overleftarrow{c}_{uv}^w \overleftarrow{\mathfrak{G}}_w$$

with only finitely many \overleftarrow{c}_{uv}^w nonzero. That is, B is a \mathbb{Z} -subalgebra of \overleftarrow{R} . Moreover,

$$(8.11) \quad (-1)^{\ell(u)+\ell(v)-\ell(w)} \overleftarrow{c}_{uv}^w \in \mathbb{Z}_{\geq 0}.$$

Proof. For sufficiently large q , we have $\overleftarrow{c}_{uv}^w = c_{\gamma^q(u)\gamma^q(v)}^{\gamma^q(w)}$; see the more general Proposition 8.25 below. Thus (8.11) follows from (8.9). Applying η_0 to (8.10), we obtain

$$G_u G_v = \sum_w \overleftarrow{c}_{uv}^w G_w.$$

By Proposition 8.1, we have a finite expansion $G_w = \sum_{\lambda} k_{\lambda}^w G_{\lambda}$. Thus

$$(8.12) \quad G_u G_v = \sum_{\mu, \nu, \lambda} k_{\mu}^u k_{\nu}^v c_{\mu\nu}^{\lambda} G_{\lambda}$$

$$(8.13) \quad \sum_w \overleftarrow{c}_{uv}^w G_w = \sum_{w, \lambda} \overleftarrow{c}_{uv}^w k_{\lambda}^w G_{\lambda}$$

$$(8.14) \quad \sum_{\mu, \nu} k_{\mu}^u k_{\nu}^v c_{\mu\nu}^{\lambda} = \sum_w \overleftarrow{c}_{uv}^w k_{\lambda}^w \quad \text{for all } \lambda.$$

By Proposition 8.1 the LHS of (8.14) is finite and equals 0 for all but finitely many λ . Since $(-1)^{|\mu|+|\nu|-|\lambda|} c_{\mu\nu}^{\lambda} \in \mathbb{Z}_{\geq 0}$ and $(-1)^{\ell(w)-|\lambda|} k_{\lambda}^w \in \mathbb{Z}_{\geq 0}$ and we have (8.11), all terms on both sides of (8.14) have the same sign $(-1)^{\ell(u)+\ell(v)-|\lambda|}$. If $\overleftarrow{c}_{uv}^w \neq 0$ for infinitely many w , then either the RHS of (8.14) is nonzero for infinitely many λ , or the RHS of (8.14) is infinite for some λ , either of which is a contradiction. \square

The following result follows from Theorem 4.15.

Proposition 8.12. *We have $\Delta(B) \subset \Gamma \otimes B$, giving B the structure of a Γ -comodule.*

8.4. Adjoining Ω . The ring B has basis $\{\overleftarrow{\mathfrak{G}}_w \mid w \in S_{\mathbb{Z}}\}$. The ring $\Gamma \otimes R^+$ has basis $\{G_{\lambda} \otimes \mathfrak{G}_v \mid \lambda \in \mathbb{Y} \text{ and } v \in S_{\neq 0}\}$. By Theorem 4.15, we have a strict containment $B \subsetneq \Gamma \otimes R$. In this section, we show that the containment becomes an equality by adjoining the element $\Omega := \Omega[-x_-]$.

Example 8.13. By (4.28) we have

$$\begin{aligned}\overleftarrow{\mathfrak{G}}_{s_1} &= G_{s_1} + \mathfrak{G}_{s_1} - G_{s_1}\mathfrak{G}_{s_1} = G_1 + (1 - G_1)x_1 \\ x_1 &= \frac{1}{1 - G_1}(\overleftarrow{\mathfrak{G}}_{s_1} - \overleftarrow{\mathfrak{G}}_{s_0}) = \Omega(\overleftarrow{\mathfrak{G}}_{s_1} - \overleftarrow{\mathfrak{G}}_{s_0}).\end{aligned}$$

where we have used (4.33) for $r = 1$ to obtain $1 - G_1 = \Omega[-x_-]$ so that $(1 - G_1)^{-1} = \Omega$. The infinite expansion $\Omega = 1 + G_1 + G_1^2 + \dots$ shows that $\Omega \notin \Gamma \otimes R$.

The computation of Example 8.13 suggests the following result.

Theorem 8.14. *Every element of $\Gamma[\Omega] \otimes R^+$ is a finite linear combination of $\Omega^k \overleftarrow{\mathfrak{G}}_w$ for $(k, w) \in \mathbb{Z}_{\geq 0} \times S_{\mathbb{Z}}$. Thus $B[\Omega] \cong \Gamma[\Omega] \otimes R^+$.*

Define

$$(8.15) \quad \hat{\partial}_i = \tilde{\pi}_i - 1 = (1 - x_i)A_i.$$

The operators $\hat{\partial}_i$ satisfy the type $A_{\mathbb{Z}}$ -braid relations. The following identity is standard.

Lemma 8.15. *For all $w \in S_n$, we have*

$$(8.16) \quad \tilde{\pi}_w = \sum_{v \leq w} \hat{\partial}_v.$$

Let $\rho^{(n)} = (n-1, n-2, \dots, 1, 0) \in \mathbb{Z}^n$. We have $\mathfrak{G}_{w_0^{(n)}} = x^{\rho^{(n)}}$, where $w_0^{(n)} \in S_n$ is the longest element. The following can also be proved using the pipedream formula for \mathfrak{G}_w .

Lemma 8.16. *We have the identity*

$$(8.17) \quad \sum_{w \in S_n} (-1)^{\ell(w)} \mathfrak{G}_w = \prod_{i=1}^{n-1} (1 - x_i)^{n-i}.$$

Proof. We have

$$(8.18) \quad \sum_{w \in S_n} (-1)^{\ell(w)} \mathfrak{G}_w = \sum_{w \in S_n} (-1)^{\ell(w)} \tilde{\pi}_{w^{-1}w_0} \mathfrak{G}_{w_0}$$

$$(8.19) \quad = (-1)^{\ell(w_0)} \left(\sum_{w \in S_n} (-1)^{\ell(w)} \tilde{\pi}_w \right) x^{\rho^{(n)}}.$$

Now,

$$\begin{aligned}
(-1)^{\ell(w_0)} \sum_{w \in S_n} (-1)^{\ell(w)} \tilde{\pi}_w &= (-1)^{\ell(w_0)} \sum_{w \in S_n} (-1)^{\ell(w)} \sum_{v \leq w} \hat{\partial}_v \\
&= (-1)^{\ell(w_0)} \sum_{v \in S_n} (-1)^{\ell(v)} \hat{\partial}_v \sum_{w \geq v} (-1)^{\ell(w) - \ell(v)} \\
&= (-1)^{\ell(w_0)} \sum_{v \in S_n} (-1)^{\ell(v)} \hat{\partial}_v \delta_{v, w_0} \\
&= \hat{\partial}_{w_0}.
\end{aligned}$$

Using $\hat{\partial}_i(x_i f) = (1 - x_i)f$ for s_i -invariant f we have

$$\hat{\partial}_{n-1} \cdots \hat{\partial}_2 \hat{\partial}_1(x^{\rho^{(n)}}) = (1 - x_1)(1 - x_2) \cdots (1 - x_{n-1})x^{\rho^{(n-1)}}.$$

Since $\prod_{i=1}^{n-1} (1 - x_i)$ is S_{n-1} -invariant, induction completes the proof. \square

Lemma 8.17. *We have the identity*

$$\sum_{w \in S_n} (-1)^{\ell(w)} G_w = \Omega[-x_-]^{n-1}.$$

Proof. For $N \gg n$, applying $\theta_{w_0^{(N)}}$ to (8.17), by Corollary 8.6 we obtain

$$\sum_{w \in S_n} (-1)^{\ell(w)} G_w(x_1, \dots, x_N) = ((1 - x_1)(1 - x_2) \cdots (1 - x_N))^{n-1}.$$

Letting $N \rightarrow \infty$ we have

$$\sum_{w \in S_n} (-1)^{\ell(w)} G_w(x_+) = \prod_{i>0} (1 - x_i)^{n-1} = \Omega[-x_+]^{n-1}.$$

Now replace x_+ by x_- . \square

We will need the left weak order \leq_L . For $v, w \in S_{\mathbb{Z}}$, we have $v \leq_L w$ if $\ell(v) + \ell(wv^{-1}) = \ell(w)$.

