



Recovery of rapidly decaying source terms from dynamical samples in evolution equations

Akram Aldroubi¹ · Le Gong¹ · Ilya Krishtal²

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Abstract

We analyze the problem of recovering a source term of the form $h(t) = \sum_j h_j \phi(t - t_j) \chi_{[t_j, \infty)}(t)$ from space-time samples of the solution u of an initial value problem in a Hilbert space of functions. In the expression of h , the terms h_j belong to the Hilbert space, while ϕ is a generic real-valued function with exponential decay at ∞ . The design of the sampling strategy takes into account noise in measurements and the existence of a background source.

Keywords Sampling theory · Forcing · Frames · Reconstruction · Semigroups · Continuous sampling

Mathematics Subject Classification 46N99 · 42C15 · 94O20

1 Introduction

Dynamical sampling refers to a set of problems in which a space-time signal u evolving in time under the action of a linear operator as in (1) below is to be sampled on a space-time set $S = \{(x, t) : x \in X, t \in T\}$ in order to recover u_0 , u , F or other information related to these functions. For example, when the goal is to recover u_0 , we get the so called space-time trade-off problems (see e.g., [3, 5–7, 14, 15, 19, 26, 27]). If the goal is to recover the unknown underlying operator A , or some of its spectral characteristics, we get the system identification problem in dynamical sampling [9, 12, 25]. In other situations, the goal is to identify the driving source term from space-time samples [8, 11]. In all dynamical sampling problems, frame theory plays a fundamental albeit, at

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✉ Le Gong
le.gong@vanderbilt.edu

¹ Department of Mathematics, Vanderbilt University, Nashville, USA

² Department of Mathematical Sciences, Northern Illinois University, DeKalb, USA

times, hidden role (see e.g. [4, 9, 10, 14]). Moreover, this important connection has also been used to develop frame theory and led to the concept of dynamical frames (see e.g. [1, 2, 13, 16–18, 21]). In this paper, we consider the problem of designing space-time sampling patterns that permit recovery of the source term of an initial value problem (IVP) or some relevant portion thereof.

1.1 Motivation

In [8], the authors introduced a new sampling technique which prescribes how one may sample the solution of an IVP to detect “bursts” in the driving force of the system. The proposed special structure of the samplers allowed one to “predict” the value of the solution at the next sampling instance provided that no burst occurred during the sampling period. Thus, if the samples at the end of the period were significantly different from the prediction, a “burst” must have occurred. In [8], the “bursts” were modeled as a linear combination of Dirac measures. In this paper, we employ a modification of the same technique to detect localized non-instantaneous sources which decay exponentially in time after activation [22]. Such sources may describe, for example, an irregular intake of rapidly degrading substances. In particular, the IVP we consider may model a complex chemical reaction contaminated by such an intake, and our goal, in this case, would be to determine when and what substances were added to the system. Many other phenomena driven by natural mechanisms, such as the dispersion of pollution, the spreading of fungal diseases and the leakage of biochemical waste, can also be described by IVP with the source terms considered here (see [22–24] and references therein). Thus, a robust sampling and reconstruction algorithm for such IVP would be beneficial for studying these real-world applications.

1.2 Problem setting

Let us give a more precise description of our setting. We consider the following abstract initial value problem:

$$\begin{cases} \dot{u}(t) = Au(t) + F(t) \\ u(0) = u_0, \end{cases} \quad t \in \mathbb{R}_+, \quad u_0 \in \mathcal{H}. \quad (1)$$

Above, the function $u : \mathbb{R}_+ \rightarrow \mathcal{H}$ is assumed to be almost everywhere differentiable with the derivative $\dot{u} \in L^2(\mathbb{R}_+, \mathcal{H})$, where $\mathcal{H}$ is some Hilbert space of functions on a subset of $\mathbb{R}^d$. Additionally, $F : \mathbb{R}_+ \rightarrow \mathcal{H}$ is a forcing term a portion of which we wish to recover and $A : D(A) \subseteq \mathcal{H} \rightarrow \mathcal{H}$ is the generator of a strongly continuous semigroup $T : \mathbb{R}_+ \rightarrow B(\mathcal{H})$. We shall also occasionally use the notation $u = u(x, t)$, where the variable $x \in \mathbb{R}^d$ is the spatial variable and $t \in \mathbb{R}_+$ represents time, with the understanding that for each fixed $t \in \mathbb{R}_+$, $u(\cdot, t)$ is a function in $\mathcal{H}$.

As in [8], we consider force terms of the form:

$$F = h + \eta$$

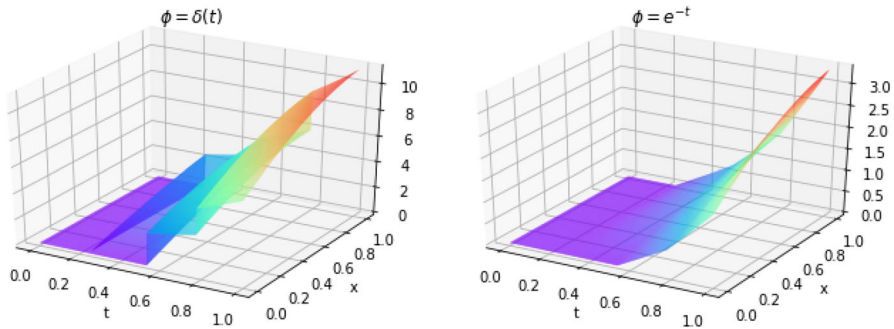


Fig. 1 The solution of the IVP (1) with the following parameters: $A = I$, $u_0 = 0$, $\eta = 0$, $h_1(x) = 3 \sin(x)$, $h_2(x) = 2.5 \cos(x)$, $h_3(x) = x + 2$, $x \in [0, 1]$, and $t_1 = 0.25$, $t_2 = 0.54$, $t_3 = 0.78$

where η is a Lipschitz continuous background source term. Unlike [8], however, the burst-like forcing term h is assumed to be given by

$$h(t) = \sum_{j=1}^N h_j \phi(t - t_j) \chi_{[t_j, \infty)}(t), \quad (2)$$

where $0 < t_1 < \dots < t_N$, $h_j \in \mathcal{H}$, and ϕ is a non-negative function with a certain prescribed decay on $[0, \infty)$. We regard t_j and h_j as the time and the shape of the j -th burst, respectively.

The goal of this paper is to provide an algorithm similar to one in [8] that recovers the “burst-like” portion h of F from space-time samples of the solution u of (1). Once again we shall choose the structure of the samplers that would allow one to detect an occurrence of a burst in a sampling period by comparing the predicted values of the samples with the actual samples. In fact, we will show that the same structure of samplers that was used in the first of the two approaches in [8] may also be used in the current situation. The recovery algorithms in this paper, however, are significantly different and are not just a straightforward tweak of the ones in [8]. The main difficulty in the current setting is the need to account for the influence of past bursts which was not an issue when those were Diracs.

To gain a visual understanding of these two cases, we set up a specific IVP with the two types of ϕ and plot the respective graphs of $u = u(x, t)$ in Fig. 1.

The model incorporating Dirac functions exhibited noticeable jumps when bursts happened, while the model with exponential decay is smoother, highlighting the challenges of detecting the bursts and mitigating the influence of past bursts.

1.3 Paper organization

The rest of the paper is organised as follows. In Sect. 2, we remind the reader of some basic properties of one-parameter operator semigroups and their use in solving IVP such as (1). We also list all model assumptions for the algorithms of this paper.

In Sect. 3, we present the main results. The section is divided into two parts. In Sect. 3.1, the decay function ϕ in (2) is assumed to be of the form $e^{-\rho t}$ for some $\rho > 0$. We present the structure of measurement functions (5) for this case and utilize discrete samples of the measurement functions to approximate the burst time and shape in the presence of background source and measurement acquisition errors. In Sect. 3.2, we consider a more general model, where the decay function ϕ does not have a concrete formula, but is rather bounded above by a decaying exponential function. Under additional assumptions that the shapes of the bursts are uniformly bounded and that the differences $t_{j+1} - t_j$, $j = 1, \dots, N - 1$, are large enough, we present a modification of the algorithm from Sect. 3.1, which solves the same problem in this more general setting. Finally, in Sect. 4, we set up specific (synthetic) dynamical systems to test the algorithms and describe the results of the testing.

2 IVP solution and model assumptions

In this section, we recall a few basic facts from the theory of one-parameter operator semigroups and summarize our model assumptions.

2.1 IVP toolkit

Definition 2.1 A strongly continuous operator semigroup is a map $T : \mathbb{R}_+ \rightarrow B(\mathcal{H})$ (where $B(\mathcal{H})$ is the space of all bounded linear operators on $\mathcal{H}$), which satisfies

- (i) $T(0) = I$,
- (ii) $T(t + s) = T(t)T(s)$ for all $t, s \geq 0$, and
- (iii) $\|T(t)h - h\| \rightarrow 0$ as $t \rightarrow 0$ for all $h \in \mathcal{H}$.

Proposition 2.2 [20] *There exist constants $a \in \mathbb{R}$ and $M \geq 1$ such that*

$$\|T(t)\| \leq Me^{at}$$

for all $t \geq 0$.