Lemma 8.18. *Let $w \in S_+$ and define $J := \{i \in \mathbb{Z} \mid s_i w < w\}$. Then*

$$\overleftarrow{\mathfrak{G}}_w \in \Omega[-x_-]^{|J|} \mathfrak{G}_w + \sum_{v <_L w} \Gamma \mathfrak{G}_v.$$

Proof. It is not hard to check that

$$\{u \in S_{\mathbb{Z}} \mid u * w = w\} = S_J = \langle s_i \mid i \in J \rangle.$$

By (4.28), we have

$$\overleftarrow{\mathfrak{G}}_w \in \left(\sum_{u \in S_J} (-1)^{\ell(u)} G_u \right) \mathfrak{G}_w + \sum_{v <_L w} \Gamma \mathfrak{G}_v.$$

But S_J is the direct product of symmetric groups, so the result follows by applying Lemma 8.17 for each factor of the product. \square

Proof of Theorem 8.14. Let \mathcal{S} be the span of $\Omega^k \overleftarrow{\mathfrak{G}}_w$ for $(k, w) \in \mathbb{Z}_{\geq 0} \times S_{\mathbb{Z}}$. We show that $\mathfrak{G}_w \in \mathcal{S}$ for all $w \in S_+$ by induction on $\ell(w)$. Clearly $\mathfrak{G}_{\text{id}} = 1 = \overleftarrow{\mathfrak{G}}_{\text{id}} \in \mathcal{S}$. Let $\text{id} \neq w \in S_+$. By Lemma 8.18 and its notation, we have

$$\mathfrak{G}_w \in \Omega^{|\mathcal{J}|} \overleftarrow{\mathfrak{G}}_w + \sum_{v <_L w} \Gamma \mathfrak{G}_v.$$

By induction, we conclude that $\mathfrak{G}_v \in \mathcal{S}$. But elements of Γ are finite linear combinations of $G_\lambda = \overleftarrow{\mathfrak{G}}_{w_\lambda}$. By Theorem 8.11, we deduce that $\mathfrak{G}_w \in \mathcal{S}$.

Applying $\tilde{\omega}$, we deduce that $\mathfrak{G}_w \in \mathcal{S}$ for $w \in S_{\neq 0}$. \square

8.5. The algebra of double K -Stanley functions.

Let

$$\Gamma(x||a) = \bigoplus_{\lambda} R(a) G_\lambda(x||a)$$

be the $R(a)$ -subspace of $\hat{\Lambda}(x||a)$ spanned by the double K -Stanleys $G_\lambda(x||a)$.

Proposition 8.19. (1) For $\lambda \in \mathbb{Y}$, we have $G_\lambda(x/a) \in \Gamma(x||a)$.

(2) For $w \in S_{\mathbb{Z}}$, we have $G_w(x||a) \in \Gamma(x||a)$.

Proof. The first statement follows from (5.34). For the second statement, Proposition 5.27 expresses $G_w(x||a)$ as a finite $R(a)$ -linear combination of the $G_v(x/a)$, and Proposition 8.1 implies that $G_v(x/a)$ is a finite \mathbb{Z} -linear combination of the $G_\lambda(x/a)$. \square

For $w \in S_{\mathbb{Z}}$ and $\lambda \in \mathbb{Y}$ let $k_\lambda^w(a) \in R(a)$ be defined by

$$(8.20) \quad G_w(x||a) = \sum_{\lambda} k_\lambda^w(a) G_\lambda(x||a).$$

Example 8.20. Let $w = s_0 s_2$. We compute the expansion of $G_w(x||a)$ in terms of the Grassmannian double K -Stanley functions $G_\lambda(x||a)$ using Proposition 5.27. First, by Remark 8.3, we have $G_{s_0 s_2} = G_2 + G_{11} - G_{21}$. By Proposition 5.27, and noting that $\mathfrak{G}_{s_2}(a) \oplus \mathfrak{G}_{s_2}(\ominus a) = 0$, we have

$$\begin{aligned} G_{s_0 s_2}(x||a) &= G_{s_0}(x/a)(\mathfrak{G}_{s_2}(a) \oplus \mathfrak{G}_{s_2}(\ominus a)) \\ &\quad + G_{s_0 s_2}(x/a)(1 - \mathfrak{G}_{s_2}(a) \oplus \mathfrak{G}_{s_2}(\ominus a)) \\ &= G_2(x/a) + G_{11}(x/a) - G_{21}(x/a). \end{aligned}$$

Using the computations in Example 5.37 we have

$$\begin{aligned} G_{s_0 s_2}(x||a) &= -(1 - (a_1 \ominus a_0))G_{21}(x||a) + (1 - (a_1 \ominus a_0))G_2(x||a) \\ &\quad + (1 - (a_1 \ominus a_0))G_{11}(x||a) - (-(a_1 \ominus a_0))G_1(x||a). \end{aligned}$$

Example 8.21. Let $w = s_2 s_1$. The coefficient of $G_2(x/a)$ in $G_{s_2 s_1}(x||a)$ is $(1 - \mathfrak{G}_{s_2}(\ominus(a)))(1 - \mathfrak{G}_{s_1}(a))$ and the coefficient of $G_2(x||a)$ in $G_2(x||a)$ is $(1 - \mathfrak{G}_{s_1}(a))$. Since these are the highest degree terms, we have

$$k_2^{s_2 s_1} = (1 - \mathfrak{G}_{s_2}(\ominus(a)))(1 - \mathfrak{G}_{s_1}(a))^2 = \frac{1 - a_1}{1 - a_2} = 1 - (a_1 \ominus a_2).$$

This agrees with Example 6.12.

Theorem 8.22. The $R(a)$ -submodule $\Gamma(x||a) \subset \hat{\Lambda}(x||a)$ is a $R(a)$ -bialgebra.

Proof. By Proposition 8.19 and (5.28), the coproduct $\Delta(G_\lambda(x||a))$ is finite and thus belongs to $\Gamma(x||a) \otimes_{R(a)} \Gamma(x||a)$. We now consider the product. By (5.33), $G_\lambda(x||a)$ is a finite $R(a)$ -linear combination of $G_\mu(x/a)$. The structure constants for the family $\{G_\mu(x/a)\}$ are the same as for $\{G_\mu\} \subset \Gamma$, and thus finite by Proposition 5.27. It follows that $G_\lambda(x||a)G_\nu(x||a) \in \Gamma(x||a)$ for any $\lambda, \nu \in \mathbb{Y}$. \square

Define the total ordering \prec on \mathbb{Z} by

$$1 \prec 2 \prec 3 \prec \cdots \prec -2 \prec -1 \prec 0.$$

Conjecture 8.23. *For $w \in S_{\mathbb{Z}}$ and $\lambda \in \mathbb{Y}$, we have*

$$(8.21) \quad (-1)^{\ell(w)-|\lambda|} k_\lambda^w(a) \in \mathbb{Z}_{\geq 0}[-(a_i \ominus a_j) \mid i \prec j].$$

Furthermore, if $d_\nu^{\lambda\mu}(a)$ denotes the coproduct structure constants of the basis $G_\lambda(x||a)$ of $\Gamma(x||a)$, then

$$(8.22) \quad (-1)^{|\lambda|+|\mu|-|\nu|} d_\nu^{\lambda\mu}(a) \in \mathbb{Z}_{\geq 0}[-(a_i \ominus a_j) \mid i \prec j].$$

For product structure constants, see (8.28). For double Stanley symmetric functions an analogous positivity is proven in [LLS21, Theorem 4.22]. Similarly to [LSS, (5.4)], we have

$$(8.23) \quad d_\nu^{\lambda\mu}(a) = \sum_{\substack{w \in S_{\mathbb{Z}} \\ w * w_\mu = w_\nu}} (-1)^{|\nu|-|\mu|-\ell(w)} k_\lambda^w(a).$$

In particular, (8.22) follows from (8.21).

Example 8.24. Example 8.20 exhibits the conjectured positivity of $k_\lambda^w(a)$ for $w = s_0s_2$. Now let $\lambda = (1)$, $\mu = (2)$, and $\nu = (21)$. By (8.23), we can calculate the coproduct structure constant in two ways:

$$\begin{aligned} d_{21}^{1,2} &= k_1^{s-1} - k_1^{s-1s_1} = 1 - (a_1 \ominus a_0) \\ d_{21}^{2,1} &= k_2^{s-1s_1} - k_2^{s-1s_1s_0} = (1 - (a_1 \ominus a_0)) - 0. \end{aligned}$$

We have used Example 8.20, and the equality $G_{s_{-1}s_1}(x||a) = G_{s_0s_2}(x||a)$ that can be verified using Proposition 5.27.