Recall [20, p. 436] that the (mild) solution of (1) can be represented as

$$u(t) = T(t)u_0 + \int_0^t T(t-s)F(s)ds. \quad (3)$$

Substituting $F = h + \eta$ with h of the form (2) yields

$$u(t) = T(t)u_0 + \sum_{t_j < t} \int_{t_j}^t T(t-s)h_j\phi(s-t_j)ds + \int_0^t T(t-s)\eta(s)ds, \quad t \geq 0. \quad (4)$$

In this paper, we will use the measurement function $m : \mathbb{R}_+ \times \mathcal{H} \rightarrow \mathbb{R}$ given by

$$m(t, g) = \langle u(t), g \rangle + v(t, g), \quad t \geq 0, \quad g \in G, \quad (5)$$

where $\langle \cdot, \cdot \rangle$ is the inner product in $\mathcal{H}$, v is the measurement acquisition noise, and G is the collection of samplers whose structure we wish to prescribe.

2.2 Model assumptions

Assumption 1 *The set of samplers G has the form $G = \tilde{G} \cup T^*(\beta)\tilde{G}$ for some countable (possibly, finite) set $\tilde{G} \subseteq \mathcal{H}$. Additionally, in the model of Sect. 3.2, the set $\tilde{G}$ is assumed to be uniformly bounded by some $R \in \mathbb{R}$, i.e. $R = \sup_{g \in \tilde{G}} \|g\| < \infty$.*

Assumption 2 *In Sect. 3.2, the burst terms are uniformly bounded, i.e. $\|h_j\| \leq H$, $j = 1, \dots, N$, for some $H \in \mathbb{R}$.*

Assumption 3 *The background source $\eta : \mathbb{R}_+ \rightarrow \mathcal{H}$ is uniformly bounded by a constant $K > 0$ and Lipschitz with a Lipschitz constant $L \geq 0$, i.e. $\sup_{t \geq 0} \|\eta(t)\| \leq K$ and $\|\eta(t+s) - \eta(t)\| \leq Ls$, $t, s \in \mathbb{R}_+$.*

Assumption 4 *The additive noise v in the measurements (5) satisfies*

$$\sup_{t > 0, h \in \mathcal{H}} |v(t, h)| \leq \sigma.$$

Assumption 5 (1) *In Sect. 3.1 we assume that the distance between two bursts is bounded below: $t_{j+1} - t_j \geq 4\beta$.*

(2) *In Sect. 3.2 we assume $t_{j+1} - t_j \geq D + 4\beta$ with some positive number D .*

3 Main results

3.1 Model with specific decay function

In this section, we consider the special case where the decay function ϕ is given by $\phi(t) = e^{-\rho t}$ with some $\rho > 0$:

$$\begin{cases} \dot{u}(t) = Au(t) + \sum_j^N h_j e^{-\rho(t-t_j)} \chi_{[t_j, \infty)}(t) + \eta(t) \\ u(0) = u_0. \end{cases} \quad (6)$$

Since the operator A generates a strongly continuous semigroup T , by Proposition 2.2, we can find real numbers M and a satisfying $\|T(t)\| \leq Me^{at}$ for all $t \geq 0$. We will use these numbers to estimate the accuracy of our recovery algorithm.

We acquire the following set of measurements:

$$m_n\left(\frac{g}{\beta}\right) = \left\langle u(n\beta), \frac{g}{\beta} \right\rangle + v\left(n\beta, \frac{g}{\beta}\right),$$

$$m_n\left(\frac{T^*(\beta)g}{\beta}\right) = \left\langle u(n\beta), \frac{T^*(\beta)g}{\beta} \right\rangle + v\left(n\beta, \frac{T^*(\beta)g}{\beta}\right), \quad g \in \tilde{G}, n \in \mathbb{N}, \quad (7)$$

where β is the time sampling step, $T^*(t)$ is the adjoint operator of $T(t)$, and v represents an additive noise (see Assumption 4).

The first of the pair of measurements in (7) serves to assess the current state of the system whereas the second one will be used as a predictor of the measurement at $t = (n+1)\beta$.

To explain our idea of the recovery algorithm, we first present what happens in the ideal case when the measurements are noiseless ($v \equiv 0$) and there is no background source ($\eta \equiv 0$). For the convenience of exposition, we define $f_n \in \mathcal{H}$ and $\tau_n \in [n\beta, (n+1)\beta)$ for each n as follows

$$\begin{cases} f_n = h_j, \tau_n = t_j, & \text{if the } j\text{-th burst occurred in } [n\beta, (n+1)\beta); \\ f_n = 0, \tau_n = n\beta, & \text{if no burst occurred in } [n\beta, (n+1)\beta). \end{cases} \quad (8)$$

There is no ambiguity in the above definition due to Assumption 5.

To reveal the predictive nature of the second measurement in (7), we first consider the difference

$$F_n = m_{n+1}\left(\frac{g}{\beta}\right) - m_n\left(\frac{T^*(\beta)g}{\beta}\right). \quad (9)$$

In the ideal case, utilizing (4) we get

$$\begin{aligned} F_n &= m_{n+1}\left(\frac{g}{\beta}\right) - m_n\left(\frac{T^*(\beta)g}{\beta}\right) \\ &= \left\langle T((n+1)\beta)u_0, \frac{g}{\beta} \right\rangle + \sum_{\tau_i < (n+1)\beta} \int_{\tau_i}^{(n+1)\beta} \left\langle T((n+1)\beta - s)f_i e^{\rho(\tau_i - s)}, \frac{g}{\beta} \right\rangle ds \\ &\quad - \left\langle T(n\beta)u_0, \frac{T^*(\beta)g}{\beta} \right\rangle - \sum_{\tau_i < n\beta} \int_{\tau_i}^{n\beta} \left\langle T(n\beta - s)f_i e^{\rho(\tau_i - s)}, \frac{T^*(\beta)g}{\beta} \right\rangle ds \\ &= \int_{\tau_n}^{(n+1)\beta} \left\langle T((n+1)\beta - s)f_n e^{d(\tau_n - s)}, \frac{g}{\beta} \right\rangle ds \\ &\quad + \sum_{\tau_i < n\beta} \int_{\tau_i}^{(n+1)\beta} \left\langle T((n+1)\beta - s)f_i e^{\rho(\tau_i - s)}, \frac{g}{\beta} \right\rangle ds \\ &\quad - \sum_{\tau_i < n\beta} \int_{\tau_i}^{n\beta} \left\langle T((n+1)\beta - s)f_i e^{\rho(\tau_i - s)}, \frac{g}{\beta} \right\rangle ds \\ &= \int_{\tau_n}^{(n+1)\beta} \left\langle T((n+1)\beta - s)f_n e^{\rho(\tau_n - s)}, \frac{g}{\beta} \right\rangle ds \\ &\quad + \sum_{\tau_i < n\beta} \int_{n\beta}^{(n+1)\beta} \left\langle T((n+1)\beta - s)f_i e^{\rho(\tau_i - s)}, \frac{g}{\beta} \right\rangle ds \end{aligned}$$

$$\begin{aligned}
&= \int_{\tau_n}^{(n+1)\beta} \left\langle T((n+1)\beta - s) f_n e^{\rho(\tau_n - s)}, \frac{g}{\beta} \right\rangle ds \\
&\quad + \sum_{\tau_i < n\beta} e^{-\rho n\beta} \int_0^\beta \left\langle T(\beta - s) f_i e^{\rho(\tau_i - s)}, \frac{g}{\beta} \right\rangle ds.
\end{aligned}$$

Remark 3.1 In the expression for F_n above, if no burst occurred in $[n\beta, (n+1)\beta)$ (i.e. $f_n = 0$), then $\int_{\tau_n}^{(n+1)\beta} \langle T((n+1)\beta - s) f_n e^{\rho(\tau_n - s)}, \frac{g}{\beta} \rangle ds = 0$. In addition, the term $\sum_{\tau_i < n\beta} e^{-\rho n\beta} \int_0^\beta \langle T(\beta - s) f_i e^{\rho(\tau_i - s)}, \frac{g}{\beta} \rangle ds$ represents the effect of the bursts that had occurred before $n\beta$.

Secondly, we calculate the difference $\Delta_n = e^{\rho\beta} F_{n+1} - F_n$, which involves the measurements in two consecutive intervals $[n\beta, (n+1)\beta)$ and $[(n+1)\beta, (n+2)\beta)$:

$$\begin{aligned}
\Delta_n &= e^{\rho\beta} F_{n+1} - F_n \\
&= e^{\rho\beta} \int_{\tau_{n+1}}^{(n+2)\beta} \left\langle T((n+2)\beta - s) f_{n+1} e^{\rho(\tau_{n+1} - s)}, \frac{g}{\beta} \right\rangle ds \\
&\quad + \sum_{\tau_i < (n+1)\beta} e^{-\rho n\beta} \int_0^\beta \left\langle T(\beta - s) f_i e^{\rho(\tau_i - s)}, \frac{g}{\beta} \right\rangle ds \\
&\quad - \int_{\tau_n}^{(n+1)\beta} \left\langle T((n+1)\beta - s) f_n e^{\rho(\tau_n - s)}, \frac{g}{\beta} \right\rangle ds \\
&\quad - \sum_{\tau_i < n\beta} e^{-\rho n\beta} \int_0^\beta \left\langle T(\beta - s) f_i e^{\rho(\tau_i - s)}, \frac{g}{\beta} \right\rangle ds \\
&= e^{\rho\beta} \int_{\tau_{n+1}}^{(n+2)\beta} \left\langle T((n+2)\beta - s) f_{n+1} e^{\rho(\tau_{n+1} - s)}, \frac{g}{\beta} \right\rangle ds \\
&\quad - \int_{\tau_n}^{(n+1)\beta} \left\langle T((n+1)\beta - s) f_n e^{\rho(\tau_n - s)}, \frac{g}{\beta} \right\rangle ds \\
&\quad + e^{-\rho n\beta} \int_0^\beta \left\langle T(\beta - s) f_n e^{\rho(\tau_n - s)}, \frac{g}{\beta} \right\rangle ds \\
&= e^{\rho\beta} \int_{\tau_{n+1}}^{(n+2)\beta} \left\langle T((n+2)\beta - s) f_{n+1} e^{\rho(\tau_{n+1} - s)}, \frac{g}{\beta} \right\rangle ds \\
&\quad + \int_{n\beta}^{\tau_n} \left\langle T((n+1)\beta - s) f_n e^{\rho(\tau_n - s)}, \frac{g}{\beta} \right\rangle ds.
\end{aligned}$$