8.6. The subring of back stable double Grothendieck polynomials. Let

$$B(x; a) = \bigoplus_{w \in S_{\mathbb{Z}}} R(a) \overleftarrow{\mathfrak{G}}_w(x; a)$$

be the $R(a)$ -submodule spanned by the back stable double Grothendieck polynomials $\overleftarrow{\mathfrak{G}}_w(x; a)$. For $u, v, w \in S_{\mathbb{Z}}$ define $\overleftarrow{c}_{uv}^w(a) \in R(a)$ by the formal expansion

$$(8.24) \quad \overleftarrow{\mathfrak{G}}_u(x; a) \overleftarrow{\mathfrak{G}}_v(x; a) = \sum_{w \in S_{\mathbb{Z}}} \overleftarrow{c}_{uv}^w(a) \overleftarrow{\mathfrak{G}}_w(x; a).$$

The existence of the expansion (8.24) follows from Theorem 5.25 and Proposition 2.5. By Proposition 5.22, we have

$$(8.25) \quad \overleftarrow{c}_{\gamma(u)\gamma(v)}^{\gamma(w)}(a) = \gamma \overleftarrow{c}_{uv}^w(a).$$

For $u, v, w \in S_+$, define $c_{uv}^w(a) \in R(a)$ by the expansion

$$(8.26) \quad \mathfrak{G}_u(x; a) \mathfrak{G}_v(x; a) = \sum_{w \in S_+} c_{uv}^w(a) \mathfrak{G}_w(x; a).$$

Proposition 8.25. *Let $u, v, w \in S_{\mathbb{Z}}$. For q such that $\gamma^q(u), \gamma^q(v), \gamma^q(w) \in S_+$, we have*

$$\overleftarrow{c}_{uv}^w(a) = \gamma^{-q} c_{\gamma^q(u)\gamma^q(v)}^{\gamma^q(w)}(a).$$

Proof. Define the GKM ring Ψ_+ to be the set of $\psi \in \text{Fun}(S_+, R(T))$ such that

$$(8.27) \quad 1 - e^\alpha \mid \psi(s_\alpha w) - \psi(w) \quad \text{for all } \alpha \in \Phi \text{ and } w \in S_+.$$

One may show that Ψ_+ is an $R(T)$ -subalgebra of $\text{Fun}(S_+, R(T))$ and that $\Psi_+ = \prod_{z \in S_+} R(T)\psi_+^z$, where ψ_+^z is the restriction of $\psi^z \in \Psi$ to $S_+ \subset S_{\mathbb{Z}}$. Using Proposition 5.24, we see that an analogue of Theorem 5.25 holds Ψ_+ replacing Ψ and $\{\mathfrak{G}_z \mid z \in S_+\}$ replacing $\{\overleftarrow{\mathfrak{G}}_z \mid z \in S_{\mathbb{Z}}\}$. By triangularity (2.10), the coefficient of ψ^w in the product $\psi^u \psi^v$ can be obtained from a finite computation that only involves the values of the values of various ψ^z -s on the lower order ideal in $S_{\mathbb{Z}}$ generated by u, v, w . When $\gamma^q(u), \gamma^q(v), \gamma^q(w) \in S_+$, this lower order ideal is contained in S_+ , so $c_{\gamma^q(u)\gamma^q(v)}^{\gamma^q(w)}(a) = \overleftarrow{c}_{\gamma^q(u)\gamma^q(v)}^{\gamma^q(w)}(a)$. The result then follows from (8.25). \square

It is shown in [AGM] that $(-1)^{\ell(u)+\ell(v)-\ell(w)} c_{uv}^w(a) \in \mathbb{Z}_{\geq 0}[-(a_j \ominus a_i) \mid i < j]$.

Example 8.26. We have $\mathfrak{G}_{s_1}(x; a) = x_1 \ominus a_1$ and $\mathfrak{G}_{s_2 s_1}(x; a) = (x_1 \ominus a_1)(x_1 \ominus a_2)$. Thus

$$\begin{aligned} \mathfrak{G}_{s_1}(x; a)^2 &= (x_1 \ominus a_1)^2 \\ &= (x_1 \ominus a_1)((x_1 \ominus a_2) \oplus (a_2 \ominus a_1)) \\ &= (1 - (a_2 \ominus a_1))\mathfrak{G}_{s_2 s_1}(x; a) + (a_2 \ominus a_1)\mathfrak{G}_{s_1}(x; a). \end{aligned}$$

We have $(1 - (a_2 \ominus a_1)) \in \mathbb{Z}_{\geq 0}[-(a_j \ominus a_i) \mid i < j]$ and $(-1)^1(a_2 \ominus a_1) \in \mathbb{Z}_{\geq 0}[-(a_j \ominus a_i) \mid i < j]$.

It follows from Proposition 8.25 that we have

$$(8.28) \quad (-1)^{\ell(u)+\ell(v)-\ell(w)} \overleftarrow{c}_{uv}^w(a) \in \mathbb{Z}_{\geq 0}[-(a_j \ominus a_i) \mid i < j].$$

Conjecture 8.27. *For fixed $u, v \in S_{\mathbb{Z}}$, only finitely many \overleftarrow{c}_{uv}^w are nonzero. That is,*

$$(8.29) \quad B(x; a) \text{ is a } R(a)\text{-subalgebra of } \overleftarrow{R}(x; a).$$

Proposition 8.28. *Suppose that the positivity (8.21) holds. Then Conjecture 8.27 holds.*

Proof. The argument is the same as the proof of Theorem 8.11. Applying η_a to (8.24), we obtain

$$G_u(x||a)G_v(x||a) = \sum_w \overleftarrow{c}_{uv}^w(a)G_w(x||a).$$

By equation (8.20) we have

$$(8.30) \quad G_u(x||a)G_v(x||a) = \sum_{\mu, \nu, \lambda} k_\mu^u(a)k_\nu^v(a)c_{\mu\nu}^\lambda(a)G_\lambda(x||a)$$

$$(8.31) \quad \sum_w \overleftarrow{c}_{uv}^w(a)G_w(x||a) = \sum_{w, \lambda} \overleftarrow{c}_{uv}^w k_\lambda^w(a)G_\lambda(x||a)$$

$$(8.32) \quad \sum_{\mu, \nu} k_\mu^u(a)k_\nu^v(a)c_{\mu\nu}^\lambda(a) = \sum_w \overleftarrow{c}_{uv}^w k_\lambda^w(a) \quad \text{for all } \lambda.$$

Let $P = \mathbb{Z}_{\geq 0}[a_i \ominus a_j \mid i < j]$. By Proposition 8.19 and Theorem 8.22, the left side of (8.32) is finite and equals 0 for all but finitely many λ . Since $c_{\mu\nu}^\lambda(a) \in (-1)^{|\mu|+|\nu|-|\lambda|}P$, and by assumption $k_\lambda^w(a) \in (-1)^{\ell(w)-|\lambda|}P$, and we have (8.28), all terms on both sides of (8.32) belong to $(-1)^{\ell(u)+\ell(v)-|\lambda|}P$. If $\overleftarrow{c}_{uv}^w(a) \neq 0$ for infinitely many w , then either the RHS of (8.32) is nonzero for infinitely many λ , or the RHS of (8.32) is infinite for some λ , either of which is a contradiction. \square

The following follows from Theorem 5.31.

Proposition 8.29. *The algebra $\Gamma(x||a)[\Omega(x/a)]$ is a Hopf algebra over $R(a)$. The $R(a)$ -algebra $B(x; a)$ is a $\Gamma(x||a)[\Omega(x/a)]$ Hopf-comodule.*

8.7. Adjoining $\Omega(x/a)$. In this subsection, we assume that Conjecture 8.27 holds, that is, $B(x; a)$ is a $R(a)$ -subalgebra of $\overleftarrow{R}(x; a)$. We then compare the two subalgebras $\Gamma(x||a) \otimes_{R(a)} R(x; a)^+$ and $B(x; a)$ by adjoining the element $\Omega(x/a)$.

Remark 8.30. The element $\Omega(x/a)$ has the following geometric interpretation: it is the class $[\mathcal{L}_{\Lambda_0}]$ (in an appropriate equivariant K -group of $\overline{\mathrm{Fl}}$) of the line bundle with fundamental weight Λ_0 . Indeed, in general the class $[\mathcal{O}_{X_{s_i}}]$ is equal to $1 - [\mathcal{L}_{-\Lambda_i}]$. Thus $[\mathcal{L}_{\Lambda_0}] = (1 - \overleftarrow{\mathfrak{G}}_{s_0}(x; a))^{-1} = \Omega(x/a)$.

Proposition 8.31. *Assume (8.29). Then*

$$(8.33) \quad \Gamma(x||a)[\Omega(x/a)] \otimes_{R(a)} R(x; a)^+ \subset \sum_{(k,w) \in \mathbb{Z}_{\geq 0} \times S_{\mathbb{Z}}} R(a)\Omega(x/a)^k \overleftarrow{\mathfrak{G}}_w(x; a).$$

Thus $B(x; a)[\Omega] \cong \Gamma(x||a)[\Omega(x/a)] \otimes_{R(a)} R(x; a)^+$.

Proof. Let R' be the right hand side of (8.33). Since $\Gamma(x||a)$ is spanned by the elements $G_\lambda(x||a) = \overleftarrow{\mathfrak{G}}_{w_\lambda}(x; a)$ and assuming (8.29) the proof reduces to showing that $\mathfrak{G}_w(x; a) \in R'$ for $w \in S_{\neq 0}$. Using $\tilde{\omega}$ one may further reduce to $w \in S_+$ which we now assume.

Arguing as in Lemma 8.18 and using its notation, let $J = \{i \mid s_i w < w\}$ and let $S_J = \langle s_j \mid j \in J \rangle$. By (5.29) we have

$$(8.34) \quad \overleftarrow{\mathfrak{G}}_w(x; a) \in \mathfrak{G}_w(x; a) \sum_{u \in S_J} (-1)^{\ell(u)} G_u(x||a) + \sum_{v <_L w} \Gamma(x||a) \mathfrak{G}_v.$$

By Proposition 5.27 we have

$$\begin{aligned} \sum_{u \in S_J} (-1)^{\ell(u)} G_u(x||a) &= \sum_{u \in S_J} (-1)^{\ell(u)} \sum_{u_1 * u_2 * u_3 = u} (-1)^{\ell(u_1) + \ell(u_2) + \ell(u_3) - \ell(u)} \times \\ &\quad G_{u_1^{-1}}(\ominus(a)) G_{u_2}(x/a) G_{u_3}(a) \\ &= \sum_{u_1 \in S_J} (-1)^{\ell(u_1)} G_{u_1^{-1}}(\ominus(a)) \sum_{u_2 \in S_J} (-1)^{\ell(u_2)} G_{u_2}(x/a) \\ &\quad \times \sum_{u_3 \in S_J} (-1)^{\ell(u_3)} \mathfrak{G}_{u_3}(a) \\ &= \sum_{u_2 \in S_J} (-1)^{\ell(u_2)} G_{u_2}(x/a) \\ &= \Omega(a/x)^{|J|} \end{aligned}$$

where in the last step we have applied the superization of Lemma 8.17. Therefore $\overleftarrow{\mathfrak{G}}_w(x; a) \in \Omega(a/x)^{|J|} \mathfrak{G}_w(x; a) + \sum_{v <_L w} \Gamma(x||a) \mathfrak{G}_v(x; a)$. The Proposition holds by induction. \square

9. DETERMINANTAL FORMULAE FOR $G_\lambda(x||a)$

In this section, we recover the equality of the Grassmannian double K -Stanley functions as Grassmannian back stable Grothendieck polynomials, with the determinantal formulae in the literature.

Proposition 9.1. *For $r \geq 1$, we have*

$$\begin{aligned} G_r(x||a) &= \gamma_a^{r-1} G_r(x/a) \\ G_{1^r}(x||a) &= \gamma_a^{1-r} G_{1^r}(x/a). \end{aligned}$$

Proof. We prove the second formula as the first follows from it by Propositions 5.34, 5.14, and 5.13.

Let $c_r := w_{(1^r)} = s_{1-r} \cdots s_{-1}s_0$ and $c_{r/p} = w_{(1^r)/(1^p)} = s_{1-r} \cdots s_{-p}$ for $0 \leq p \leq r$. The 0-Hecke factorizations of c_r with left factor in $S_{\neq 0}$ are given by $c_{r/p} * c_p$ for $1 \leq p \leq r$ and $c_{r/(p-1)} * c_p$ for $2 \leq p \leq r$. Letting $A = (a_0, a_{-1}, \dots, a_{2-r})$ we have

$$\mathfrak{G}_{c_{r/p}^{-1}}(\ominus a) = \mathfrak{G}_{s_p s_{p+1} \cdots s_{r-1}}[A] = G_{r-p}[A].$$

By (5.18) we have

$$\begin{aligned} G_{(1^r)}(x||a) &= \sum_{p=1}^r \mathfrak{G}_{c_{r/p}^{-1}}(\ominus a) G_{(1^p)}(x/a) - \sum_{p=2}^r \mathfrak{G}_{c_{r/(p-1)}^{-1}}(\ominus a) G_{(1^p)}(x/a) \\ &= \sum_{p=1}^r (G_{r-p}[A] - G_{r-p+1}[A]) G_{(1^p)}(x/a) \end{aligned}$$

using the fact that $G_{1^r}[A] = 0$ because A consists of $r-1$ variables. We have $\gamma_a^{1-r}(p_1(x||a)) = p_1(x||a) + A$ and

$$\gamma_a^{1-r} G_{1^r}(x/a) = G_{1^r}[x_{\leq 0} - a_{\leq 0} + A].$$

By [Buc02a] we have

$$\Delta(G_{1^r}) = \sum_{p=0}^r G_{1^p} \otimes G_{1^{r-p}} - \sum_{p=1}^r G_{1^p} \otimes G_{1^{r+1-p}}.$$

Since we are using the coproduct such that $p_r(x/a)$ are primitive we may superize this formula, replacing K -Stanleys by superized K -Stanleys. We obtain

$$\begin{aligned} \gamma_a^{1-r} G_{1^r}(x/a) &= \sum_{p=0}^r G_{1^p}(x/a) G_{1^{r-p}}[A] - \sum_{p=1}^r G_{1^p}(x/a) G_{1^{r+1-p}}[A] \\ &= \sum_{p=1}^r G_{1^p}(x/a) G_{1^{r-p}}[A] - \sum_{p=1}^r G_{1^p}(x/a) G_{1^{r+1-p}}[A] \end{aligned}$$

as required. \square

Lemma 9.2. *If $s_i^a g = g$ then $\tilde{\pi}_i^{a,\ominus}(fg) = \tilde{\pi}_i^{a,\ominus}(f)g$.*

Lemma 9.3. *For $p, q \in \mathbb{Z}$ and $r \in \mathbb{Z}_{>0}$, we have*

$$(9.1) \quad \tilde{\pi}_p^{a,\ominus} \gamma_a^q G_r(x/a) = \begin{cases} \gamma_a^q G_r(x/a) & \text{if } p \neq q \\ \gamma_a^{q-1} G_{r-1}(x/a) & \text{if } p = q. \end{cases}$$

Proof. We have

$$\tilde{\pi}_p^{a,\ominus} \gamma_a^q G_r(x/a) = \gamma_a^q \tilde{\pi}_{p-q}^{a,\ominus} G_r(x/a).$$

If $p \neq q$ then $s_{p-q}^a G_r(x/a) = G_r(x/a)$ and $\gamma_a^q \tilde{\pi}_{p-q}^{a,\ominus} G_r(x/a) = \gamma_a^q G_r(x/a)$. If $p = q$ then

$$\begin{aligned} \gamma_a^q \tilde{\pi}_0^{a,\ominus} G_r(x/a) &= \gamma_a^q \tilde{\pi}_0^{a,\ominus} \gamma_a^{1-r} \gamma_a^{r-1} G_r(x/a) \\ &= \gamma_a^{q+1-r} \tilde{\pi}_{r-1}^{a,\ominus} G_r(x||a) \\ &= \gamma_a^{q+1-r} G_{r-1}(x||a) \\ &= \gamma_a^{q-1} G_{r-1}(x/a). \end{aligned} \quad \square$$

In the following, we denote by $\binom{n}{k}$ the signed binomial coefficient, the coefficient of x^k in the power series $(1+x)^n$ for any integer n . In particular for $n \geq 0$, $\binom{-n}{k} = (-1)^k \binom{k+n-1}{n-1}$.

Formulas (9.2) and (9.3) are due to [HIMN] and [And17] respectively.