The above calculation leads us to the key observation

$$\Delta_n = 0 \quad \text{in the ideal case if no burst occurred in } [n\beta, (n+2)\beta). \quad (10)$$

In case the j -th burst did occur in the interval $[(n+1)\beta, (n+2)\beta)$, we use the following calculation to estimate the inner products $\langle h_j, g \rangle$. In view of Assumption 5, we have

$f_{n-1} = f_n = f_{n+2} = 0$ and, for β sufficiently small, we get

$$\begin{aligned}
 & e^{3\rho\beta} F_{n+2} - F_{n-1} \\
 &= e^{3\rho\beta} \int_{\tau_{n+2}}^{(n+3)\beta} \left\langle T((n+3)\beta - s) f_{n+2} e^{\rho(\tau_{n+2}-s)}, \frac{g}{\beta} \right\rangle ds \\
 &+ \sum_{\tau_i < (n+2)\beta} e^{-\rho(n-1)\beta} \int_0^\beta \left\langle T(\beta - s) f_i e^{\rho(\tau_i-s)}, \frac{g}{\beta} \right\rangle ds \\
 &- \int_{\tau_{n-1}}^{n\beta} \left\langle T(n\beta - s) f_{n-1} e^{\rho(\tau_{n-1}-s)}, \frac{g}{\beta} \right\rangle ds \\
 &- \sum_{\tau_i < (n-1)\beta} e^{-\rho(n-1)\beta} \int_0^\beta \left\langle T(\beta - s) f_i e^{\rho(\tau_i-s)}, \frac{g}{\beta} \right\rangle ds \\
 &= \int_0^\beta \left\langle T(\beta - s) f_{n+1} e^{\rho(\tau_{n+1}-(n-1)\beta-s)}, \frac{g}{\beta} \right\rangle ds \\
 &= \int_0^\beta \left\langle T(\beta - s) h_j e^{\rho(t_j-(n-1)\beta-s)}, \frac{g}{\beta} \right\rangle ds \approx \langle h_j, g \rangle,
 \end{aligned}$$

where the last assertion is (essentially) yielded by the following lemma.

Lemma 3.2 Assume that $t_j \in [(n+1)\beta, (n+2)\beta]$ and

$$v_k(h_j, g, \beta) = \left| \int_0^\beta \left\langle T(\beta - s) h_j e^{\rho(t_j-(n-k)\beta-s)}, \frac{g}{\beta} \right\rangle ds - \langle h_j, g \rangle \right|, \quad k = 0, 1. \quad (11)$$

Then

$$v_k(h_j, g, \beta) \leq \|g\| \left(M \|h_j\| (e^{(k+2)\rho\beta} - 1) \mathbf{e}(a\beta) + \sup_{s \in [0, \beta]} \|T(s)h_j - h_j\| \right), \quad (12)$$

where M and a are as in Proposition 2.2 and

$$\mathbf{e}(t) = \begin{cases} \frac{e^t - 1}{t}, & t \neq 0; \\ 1, & t = 0. \end{cases} \quad (13)$$

In particular, $v_k(h_j, g, \beta) \rightarrow 0$ as $\beta \rightarrow 0$.

Proof Observe that

$$v_k(h_j, g, \beta) = \left| \int_0^\beta \left\langle T(\beta - s) h_j e^{\rho(t_j-(n-k)\beta-s)}, \frac{g}{\beta} \right\rangle ds - \langle h_j, g \rangle \right|$$

$$\begin{aligned}
&= \left| \int_0^\beta \left\langle T(\beta-s)h_j e^{\rho(t_j-(n-k)\beta-s)} - h_j, \frac{g}{\beta} \right\rangle ds \right| \\
&\leq \int_0^\beta \|T(\beta-s)h_j e^{\rho(t_j-(n-k)\beta-s)} - h_j\| \frac{\|g\|}{\beta} ds \\
&\leq \frac{\|g\|}{\beta} \int_0^\beta \|T(\beta-s)h_j e^{\rho(t_j-(n-k)\beta-s)} - T(\beta-s)h_j\| ds \\
&\quad + \frac{\|g\|}{\beta} \int_0^\beta \|T(\beta-s)h_j - h_j\| ds \\
&= I_1 + I_2.
\end{aligned} \tag{14}$$

Using Proposition 2.2, i.e. the inequality $\|T(t)\| \leq Me^{at}$, we get

$$\begin{aligned}
I_1 &= \frac{\|g\|}{\beta} \int_0^\beta \|T(\beta-s)h_j e^{\rho(t_j-(n-k)\beta-s)} - T(\beta-s)h_j\| ds \\
&\leq \frac{\|g\|}{\beta} \int_0^\beta \|T(\beta-s)\| \|h_j\| (e^{(k+2)\rho\beta} - 1) ds \\
&\leq \|g\| M \|h_j\| (e^{(k+2)\rho\beta} - 1) \frac{1}{\beta} \int_0^\beta e^{a(\beta-s)} ds \\
&\leq \|g\| \|h_j\| M (e^{(k+2)\rho\beta} - 1) \mathbf{e}(a\beta)
\end{aligned} \tag{15}$$

and

$$\begin{aligned}
I_2 &= \frac{\|g\|}{\beta} \int_0^\beta \|T(\beta-s)h_j - h_j\| ds \\
&= \frac{\|g\|}{\beta} \int_0^\beta \|T(s)h_j - h_j\| ds \\
&\leq \|g\| \sup_{s \in [0, \beta]} \|T(s)h_j - h_j\|.
\end{aligned} \tag{16}$$

Estimate (12) immediately follows from (14), (15), and (16). We get $\lim_{\beta \rightarrow 0} v_k(h_j, g, \beta) = 0$ since $\lim_{\beta \rightarrow 0} (e^{(k+2)\rho\beta} - 1)\mathbf{e}(a\beta) = 0$ and $\lim_{\beta \rightarrow 0} \sup_{s \in [0, \beta]} \|T(s)h_j - h_j\| = 0$ due to the strong continuity of the semigroup T . $\square$

Equipped with the above observations, we are naturally led to Algorithm 1 below. The algorithm turns out to be robust both with respect to the considered additive measurement noise and introduction of a background source as described in the following Theorem 3.3.

Algorithm 1. Pseudo-code for approximating the time and shape of a possible burst with prescribed exponential decay

```

1: Input: Measurements:  $m_n(\frac{g}{\beta})$ ,  $m_n(\frac{T^*(\beta)g}{\beta})$  and threshold:  $Q(g, \beta)$ ,  $g \in \tilde{G}$ 
2: Compute  $F_i = m_{i+1}(\frac{g}{\beta}) - m_i(\frac{T^*(\beta)g}{\beta})$ 
3: Compute  $e^{\rho\beta} F_{i+1} - F_i$ 
4: For  $g \in \tilde{G}$  do
5:    $i=1$ 
6:   while  $i\beta < T$ 
7:     if  $e^{\rho\beta} F_{i+1} - F_i > Q(g, \beta)$  then
8:        $f(g) := e^{3\rho\beta} F_{i+2} - F_{i-1}$ 
9:        $t := (i+1)\beta$ 
10:       $i = i+3$ 
11:     else
12:       if  $e^{\rho\beta} F_{i+2} - F_{i+1} > Q(g, \beta)$  then
13:          $f(g) := e^{3\rho\beta} F_{i+3} - F_i$ 
14:          $t := (i+2)\beta$ 
15:          $i = i+3$ 
16:       else
17:          $i = i+1$ 
18:   Output:  $t$  and  $f(g)$  for all  $g \in \tilde{G}$ .

```

Theorem 3.3 Under Assumptions 1, 3, 4 and 5, and M, a as in Proposition 2.2, let

$$Q(g, \beta) = e^{(\rho+a)\beta} ML\|g\|\beta + e^{a\beta}(e^\beta - 1)MK\|g\| + 4e^{\rho\beta}\sigma \quad (17)$$

be the threshold in Algorithm 1. Let also t_j and $f_j(g)$ be the outputs of Algorithm 1. Then $|t_j - t_j| \leq \beta$ and

$$\begin{aligned}
& |f_j(g) - \langle h_j, g \rangle| \\
& \leq 3e^{(3\rho+a)\beta} ML\|g\|\beta + e^{a\beta}(e^{3\rho\beta} - 1)MK\|g\| + 4e^{3\rho\beta}\sigma \\
& \quad + 2e^{\rho\beta} Q(g, \beta) + \max\{v_0(h_j, g, \beta), v_1(h_j, g, \beta)\}, \quad (18)
\end{aligned}$$

where $v_k, k = 0, 1$, are given by (12). In particular, for sufficiently small $\beta > 0$, one has $|f_j(g) - \langle h_j, g \rangle| \leq 13\sigma$ as long as $\sigma > 0$.

Proof Adjusting the computations in the ideal case to account for the noise and the background source, we get

$$\begin{aligned}
 F_n &= m_{n+1} \left(\frac{g}{\beta} \right) - m_n \left(\frac{T^*(\beta)g}{\beta} \right) \\
 &= \int_{\tau_n}^{(n+1)\beta} \left\langle T((n+1)\beta - s) f_n e^{\rho(\tau_n - s)}, \frac{g}{\beta} \right\rangle ds \\
 &\quad + \sum_{\tau_i < n\beta} \int_0^\beta \left\langle T(\beta - s) f_i e^{\rho(\tau_i - n\beta - s)}, \frac{g}{\beta} \right\rangle ds \\
 &\quad + \int_0^\beta \left\langle T(\beta - s) \eta(n\beta + s), \frac{g}{\beta} \right\rangle ds + v \left((n+1)\beta, \frac{g}{\beta} \right) - v \left(n\beta, \frac{T^*(\beta)g}{\beta} \right).
 \end{aligned}$$

The difference $\Delta_n = e^{\rho\beta} F_{n+1} - F_n$ not only allows us to detect the burst in the ideal case (where it kills the effect of the past bursts according to (10)), but also, as we shall see presently, mitigates the effect of the background source. Once again, adjusting the previous computations for noise and background source, we get

$$\begin{aligned}
 \Delta_n &= e^{\rho\beta} F_{n+1} - F_n \\
 &= \int_{\tau_{n+1}}^{(n+2)\beta} \left\langle T((n+2)\beta - s) f_{n+1} e^{\rho(\tau_{n+1} + \beta - s)}, \frac{g}{\beta} \right\rangle ds \\
 &\quad + \int_{n\beta}^{\tau_n} \left\langle T((n+1)\beta - s) f_n e^{\rho(\tau_n - s)}, \frac{g}{\beta} \right\rangle ds \\
 &\quad + e^{\rho\beta} \int_0^\beta \left\langle T(\beta - s) \eta((n+1)\beta + s), \frac{g}{\beta} \right\rangle ds \\
 &\quad - \int_0^\beta \left\langle T(\beta - s) \eta(n\beta + s), \frac{g}{\beta} \right\rangle ds + \alpha_n,
 \end{aligned}$$

where $\alpha_n = e^{\rho\beta} v \left((n+2)\beta, \frac{g}{\beta} \right) - e^{\rho\beta} v \left((n+1)\beta, \frac{T^*(\beta)g}{\beta} \right) - v \left((n+1)\beta, \frac{g}{\beta} \right) + v \left(n\beta, \frac{T^*(\beta)g}{\beta} \right)$.

We remark that Assumption 5 ($t_{j+1} - t_j > 4\beta$) implies that, at most one of the terms f_n, f_{n+1} is non-zero in the expression of $e^{\rho\beta} F_{n+1} - F_n$ above.

Firstly, we prove that if no burst occurred in $[n\beta, (n+2)\beta]$ (i.e. $f_n = f_{n+1} = 0$), then $|\Delta_n|$ is below our chosen threshold (17). The proof is achieved via the following computations that make use of Assumptions 3 and 4:

$$\begin{aligned}
 |\Delta_n| &= |e^{\rho\beta} F_{n+1} - F_n| \\
 &= \left| e^{\rho\beta} \int_0^\beta \left\langle T(\beta - s) \eta((n+1)\beta + s), \frac{g}{\beta} \right\rangle ds - \int_0^\beta \left\langle T(\beta - s) \eta(n\beta + s), \frac{g}{\beta} \right\rangle ds + \alpha_n \right| \\
 &\leq \left| e^{\rho\beta} \int_0^\beta \left\langle T(\beta - s) \eta((n+1)\beta + s), \frac{g}{\beta} \right\rangle ds - e^{\rho\beta} \int_0^\beta \left\langle T(\beta - s) \eta(n\beta + s), \frac{g}{\beta} \right\rangle ds \right|
 \end{aligned}$$

$$\begin{aligned}
& + \left| e^{\rho\beta} \int_0^\beta \left\langle T(\beta-s)\eta(n\beta+s), \frac{g}{\beta} \right\rangle ds - \int_0^\beta \left\langle T(\beta-s)\eta(n\beta+s), \frac{g}{\beta} \right\rangle ds \right| + |\alpha_n| \\
& \leq e^{\rho\beta} \int_0^\beta \|\eta((n+1)\beta+s) - \eta(n\beta+s)\| \|T^*(\beta-s)\frac{g}{\beta}\| ds \\
& \quad + (e^{\rho\beta} - 1) \int_0^\beta \|\eta(n\beta+s)\| \|T^*(\beta-s)\frac{g}{\beta}\| ds + 4e^{\rho\beta}\sigma \\
& \leq e^{\rho\beta} CL\|g\|\beta + (e^{\rho\beta} - 1)CK\|g\| + 4e^{\rho\beta}\sigma = Q(g, \beta)
\end{aligned} \tag{19}$$

where $C = Me^{a\beta}$ so that $\|T^*(\beta-s)\| \leq C$.

Secondly, assume that the j -th burst with the shape h_j occurred in the interval $[(n+1)\beta, (n+2)\beta)$. To analyze this situation, we will look at two cases: (1) the j -th burst is detected by our algorithm; and (2) the j -th burst is not detected. For Case 1, the burst is detected if and only if $|\Delta_n| > Q(g, \beta)$ or $|\Delta_{n+1}| > Q(g, \beta)$ and we need to prove (18). For Case 2, when the j -th burst is not detected, we will show that $\langle h_j, g \rangle$ is small, i.e. (18) holds with $f_j(g) = 0$.

Case 1. The j -th burst is detected in $[(n+1)\beta, (n+2)\beta)$.

Assume that $|\Delta_n| > Q(g, \beta)$. Then $\tau_{n+1} = t_j$, $f_{n+1} = h_j$, $f_{n-1} = f_n = f_{n+2} = 0$ and Algorithm 1 returns $t_j = (n+1)\beta$ and $f_j(g) = e^{3\rho\beta}F_{n+2} - F_{n-1}$. We need to establish (18), i.e. show that $\langle h_j, g \rangle \approx e^{3\rho\beta}F_{n+2} - F_{n-1}$ for small β . We get

$$\begin{aligned}
& e^{3\rho\beta}F_{n+2} - F_{n-1} \\
& = \int_0^\beta \left\langle T(\beta-s)h_j e^{\rho(t_j-(n-1)\beta-s)}, \frac{g}{\beta} \right\rangle ds \\
& \quad + e^{3\rho\beta} \int_0^\beta \left\langle T(\beta-s)\eta((n+2)\beta+s), \frac{g}{\beta} \right\rangle ds \\
& \quad - \int_0^\beta \left\langle T(\beta-s)\eta((n-1)\beta+s), \frac{g}{\beta} \right\rangle ds + \alpha'_{n-1}
\end{aligned}$$

where $\alpha'_{n-1} = e^{3\rho\beta}v\left((n+3)\beta, \frac{g}{\beta}\right) - e^{3\rho\beta}v\left((n+2)\beta, \frac{T^*(\beta)g}{\beta}\right) - v\left(n\beta, \frac{g}{\beta}\right) + v\left((n-1)\beta, \frac{T^*(\beta)g}{\beta}\right)$. Therefore,

$$\begin{aligned}
& |f_j(g) - \langle h_j, g \rangle| \\
& = |e^{3\rho\beta}F_{n+2} - F_{n-1} - \langle h_j, g \rangle| \\
& \leq \left| \int_0^\beta \left\langle T(\beta-s)h_j e^{\rho(t_j-(n-1)\beta-s)}, \frac{g}{\beta} \right\rangle ds - \langle h_j, g \rangle \right| + 4e^{3\rho\beta}\sigma \\
& \quad + \left| e^{3\rho\beta} \int_0^\beta \left\langle T(\beta-s)\eta((n+2)\beta+s), \frac{g}{\beta} \right\rangle ds \right. \\
& \quad \left. - \int_0^\beta \left\langle T(\beta-s)\eta((n-1)\beta+s), \frac{g}{\beta} \right\rangle ds \right| \\
& \leq v_1(h_j, g, \beta) + 3e^{3\rho\beta}CL\|g\|\beta + (e^{3\rho\beta} - 1)CK\|g\| + 4e^{3\rho\beta}\sigma,
\end{aligned}$$

where $C = Me^{a\beta}$ and $v_1(h_j, g, \beta)$ is given by (12). Thus, estimate (18) is established for the case when $|\Delta_n| > Q(g, \beta)$.