Proposition 9.4. *With $\ell = \ell(\lambda)$,*

$$(9.2) \quad G_\lambda(x||a) = \det \left(\gamma_a^{\lambda_i - i} \sum_{k \geq 0} (-1)^k \binom{i-j}{k} G_{\lambda_i - i + j + k}(x/a) \right)_{1 \leq i, j \leq \ell}$$

$$(9.3) \quad G_\lambda(x||a) = \det \left(\gamma_a^{i - \lambda'_i} \sum_{p \geq 0} (-1)^p \binom{p + \lambda'_i - 1}{\lambda'_i - 1} e_{\lambda'_i - i + j + p}(x/a) \right)_{1 \leq i, j \leq \lambda_1}.$$

Proof. Assuming (9.2) we derive (9.3) as follows. Applying $\tilde{\omega}$ to (9.2) for $G_{\lambda'}(x||a)$, by Propositions 5.34, 5.13, 4.19, and 4.20 we have

$$\begin{aligned}
G_{\lambda}(x||a) &= \tilde{\omega} \det \left(\gamma_a^{\lambda'_i - i} \sum_{k \geq 0} (-1)^k \binom{i-j}{k} G_{\lambda'_i - i + j + k}(x/a) \right) \\
&= \det \left(\gamma_a^{i - \lambda'_i} \sum_{k \geq 0} (-1)^k \binom{i-j}{k} G_{1^{\lambda'_i - i + j + k}}(x/a) \right) \\
&= \det \left(\gamma_a^{i - \lambda'_i} \sum_{k \geq 0} (-1)^k \binom{i-j}{k} \times \right. \\
&\quad \left. \sum_{m \geq 0} (-1)^m \binom{m + \lambda'_i - i + j + k - 1}{m} e_{m + \lambda'_i - i + j + k}(x/a) \right) \\
&= \det \left(\gamma_a^{i - \lambda'_i} \sum_{p \geq 0} (-1)^p e_{\lambda'_i - i + j + p}(x/a) \times \right. \\
&\quad \left. \sum_{k+m=p} \binom{i-j}{k} \binom{p + \lambda'_i - i + j - 1}{m} \right) \\
&= \det \left(\gamma_a^{i - \lambda'_i} \sum_{p \geq 0} (-1)^p e_{\lambda'_i - i + j + p}(x/a) \binom{p + \lambda'_i - 1}{p} \right).
\end{aligned}$$

To prove (9.2) let

$$L_{ij}^{(m)} = \gamma_a^{m-i} \sum_{k \geq 0} (-1)^k \binom{i-j}{k} G_{m-i+j+k}(x/a)$$

so that $H_{\lambda} := \det L_{ij}^{(\lambda_i)}$ is the right hand side of (9.2). By induction it suffices to show that

$$(9.4) \quad \tilde{\pi}_p^{a,\ominus} H_{\lambda} = \begin{cases} H_{\mu} & \text{if } p = \lambda_i - i \text{ and } \lambda/\mu \text{ is a box in the } i\text{-th row} \\ H_{\lambda} & \text{otherwise.} \end{cases}$$

By Lemma 9.3, every entry $L_{ij}^{(\lambda_i)}$ in the i -th row is s_p^a -invariant for all p except $p = \lambda_i - i$.

Suppose $p \neq \lambda_i - i$ for all i . We have $s_p^a(H_{\lambda}) = H_{\lambda}$ and $\tilde{\pi}_p^{a,\ominus} H_{\lambda} = H_{\lambda}$ which agrees with (9.4).

Otherwise let $p = \lambda_i - i$ for some i . Since this p is necessarily unique it follows that $\tilde{\pi}_p^{a,\ominus}$ fixes all of the entries in the rows of the determinant other than the i -th. By Lemma 9.3 we have

$$(9.5) \quad \tilde{\pi}_{\lambda_i - i}^{a,\ominus} L_{ij}^{(\lambda_i)} = L_{ij}^{(\lambda_i - 1)}.$$

If λ has a removable box in the i -th row and μ is the partition obtained by removing a box in row i then we have $L_{qj}^{(\mu_q)} = L_{qj}^{(\lambda_q)}$ for $q \neq i$. By multilinearity of the determinant, (9.4) follows from (9.5).

Otherwise, let $\ell := \lambda_{i+1} = \lambda_i$. Define $F_m := \gamma_a^{-1} G_m(x/a)$. Then by (9.6) with $d = 1$, and using $G_r[a_0] = a_0^r$, we have

$$G_m(x/a) = \frac{1}{1 - a_0} (F_m - a_0 F_{m-1}).$$

Thus

$$\begin{aligned} \gamma_a^{i-\ell} L_{ij}^{(\ell)} &= \sum_{k \geq 0} (-1)^k \binom{i-j}{k} G_{\ell-i+j+k}(x/a) \\ &= \frac{1}{1 - a_0} \sum_{k \geq 0} (-1)^k \binom{i-j}{k} (F_{\ell-i+j+k} - a_0 F_{\ell-i+j+k-1}) \\ \gamma_a^{i-\ell} L_{i+1,j}^{(\ell)} &= \gamma_a^{-1} \sum_{k \geq 0} (-1)^k \binom{i+1-j}{k} G_{\ell-i+j+k-1}(x/a) \\ &= \sum_{k \geq 0} (-1)^k \binom{i+1-j}{k} F_{\ell-i+j+k-1}. \end{aligned}$$

We have

$$\begin{aligned} \gamma_a^{i-\ell} L_{ij}^{(\ell)} + \frac{1}{1 - a_0} \gamma_a^{i-\ell} L_{i+1,j}^{(\ell)} &= \frac{1}{1 - a_0} \sum_{k \geq 0} (-1)^k \left(\binom{i-j}{k} (F_{\ell-i+j+k} - a_0 F_{\ell-i+j+k-1}) \right. \\ &\quad \left. + \binom{i+1-j}{k} F_{\ell-i+j+k-1} \right) \\ &= \frac{-a_0 F_{\ell-i+j-1} + F_{\ell-i+j-1}}{1 - a_0} \\ &\quad + \frac{1}{1 - a_0} \sum_{k \geq 0} (-1)^k \left(\binom{i-j}{k} + a_0 \binom{i-j}{k+1} - \binom{i+1-j}{k+1} \right) F_{\ell-i+j+k} \\ &= F_{\ell-i+j-1} + \sum_{k \geq 0} (-1)^{k+1} \binom{i-j}{k+1} F_{\ell-i+j+k} \\ &= \gamma_a^{i-\ell} L_{ij}^{(\ell-1)}. \end{aligned}$$

Applying $\gamma_a^{\ell-i}$, we obtain $L_{ij}^{(\ell-1)} = L_{ij}^{(\ell)} + (\gamma_a^{\ell-i} \frac{1}{1-a_0} \gamma_a^{i-\ell}) L_{i+1,j}^{(\ell)}$, showing that $\tilde{\pi}_p^{a,\oplus} H_\lambda = H_\lambda$ in this case. \square

Example 9.5. Let $\lambda = (1, 1)$. We have

$$G_{11}(x||a) = \begin{vmatrix} G_1 & G_2 + G_3 + G_4 + \cdots \\ \gamma_a^{-1}(1 - G_1) & \gamma_a^{-1}(G_1) \end{vmatrix}.$$

Note that 0 is not the residue of a corner box. We have

$$\tilde{\pi}_0^{a,\oplus} G_{11}(x||a) = \begin{vmatrix} 1 & \gamma_a^{-1}(G_1 + G_2 + G_3 + \cdots) \\ \gamma_a^{-1}(1 - G_1) & \gamma_a^{-1}(G_1) \end{vmatrix}.$$

The proof of Proposition 9.4 shows that

$$1 = G_1 + \frac{1}{1-a_0} \gamma_a^{-1}(1-G_1)$$

$$\gamma_a^{-1}(G_1 + G_2 + G_3 + \dots) = (G_2 + G_3 + G_4 + \dots) + \frac{1}{1-a_0} \gamma_a^{-1}(G_1)$$

so $\tilde{\pi}_0^{a,\ominus} G_{11}(x||a) = G_{11}(x||a)$.

Example 9.6. Let $\lambda = (1, 1)$. We check (9.2). First, suppose that $a = 0$. Using e.g. [Buc02a, Thm. 5.4] we have $G_r G_1 = G_{r+1} + G_{r,1} - G_{r+1,1}$ for all $r \geq 1$.

$$\begin{aligned} & \det \begin{pmatrix} G_1 & G_2 + G_3 + \dots \\ 1 - G_1 & G_1 \end{pmatrix} \\ &= G_1^2 - (1 - G_1)(G_2 + G_3 + \dots) \\ &= (G_2 + G_{11} - G_{21}) - (G_2 + G_3 + \dots) \\ &+ ((G_3 + G_{21} - G_{31}) + (G_4 + G_{31} - G_{41}) + \dots) \\ &= G_{11}. \end{aligned}$$

Now consider the general double case. By e.g. [Buc02a, §6] and superizing we have

$$\Delta(G_r(x/a)) = \sum_{p=0}^r G_p(x/a) \otimes G_{r-p}(x/a) - \sum_{p=1}^r G_p(x/a) \otimes G_{r+1-p}(x/a).$$

For $d \geq 0$ and letting $A = a_0 + a_{-1} + \dots + a_{1-d}$ we have

$$(9.6) \quad \gamma_a^{-d}(G_r(x/a)) = \sum_{p=0}^r G_p(x/a) G_{r-p}[A] - \sum_{p=1}^r G_p(x/a) G_{r+1-p}[A].$$

For $d = 1$ we have

$$\begin{aligned} \gamma_a^{-1}(G_1(x/a)) &= G_1[a_0] + G_1(x/a) - G_1(x/a)G_1[a_0] \\ &= a_0 + (1 - a_0)G_1(x/a). \\ & \det \begin{pmatrix} G_1(x/a) & (G_2(x/a) + G_3(x/a) + \dots) \\ \gamma_a^{-1}(1 - G_1(x/a)) & \gamma_a^{-1}(G_1(x/a)) \end{pmatrix} \\ &= G_1(x/a)(a_0 + (1 - a_0)G_1(x/a)) \\ &- (1 - a_0 - (1 - a_0)G_1(x/a))(G_2(x/a) + G_3(x/a) + \dots) \\ &= a_0 G_1(x/a) + (1 - a_0)G_1(x/a)^2 \\ &- (1 - a_0)(1 - G_1(x/a))(G_2(x/a) + G_3(x/a) + \dots) \\ &= a_0 G_1(x/a) + (1 - a_0)G_{11}(x/a). \end{aligned}$$

Let us compare with Proposition 9.1.

$$\Delta(G_{1^r}(x/a)) = \sum_{p=0}^r G_{1^p}(x/a) \otimes G_{1^{r-p}}(x/a) - \sum_{p=1}^r G_{1^p}(x/a) \otimes G_{1^{r+1-p}}(x/a).$$

$$\begin{aligned} G_{11}(x||a) &= \gamma_a^{-1} G_{11}(x/a) \\ &= G_1(x/a) G_1[a_0] + G_{11}(x/a) - G_1(x/a) G_{11}[a_0] - G_{11}(x/a) G_1[a_0] \\ &= a_0 G_1(x/a) + (1 - a_0)G_{11}(x/a) \end{aligned}$$

which agrees with the determinant.

10. DEGENERACY LOCI

By [Buc02b, Theorem 2.1] it is known that double Grothendieck polynomials are the universal formulas for K -classes of certain degeneracy loci based on quivers. Since back stable double Grothendiecks are certain limits of double Grothendieck polynomials, they can also be computed by universal quiver locus formulas. Following the suggestion of Buch [Buc+] we apply such a quiver formula and recover one of our formulas for back stable Grothendieck polynomials.

Without loss of generality we take $w \in S_+$ and then $w \in S_n$. Let $m \geq 0$ be a non-negative integer and let $x_-^{(m)} = (x_{1-m}, \dots, x_{-1}, x_0)$ and $a_-^{(m)} = (a_{1-m}, \dots, a_{-1}, a_0)$ be sets of m variables with nonpositive indices. Let $\gamma^m(w)$ be the m -fold forward shift of w . By definition

$$(10.1) \quad \overleftarrow{\mathfrak{G}}_w = \lim_{m \rightarrow \infty} \mathfrak{G}_{\gamma^m(w)}(x_-^{(m)}, x; a_-^{(m)}, a).$$

We now apply a formula for double Grothendieck polynomials which is a variant of [BKY05, Theorem 4] but whose overall form more closely follows [BKY04, Theorem 4]. We use a set of m variables followed by $n-1$ sets of one variable each. For the x variables we use $x_-^{(m)}$ and then x_1, x_2, \dots, x_{n-1} . The equivariant a variables are similarly grouped. This grouping of the variables is *compatible* with $\gamma^m(w)$ and $\gamma^m(w^{-1}) = (\gamma^m(w))^{-1}$ in the language of [BKY04]. Our variant of [BKY05, Theorem 4] states that

$$(10.2) \quad \mathfrak{G}_{\gamma^m(w)}(x; a) = \sum_{\lambda^\bullet} (-1)^{|\lambda^\bullet| - \ell(w)} c_{\lambda^\bullet}^w G_{\lambda^{(1-n)}}[-a_{n-1}] \cdots G_{\lambda^{(-1)}}[-a_1] \times \\ G_{\lambda^{(0)}}[x_-^{(m)}/a_-^{(m)}] G_{\lambda^{(1)}}[x_1] \cdots G_{\lambda^{(n-1)}}[x_{n-1}]$$

where $\lambda^\bullet = (\lambda^{(j)} \mid 1-n \leq j \leq n-1)$ runs over tuples of partitions and $c_{\lambda^\bullet}^w$ is the number of tableau tuples $T^\bullet = (T^{(1-n)}, \dots, T^{(n-1)})$ such that $T^{(j)}$ is a *decreasing* tableau (one whose rows strictly decrease from left to right and whose columns strictly decrease from top to bottom) of shape $\lambda^{(j)}$ for $1-n \leq j \leq n-1$ such that the juxtaposition of the column-reading words $T^{(1-n)} \cdots T^{(0)} \cdots T^{(n-1)}$ is 0-Hecke equivalent to w , and for $1 \leq i \leq n-1$, the entries of $T^{(\pm i)}$ are at least i .

Remark 10.1.

- To compare the formulas it is better to look at [BKY04, Theorem 4]. Here we use decreasing tableaux instead of increasing, which allows us to avoid transposing the shapes of the tableaux. We are using a different form of double Grothendieck polynomial than [BKY05]; see Remark 5.2.
- The tuples of tableaux are in straightforward bijection with pipedreams; the labels in a row of a decreasing tableau tell where to place the crossings in a row of the corresponding pipedream.

We observe that for a single variable z , $G_\lambda[z] = 0$ unless λ is a single row, say (r) , in which case $G_r[z] = z^r$. Similarly, $G_\lambda[-z] = 0$ unless λ is a single column, say (1^r) , in which case one may show that $G_{1^r}[-z] = (\ominus z)^r$. Therefore we may assume that $\lambda^{(j)}$ is a single row for $1 \leq j \leq n-1$ and a single column for $1-n \leq j \leq -1$. Next we observe that the Fomin-Kirillov formula for Grothendieck polynomials can

be rewritten as follows. For $z \in S_n$ we have

$$\mathfrak{G}_z(x_1, \dots, x_{n-1}) = \sum_{T^\bullet} (-1)^{\ell(T^\bullet) - \ell(z)} \prod_{j=1}^{n-1} x_j^{\ell(T^{(j)})}$$

where the sum runs over $T^\bullet = (T^{(1)}, \dots, T^{(n-1)})$ where $T^{(j)}$ is a single row decreasing tableau whose entries are at least j , and such that $T^{(1)} \dots T^{(n-1)}$ is Hecke equivalent to z .

Applying inverses and evaluating at $(\ominus(a_1), \dots, \ominus(a_{n-1}))$, for $u \in S_n$ we see that

$$\mathfrak{G}_{u^{-1}}(\ominus(a_1), \dots, \ominus(a_{n-1})) = \sum_{U^\bullet} (-1)^{\ell(U^\bullet) - \ell(u)} \prod_{j=1}^{n-1} (\ominus a_j)^{\ell(U^{(j)})}$$

where $U^\bullet = (U^{(1)}, \dots, U^{(n-1)})$ runs over tuples with $U^{(j)}$ a decreasing tableau of single column shape with entries at least j , with $U^{(n-1)} \dots U^{(1)}$ Hecke equivalent to u .

Finally we use Remark 8.3 for the expansion of a super K -Stanley function $G_v(x_-^{(m)}/a_-^{(m)})$ into Grassmannian super K -Stanleys $G_\lambda(x_-^{(m)}/a_-^{(m)})$.