Assume now that $|\Delta_n| \leq Q(g, \beta)$ and $|\Delta_{n+1}| > Q(g, \beta)$. In this case, Algorithm 1 returns $t_j = (n+2)\beta$ and $f_j(g) = e^{3\rho\beta}F_{n+3} - F_n$. In particular,

$$\begin{aligned} & |f_j(g) - \langle h_j, g \rangle| \\ & \leq \left| \int_0^\beta \left\langle T(\beta-s)h_j e^{\rho(t_j-n\beta-s)}, \frac{g}{\beta} \right\rangle ds - \langle h_j, g \rangle \right| \\ & \quad + 3e^{3\rho\beta}CL\|g\|\beta + (e^{3\rho\beta} - 1)CK\|g\| + 4e^{3\rho\beta}\sigma, \end{aligned}$$

where $|\int_0^\beta \langle T(\beta-s)h_j e^{\rho(t_j-n\beta-s)}, \frac{g}{\beta} \rangle ds - \langle h_j, g \rangle| = v_0(h_j, g, \beta)$ is given by (12). Thus, estimate (18) holds when $|\Delta_{n+1}| > Q(g, \beta)$ as well, and Case 1 is covered.

Case 2. The j -th burst is in $[(n+1)\beta, (n+2)\beta)$, but $\langle h_j, g \rangle$ is too small to be detected.

We need to show that (18) holds with $f_j(g) = 0$. We have

$$\begin{aligned} & e^{2\rho\beta}F_{n+2} - F_n \\ & = \int_0^\beta \left\langle T(\beta-s)h_j e^{\rho(t_j-n\beta-s)}, \frac{g}{\beta} \right\rangle ds + e^{2\rho\beta} \int_0^\beta \left\langle T(\beta-s)\eta((n+2)\beta+s), \frac{g}{\beta} \right\rangle ds \\ & \quad - \int_0^\beta \left\langle T(\beta-s)\eta(n\beta+s), \frac{g}{\beta} \right\rangle ds + \alpha_n'' \end{aligned} \quad (20)$$

where $\alpha_n'' = e^{2\rho\beta}v\left((n+3)\beta, \frac{g}{\beta}\right) - e^{2\rho\beta}v\left((n+2)\beta, \frac{T^*(\beta)g}{\beta}\right) - v\left((n+1)\beta, \frac{g}{\beta}\right) + v\left(n\beta, \frac{T^*(\beta)g}{\beta}\right)$. Using (20) to estimate $\langle h_j, g \rangle$, we get

$$\begin{aligned} & |\langle h_j, g \rangle| \\ & \leq \left| - \int_0^\beta \left\langle T(\beta-s)h_j e^{\rho(t_j-n\beta-s)}, \frac{g}{\beta} \right\rangle ds \right| \\ & \quad + \left| \int_0^\beta \left\langle T(\beta-s)h_j e^{\rho(t_j-n\beta-s)}, \frac{g}{\beta} \right\rangle ds - \langle h_j, g \rangle \right| \\ & \leq \left| - \int_0^\beta \left\langle T(\beta-s)h_j e^{\rho(t_j-n\beta-s)}, \frac{g}{\beta} \right\rangle ds + (e^{2\rho\beta}F_{n+2} - F_n) \right| \\ & \quad + |F_n - e^{2\rho\beta}F_{n+2}| + v_0(h_j, g, \beta) \\ & \leq \left| e^{2\rho\beta} \int_0^\beta \left\langle T(\beta-s)\eta((n+2)\beta+s), \frac{g}{\beta} \right\rangle ds \right. \\ & \quad \left. - \int_0^\beta \left\langle T(\beta-s)\eta(n\beta+s), \frac{g}{\beta} \right\rangle ds + \alpha_n'' \right| \\ & \quad + e^{\rho\beta}|e^{\rho\beta}F_{n+2} - F_{n+1}| + |e^{\rho\beta}F_{n+1} - F_n| + v_0(h_j, g, \beta) \\ & \leq 2e^{2\rho\beta}CL\|g\|\beta + (e^{2\rho\beta} - 1)CK\|g\| + 4e^{2\rho\beta}\sigma + 2e^{\rho\beta}Q(g, \beta) + v_0(h_j, g, \beta), \end{aligned}$$

where we have used the fact that $|e^{\rho\beta}F_{n+2} - F_{n+1}| \leq Q(g, \beta)$, $|e^{\rho\beta}F_{n+1} - F_n| \leq Q(g, \beta)$, and estimated the term $|e^{2\rho\beta} \int_0^\beta \left\langle T(\beta-s)\eta((n+2)\beta+s), \frac{g}{\beta} \right\rangle ds - \int_0^\beta \left\langle T(\beta-s)\eta(n\beta+s), \frac{g}{\beta} \right\rangle ds + \alpha_n''|$ in a similar way as (19). The above estimates establish (18) in Case 2, and the theorem is proved. $\square$

3.2 Model with general decay function

In this section, we consider the same dynamical system, but we discuss a more general situation. Here the decay function ϕ does not have a concrete formula, but its decay velocity is restricted. The model is as follows:

$$\begin{cases} \dot{u}(t) = Au(t) + \sum_j^N h_j \phi(t - t_j) \chi_{[t_j, \infty)}(t) + \eta, \\ u(0) = u_0, \end{cases} \quad (21)$$

where the function ϕ is continuous on $[0, \infty)$ and satisfies $\phi(0) = 1$ and

$$0 < \phi(t) \leq e^{-\rho t} \quad (22)$$

for some $\rho > 0$.

In this model, we continue to use the constants introduced in Proposition 2.2 and Assumptions 1 to 5, as well as $C = Me^{a\beta}$. We also assume that the bursts are uniformly bounded as mentioned in Assumption 2. The constant D in Assumption 5 that controls the time gap between the bursts ($t_{j+1} - t_j \geq D + 4\beta$) is chosen in a way that

$$\epsilon = \frac{2}{e^{\rho D} - 1} CHR \quad (23)$$

is a small quantity. We shall also need the following modification of the technical Lemma 3.2.

Lemma 3.4 Assume that $t_j \in [(n+1)\beta, (n+2)\beta)$ and

$$v_{k,i}(h_j, g, \beta) = \left| \int_0^\beta \left\langle T(\beta-s)h_j e^{k\rho\beta} \phi((n+i)\beta+s-t_j), \frac{g}{\beta} \right\rangle ds - \left\langle h_j, g \right\rangle \right|, \quad k, i = 2, 3. \quad (24)$$

Then

$$v_{k,i}(h_j, g, \beta) \leq \|g\| \left(\|h_j\| M \max_{s \in [(i-2)\beta, i\beta]} |e^{k\rho\beta} \phi(s) - 1| e(a\beta) + \sup_{s \in [0, \beta]} \|T(s)h_j - h_j\| \right), \quad (25)$$

where $\mathbf{e}$ is given by (13). In particular, $v_{k,i}(h_j, g, \beta) \rightarrow 0$ as $\beta \rightarrow 0$, $k, i = 2, 3$.

Proof Similarly to (14), we separate each $v_{k,i}(h_j, g, \beta)$, $k, i = 2, 3$, into two parts:

$$\begin{aligned} & \left| \int_0^\beta \left\langle T(\beta - s)h_j e^{k\rho\beta} \phi((n+i)\beta + s - t_j), \frac{g}{\beta} \right\rangle ds - \langle h_j, g \rangle \right| \\ & \leq \frac{\|g\|}{\beta} \int_0^\beta \|T(\beta - s)h_j e^{k\rho\beta} \phi((n+i)\beta + s - t_j) - T(\beta - s)h_j\| ds \\ & \quad + \frac{\|g\|}{\beta} \int_0^\beta \|T(\beta - s)h_j - h_j\| ds \\ & = I_1 + I_2. \end{aligned}$$

Estimate for I_2 is still given by (16). For I_1 , by $\|T(t)\| \leq Me^{at}$, we have

$$\begin{aligned} I_1 &= \frac{\|g\|}{\beta} \int_0^\beta \|T(\beta - s)h_j e^{k\rho\beta} \phi((n+i)\beta + s - t_j) - T(\beta - s)h_j\| ds \\ &\leq \frac{\|g\|}{\beta} \int_0^\beta \|T(\beta - s)\| \|h_j\| |e^{k\rho\beta} \phi((n+i)\beta + s - t_j) - 1| ds \\ &= \|g\| \|h_j\| M \max_{s \in [(i-2)\beta, i\beta]} |e^{k\rho\beta} \phi(s) - 1| \mathbf{e}(a\beta). \end{aligned}$$

By the assumption on ϕ , $I_1 \rightarrow 0$ as $\beta \rightarrow 0$. □

Algorithm 2 Pseudo-code for approximating the time and shape of a possible burst with varying decay

- 1: **Input:** Measurements: $m_n(\frac{g}{\beta})$, $m_n(\frac{T^*(\beta)g}{\beta})$; threshold: $Q_1(g, \beta)$, for $g \in \tilde{G}$; a parameter $D > 0$
 - 2: Compute $F_i = m_{i+1}(\frac{g}{\beta}) - m_i(\frac{T^*(\beta)g}{\beta})$
 - 3: Compute $e^{\rho\beta} F_{i+1} - F_i$
 - 4: **For** $g \in \tilde{G}$ **do**
 - 5: $i=1$
 - 6: **while** $i\beta < T$ **do**
 - 7: **if** $e^{\rho\beta} F_{i+1} - F_i > Q_1(g, \beta)$ **then**
 - 8: $f(g) := e^{3\rho\beta} F_{i+2} - F_{i-1}$
 - 9: $t := (i+1)\beta$
 - 10: $i = i + 3 + \lfloor \frac{D}{\beta} \rfloor$
 - 11: **else**
 - 12: **if** $e^{\rho\beta} F_{i+2} - F_{i+1} > Q_1(g, \beta)$ **then**
 - 13: $f(g) := e^{3\rho\beta} F_{i+3} - F_i$
 - 14: $t := (i+2)\beta$
 - 15: $i = i + 3 + \lfloor \frac{D}{\beta} \rfloor$
 - 16: **else**
 - 17: $i = i + 1$
 - 18: **Output:** t and $f(g)$ for all $g \in \tilde{G}$.
-