Combining the above, we see that (10.2) becomes

$$\begin{aligned} & \mathfrak{G}_{\gamma^m(w)}(x_-^{(m)}, x; a_-^{(m)}, a) \\ &= \sum_{u*v*z=w} (-1)^{\ell(u) + \ell(v) + \ell(z) - \ell(w)} \mathfrak{G}_{u^{-1}}(\ominus(a)) G_v(x_-^{(m)}/a_-^{(m)}) \mathfrak{G}_z(x). \end{aligned}$$

Sending m to infinity we recover the triple coproduct formula Proposition 5.17.

11. FURTHER DIRECTIONS

11.1. Ideal sheaf basis.

For $w \in S_n$, define

$$\mathfrak{G}_w^\partial(x; a) := \sum_{\substack{v \in S_n \\ v \geq w}} (-1)^{\ell(vw)} \mathfrak{G}_v(x; a).$$

These polynomials represent the ideal sheaf basis in the equivariant cohomology of finite flag varieties, and depend on n . They can be generated by the operators $\tilde{\pi}_i^x - 1$, with $\mathfrak{G}_{w_0}^\partial(x; a) := \prod_{i+j \leq n} (x_i \ominus a_j - 1)$. For $w \in S_{\mathbb{Z}}$, define the ideal sheaf backstable classes

$$\tilde{\psi}_\partial^w := \sum_{v \geq w} (-1)^{\ell(vw)} \tilde{\psi}^v \in \tilde{\Psi}.$$

Note that the value $\tilde{\psi}_\partial^w(u)$ is well-defined for $u \in S_{\mathbb{Z}}$ since by (2.10), $\tilde{\psi}^v(u) = 0$ unless $u \geq v$. We expect many of the results of this work to have analogues for the classes $\tilde{\psi}_\partial^w$. We do not know whether $\tilde{\psi}_\partial^w$ is the image of an element of $\tilde{R}(x; a)$ under the res map of Theorem 5.25.

11.2. K -Peterson subalgebra. In [LLS21], we defined a commutative subalgebra of the infinite nilHecke algebra and showed that it provided a model for the equivariant homology of the infinite Grassmannian. This commutative subalgebra is an analogue of the *Peterson subalgebra* of an affine nilHecke algebra. We expect that this construction can be extended to K -homology of the infinite Grassmannian. The analogous *K -Peterson subalgebra* for modeling the equivariant K -homology of the affine Grassmannian is constructed in [LSS]. We remark that recent work of

Kato [Kat] (see also [LLMS]) relate the K -homology of the affine Grassmannian with quantum K -theory of flag varieties.

11.3. Relation to back stable Schubert polynomials. Lenart [Len99] showed that Grothendieck polynomials expand into Schubert polynomials with alternating coefficients, and gave a combinatorial interpretation of the coefficients. It would be interesting to study the expansion of back stable (double) Grothendieck polynomials $\overleftarrow{\mathfrak{G}}_w$ into back stable (double) Schubert polynomials $\overleftarrow{\mathfrak{S}}_w$, and the relation to our other expansion formulae such as the coproduct formula (Theorem 4.15).

11.4. Relation to K -theory affine Schubert calculus. Affine Schubert calculus is one of our main motivations to study back stable Schubert calculus. We expect many interesting relations between these subjects. In particular, we expect that a wealth of combinatorics can be found in the expansion coefficients of Schubert classes of the infinite flag variety (or infinite Grassmannian) in terms of Schubert classes of the affine flag variety (or affine Grassmannian) [LLS+, LSS, Mor]. In cohomology, these expansion coefficients are known as *k*-branching coefficients.

APPENDIX A. GROTHENDIECK INVERSION

Let $J \subset \mathbb{Z} \setminus \{0\}$ and let S_J be the subgroup of $S_{\mathbb{Z}}$ generated by s_j for $j \in J$. Let $v, w \in S_{\mathbb{Z}}$ be such that $S_J v = S_J w$. Say $v^J \leq w$ if there is a $u \in S_J$ such that $u * v = w$.

Proposition A.1. *Let W' be a $J \leq$ interval in $S_{\mathbb{Z}}$ and let $\{f_w \mid w \in W'\}$ and $\{g_w \mid w \in W'\}$ be families of elements. Then*

$$(A.1) \quad g_w = \sum_{u * v = w} (-1)^{\ell(u) + \ell(v) - \ell(w)} \mathfrak{G}_u(a) f_v$$

holds if and only if

$$(A.2) \quad f_w = \sum_{u * v = w} (-1)^{\ell(u) + \ell(v) - \ell(w)} \mathfrak{G}_{u^{-1}}(\ominus a) g_v$$

does.

This is equivalent to the following. Let W' be as above. Define the $W' \times W'$ matrices

$$(A.3) \quad A_{vw} = \sum_{\substack{u \in S_J \\ u * v = w}} (-1)^{\ell(u) + \ell(v) - \ell(w)} \mathfrak{G}_u(a)$$

$$(A.4) \quad B_{vw} = \sum_{\substack{u \in S_J \\ u * v = w}} (-1)^{\ell(u) + \ell(v) - \ell(w)} \mathfrak{G}_{u^{-1}}(\ominus(a)).$$

Then A and B are inverse.

Let

$$(A.5) \quad G = \sum_{u \in S_J} (-1)^{\ell(u)} \mathfrak{G}_u(a) \tilde{\pi}_u$$

$$(A.6) \quad H = \sum_{u \in S_J} (-1)^{\ell(u)} \mathfrak{G}_{u^{-1}}(\ominus(a)) \tilde{\pi}_u.$$

Here we are working in the 0-Hecke algebra with coefficients in the field $\mathbb{Q}(a_i \mid i \in \mathbb{Z})$ where the field elements commute with the operators $\tilde{\pi}_i$. The statement that A and B are inverse, is equivalent to

$$(A.7) \quad \langle \pi_w | GH | \tilde{\pi}_v \rangle = \delta_{v,w}$$

$$(A.8) \quad \langle \pi_w | HG | \tilde{\pi}_v \rangle = \delta_{v,w}$$

for all $v \leq^J w$. The notation $\langle f | g | h \rangle$ means apply g to h and take the coefficient of f .

Working on all cosets at once, this is further equivalent to

$$(A.9) \quad \langle \tilde{\pi}_w | GH | \tilde{\pi}_{\text{id}} \rangle = \delta_{\text{id},w}$$

$$(A.10) \quad \langle \tilde{\pi}_w | HG | \tilde{\pi}_{\text{id}} \rangle = \delta_{\text{id},w}$$

for all $w \in S_J$ since

$$\langle \tilde{\pi}_w | GH | \tilde{\pi}_v \rangle = \sum_u_{u*v=w} \langle \tilde{\pi}_u | GH | \tilde{\pi}_{\text{id}} \rangle$$

and similarly for HG instead of GH .

Thus it suffices to prove (A.10), as (A.9) holds by formal properties of inverses. But by Proposition 5.8 equation (A.10) is equivalent to

$$(A.11) \quad \delta_{\text{id},w} = \mathfrak{G}_w(a; a).$$

Equation (A.11) follows from the fact that $\Psi^{-1}(\mathfrak{G}_w(va; a)) = [\mathcal{O}^w]|_v$ is the localization at v , of the equivariant class of the structure sheaf \mathcal{O}^w of the opposite Schubert variety $\overline{B_- w B / B}$ in $K_T^*(\text{Fl}_n)$.

We state a “right-handed” version of Proposition A.1. With J as above let $v, w \in S_{\mathbb{Z}}$ be such that $vS_J = wS_J$. Say $v \leq^J w$ if there is a $u \in S_J$ such that $v * u = w$.

Proposition A.2. *Let W' be a \leq^J interval in $S_{\mathbb{Z}}$ and let $\{f_w \mid w \in W'\}$ and $\{g_w \mid w \in W'\}$ be families of elements. Then*

$$(A.12) \quad g_w = \sum_{v*u=w} (-1)^{\ell(u)+\ell(v)-\ell(w)} \mathfrak{G}_u(a) f_v$$

holds if and only if

$$(A.13) \quad f_w = \sum_{v*u=w} (-1)^{\ell(u)+\ell(v)-\ell(w)} \mathfrak{G}_{u^{-1}}(\ominus a) g_v$$

does.

APPENDIX B. SOME COMPUTATIONS

Recall that $x_- = (x_0, x_{-1}, x_{-2}, \dots)$. We write X_i for (x_1, x_2, \dots, x_i) .