Theorem 3.5 Under Assumptions 1 to 5, $Q(g, \beta)$ given by (17), and ϵ —by (23), let

$$Q_1(g, \beta) = Q(g, \beta) + \epsilon \quad (26)$$

be the threshold in Algorithm 2. Let also t_j and $f_j(g)$ be the outputs of Algorithm 2. Then $|t_j - t_j| \leq \beta$ and

$$\begin{aligned} & |f_j(g) - \langle h_j, g \rangle| \\ & \leq \epsilon + 3e^{(3\rho+a)\beta} ML \|g\| \beta + e^{a\beta} (e^{3\rho\beta} - 1) MK \|g\| + 4e^{3\rho\beta} \sigma \\ & \quad + 2e^{\rho\beta} Q_1(g, \beta) + \max\{v_{3,2}(h_j, g, \beta), v_{3,3}(h_j, g, \beta), v_{2,2}(h_j, g, \beta)\} \end{aligned} \quad (27)$$

where $v_{k,i}$, $k, i = 2, 3$, are defined by (24) so that $v_{k,i}(h_j, g, \beta) \rightarrow 0$ as $\beta \rightarrow 0$ (by Lemma 3.4). In particular, for sufficiently small $\beta > 0$, one has $|f_j(g) - \langle h_j, g \rangle| \leq 13\sigma + 4\epsilon$ as long as σ and ϵ are not both 0.

Proof Suppose that we have detected the $(j-1)$ -th burst in the time interval $[m\beta, (m+2)\beta)$ for some $m \in \mathbb{N}$. By Assumption 5, the next nonzero burst h_j must happen no sooner than $m\beta + D + 4\beta$, thus we just need to continue our detection from $(m+3 + \lfloor \frac{D}{\beta} \rfloor)\beta$. Now we simply denote $(m+3 + \lfloor \frac{D}{\beta} \rfloor)\beta$ by n and analyze the occurrence of a burst in the interval $[n\beta, (n+2)\beta)$. To do that, we first evaluate the quantities F_n and $\Delta_n = e^{\rho\beta} F_{n+1} - F_n$ from the measurements (7):

$$\begin{aligned} F_n &= m_{n+1} \left(\frac{g}{\beta} \right) - m_n \left(\frac{T^*(\beta)g}{\beta} \right) \\ &= \int_{\tau_n}^{(n+1)\beta} \left\langle T((n+1)\beta - s) f_n \phi(s - \tau_n), \frac{g}{\beta} \right\rangle ds \\ &\quad + \sum_{\tau_i < n\beta} \int_0^\beta \left\langle T(\beta - s) f_i \phi(n\beta + s - \tau_i), \frac{g}{\beta} \right\rangle ds \\ &\quad + \int_0^\beta \left\langle T(\beta - s) \eta(n\beta + s), \frac{g}{\beta} \right\rangle ds + v \left((n+1)\beta, \frac{g}{\beta} \right) - v \left(n\beta, \frac{T^*(\beta)g}{\beta} \right), \end{aligned}$$

and

$$\begin{aligned} \Delta_n &= e^{\rho\beta} F_{n+1} - F_n \\ &= e^{\rho\beta} \int_{\tau_{n+1}}^{(n+2)\beta} \left\langle T((n+2)\beta - s) f_{n+1} \phi(s - \tau_{n+1}), \frac{g}{\beta} \right\rangle ds \\ &\quad + \int_0^\beta \left\langle T(\beta - s) f_n e^{\rho\beta} \phi((n+1)\beta + s - \tau_n), \frac{g}{\beta} \right\rangle ds \\ &\quad - \int_{\tau_n}^{(n+1)\beta} \left\langle T((n+1)\beta - s) f_n \phi(s - \tau_n), \frac{g}{\beta} \right\rangle ds \\ &\quad + \sum_{\tau_i < n\beta} \int_0^\beta \left\langle T(\beta - s) f_i (e^{\rho\beta} \phi((n+1)\beta + s - \tau_i) - \phi(n\beta + s - \tau_i)), \frac{g}{\beta} \right\rangle ds \end{aligned}$$

$$+e^{\rho\beta} \int_0^\beta \left\langle T(\beta-s)\eta((n+1)\beta+s), \frac{g}{\beta} \right\rangle ds - \int_0^\beta \left\langle T(\beta-s)\eta(n\beta+s), \frac{g}{\beta} \right\rangle ds + \alpha_n$$

where $\alpha_n = e^{\rho\beta} v\left((n+2)\beta, \frac{g}{\beta}\right) - e^{\rho\beta} v\left((n+1)\beta, \frac{T^*(\beta)g}{\beta}\right) - v\left((n+1)\beta, \frac{g}{\beta}\right) + v\left(n\beta, \frac{T^*(\beta)g}{\beta}\right)$. From the expression above, we note that since we don't have a concrete formula for $\phi(t)$, we are unable to use the technique in Sect. 3.1 to cancel the effect of the bursts that occurred prior to $n\beta$. However, by Assumption 5, the requirement that the distance $|t_{j+1} - t_j|$ between two bursts is large enough, ensures that if no burst occurred in $[n\beta, (n+2)\beta]$ (i.e. $f_n = f_{n+1} = 0$), then $|e^{\rho\beta} F_{n+1} - F_n|$ is below our chosen threshold (26). We will show that via the calculations below, where we use (19), (22) and Assumption 5.

$$\begin{aligned} |\Delta_n| &= |e^{\rho\beta} F_{n+1} - F_n| \\ &\leq \sum_{\tau_i < n\beta} \left| \int_0^\beta \left\langle T(\beta-s) f_i(e^{\rho\beta} \phi((n+1)\beta+s-\tau_i) - \phi(n\beta+s-\tau_i)), \frac{g}{\beta} \right\rangle ds \right| \\ &\quad + \left| \int_0^\beta \left\langle T(\beta-s)(e^{\rho\beta} \eta((n+1)\beta+s) - \eta(n\beta+s)), \frac{g}{\beta} \right\rangle ds \right| + |\alpha_n| \\ &\leq \sum_{\tau_i < n\beta} \int_0^\beta |e^{\rho\beta} \phi((n+1)\beta+s-\tau_i) - \phi(n\beta+s-\tau_i)| \|f_i\| \left\| T^*(\beta-s) \frac{g}{\beta} \right\| ds \\ &\quad + e^{\rho\beta} CL \|g\| \beta + (e^{\rho\beta} - 1) CK \|g\| + 4e^{\rho\beta} \sigma \\ &\leq \sum_{\tau_i < n\beta} 2e^{-\rho(n\beta-\tau_i)} C \|f_i\| \|g\| + w(C, L, g, \beta, K, \sigma) \\ &= \sum_{k=1}^{j-1} 2e^{-\rho(n\beta-t_k)} C \|h_k\| \|g\| + w(C, L, g, \beta, K, \sigma) \\ &\leq \sum_{k=1}^{\infty} 2e^{-k\rho D} CHR + w(C, L, g, \beta, K, \sigma) \\ &\leq \frac{2}{e^{\rho D} - 1} CHR + w(C, L, g, \beta, K, \sigma) \\ &\leq \epsilon + w(C, L, g, \beta, K, \sigma) = Q_1(g, \beta) \end{aligned} \quad (28)$$

where $w(C, L, g, \beta, K, \sigma) = e^{\rho\beta} CL \|g\| \beta + (e^{\rho\beta} - 1) CK \|g\| + 4e^{\rho\beta} \sigma$. Recall that in the above calculation $C = Me^{a\beta}$, $\epsilon = \frac{2}{e^{\rho D} - 1} CHR$ as defined by (23), H is the upper bound constant in Assumption 2, L, K are the Lipschitz constant and the background source upper bound, respectively, in Assumption 3, and $R = \sup_{g \in \tilde{G}} \|g\|$ as in Assumption 1. $\square$

Remark 3.6 In this case, the time difference D between every pair of adjacent non-zero bursts will influence the error estimate. When $\epsilon < \sigma$, the past bursts only have a very weak impact on the subsequent bursts and their influence together is even smaller than the noise level σ .

We now assume that the j -th burst occurred in the interval $[(n+1)\beta, (n+2)\beta)$. Similarly to our discussion in Sect. 3.1, we consider two cases: (1) the burst is detected; and (2) the burst is not detected. As before, for Case 1, the burst is detected if and only if $|\Delta_n| > Q_1(g, \beta)$ or $|\Delta_{n+1}| > Q_1(g, \beta)$. We need to establish (27) for each of the cases (assuming $f_j(g) = 0$ in Case 2).

Case 1. The j -th burst is detected in $[(n+1)\beta, (n+2)\beta)$.

Assume that $|\Delta_n| > Q_1(g, \beta)$. Then $\tau_{n+1} = t_j$, $f_{n+1} = h_j$, $f_{n-1} = f_n = f_{n+2} = 0$ and Algorithm 2 returns $t_j = (n+1)\beta$ and $f_j(g) = e^{3\rho\beta} F_{n+2} - F_{n-1}$. We get

$$\begin{aligned} & e^{3\rho\beta} F_{n+2} - F_{n-1} \\ &= \int_0^\beta \left\langle T(\beta-s) h_j e^{3\rho\beta} \phi((n+2)\beta + s - t_j), \frac{g}{\beta} \right\rangle ds \\ &+ \sum_{\tau_i < (n-1)\beta} \int_0^\beta \left\langle T(\beta-s) f_i(e^{3\rho\beta} \phi((n+2)\beta + s - \tau_i) - \phi((n-1)\beta + s - \tau_i)), \frac{g}{\beta} \right\rangle ds \\ &+ e^{3\rho\beta} \int_0^\beta \left\langle T(\beta-s) \eta((n+2)\beta + s), \frac{g}{\beta} \right\rangle ds \\ &- \int_0^\beta \left\langle T(\beta-s) \eta((n-1)\beta + s), \frac{g}{\beta} \right\rangle ds + \alpha'_{n-1} \end{aligned}$$

where $\alpha'_{n-1} = e^{3\rho\beta} v\left((n+3)\beta, \frac{g}{\beta}\right) - e^{3\rho\beta} v\left((n+2)\beta, \frac{T^*(\beta)g}{\beta}\right) - v\left(n\beta, \frac{g}{\beta}\right) + v\left((n-1)\beta, \frac{T^*(\beta)g}{\beta}\right)$.

Computing the error gives

$$\begin{aligned} & |f_j(g) - \langle h_j, g \rangle| \\ &= |e^{3\rho\beta} F_{n+2} - F_{n-1} - \langle h_j, g \rangle| \\ &\leq \left| \int_0^\beta \left\langle T(\beta-s) h_j e^{3\rho\beta} \phi((n+2)\beta + s - t_j), \frac{g}{\beta} \right\rangle ds - \langle h_j, g \rangle \right| + |\alpha'_{n-1}| \\ &+ \left| \sum_{\tau_i < (n-1)\beta} \int_0^\beta \left\langle T(\beta-s) f_i(e^{3\rho\beta} \phi((n+2)\beta + s - \tau_i) - \phi((n-1)\beta + s - \tau_i)), \frac{g}{\beta} \right\rangle ds \right| \\ &+ \left| \int_0^\beta \left\langle T(\beta-s) e^{3\rho\beta} \eta((n+2)\beta + s), \frac{g}{\beta} \right\rangle ds - \int_0^\beta \left\langle T(\beta-s) \eta((n-1)\beta + s), \frac{g}{\beta} \right\rangle ds \right| \\ &\leq v_{3,2}(h_j, g, \beta) + \epsilon + 3e^{3\rho\beta} CL \|g\| \beta + (e^{3\rho\beta} - 1)CK \|g\| + 4e^{3\rho\beta} \sigma, \end{aligned} \quad (29)$$

where $v_{3,2}$ is given by (24) and we estimated the last two terms of the first inequality similarly to (28).

Now assume that $|\Delta_n| \leq Q_1(g, \beta)$ and $|\Delta_{n+1}| > Q_1(g, \beta)$. Then Algorithm 2 returns $t_j = (n+2)\beta$ and $f_j(g) = e^{3\rho\beta} F_{n+3} - F_n$. We then have

$$\begin{aligned}
& |\langle f_j(g) - \langle h_j, g \rangle| \\
& \leq \left| \int_0^\beta \left\langle T(\beta - s) h_j e^{3\rho\beta} \phi((n+3)\beta + s - t_j), \frac{g}{\beta} \right\rangle ds - \langle h_j, g \rangle \right| \\
& \quad + \epsilon + 3e^{3\rho\beta} CL \|g\| \beta + (e^{3\rho\beta} - 1)CK \|g\| + 4e^{3\rho\beta} \sigma,
\end{aligned}$$

where $|\int_0^\beta \langle T(\beta - s) h_j e^{3\rho\beta} \phi((n+3)\beta + s - t_j), \frac{g}{\beta} \rangle ds - \langle h_j, g \rangle| = v_{3,3}(h_j, g, \beta)$ is given by (24).

Case 2. The j -th burst is in $[(n+1)\beta, (n+2)\beta)$, but $\langle h_j, g \rangle$ it is not detected.

If the j -th burst occurred in $[(n+1)\beta, (n+2)\beta)$ but was not detected by Algorithm 2, we use the fact that $e^{2\rho\beta} F_{n+2} - F_n$ is small to show that $\langle h_j, g \rangle \approx 0$.

$$\begin{aligned}
& e^{2\rho\beta} F_{n+2} - F_n \\
& = \int_0^\beta \left\langle T(\beta - s) h_j e^{2\rho\beta} \phi((n+2)\beta + s - t_j), \frac{g}{\beta} \right\rangle ds \\
& \quad + \sum_{\tau_i < n\beta} \int_0^\beta \left\langle T(\beta - s) f_i(e^{2\rho\beta} \phi((n+2)\beta + s - \tau_i) - \phi(n\beta + s - \tau_i)), \frac{g}{\beta} \right\rangle ds \\
& \quad + e^{2\rho\beta} \int_0^\beta \left\langle T(\beta - s) \eta((n+2)\beta + s), \frac{g}{\beta} \right\rangle ds - \int_0^\beta \left\langle T(\beta - s) \eta(n\beta + s), \frac{g}{\beta} \right\rangle ds + \alpha_n''
\end{aligned}$$

where $\alpha_n'' = e^{2\rho\beta} v\left((n+3)\beta, \frac{g}{\beta}\right) - e^{2\rho\beta} v\left((n+2)\beta, \frac{T^*(\beta)g}{\beta}\right) - v\left((n+1)\beta, \frac{g}{\beta}\right) + v\left(n\beta, \frac{T^*(\beta)g}{\beta}\right)$. Now we estimate $|\langle h_j, g \rangle|$:

$$\begin{aligned}
& |\langle h_j, g \rangle| \\
& \leq \left| - \int_0^\beta \left\langle T(\beta - s) h_j e^{2\rho\beta} \phi((n+2)\beta + s - t_j), \frac{g}{\beta} \right\rangle ds \right| \\
& \quad + \left| \int_0^\beta \left\langle T(\beta - s) h_j e^{2\rho\beta} \phi((n+2)\beta + s - t_j), \frac{g}{\beta} \right\rangle ds - \langle h_j, g \rangle \right| \\
& \leq \left| - \int_0^\beta \left\langle T(\beta - s) h_j e^{2\rho\beta} \phi((n+2)\beta + s - t_j), \frac{g}{\beta} \right\rangle ds + (e^{2\rho\beta} F_{n+2} - F_n) \right| \\
& \quad + |F_n - e^{2\rho\beta} F_{n+2}| + v_{2,2}(h_j, g, \beta) \\
& \leq \epsilon + 2e^{2\rho\beta} CL \|g\| \beta + (e^{2\rho\beta} - 1)CK \|g\| \\
& \quad + 4e^{2\rho\beta} \sigma + 2e^{2\rho\beta} Q_1(g, \beta) + v_{2,2}(h_j, g, \beta)
\end{aligned} \tag{30}$$

where we have used $|e^{\rho\beta} F_{n+2} - F_{n+1}| \leq Q_1(g, \beta)$, $|e^{\rho\beta} F_{n+1} - F_n| \leq Q_1(g, \beta)$, and estimated the term $\left| - \int_0^\beta \left\langle T(\beta - s) h_j e^{2\rho\beta} \phi((n+2)\beta + s - t_j), \frac{g}{\beta} \right\rangle ds + (e^{2\rho\beta} F_{n+2} - F_n) \right|$ as in (28). The theorem is proved. $\square$

4 Simulation

In order to evaluate the performance of our algorithms, we apply them to the following specific IVP:

$$\begin{cases} \dot{u}(t) = u(t) + \sum_{i=1} h_i e^{-\rho(t-t_i)} \chi_{[t, \infty)}(t) + \eta \\ u(0) = 0 \end{cases}$$

with $h_1(x) = 3 \sin(x)$, $h_2(x) = 2.5 \cos(x)$, $h_3(x) = x + 2$, $x \in [0, 1]$, $t_1 = 0.25$, $t_2 = 0.54$, $t_3 = 0.78$, $t \in [0, 1]$, and one of the two different types of background sources: $\eta = xe^{-Lt}$ or $\eta = x \sin(Lt)$.

Let $g_1(x) = 1$, $g_2(x) = x$, and $g_3(x) = x^2$ be the sensor functions and compute the ground truth $\langle h_i, g_j \rangle$ for $i, j = 1, 2, 3$. In the simulation, we let $\rho = 1$, $L = 10^{-2}$ and the noise level $\sigma = 10^{-3}$. The goal is to find the burst times $\{0.25, 0.76, 1.1\}$ and compare the output $f_i(g_j)$ with the ground truth $\langle h_i, g_j \rangle$ ($i, j = 1, 2, 3$) for different time steps $\beta = 0.015$ and $\beta = 0.01$, respectively. We acquire the measurements (7) and use the algorithm in Sect. 3.1. The results are shown in Fig. 2.

To test the algorithms for the model in Sect. 3.2, we use the same burst and sensor functions. We also test on the same background sources and let $L = 10^{-2}$. But here

Fig. 2 Plot for the bursts: model with $\phi(t) = e^{-t}$, $L = 10^{-2}$ and $\sigma = 10^{-3}$. The results for h_i lie in the i -th column. Red circles stand for the ground truth $\langle h_i, g_j \rangle$, black squares stand for the output $f_i(g_j)$ when $\beta = 0.015$ and blue triangles stand for the output $f_i(g_j)$ when $\beta = 0.01$.