B.1. Back stable Grothendieck polynomials. We compute $\overleftarrow{\mathfrak{G}}_{s_0}$ from the definition.

$$\mathfrak{G}_{s_1} = e_1[X_1]$$

$$\mathfrak{G}_{s_2} = e_1[X_2] - e_2[X_2]$$

$$\mathfrak{G}_{s_3} = e_1[X_3] - e_2[X_3] + e_3[X_3]$$

$$\mathfrak{G}_{s_n} = e_1[X_n] - e_2[X_n] + e_3[X_n] - \dots + (-1)^{n-1} e_n[X_n].$$

Therefore,

$$\overleftarrow{\mathfrak{G}}_{s_0} = G_{s_0} = \lim_{n \rightarrow -\infty} \mathfrak{G}_{s_n}(x_{1-n}, \dots, x_{-1}, x_0) = e_1 - e_2 + e_3 - \dots = 1 - \Omega[-x].$$

We compare Proposition 4.6 and Theorem 4.15 for $\overleftarrow{\mathfrak{G}}_{s_1}, \overleftarrow{\mathfrak{G}}_{s_2}, \overleftarrow{\mathfrak{G}}_{s_{-1}}$.

$$\begin{aligned} \overleftarrow{\mathfrak{G}}_{s_1} &= \gamma(\overleftarrow{\mathfrak{G}}_{s_0}) \\ &= e_1[x_- + x_1] - e_2[x_- + x_1] + e_3[x_- + x_1] - \dots \\ &= (x_1 + e_1) - (x_1 e_1 + e_2) + (x_1 e_2 + e_3) - \dots \\ &= x_1 - x_1(e_1 - e_2 + e_3 - \dots) + (e_1 - e_2 + e_3 - \dots) \\ &= \mathfrak{G}_{s_1} - G_{s_1} \mathfrak{G}_{s_1} + G_{s_1}. \\ \overleftarrow{\mathfrak{G}}_{s_2} &= \gamma^2(\overleftarrow{\mathfrak{G}}_{s_0}) \\ &= e_1[x_- + X_2] - e_2[x_- + X_2] + e_3[x_- + X_2] - \dots \\ &= (e_1[X_2] + e_1) - (e_2[X_2] + e_1[X_2]e_1 + e_2) \\ &\quad + (e_2[X_2]e_1 + e_1[X_2]e_2 + e_3) - \dots \\ &= (e_1[X_2] - e_2[X_2]) + (1 - e_1[X_2] + e_2[X_2])e_1 \\ &\quad + (-1 + e_1[X_2] - e_2[X_2])e_2 + (1 - e_1[X_2] + e_2[X_2])e_3 + \dots \\ &= \mathfrak{G}_{s_2} + G_{s_2} - G_{s_2} \mathfrak{G}_{s_2}. \\ \overleftarrow{\mathfrak{G}}_{s_{-1}} &= \gamma^{-1}(\overleftarrow{\mathfrak{G}}_{s_0}) \\ &= e_1[x_- - x_0] - e_2[x_- - x_0] + e_3[x_- - x_0] - \dots \\ &= (e_1 - x_0) - (e_2 - x_0 e_1 + x_0^2) + (e_3 - x_0 e_2 + x_0^2 e_1 - x_0^3) - \dots \\ &= G_{s_{-1}} + \frac{-x_0}{1 - x_0} - \frac{-x_0}{1 - x_0} G_{s_{-1}} \\ &= G_{s_{-1}} + \mathfrak{G}_{s_{-1}} - G_{s_{-1}} \mathfrak{G}_{s_{-1}}. \end{aligned}$$

Here,

$$\mathfrak{G}_{s_{-1}} = \omega(G_{s_1}) = \omega(x_1) = \frac{-x_0}{1 - x_0}.$$

In particular for $k > 0$ we have

$$\begin{aligned} \overleftarrow{\mathfrak{G}}_{s_k} &= G_1 \oplus x_1 \oplus x_2 \oplus \dots \oplus x_k \\ \overleftarrow{\mathfrak{G}}_{s_{-k}} &= G_1 \ominus x_0 \ominus x_1 \ominus \dots \ominus x_{1-k}. \end{aligned}$$

We compute $\overleftarrow{\mathfrak{G}}_{s_{-1}s_0}$ from the limit definition:

$$\begin{aligned} \mathfrak{G}_{s_1 s_2} &= e_2[X_2] \\ \mathfrak{G}_{s_2 s_3} &= e_2[X_3] - 2e_3[X_3] \\ \mathfrak{G}_{s_3 s_4} &= e_2[X_4] - 2e_3[X_4] + 3e_4[X_4]. \end{aligned}$$

Thus,

$$\overleftarrow{\mathfrak{G}}_{s_{-1}s_0} = G_{s_{-1}s_0} = G_{11} = e_2 - 2e_3 + 3e_4 - \dots.$$

Using Proposition 4.6, we have

$$\begin{aligned}
\overleftarrow{\mathfrak{G}}_{s_0s_1} &= e_2[x_- + x_1] - 2e_3[x_- + x_1] + 3e_4[x_- + x_1] \\
&= (e_2 + x_1e_1) - 2(e_3 + x_1e_2) + 3(e_4 + x_1e_3) - \dots \\
&= x_1(e_1 - 2e_2 + 3e_3 - \dots) + (e_2 - 2e_3 + 3e_4 - \dots) \\
&= x_1G_{s_0} - x_1(e_2 - 2e_3 + \dots) + G_{s_0s_1} \\
&= G_{s_0}\mathfrak{G}_{s_1} - G_{s_0s_1}\mathfrak{G}_{s_1} + G_{s_0s_1},
\end{aligned}$$

agreeing with Theorem 4.15.

With s_λ denoting a Schur polynomial, we compute $\overleftarrow{\mathfrak{G}}_{s_1s_0}$ from the limit definition:

$$\begin{aligned}
\mathfrak{G}_{s_2s_1} &= s_2[X_1] \\
\mathfrak{G}_{s_3s_2} &= s_2[X_2] - s_{21}[X_2] \\
\mathfrak{G}_{s_4s_3} &= s_3[X_3] - s_{21}[X_3] + s_{211}[X_3] \\
\overleftarrow{\mathfrak{G}}_{s_1s_0} &= G_{s_1s_0} = s_2 - s_{21} + s_{211} - s_{2111} + \dots
\end{aligned}$$

Therefore, using Proposition 4.6, we have

$$\begin{aligned}
\overleftarrow{\mathfrak{G}}_{s_0s_{-1}} &= \gamma^{-1}(\overleftarrow{\mathfrak{G}}_{s_1s_0}) \\
&= s_2[x_- - x_0] - s_{21}[x_- - x_0] + s_{211}[x_- - x_0] - s_{2111}[x_- - x_0] + \dots \\
&= (s_2 - x_0s_1) - (s_{21} - x_0s_2 - x_0s_{11} + x_0^2s_1) \\
&\quad + (s_{211} - x_0s_{21} - x_0s_{111} + x_0^2s_2 + x_0^2s_{11} - x_0^3s_1) \\
&\quad - (s_{2111} - x_0s_{211} - x_0s_{1111} + x_0^2s_{21} + x_0^2s_{111} - x_0^3s_2 - x_0^3s_{11} + x_0^4s_1) \\
&\quad + \dots \\
&= G_{s_0s_{-1}} + G_{s_0}\mathfrak{G}_{s_{-1}} - G_{s_0s_{-1}}\mathfrak{G}_{s_{-1}}.
\end{aligned}$$

B.2. K -Stanley polynomials.

We give some formulae for various G_λ . See also Proposition 9.4 and Proposition 4.20.

$$\begin{aligned}
G_r &= s_r - s_{r,1} + s_{r,1,1} - s_{r,1,1,1} + \dots \\
G_{1^r} &= e_r - \binom{r}{r-1}e_{r+1} + \binom{r+1}{r-1}e_{r+2} - \dots \\
G_{21} &= s_{21} - s_{22} - 2s_{211} + 2s_{221} - s_{222} + 3s_{2111} - 3s_{2211} \\
&\quad + 2s_{2221} - s_{2222} + \dots \\
G_{31} &= s_{31} - s_{32} - 2s_{311} + 2s_{321} - s_{322} + \dots \\
G_{211} &= s_{211} - s_{221} + s_{222} - 3s_{2111} + 3s_{2211} - 3s_{2221} + 2s_{2222} + \dots
\end{aligned}$$

We give some products of K -Stanley functions; see Proposition 8.1.

$$\begin{aligned}
G_1G_1 &= G_2 + G_{11} - G_{21} \\
G_2G_1 &= G_3 + G_{21} - G_{31} \\
G_{11}G_1 &= G_{21} + G_{111} - G_{211}.
\end{aligned}$$

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