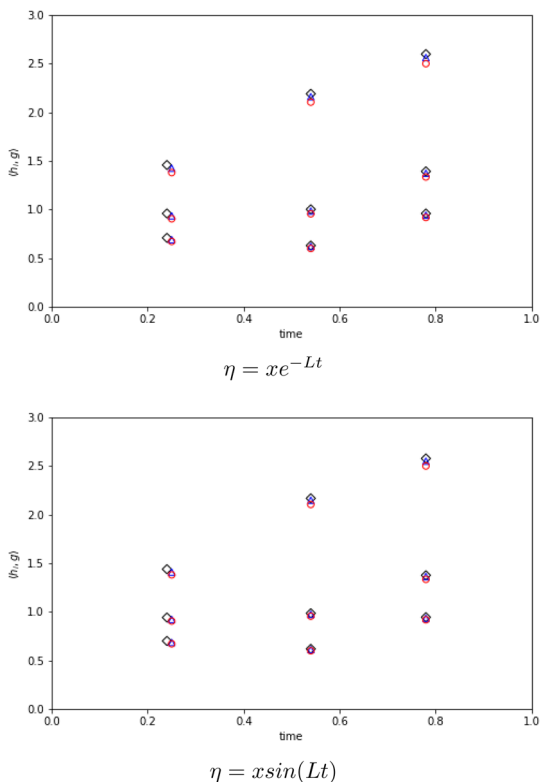
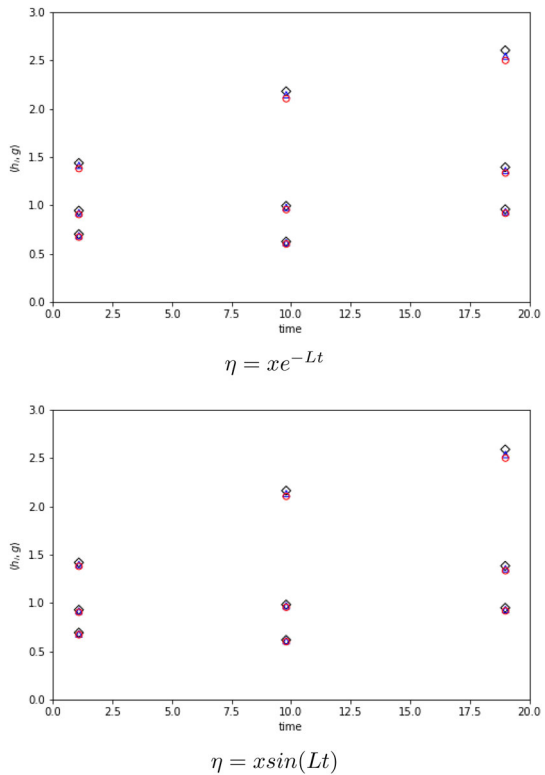


Fig. 3 Plot for the bursts: model with $\phi(t) = \frac{1}{2}(e^{-2t} + e^{-t})$, $L = 10^{-2}$ and $\sigma = 10^{-3}$. The results for h_i lie in the i -th column. Red circles stand for the ground truth $\langle h_i, g_j \rangle$, black squares stand for the output $\hat{f}_i(g_j)$ when $\beta = 0.015$ and blue triangles stand for the output $\hat{f}_i(g_j)$ when $\beta = 0.01$



we let $\phi(t) = \frac{1}{2}(e^{-2t} + e^{-t})$, thus $0 < \phi(t) \leq e^{-t}$. For other parameters, we let $t_1 = 1.1$, $t_2 = 9.8$, $t_3 = 19$, $D = 8.6$ and $\sigma = 10^{-3}$ ($\epsilon < \sigma$). The goal is still to find out the bursts and compare the output with the ground truth for $\beta = 0.015$ and $\beta = 0.01$, respectively. We utilize the algorithm in Sect. 3.2 and the results are shown in Fig. 3.

In Figs. 2 and 3, we plot estimates and ground truth in the same figure. The test shows that our algorithms can find out all bursts and the error gets smaller when the time step β gets shorter. To gain deeper insight into the impact of parameters β , L and σ on our algorithm, we conducted simulations on the model with $\phi(t) = e^{-t}$ where we varied one parameter and fixed others. In our simulation, we evaluated the accuracy of the estimates of $\langle h_i, g_2 \rangle$ by calculating the relative error:

$$\frac{\sqrt{\sum_{i=1}^3 \left| \langle h_i, g_2 \rangle - \hat{f}_i(g_2) \right|^2}}{\sqrt{\sum_{i=1}^3 \left| \langle h_i, g_2 \rangle \right|^2}}.$$

In Fig. 4, we plot the relation between the errors on $\langle h_i, g_2 \rangle$ and the sampling time step β by fixing the Lipschitz constant of the background source $L = 0.01$ and the

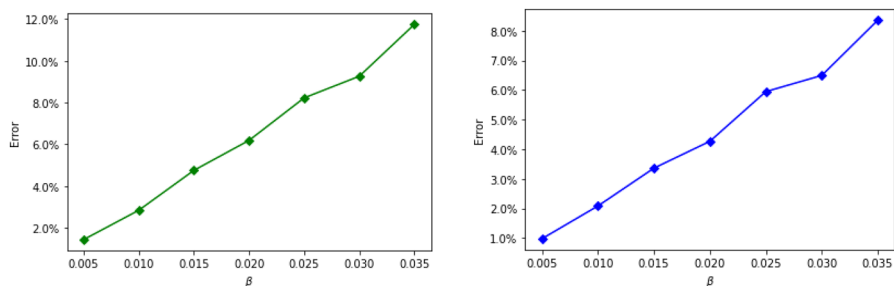


Fig. 4 The error estimate of $\langle h_i, g_2 \rangle$ vs. β : $L = 10^{-2}$, $\sigma = 10^{-3}$. The background sources are $\eta(x, t) = xe^{-Lt}$ and $\eta(x, t) = x \sin(Lt)$ for the first and second columns, respectively

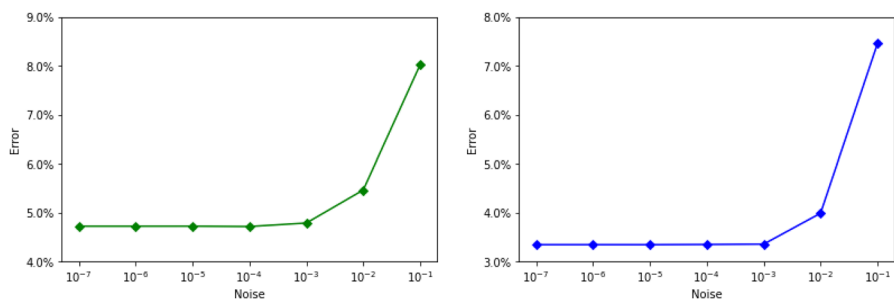


Fig. 5 The error estimate of $\langle h_i, g_2 \rangle$ vs. Noise: $L = 10^{-2}$, $\beta = 0.015$. The background sources are $\eta(x, t) = xe^{-Lt}$ and $\eta(x, t) = x \sin(Lt)$ for the first and second columns, respectively

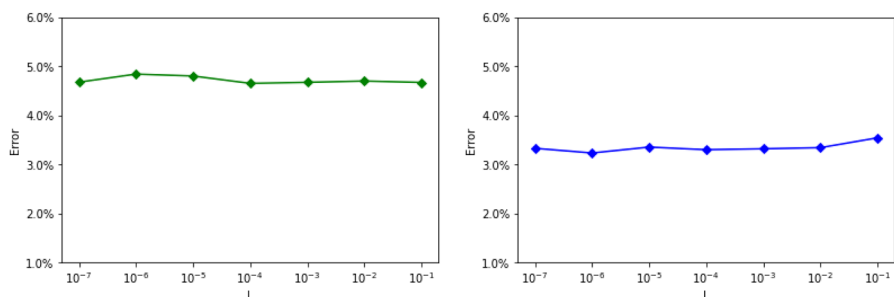


Fig. 6 The error estimate of $\langle h_i, g_2 \rangle$ vs. L : $\beta = 0.015$, $\sigma = 10^{-3}$. The background sources are $\eta(x, t) = xe^{-Lt}$ and $\eta(x, t) = x \sin(Lt)$ for the first and second columns, respectively

noise level $\sigma = 10^{-3}$. The results indicate that the error is very low when β is sufficiently small, which demonstrates the accuracy of our algorithm. Additionally, the error appears to grow linearly as β gets bigger.

In Fig. 5, we plot the relation between the errors and the noise level σ by fixing the time step β and the Lipschitz constant L . We notice that in this case the noise level has little influence on the error when it is less than 10^{-3} . When the standard deviation of the additive noise is $\sigma = 10^{-1}$, it constitutes roughly 10% of the signal values. Consequently, the error is primarily determined by the additive noise rather

than the time step β , and the Lipschitz constant L of the underlying background η . This phenomenon accounts for the sudden spikes in the relative recovery error.

In Fig. 6, we vary the Lipschitz constant L and fix β and σ . The test shows that for this IVP the error is almost independent of the variance of the Lipschitz constant when $L < 10^{-1}$.

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Data availability The data that support the findings of this study are available from the corresponding author, Le Gong, upon reasonable request.

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